2D Stellar convection

Candidate number: 24
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To better understand and simulate stellar convection, this project aims to utilize hydrodynamics which consists of a few equations describing complex behaviour. These equations will be solved numerically with explicit methods such as forwards time centred space scheme and upwind differencing scheme where initial conditions are given from the photosphere. Due to a numerical bug, results have not been produced. This paper includes discussions around the model, trouble which occurred and troubleshooting.

I. INTRODUCTION

In an attempt to better understand how stellar convection works, hydrodynamics must be understood as well. Hydrodynamics is the science of fluids, i.e. the stellar motion will be described through the motion of fluid in this project. Hydrodynamic equations describe the internal solar motion through a set of variables. This project aims to produce a cross-section of the solar interior as a rectangular box, with x along its horizontal direction and y along its vertical. The purpose is to model two-dimension stellar convection within this rectangular box.

II. THEORY

A. 2D hydrodynamic equations

The hydrodynamic equations consist of a few equations which describe a complex behaviour. The continuity equation with no sources or sinks is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

where $\mathbf{u}$ is 2D velocity defined as $\mathbf{u} = \{u_x, u_y\}$ and ρ is density. The momentum equation for both x- and y-direction, excluding the viscous stress tensor but including gravity is

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla P + \rho \mathbf{g} \tag{2}$$

where P is pressure and $\mathbf{g}$ is gravitational acceleration. The energy equation is

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{u}) = -P \nabla \cdot \mathbf{u} \tag{3}$$

where e is the internal energy (units of energy per volume).

B. Model

The solar interior will be modelled in a rectangular box where the y-direction encloses 4 Mm with the upper boundary at the solar surface and the lower boundary inside the Sun. The x-axis encloses 12 Mm. Considering that a star is approximately a sphere, the vertical direction is also referred to as the radial direction.
The grid of this rectangular box is defined with 300 boxes in the x-direction and 100 boxes in the y-direction.

We assume the gravity in the box is constant so that

$$|\mathbf{g}| = GM_\odot / R_\odot^2 \tag{4}$$

where G is the Newtonian gravitational constant, $M_\odot$ is the solar mass, and $R_\odot$ is the solar radius. Additionally, we assume an ideal gas with mean molecular weight $\mu = 0.61$ and the equation of state is given by

$$P = (\gamma - 1)e \tag{5}$$

Internal energy is defined as

$$e = \frac{1}{(\gamma - 1)} n k_b T = \frac{1}{(\gamma - 1)} \frac{\rho}{\mu m_u} k_B T \tag{6}$$

where m_u is the atomic mass unit, μ is the mean molecular weight, k_B is Boltzmann's constant, and T is temperature. $\gamma = c_P/c_V$ is the heat capacity ratio and has the value 5/3 for an ideal gas [2].

C. Initial conditions

Initial conditions for temperature, pressure, density, and energy follow two requirements

- The gas needs to be in hydrostatic equilibrium
- The double logarithmic gradient is defined as

  $$\nabla = \frac{\partial \ln T}{\partial \ln P} \tag{7}$$

  and must have a slightly higher value than 2/5. The temperature and pressure in the box can be computed for the first time step from the temperature gradient and values at the top of the box ("Solar photosphere" in appendix B of ref. [1]), given the requirements.

From the initial conditions, we can integrate from the top down to calculate values for the entire box using the governing equations for the internal structure

$$\begin{aligned} dM &= 4\pi R_\odot^2 \rho \\ dP &= -g\rho \\ dT &= \nabla \frac{T}{P} dP \end{aligned} \tag{8}$$

D. Boundary conditions

A two-dimensional system has 4 boundary conditions: 2 vertical (top and bottom of the box) and 2 horizontal (left and right). The horizontal boundary conditions will be implemented periodically, but the vertical is computed using approximating schemes. The conditions for the vertical boundaries conditions are the following:

- Vertical velocity is zero at both the upper and lower boundary.
- Vertical gradient of the horizontal component of the velocity is zero at the boundary.
- For density and energy the hydrostatic equilibrium must be fulfilled, meaning the pressure gradient is given

$$\nabla P = -g\rho = -\frac{GM_\odot \rho}{R_\odot^2} \tag{9}$$

E. 2D Gaussian perturbation

From hydrostatic equilibrium, the gas can be provoked to create convective instability. This is done by implementing a 2D Gaussian perturbation in the initial temperature. The equation for a Gaussian distribution is

$$y = A \cdot \exp\left[\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 \cdot \frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] \tag{10}$$

for data with a mixture of values sampled from two Gaussian distributions [3]. A is the amplitude of the distribution, μ is the x-value at the centre of the distributions (mean) and σ is the measured width of the distribution (standard deviation).

F. Discretisation

The hydrodynamic equations must be solved in a specific way to obtain stability in numerical solutions. This can be done by discretising the partial derivatives using finite difference methods. These methods are explicit, i.e. parameters in the present time are used to calculate the parameters at the next time step. We define an arbitrary two-dimensional variable ρ

$$\rho_{i,j}^n \equiv \rho(x_i, y_j, t_n) \tag{11}$$

where subscripts i, j are indices of spatial coordinates and n is the time step.

1. *Forward Time Centred Space (FTCS)*

This method can be divided into forwards differencing (Forward Time in FTCS) and central differencing (centred space in FTCS). By forward differencing we are taking the difference with the next instance such as time or position. Central differencing takes the difference between neighbouring cells.
The forward differencing for an arbitrary two-dimensional variable ρ advanced in time looks like

$$\rho_{i,j}^{n+1} \approx \frac{\rho_{i,j}^{n+1} - \rho_{i,j}^n}{\Delta t} \tag{12}$$

The central differencing for an arbitrary two-dimensional variable ρ in the x-direction has the form

$$\left[\frac{\partial \rho}{\partial x}\right]_{i,j}^n \approx \frac{\rho_{i+1,j}^n - \rho_{i-1,j}^n}{2\Delta x} \tag{13}$$

and in y-direction

$$\left[\frac{\partial \rho}{\partial y}\right]_{i,j}^n \approx \frac{\rho_{i,j+1}^n - \rho_{i,j-1}^n}{2\Delta y} \tag{14}$$

2. *Upwind differencing*

The upwind differencing is used to model the transport properties, i.e. velocity dependent. This is used to avoid numerical errors, which FTCS could implement. The upwind considers the direction of the flow and takes the difference from the previous instance. We define positive x-direction to the right and positive y-direction upwards. For a flow in the x-direction, i.e. $u_{i,j}^n$, the scheme looks like

$$\left[\frac{\partial \rho}{\partial x}\right]_{i,j}^n \approx \begin{cases} \frac{\rho_{i,j}^n - \rho_{i-1,j}^n}{\Delta x}, & u_{i,j}^n \geq 0 \\ \frac{\rho_{i+1,j}^n - \rho_{i,j}^n}{\Delta x}, & u_{i,j}^n < 0 \end{cases} \tag{15}$$

and in the y-direction where velocity is $w_{i,j}^n$ we have

$$\left[\frac{\partial \rho}{\partial y}\right]_{i,j}^n \approx \begin{cases} \frac{\rho_{i,j}^n - \rho_{i,j-1}^n}{\Delta y}, & w_{i,j}^n \geq 0 \\ \frac{\rho_{i,j+1}^n - \rho_{i,j}^n}{\Delta y}, & w_{i,j}^n < 0 \end{cases} \tag{16}$$

III. METHOD

A. Discretisation

The discretisation of the continuity and momentum equation for the horizontal velocity can be found in the project description [4]. Appendix B 1 describes the discretisation of the momentum equation for the vertical velocity, and Appendix B 2 describes the energy equation.

B. Initial conditions

Given the double gradient in eq. 7, and the values on top of the box we can solve the temperature and pressure of the box for the first time step.
To solve for pressure, we express the double gradient in terms of P

$$\nabla = \frac{\partial \ln T}{\partial \ln P} \tag{17}$$
$$\nabla \partial(\ln P) = \partial(\ln T) \tag{18}$$
$$\nabla \ln \frac{P}{P_0} = \ln \frac{T}{T_0} \tag{19}$$
$$P^0_{i,j} = P_0 \left(\frac{T}{T_0}\right)^{1/\nabla} \tag{20}$$

where T_0 and P_0 are the temperature and pressure on top of the box.
For the temperature, we do the same procedure, but solving for T

$$\nabla = \frac{\partial \ln T}{\partial \ln P} = \frac{P}{T}\frac{\partial T}{\partial P}\bigg| \cdot \frac{\partial y}{\partial y} \tag{21}$$
$$= \frac{P}{T}\frac{\partial T}{\partial y}\frac{\partial y}{\partial P} \tag{22}$$
$$\tag{23}$$

For $\frac{\partial P}{\partial y}$ we insert pressure gradient given hydrostatic equilibrium in eq. 9

$$\nabla = -\frac{P}{T}\frac{1}{\rho|g_y|}\frac{\partial T}{\partial y} \tag{24}$$
$$\tag{25}$$

P can be expressed as $P = (\rho k_B T)/(\mu m_u)$

$$\nabla = -\frac{\cancel{\rho} k_B \cancel{T}}{\cancel{T}\cancel{\rho}|g_y|}\frac{\partial T}{\partial y} = \frac{k_B}{\mu m_u |g_y|}\frac{\partial T}{\partial y} \tag{26}$$
$$\partial T = -\frac{\mu m_u |g_y|}{k_B}\nabla \partial y \tag{27}$$
$$\int_T^{T_0} dT = -\frac{\mu m_u |g_y| \nabla}{k_B}\int_y^{y_0} dy \tag{28}$$
$$\tag{29}$$

Integrating from the bottom of the box and upwards yields the following expression

$$T^0_{i,j} = T_0 - \frac{\mu m_u |g_y| \nabla (y - y_0)}{k_B} \tag{30}$$

Initial energy and density are calculated from temperature and pressure

$$\begin{aligned} e^0_{i,j} &= \frac{P_0}{(\gamma - 1)} \\ \rho^0_{i,j} &= \frac{2}{3}\frac{e^0_{i,j}\mu m_u}{k_B T_0} \end{aligned} \tag{31}$$

C. Boundary conditions

Horizontal boundaries are implemented as periodic boundary conditions such that

$$\phi^n_{-1,j} = \phi^n_{Nx-1,j} \tag{32}$$
$$\phi_{Nx,j} = \phi^n_{0,j} \tag{33}$$
$$\tag{34}$$

where ϕ is some function. The vertical boundaries are found using 3-point forward/backward difference approximations. The 3-point forward difference approximation is defined as

$$\left[\frac{\partial \phi}{\partial y}\right]^n_{i,j} = \frac{-\phi^n_{i,j+2} + 4\phi^n_{i,j+1} - 3\phi^n_{i,j}}{2\Delta y} \tag{35}$$

and the 3-point backward difference approximation

$$\left[\frac{\partial \phi}{\partial y}\right]^n_{i,j} = \frac{-\phi^n_{i,j} + 4\phi^n_{i,j-1} - 3\phi^n_{i,j-2}}{2\Delta y} \tag{36}$$

1. Vertical boundary: Velocity

From the conditions given in the section III C, the vertical velocity is set to zero like so

$$w_{i,0} = w_{i,Ny-1} = 0 \tag{37}$$

For the horizontal velocity, we set the horizontal component of the vertical velocity to zero using the 3-point forward difference approximation (eq. 35) and solve for $u^n_{i,j}$. The boundary condition for $u^n_{i,0}$ is

$$\left[\frac{\partial \phi}{\partial y}\right]^n_{i,j} = \frac{-\phi^n_{i,j+2} + 4\phi^n_{i,j+1} - 3\phi^n_{i,j}}{2\Delta y} \tag{38}$$
$$\Rightarrow 0 = \frac{-u^n_{i,j+2} + 4u^n_{i,j+1} - 3u^n_{i,j}}{2\Delta y} \tag{39}$$
$$u^n_{i,j} = \frac{-u^n_{i,j+2} + 4u^n_{i,j+1}}{3} \tag{40}$$
$$u^n_{i,0} = \frac{-u^n_{i,2} + 4u^n_{i,1}}{3} \tag{41}$$
$$\tag{42}$$

and similarly for the $u^n_{i,Ny-1}$ using 3-point backward difference approximation given in eq. 36

$$u^n_{i,j} = \frac{4u^n_{i,j-1} - u^n_{i,j-2}}{3} \tag{43}$$
$$u^n_{i,N_y-1} = \frac{4u^n_{i,N_y-2} - u^n_{i,N_y-3}}{3} \tag{44}$$
$$\tag{45}$$

2. Vertical boundary: Energy

The boundary condition for internal energy gradient $\frac{\partial e}{\partial y}$ is found using hydrostatic equilibrium. From eq. 3, we find the gradient as

$$\frac{\partial e}{\partial y} = \frac{1}{(\gamma - 1)} \frac{\partial P}{\partial y} \tag{46}$$

From hydrostatic equilibrium and equation of state for an ideal gas, we can find an expression for $\frac{\partial P}{\partial y}$

$$\nabla P = -g\rho \tag{47}$$

$$\frac{\partial P}{\partial y} = -g \frac{\mu m_u}{k_B T} P \tag{48}$$

$$\tag{49}$$

Inserting the expression for $\frac{\partial P}{\partial y}$ into $\frac{\partial e}{\partial y}$ gives

$$\frac{\partial e}{\partial y} = -g \frac{1}{(\gamma - 1)} \frac{\mu m_u}{k_B T} \underbrace{P}_{eq.3} \tag{50}$$

$$= -g \frac{1}{(\gamma - 1)} \frac{\mu m_u}{k_B T} (\gamma - 1) e \tag{51}$$

$$\left[\frac{\partial e}{\partial y}\right]_{i,j}^{n} = -g \frac{\mu m_u e}{k_B T} \tag{52}$$

$$\tag{53}$$

Now using 3-point forward difference approximation for $e^n_{i,0}$ we get

$$e^n_{i,0} = \frac{-e^n_{i,2} + 4^n_{i,1}}{3} \cdot \left(1 - \frac{2\Delta y C}{3T^n_{i,0}}\right)^{-1} \tag{54}$$

$$= \frac{-e^n_{i,2} - 4e^n_{i,1}}{3 - 2\Delta y C / T^n_{i,0}} \tag{55}$$

where C is defined for convenience as $C = -\mu m_u g$. For e^n_{i,N_y-1} we have

$$e^n_{i,N_y-1} = \frac{-e^n_{i,N_y-3} + 4^n_{i,N_y-2}}{3} \cdot \left(1 + \frac{2\Delta y C}{3T^n_{i,N_y-1}}\right)^{-1} \tag{56}$$

$$= \frac{-e^n_{i,N_y-3} - 4e^n_{i,N_y-2}}{3 + 2\Delta y C / T^n_{i,N_y-1}} \tag{57}$$

3. Density

For density, we use the equation of state for the ideal gas

$$\rho^n_{i,j} = \frac{(\gamma - 1)\mu m_u}{k_B} \frac{e^n_{i,j}}{T^n_{i,j}} \tag{58}$$

D. The code

The code is set up like a class consisting of the following functions:

- _init_()
  Necessary constants are defined based on the model and equations needed to describe the model (found in section II). Empty arrays for values to be computed in the code were defined in this function as well. These values include temperature, pressure, density, horizontal velocity, vertical velocity, internal energy, and Gaussian perturbation.

- initialise()
  This function sets the initial conditions for the values and integrates downwards the box using Euler's method. It is important to make sure temperature and pressure are updated first, as internal energy and density depend on them. In the function, there is an option to turn on/off the perturbation through a simple boolean.

- timestep()
  For each primary variable, the relative change is computed for each time step.

  $$rel(\phi) = \left| \frac{\partial \phi}{\partial t} \cdot \frac{1}{\phi} \right| \tag{59}$$

  The relevant variables to compute the relative change for are ρ, ρu (density · horizontal velocity), ρw (density · vertical velocity), e, x (horizontal position), and y (vertical position). To make sure all quantities satisfy our condition, we choose the largest of these values

  $$\delta = \max(\max(rel(\phi))) \tag{60}$$

  The time step length Δt is found by the following relation

  $$\Delta t = \frac{p}{\delta} \tag{61}$$

  where p is typically around 0.1.

- boundary_conditions()
  Applying boundary conditions that set the value of ϕ or the derivate of ϕ are the endpoints of the numerical grid. This is further described in section III C.

- central_x()
  The discretisation of the central difference scheme in the x-direction can be found in eq. 13. `numpy.roll` is a useful tool for implementing this.

- central_y()
  The discretisation of the central difference scheme in the y-direction can be found in eq. 14. `numpy.roll` is a useful tool for implementing this.

- upwind_x()
  The discretisation of the upwind difference scheme in the x-direction can be found in eq. 15. In addition to `numpy.roll`, the `numpy.ma` module was useful for separating the positive and negative velocities.

- upwind_y()
  The discretisation of the upwind difference scheme in the y-direction can be found in eq. 16. In addition to `numpy.roll`, the `numpy.ma` module was useful for separating the positive and negative velocities.

- hydro_solver()
  In this function, the hydrodynamic equations are solved. The timestep() function is called first to retrieve the timestep Δt. Then ρ, u, w, and e are solved before calling boundary_conditions(). Lastly, we solve for P and T which depend on e, ρ and P . The reason boundary_condition() is called after density, velocity, and internal energy is that the function only applies to these values. We make sure these values are correct at the boundary before calculating P and T.

IV. RESULTS

The continuity equation (eq. 1), momentum equation (eq. 2), and energy equation (eq. 3) can be rewritten from vector form to differential form as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial y} \tag{62}$$

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial \rho u^2}{\partial x} - \frac{\partial \rho u w}{\partial y} - \frac{\partial P}{\partial x} \tag{63}$$

$$\frac{\partial \rho w}{\partial t} = -\frac{\partial \rho w^2}{\partial y} - \frac{\partial \rho u w}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \tag{64}$$

$$\frac{\partial e}{\partial t} = -\frac{\partial e u}{\partial x} - \frac{\partial e w}{\partial y} - P\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial y}\right) \tag{65}$$

$$\tag{66}$$

Calculations can be found in appendix A.

V. DISCUSSION AND CONCLUSION

A. The momentum equation

The 2D hydrodynamic equations are presented at the beginning of section . The momentum equation for the horizontal direction can be found in eq. 63 and vertical direction in eq. 64. Section II B describes the model setup where the vertical direction of the box is also referred to as the radial direction. Thus, gravity is only in the y-direction as the force works radially.

B. Discretisation

Spatial discretisation of the hydrodynamic equations is done either through central differencing or upwind differencing. Upwind differencing is used to model transport, thus the direction must be taken into account. This can be seen from the algorithm, which uses the previous cell to compute the next. The direction of the flow determines if the previous cell is to the right or left of the next cell we want to compute. When writing out the algorithm, the terms with velocity dependency are calculated using upwind differencing.

C. Troubleshooting

Unfortunately, my program had a bug which only produced a black animation window. Thus being unable to produce the movie to verify hydrostatic equilibrium and capture snapshots of the system's evolution after the gas had been provoked. It is unsure what caused this issue. Initially, there was a suspicion that the arrays were not updating with each iteration. However, that was not the case. Perhaps, the system did not evolve as expected which could have caused this mishap. In an attempt at locating and fixing the bug, the code was compared with fellow students. This also sparked a discussion of the code's structure in the sense of different techniques which could be used to solve the task and what to think about when setting it up.

D. Thoughts

Through the discussion with fellow students, some interesting thoughts and questions emerged.
Rather than modelling the convection from inside the Sun and towards the surface, it was modelled from the surface down. This makes it not possible to gather quantitative information about the energy production in the core, which is then transported by convection (among other transport methods. The model is therefore limited to only visualising the convective motion, not giving information on how much energy was produced in the core.
Due to numerical limitations, we have no information past the boundaries of the box. Boundary conditions are set values to avoid outflow past boundaries. Due to this, the model is unable to describe what happens at the surface. It would be particularly interesting to observe solar prominences as a consequence of convection. Though, such a model would probably introduce more complexity through magnetohydrodynamics. However, would introducing boundary conditions introduce information loss? In reality, some of the hot plasma which rises to the surface will radiate energy at the surface, but some will cool and flow back down into the Sun. Seeing our model has no further information past the

boundaries, will data about energy radiated at the surface be lost or will all plasma cool and flow back down? These are some things to be aware of when studying the videoes produced in this project, but this situation brings on another question. For instance, would energy be conserved in this system? A star's life cycle satisfies the conservation of energy law [5]. It is difficult to say for sure if the energy in our model is conserved seeing we have no information on energy production in the core and energy release at the surface.

However, this project only aims to model the Sun in an attempt at understanding it better. All models have their limitations and improving models will introduce more complexity. Though, these questions are good ways to perhaps extend the project.

There are some ways to verify the model based on the information available by

- comparing the temperature in the granulation pattern with scientific papers. The values should somewhat correspond with the Sun.
- making physical sense of the values inside the star versus the surface, e.g. temperature should be higher inside the star than at the surface.
- utilising arrows in the videoes to make sure everything is rising towards the surface.

Even though I was unable to finish the project, this was a great learning experience. I have gained a better understanding of the hydrodynamic equations, both mathematically and physically. I have been introduced to methods to discretise in two-dimensional systems and benefits of different algorithms and how to implement them. Lastly, I have been more aware of the limitations of a model through discussing it with fellow students.

Appendix A: 2D hydrodynamic equations

The 2D hydrodynamic equations can be split into components in x and y.

1. The continuity equation

We begin with the continuity equation, eq. 1.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u_i) \end{aligned} \tag{A1}$$

where we use $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ in two dimensions and $i = x, y$. Thus, we have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_x}{\partial x} - \frac{\partial \rho u_y}{\partial y} \tag{A2}$$

2. The momentum equation

The momentum equation is found in eq. 2

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla P + \rho \mathbf{g} \tag{A3}$$

where the gravitational force is $\mathbf{g} = -\rho g(r)\hat{\mathbf{r}}$.

$$\begin{aligned} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) &= \mathbf{u}(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} \\ &= \nabla(\mathbf{uu}) \end{aligned} \tag{A4}$$

where ref. [6] is used for the first identity and ref. [7] for the second identity. We can rewrite the momentum equation as

$$\begin{aligned} \frac{\partial \rho \mathbf{u}}{\partial t} &= -\frac{\partial}{\partial \nabla} \cdot (\rho \mathbf{uu}) - \nabla P - \rho \mathbf{g} \\ \frac{\partial \rho u_i}{\partial t} &= -\frac{\partial \rho u_i u_k}{\partial x_k} - \nabla P - \rho \mathbf{g} \end{aligned} \tag{A5}$$

The x-component of the momentum equation for $i = x$

$$\begin{aligned} \frac{\partial \rho u_x}{\partial t} &= -\frac{\partial \rho u_x u_k}{\partial x_k} - \frac{\partial P}{\partial x_k} \\ k = x, y \rightarrow &= -\frac{\partial \rho u_x^2}{\partial x} - \frac{\partial \rho u_x u_y}{\partial y} - \frac{\partial P}{\partial x} \end{aligned} \tag{A6}$$

and the y-component for $i = y$

$$\begin{aligned} \frac{\partial \rho u_y}{\partial t} &= -\frac{\partial \rho u_y u_k}{\partial x_k} - \frac{\partial P}{\partial x_k} \\ k = x, y \rightarrow &= -\frac{\partial \rho u_y^2}{\partial y} - \frac{\partial \rho u_x u_y}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \end{aligned} \tag{A7}$$

3. The energy equation

The energy equation (eq. 3) is

$$\begin{aligned} \frac{\partial e}{\partial t} + \nabla\cdot(e\mathbf{u}) &= -P\nabla\cdot\mathbf{u} \\ \Rightarrow \frac{\partial e}{\partial t} &= -\nabla\cdot(eu_i) - P\nabla\cdot u_i \end{aligned} \tag{A8}$$

and rewrite this in two components similar to the continuity equation

$$\begin{aligned} \frac{\partial e}{\partial t} &= -\frac{\partial eu_x}{\partial x} - \frac{\partial eu_y}{\partial y} - P\frac{\partial u_x}{\partial x} - P\frac{\partial u_y}{\partial y} \\ &= -\frac{\partial eu_x}{\partial x} - \frac{\partial eu_y}{\partial y} - P\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) \end{aligned} \tag{A9}$$

4. Summary

The 2D hydrodynamic equations become

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u_x}{\partial x} - \frac{\partial \rho u_y}{\partial y} \\ \frac{\partial \rho u_x}{\partial t} &= -\frac{\partial \rho u_x^2}{\partial x} - \frac{\partial \rho u_x u_y}{\partial y} - \frac{\partial P}{\partial x} \\ \frac{\partial \rho u_y}{\partial t} &= -\frac{\partial \rho u_y^2}{\partial y} - \frac{\partial \rho u_x u_y}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \\ \frac{\partial e}{\partial t} &= -\frac{\partial eu_x}{\partial x} - \frac{\partial eu_y}{\partial y} - P\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) \end{aligned} \tag{A10}$$

Substituting $u_x \to u$ and $u_y \to w$ gives

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial y} \\ \frac{\partial \rho u}{\partial t} &= -\frac{\partial \rho u^2}{\partial x} - \frac{\partial \rho uw}{\partial y} - \frac{\partial P}{\partial x} \\ \frac{\partial \rho w}{\partial t} &= -\frac{\partial \rho w^2}{\partial y} - \frac{\partial \rho uw}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \\ \frac{\partial e}{\partial t} &= -\frac{\partial eu}{\partial x} - \frac{\partial ew}{\partial y} - P\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial y}\right) \end{aligned} \tag{A11}$$

Appendix B: Discretisation

In order to solve the 2D hydrodynamic equations obtained in appendix A, each partial derivative needs to be discretized in a specific way. Common for all the equations is that the left-hand side is discretized using forward time. The flow (u, w) is not assumed to be constant.
discretisation of the continuity equation $\partial\rho/\partial t$ and the horizontal momentum equation $\partial\rho u/\partial t$ is given in the project description. We need to find the discretisation of the vertical momentum equation $\partial\rho w/\partial t$ and the energy equation $\partial e/\partial t$ using the same method.

1. The vertical momentum equation

The vertical momentum equation is

$$\frac{\partial \rho w}{\partial t} = -\frac{\partial \rho w^2}{\partial y} - \frac{\partial \rho uw}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \tag{B1}$$

w is assumed to not be constant and the vertical momentum equation is split into additional terms due to the product rule.

$$\begin{aligned} \frac{\partial \rho w}{\partial t} &= -\frac{\partial \rho w^2}{\partial y} - \frac{\partial \rho uw}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \\ &= -\rho w\left(\frac{\partial w}{\partial y} + \frac{\partial u}{\partial x}\right) - w\frac{\partial \rho w}{\partial y} - u\frac{\partial \rho w}{\partial x} - \frac{\partial P}{\partial y} + \rho g_y \end{aligned} \tag{B2}$$

Further expressed as

$$\left[\frac{\partial \rho w}{\partial t}\right]_{i,j}^n = -[\rho w]_{i,j}^n\left(\left[\frac{\partial w}{\partial y}\right]_{i,j}^n + \left[\frac{\partial u}{\partial x}\right]_{i,j}^n\right) - w_{i,j}^n\left[\frac{\partial \rho w}{\partial y}\right]_{i,j}^n - u_{i,j}^n\left[\frac{\partial \rho w}{\partial x}\right]_{i,j}^n - \left[\frac{\partial P}{\partial y}\right]_{i,j}^n + [\rho g_y]_{i,j}^n \tag{B3}$$

All spatial derivates are approximated using upwind differencing, except for the horizontal velocity gradient ($\partial u/\partial x$) and the pressure gradient ($\partial P/\partial y$) which are approximated using central differencing. The terms on the right-hand side of the horizontal momentum equation are discretised as

$$\begin{aligned}
\left[\frac{\partial u}{\partial x}\right]_{i,j}^{n} &\approx \frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2\Delta x} \\
\left[\frac{\partial P}{\partial y}\right]_{i,j}^{n} &\approx \frac{P_{i,j+1}^{n} - P_{i,j-1}^{n}}{2\Delta y} \\
\left[\frac{\partial w}{\partial y}\right]_{i,j}^{n} &\approx \begin{cases} \frac{w_{i,j}^{n} - w_{i,j-1}^{n}}{\Delta y}, & w_{i,j}^{n} \geq 0 \\ \frac{w_{i,j+1}^{n} - w_{i,j}^{n}}{\Delta y}, & w_{i,j}^{n} < 0 \end{cases} \\
\left[\frac{\partial \rho w}{\partial y}\right]_{i,j}^{n} &\approx \begin{cases} \frac{[\rho w]_{i,j}^{n} - [\rho w]_{i,j-1}^{n}}{\Delta y}, & w_{i,j}^{n} \geq 0 \\ \frac{[\rho w]_{i,j+1}^{n} - [\rho w]_{i,j}^{n}}{\Delta y}, & w_{i,j}^{n} < 0 \end{cases} \\
\left[\frac{\partial \rho w}{\partial x}\right]_{i,j}^{n} &\approx \begin{cases} \frac{[\rho w]_{i,j}^{n} - [\rho w]_{i-1,j}^{n}}{\Delta x}, & u_{i,j}^{n} \geq 0 \\ \frac{[\rho w]_{i+1,j}^{n} - [\rho w]_{i,j}^{n}}{\Delta x}, & u_{i,j}^{n} < 0 \end{cases}
\end{aligned} \tag{B4}$$

The left-hand side is discretised using forward time

$$\begin{aligned}
\left[\frac{\partial \rho w}{\partial t}\right]_{i,j}^{n+1} &= \frac{[\rho w]_{i,j}^{n+1} - [\rho w]_{i,j}^{n}}{\Delta t} \\
&= \frac{\rho_{i,j}^{n+1} w_{i,j}^{n+1} - [\rho w]_{i,j}^{n}}{\Delta t} \\
w_{i,j}^{n+1} &= \frac{\left[\frac{\partial \rho w}{\partial t}\right]_{i,j}^{n+1} \Delta t + [\rho w]_{i,j}^{n}}{\rho_{i,j}^{n+1}}
\end{aligned} \tag{B5}$$

since $\rho_{i,j}^{n+1}$ has already been calculated, we only concern ourselves with advancing $w_{i,j}^{n+1}$ in time.

2. The energy equation

The energy equation is

$$\frac{\partial e}{\partial t} = -\frac{\partial e u_x}{\partial x} - \frac{\partial e u_y}{\partial y} - P\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) \tag{B6}$$

Since u and w are not constant, the energy equation is also split into additional terms

$$\frac{\partial e}{\partial t} = -u\frac{\partial e}{\partial x} - e\frac{\partial u}{\partial x} - w\frac{\partial e}{\partial y} - e\frac{\partial w}{\partial y} - P\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\right) \tag{B7}$$

Further expressed as

$$\left[\frac{\partial e}{\partial t}\right]_{i,j}^{n} = -u\left[\frac{\partial e}{\partial x}\right]_{i,j}^{n} - e\left[\frac{\partial u}{\partial x}\right]_{i,j}^{n} - w\left[\frac{\partial e}{\partial y}\right]_{i,j}^{n} - e\left[\frac{\partial w}{\partial y}\right]_{i,j}^{n} - P\left(\left[\frac{\partial u}{\partial x}\right]_{i,j}^{n} + \left[\frac{\partial w}{\partial y}\right]_{i,j}^{n}\right) \tag{B8}$$

The spatial derivatives of e are approximated using upwind differencing, while the derivatives of u and w are approximated using a central scheme. The right-hand side of the energy equation is discretised as

$$\begin{aligned}
\left[\frac{\partial u}{\partial x}\right]_{i,j}^{n} &\approx \frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2\Delta x} \\
\left[\frac{\partial w}{\partial y}\right]_{i,j}^{n} &\approx \frac{w_{i,j+1}^{n} - w_{i,j-1}^{n}}{2\Delta y} \\
\left[\frac{\partial e}{\partial x}\right]_{i,j}^{n} &\approx \begin{cases} \frac{e_{i,j}^{n} - e_{i-1,j}^{n}}{\Delta x}, & u_{i,j}^{n} \geq 0 \\ \frac{e_{i+1,j}^{n} - e_{i,j}^{n}}{\Delta x}, & u_{i,j}^{n} < 0 \end{cases} \\
\left[\frac{\partial e}{\partial y}\right]_{i,j}^{n} &\approx \begin{cases} \frac{e_{i,j}^{n} - e_{i-1,j}^{n}}{\Delta y}, & w_{i,j}^{n} \geq 0 \\ \frac{e_{i+1,j}^{n} - e_{i,j}^{n}}{\Delta y}, & w_{i,j}^{n} < 0 \end{cases}
\end{aligned} \tag{B9}$$

The left-hand side discretised using forward time gives

$$\begin{aligned}
\left[\frac{\partial e}{\partial t}\right]_{i,j}^{n+1} &= \frac{e_{i,j}^{n+1} - e_{i,j}^{n}}{\Delta t} \\
e_{i,j}^{n+1} &= \left[\frac{\partial e}{\partial t}\right]_{i,j}^{n+1} \Delta t + e_{i,j}^{n}
\end{aligned} \tag{B10}$$

Appendix C: References

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