Stellar modelling

Candidate number 24
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This abstract is abstract.

I. INTRODUCTION

The central parts of stars is crucial for their energy production. This is where the vast majority of energy is produced and transported throughout the star. The two main transport mechanisms are radiative transport and convective transport. The purpose of this project is to model the central parts of a Sun-like star including both radiative and convective energy transport. The model should fulfill the following goals:

- Has luminosity (L), mass (m), and radius (r) all going to 0 or at least within 5% of given initial conditions for L, m, and r.
- Has a core ($L < 0.995$) reaching out to at least 10% of R_0.
- Has a continuous convection zone near the surface of the star. The width of this convection zone should be at least 15% of R_0. A small radiation zone at the edge and/or a second convection zone closer to the center is acceptable, but the convective flux should be small compared to the "main" convection zone near the surface.

II. THEORY

In this model, we have the following assumptions:

- Mass fraction of each atomic species is independent of radius and given in table II.
- All elements are fully ionized.
- Produced $^2_1$D is immediately consumed by the next step to produce $^3_2$He.
- We do not consider changes over time, as we are looking at a snapshot of a star at a particular moment in time.
- Ideal gas
- No heat conduction in the star, only radiation and convection.
- $\alpha_{lm} = 1$ in eq. 16.

The initial conditions for the model can be found in table I. The governing equations for solving the internal structure of the radiative zone of the Sun are as follows

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

Table I. Initial conditions for our stellar model. Solar parameters are found in Appendix B of ref. [1]. $\bar{\rho}_\odot = 1.408 \cdot 10^3$ kg m^{-3} is the average density of the Sun.

Symbol	Name	Value
L_0	Luminosity	$1.0 \cdot L_\odot$
R_0	Radius	$1.0 \cdot R_\odot$
M_0	Mass	$1.0 \cdot M_\odot$
ρ_0	Density	$1.42 \cdot 10^{-7} \cdot \bar{\rho}_\odot$
$T0$	Temperature	5770 K

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^2} \tag{2}$$

$$\frac{\partial L}{\partial m} = \varepsilon \tag{3}$$

$$\frac{\partial T}{\partial m} = \nabla \frac{T}{P}\frac{\partial P}{\partial m} \tag{4}$$

where ρ is density, r is the stellar radius, G is Newton´s gravitational constant, T is temperature and ε is the full energy generation per unit mass in a star. Before the four partial differential equations can be solved numerically, there are additional equations needed to be defined. In a star, there are several forces that may contribute to the pressure in stars. In the case of our Sun, it is mainly gas pressure and radiative pressure to a small degree. The radiative pressure depends only on temperature and not density

$$P_{\text{rad}} = \frac{a}{3}T^4 \tag{5}$$

P_{rad} is radiative pressure, T is temperature and a is the radiation density constant defined as

$$a = \frac{4\sigma}{c} \tag{6}$$

where σ is Stefan-Boltzmann's constant and c is the speed of light. Gas pressure can be expressed through the equation of state for an ideal gas

$$P_{\text{gas}} = \frac{\overbrace{N}^{m/(\mu m_u)}}{V} k_B T = \underbrace{\frac{m}{V}}_{\rho} \frac{k_B T}{\mu m_u} = \frac{\rho k_B T}{\mu m_u} \tag{7}$$

where P_{gas} is gas pressure, V is volume, N is the number of particles, k_B is the Boltzmann constant, m_u is atomic mass unit, μ is mean molecular weight and T, once again, is temperature. Thus, giving us the total pressure in a star to be

$$P = P_{\text{gas}} + P_{\text{rad}} = \frac{\rho k_B T}{\mu m_u} + \frac{a}{3}T^4 \tag{8}$$

We see that gas pressure in a star is related to the microphysics through the average weight of the particles μ. The mean molecular weight per particle is calculated with the following expression

$$\mu = \frac{1}{\left(\sum \frac{\text{free particles per ion}}{\text{nucleons per ion}} \cdot \text{ion mass fraction}\right)} \tag{9}$$

A check for convective instability at each mass shell is required in order to determine the expression for $\frac{\partial T}{\partial m}$. In the case of a convective stable shell and we are dealing with radiative transport only, the equation becomes

$$\frac{\partial T}{\partial m} = -\frac{-3\kappa L}{256\pi^2\sigma r^4 T^3} \tag{10}$$

where κ is the opacity (units of $\text{m}^2\ \text{kg}^{-1}$) and L is the luminosity of a star. If the shell is not convective stable, we use eq. 4 for convective transport. The instability criterion is as follows

$$\nabla > \nabla_{\text{ad}} \tag{11}$$

where ∇ is the temperature gradient of the star (often referred to as ∇^* as well) and ∇_{ad} is the adiabatic temperature gradient (the case where parcel temperature, density, and pressure are the exact same as the surroundings after it has moved). The adiabatic temperature gradient is given by

$$\nabla_{\text{ad}} = \frac{P\delta}{T\rho c_P} \tag{12}$$

where δ simplifies to 1 using ideal gas law and c_P is the specific heat capacity given by

$$c_P = \frac{5}{2}\frac{k_B}{\mu m_u} \tag{13}$$

The expression for ∇^* can be found derived in eq. A10 in appendix A 3

$$\nabla^* = \xi^2 + k\xi + \nabla_{\text{ad}} \tag{14}$$

where ξ is found by obtaining the real root from the cubic polynomial in eq. A15 in appendix A 4:

$$\xi^3 + l_m^{-2}U\xi^2 + l_m^{-2}Uk\xi - l_m^{-2}U(\nabla_{\text{stable}} - \nabla_{\text{ad}}) = 0 \tag{15}$$

We assume a spherical parcel which allows us to use the geometric factor obtained in A 1 in the expression for k, l_m is the mixing length defined as

$$l_m = \alpha_{lm} H_P \tag{16}$$

where H_P is the pressure scale height found using the equation below

$$H_P = \frac{kT}{\mu m_u g} \tag{17}$$

g is the gravitational acceleration found by equating Newton's second law with Newton's gravitational law. ∇_{stable} is the temperature gradient needed for all the energy to be carried by radiation. To define that quantity we need to define the flux. Radiative flux is

$$F_{\text{rad}} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* \tag{18}$$

Generally, the total energy flux at any point in the star has to obey

$$F_{\text{rad}} + F_{\text{con}} = \frac{L}{4\pi r^2} \tag{19}$$

but the total flux is also defined as

$$F_{\text{rad}} + F_{\text{con}} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{\text{stable}} \tag{20}$$

By equating eq. 19 and eq. 20 we obtain the following expression for ∇_{stable}

$$\nabla_{\text{stable}} = \frac{L 3\kappa\rho H_P}{4\pi r^2 16\sigma T^4} \tag{21}$$

Lastly, ∇_p is defined in eq. A8

$$\begin{aligned}(\nabla_p - \nabla_{\text{ad}}) &= \frac{32\sigma T^3}{3\kappa\rho^2 c_p v}\frac{S}{Qd}(\nabla^* - \nabla_p) \\ \Rightarrow \nabla_p &= \frac{4U}{l_m}\xi^2 + \nabla_{\text{ad}}\end{aligned} \tag{22}$$

The relation between the temperature gradients are

$$\nabla_{\text{ad}} < \nabla_{\text{p}} < \nabla^* < \nabla_{\text{stable}} \tag{23}$$

III. METHOD

IV. MEAN MOLECULAR WEIGHT

Table II. Mass fraction of elements in our model

Symbol	Element	Value
X	Hydrogen	0.7
Y_2^3He	Helium-3	10^{10}
Y	Helium-4	0.29
Z_3^7Li	Lithium-7	10^{-7}
Z_4^7Be	Beryllium-7	10^{-7}
Z_7^{14}N	Nitrogen-14	10^{-11}

The mean molecular weight μ is calculated through eq 9. We consider the fully ionized case for all elements.

Thus, ”ion mass fraction” is 1 for all elements, and ”free particles” become the number of atoms (= 1 per element) plus the number of free electrons.

$$\mu = \frac{1}{2X + \mathrm{Y}^3_2\mathrm{He} + \frac{3}{4}Y + \frac{4}{7}\mathrm{Z}^7_3\mathrm{Li} + \frac{5}{7}\mathrm{Z}^7_4\mathrm{Be} + \frac{8}{14}\mathrm{Z}^{14}_7\mathrm{N}} \tag{24}$$

V. OPACITY

Opacity, κ is obtained from `opacity.txt`. The structure of of the .txt file is as follows:

- The top row is $\log_{10}(\mathrm{R})$, where $R \equiv \frac{\rho}{(T/10^6)^3}$ and ρ is given in cgs units [g/cm^3].

- The first column is $\log_{10}(\mathrm{T})$, with T given in K.

- The rest of the table is $\log_{10}(\kappa)$ given in cgs units [cm^2/g].

VI. INTEGRATION

The four partial differential equations are numerically solved using Euler's method

$$\begin{aligned} dV &= f\ dm \\ V_{i+1} &= V_i + dV \end{aligned} \tag{25}$$

where V is the variable we are interested in calculating for each iteration, f is the absolute value of the governing equations (in our case: equations 1, 2, 3 and 4). dm is a variable step length defined as

$$dm = \mathrm{MIN}\left[\frac{pV}{f}\right] \tag{26}$$

where p is a fraction that V is allowed to change and

VII. RESULTS

VIII. DISCUSSION

IX. CONCLUSION

ACKNOWLEDGMENTS

I would like to thank myself for writing this beautiful document.

Appendix A: Exercises

1. Exercise 5.10

The geometric factor is defined as $S/(Qd)$ where S is the surface area, d is diameter, and Q is the surface normal to velocity v. For a spherical parcel with radius r_p, the geometric factor becomes

$$\frac{S}{Qd} = \frac{4\pi r_p^2}{2r_p \cdot \pi r_p^2} = \frac{2}{r_p} \tag{A1}$$

and $r_p = l_m/2$ where l_m is mixing length.

2. Exercise 5.11

We wish to obtain an expression for $(\nabla^* - \nabla_p)$ as a function of ∇^* and ∇_{stable} by inserting equation 5.80 and 5.82 into 5.81 from AST3310 lecture notes.
Equation 5.80 from lecture notes is

$$F_{\mathrm{con}} = \rho c_p T\sqrt{g\delta} H_P^{-3/2}\left(\frac{l_m}{2}\right)^2 (\nabla^* - \nabla_p)^{3/2} \tag{A2}$$

Equation 5.82 is

$$F_{\mathrm{rad}} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* \tag{A3}$$

Last we have equation 5.81 (hereby: eq. A4), in which 5.80 (hereby: eq. A2) and 5.82 (hereby: eq. A3) will be inserted into

$$F_{\text{rad}} + F_{\text{con}} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{\text{stable}} \tag{A4}$$

Inserting eq. A2 and eq. A4 in . A4 we obtain

$$\begin{aligned}
\rho c_p T\sqrt{g\delta}H_P^{-3/2}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{3/2} + \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* &= \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{\text{stable}} \\
\rho c_p T\sqrt{g\delta}H_P^{-\cancel{3}^{1}/2}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{3/2} &= \frac{16\sigma T^4}{3\kappa\rho \cancel{H_P}}(\nabla_{\text{stable}-\nabla^*}) \\
(\nabla^* - \nabla_p)^{3/2} &= \underbrace{\frac{64\sigma T^{\cancel{4}^{3}}}{3\kappa\rho^2 c_p \cancel{T}}\sqrt{\frac{H_P}{g\delta}}}_{U}\, l_m^{-2}(\nabla_{\text{stable}-\nabla^*}) \\
(\nabla^* - \nabla_p)^{3/2} &= l_m^{-2}U(\nabla_{\text{stable}} - \nabla^*)
\end{aligned} \tag{A5}$$

where

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}} \tag{A6}$$

3. Exercise 5.12

To get a second order equation for $(\nabla^* - \nabla_p)^{1/2}$, we insert equation 5.74 from lecture notes [1] into

$$(\nabla_p - \nabla_{\text{ad}}) = (\nabla^* - \nabla_{\text{ad}}) - (\nabla^* - \nabla_p) \tag{A7}$$

where eq. 5.74 (hereby, eq. A8) is

$$(\nabla_p - \nabla_{\text{ad}}) = \frac{32\sigma T^3}{3\kappa\rho^2 c_p v}\frac{S}{Qd}(\nabla^* - \nabla_p) \tag{A8}$$

From equation 5.70 in lecture notes (hereby, eq. A9) we obtain the expression for v

$$v = \sqrt{\frac{g\delta}{H_P}}\frac{l_m}{2}\underbrace{(\nabla^* - \nabla_p)^{1/2}}_{\xi} = \sqrt{\frac{g\delta}{H_P}}\frac{l_m}{2}\xi \tag{A9}$$

where $\xi = (\nabla^* - \nabla_p)^{1/2}$.
By inserting eq. A8 into eq. A7 we get the following

$$\begin{aligned}
\frac{32\sigma T^3}{3\kappa\rho^2 c_p \underbrace{v}_{eq.A9}}\frac{S}{Qd}\underbrace{(\nabla^* - \nabla_p)}_{\xi^2} &= (\nabla^* - \nabla_{\text{ad}}) - \underbrace{(\nabla^* - \nabla_p)}_{\xi^2} \\
\underbrace{\frac{64\sigma T^3}{3\kappa\rho^2 c_p\sqrt{\frac{g\delta}{H_P}}\cancel{\xi}}}_{U}\frac{S}{Qdl_m}\xi^{\cancel{2}} &= (\nabla^* - \nabla_{\text{ad}}) - \xi^2 \\
\xi^2 + \underbrace{U\frac{S}{Qdl_m}}_{k}\xi - (\nabla^* - \nabla_{\text{ad}}) &= 0 \\
\xi^2 + k\xi - (\nabla^* - \nabla_{\text{ad}}) &= 0
\end{aligned} \tag{A10}$$

Roots of ξ are

$$\Rightarrow \xi = \frac{-k \pm \sqrt{k^2 + 4(\nabla^* - \nabla_{\rm ad})}}{2} \tag{A11}$$

To find the viable solution(s) we need to look at the definition of ξ. For the gas parcel to rise, ∇_p must be smaller than ∇^*, i.e. $\nabla_p < \nabla^* \Rightarrow \xi = (\nabla^* - \nabla_p)^{1/2} > 0$. The only viable solution for ξ is

$$\xi = \frac{-k + \sqrt{k^2 + 4(\nabla^* - \nabla_{\rm ad})}}{2} \tag{A12}$$

4. Exercise 5.13

To eliminate ∇^* in eq. A5, we use our results from eq. A10.
We begin by expressing eq. A10 in terms of ∇^*

$$\begin{gathered} \xi^2 + k\xi - (\nabla^* - \nabla_{\rm ad}) = 0 \\ \nabla^* = \xi^2 + k\xi + \nabla_{\rm ad} \end{gathered} \tag{A13}$$

followed by rewriting eq. A5 and insert eq. A13

$$\begin{aligned} \underbrace{(\nabla^* - \nabla_p)^{3/2}}_{\xi^3} &= l_m^{-2} U (\nabla_{\rm stable} - \nabla^*) \\ \xi^3 &= l_m^{-2} U (\nabla_{\rm stable} - \underbrace{\nabla^*}_{eq. A13}) \\ \xi^3 &= l_m^{-2} U (\nabla_{\rm stable} - \xi^2 - k\xi - \nabla_{\rm ad}) \end{aligned} \tag{A14}$$

$$\xi^3 + l_m^{-2} U \xi^2 + l_m^{-2} U k \xi - l_m^{-2} U (\nabla_{\rm stable} - \nabla_{\rm ad}) = 0 \tag{A15}$$

This cubic polynomial will have three roots where two of which are complex and the third one is real. However, since $\xi > 0$, only the real root is a viable solution.

Appendix B: Sanity checks

Table III. A check to see if the interpolation of opacity values give reasonable outputs.

$\log_{10}T$	$\log_{10}R$ (cgs)	$\log_{10}\kappa$ (cgs) (expected)	κ (SI) (computed)	$\log_{10}\kappa$ (cgs) (expected)	κ (SI) (computed)
3.750	-6.00	-1.55	-1.546000	0.00284	0.002844
3.755	- 5.95	-1.51	-1.502573	0.00311	0.003144
3.755	- 5.80	-1.57	-1.567107	0.00268	0.002710
3.755	- 5.70	-1.61	-1.609850	0.00246	0.002456
3.755	- 5.55	-1.67	-1.673324	0.00212	0.002122
3.770	- 5.95	-1.33	-1.312074	0.00470	0.004874
3.780	- 5.95	-1.20	-1.190270	0.00625	0.006453
3.795	- 5.95	-1.02	-1.018097	0.00945	0.009592
3.770	- 5.80	-1.39	-1.374133	0.00405	0.004225
3.775	- 5.75	-1.35	-1.331313	0.00443	0.004663
3.780	- 5.70	-1.31	-1.288318	0.00494	0.005149
3.795	- 5.55	-1.16	-1.157945	0.00689	0.006951
3.800	- 5.50	-1.11	-1.114000	0.00769	0.007691

Table IV. Sanity check from example 5.1 in ref. [1]

	∇_{stable}	∇_{ad}	∇^*	H_P	U	ξ	v	$\frac{F_{\text{con}}}{F_{\text{con}}+F_{\text{rad}}}$	$\frac{F_{\text{rad}}}{F_{\text{con}}+F_{\text{rad}}}$
Expected	3.26	0.4	0.4	32.4	594000	0.001173	65.6	0.88	0.12
Computed	3.16053	0.4	0.400001	31.4	603441	0.001189	65.407	0.873438	0.126562

```
nabla_stable > nabla* > nabla_p > nabla_ad satisfied?: True
```

Appendix C: References

[1] Gudiksen, B. AST3310: Astrophysical plasma and stellar interiors. [PDF].