Big Bang Nucleosynthesis - Predicting the abundance of light elements

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This abstract is abstract.

Big Bang Nucleosynthesis (BBN) describes the production of the lightest elements in the first few moments after the Big Bang. This paper aims to use the Boltzmann equation to compute the abundance of the lightest elements.

Before the BBN begin, the baryonic matter in the Universe is almost entirely in the form of free protons and neutrons which interact through the weak nuclear force. This interaction transforms protons into neutrons and vice versa.

We use the Boltzmann equation to describe the change of particle specie "i" from the interaction via the weak nuclear force, both in and out of equilibrium

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = J_i \tag{1}$$

$3Hn_i$ is the dilution of number density due to expansion, while J_i is the reaction rate of particle i. In this model, we look at all the weak force reactions between proton and neutron as decays, with the effect of the electrons and neutrinos in the decay rates. This gives us the following expression for J_i

$$J_i = n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j} \tag{2}$$

We obtain the Boltzmann equation to be the following

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j} \tag{3}$$

H is the Hubble parameter defined as

$$H \equiv \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \tag{4}$$

Relative number density for particle specie "i" is defined as

$$Y_i = \frac{n_i}{n_b} \tag{5}$$

where the total baryon nucleon density, n_b is

$$n_b = n_{b0}a^{-3} = \frac{\rho_{b0}a^{-3}}{m_p} \tag{6}$$

and the total baryon mass density, ρ_{b0} is

$$\rho_{b0} = \Omega_{b0}\rho_{c0} \tag{7}$$

For simplicity, we use the logarithm of temperature as our time variable

$$T = T_0a^{-1} \tag{8}$$

Table I. Values for different parameters which will be used in this project

Symbol	Value	Units
h	0.7	
N_{eff}	3	
Ω_{b0}	0.05	
H_0	$100h$	km s^{-1} Mpc^{-1}
ρ_{c0}	$9.2\cdot10^{-27}$	kg m^{-3}

A.

We want to show the equations for dY$_i$/d(ln T) starting from equation 3 is

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = -\frac{1}{H}[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}] \tag{9}$$

To show this we begin by finding the time derivative of Y_i given in equation 5

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y_i &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_i}{n_b}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(n_in_b^{-1}\right)\\ &= \frac{1}{n_b}\frac{\mathrm{d}}{\mathrm{d}t}\left(n_i\right) - \frac{n_i}{n_b^2}\frac{d}{dt}(\underbrace{n_b}_{eq.6})\end{aligned} \tag{10}$$

where the time derivative of n_b is

$$\begin{aligned}\frac{d}{dt}(n_{b0}a^{-3}) &= \frac{n_{b0}a^{-3}}{da}\frac{da}{dt}\\ &= -3\frac{\overbrace{n_{b0}}^{=n_ba^3,eq.6}}{a^4}\frac{da}{dt}\\ &= -3\frac{n_ba^3}{a^4}\frac{da}{dt}\\ &= -3n_b\underbrace{\frac{1}{a}\frac{da}{dt}}_{=H,eq.4}\\ &= -3n_bH\end{aligned} \tag{11}$$

Inserting eq. 11 back into eq. 10 we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y_i &= \frac{1}{n_b}\frac{\mathrm{d}}{\mathrm{d}t}\left(n_i\right) - \frac{n_i}{n_b^2}(-3n_bH)\\ &= \frac{1}{n_b}\frac{\mathrm{d}}{\mathrm{d}t}\left(n_i\right) + \frac{n_i}{n_b}(3H)\end{aligned} \tag{12}$$

We are not interested in dY$_i$/dt, but rather dY$_i$/d(ln T). So we further manipulate the expression we already have by multiplying with $\frac{dt}{d(\ln T)}\frac{dT}{dT}$

$$
\begin{aligned}
\frac{dY_i}{d(\ln T)} &= \underbrace{\frac{\mathrm{d}Y_i}{\mathrm{d}t}}_{eq.12} \cdot \underbrace{\frac{dt}{dT}}_{T \text{ from } eq.8} \underbrace{\frac{dT}{d(\ln T)}}_{=T} \\
&= \left(\frac{1}{n_b}\frac{\mathrm{d}}{\mathrm{d}t}\left(n_i\right) + \frac{n_i}{n_b}(3H)\right)\right)\left(\frac{d}{dt}(T_0 a^{-1})\right)^{-1} \cdot T \\
&= \frac{1}{n_b}\left(\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(n_i\right) + n_i 3H}_{eq.3}\right)\left(-\frac{1}{HT}\right) \cdot T \\
&= -\frac{1}{HT}\left(\underbrace{\frac{n_j}{n_b}}_{=Y_j, eq.5}\Gamma_{j\to i} - \underbrace{\frac{n_i}{n_b}}_{=Y_i, eq.5}\Gamma_{i\to j}\right) \\
&= -\frac{1}{H}\left[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}\right]
\end{aligned}
\tag{13}
$$

I. B.

When temperature reach the rest energy of an electron, i.e. $k_B T < m_c^2 = 0.511\ MeV$, the electron and positron are no longer relativistic and will annihilate [1]. Before this point, the relativistic particles contributing to effective number of degrees of freedom, g_{*s}, are electrons, positrons and photons. Generally, $g_{*s} \neq g_*$, but in the early Universe the difference is so small and of little significance [1].

$$
\begin{aligned}
[g_{*s}]_{\text{before}} &= \sum_{i=\text{boson}} g_i + \frac{7}{8}\sum_{i=\text{fermions}} g_i \\
&= \underbrace{2}_{\gamma} + \frac{7}{8} \times \underbrace{2}_{e^+} \times \underbrace{2}_{e^-} \\
&= \frac{11}{2}
\end{aligned}
\tag{14}
$$

After the annihilation, only photons are contributing to g_{*s}

$$
[g_{*s}]_{\text{after}} = \sum_{i=\text{boson}} g_i = 2 \tag{15}
$$

The consequence of the annihilation is the fact that the entropy in a comoving volume must be conserved [1]. In our case, the entropy of e^+e^- is transferred to the photon gas and leads to temperature increase.
We find the conservation of entropy by equating eq. 14 and eq. 15

$$
\begin{aligned}
&[g_{*s}(aT^3)]_{\text{before}} = [g_{*s}(aT^3)]_{\text{after}} \\
&\frac{11}{2} a_{\text{before}} T^3_{\text{before}} = 2 a_{\text{after}} T^3_{\text{after}} \\
&\left(\frac{11}{4}\right)^{1/3} T_{\text{before}} = T_{\text{after}} = T_{\nu,\text{after}} \\
&\qquad T_{\text{before}} = T_{\nu,\text{after}} = \left(\frac{4}{11}\right)^{1/3} T_{\text{after}}
\end{aligned}
\tag{16}
$$

where the scale factor a is essentially the same before and after, due to annihilation occurring too fast for a to evolve. We also know neutrinos are thermally decoupled from the photons gas and take no part in the temperature increase [1]. The relation in eq. 16 states the photon temperature increase a bit as a result of the annihilation and neutrinos have lower temperature today than cosmic photons.

C.

We assume that photons and N_{eff} number of neutrino species make up all of the radiation in our Universe. The density parameter of that universe today is defined as

$$
\Omega_{r0} = \rho_{r0}/\rho_{c0} \tag{17}
$$

where r_{c0} is the critical energy density of the universe today given by

$$
\begin{aligned}
\rho_{c0} = \frac{3H_0^2}{8\pi G} &\approx 9.2 \times 10^{-27} \text{ kg m}^{-3} \\
&\approx 9.2 \times 10^{-30} \text{g cm}^{-3}
\end{aligned}
\tag{18}
$$

As our Universe only consists of photons we use eq. 4.13 from AST3220 lecture notes [1], which provides us with the expression for the energy density of gas of ultrarelativistic bosons

$$
\rho_{r0}c^2 = \frac{\pi^2}{30} g_* \frac{(k_b T_0)^4}{(\hbar c)^3} \tag{19}
$$

where the effective number of relativistic degrees of freedom is defined in eq. 4.22 of lecture notes [1]

$$
g_* = \sum_{\text{i = boson}} g_i \left(\frac{T_i}{T}\right) + \frac{7}{8}\sum_{\text{i = fermions}} g_i \left(\frac{T_i}{T}\right) \tag{20}
$$

where T is photon temperature and i is particle specie. In our case, we have

- Photons $\to g_\gamma = 2$
- Number of neutrino species $\to N_{\text{eff}} = 3$ $(\nu_e, \nu_\mu, \nu_\tau)$.

Thus, we have

$$g_* = 2\left(\frac{\overbrace{T_\gamma}^{=T}}{T}\right)^4 + \frac{7}{8}\cdot 2N_{\text{eff}}\left(\frac{\overbrace{T_\nu}^{eq.16}}{T}\right)^4 \\ = 2 + \frac{7}{4}N_{\text{eff}}\left(\frac{4}{11}\right)^{4/3} \tag{21}$$

where we have multiplied N_{eff} with 2 for the antiparticles. Inserting eq. 21 back into eq. 19 we get the following expression

$$\rho_{r0}c^2 = \frac{\pi^2}{30}\left[2 + \frac{7}{4}N_{\text{eff}}\left(\frac{4}{11}\right)^{4/3}\right]\frac{(k_BT_0)^4}{(\hbar c)^3} \tag{22}$$

Lastly, we insert eq. 22 into eq. 17 to obtain an expression for Ω_{ro}

$$\begin{aligned}\Omega_{r0} &= \frac{\rho_{r0}c^2}{\rho_{c0}} = \frac{\frac{\pi^2}{30}\left[2 + \frac{7}{4}N_{\text{eff}}\left(\frac{4}{11}\right)^{4/3}\right]\frac{(k_BT)^4}{(\hbar c)^3}}{\frac{3H_0^2}{8\pi G}} \\ &= \frac{8\pi^3 G}{90H_0^2}\frac{(k_BT_0)^4}{\hbar^3c^3}2\left[1 + \frac{7}{8}N_{\text{eff}}\left(\frac{4}{11}\right)^{4/3}\right] \\ &= \frac{\rho_{r0}}{\rho_{c0}} = \frac{8\pi^3}{45}\frac{G}{H_0^2}\frac{(k_BT_0)^4}{\hbar^3c^5}\left[1 + \frac{7}{8}N_{\text{eff}}\left(\frac{4}{11}\right)^{4/3}\right]\end{aligned} \tag{23}$$

D.

The Friedman equation at the time of BBN where our Universe was completely dominated by radiation is

$$H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = H_0\sqrt{\Omega_{r0}}a^{-2} \tag{24}$$

where H_0 is the Hubble parameter defined as $H_0 = 100h\ km\ s^{-1}\ Mpc^{-1}$. We rearrange the terms and integrate both sides to obtain an expression for $a(t)$ and $t(T)$.

$$\begin{aligned}\frac{1}{a}a^2da &= H_0\sqrt{\Omega_{r0}}dt \\ \int_{a=0}^{a}\frac{1}{a}a^2da &= \int_{t=0}^{a}H_0\sqrt{\Omega_{r0}}dt \\ \frac{1}{2}a^2 &= H_0\sqrt{\Omega_{r0}}t \\ a(t) &= \sqrt{2H_0\sqrt{\Omega_{r0}}t}\end{aligned} \tag{25}$$

Then we use the relation $a = T_0/T$ from eq. 8, to find an expression for t(T).

$$t(T) = \frac{T_0^2}{T^2 2H_0\sqrt{\Omega_{r0}}} \tag{26}$$

T_0 is the temperature today of the cosmic microwave background (CMB) with the value of $2.72548\pm0.00057\ K$ [2].
The age of the Universe at

- $t(10^{10}\ K) = 1.06128\ s$
- $t(10^9\ K) = 10612.89405\ s$
- $t(10^8\ K) = 106128940.58165\ s$

E.

Before BBN starts, we assume the baryonic matter in the universe consists of neutrons and protons at thermal equilibrium, i.e. $T_i = T_p = T_n$. The relation between equilibrium number densities for neutrons and protons is given in the task description as

$$\frac{n_n^{(0)}}{n_p^{(0)}} = \left(\frac{m_p}{m_n}\right)^{3/1} e^{-(m_n-m_p)c^2/k_BT_i} \tag{27}$$

where $n^{(0)}$ denotes the equilibrium number densities. We use that $m_p \approx m_n$ outside of exponentials so that our expression becomes

$$\frac{n_n^{(0)}}{n_p^{(0)}} = e^{-(m_n-m_p)c^2/k_BT_i} \tag{28}$$

Relative number density is defined in eq. 5, which will be used to find the relative number densities for neutrons and protons at the initial temperature T_i. We also know that $n_b = n_p + n_n$.

$$Y_n(T_i) = \frac{n_p}{n_b} = \frac{n_p}{n_p + n_n} = \frac{\cancel{n_p}}{\cancel{n_p}\left(1 + \frac{n_n}{n_p}\right)} \tag{29}$$

We can insert eq. 28 into eq. 32

$$Y_n(T_i) = [1 + e^{-(m_n-m_p)c^2/k_BT_i}]^{-1} \tag{30}$$

The relative number density for neutrons at the initial temperature T_i is

$$Y_n(T_i) = \frac{n_n}{n_p + n_n} \tag{31}$$

and we find that the sum of eq. 32 and eq. 31 is

$$\begin{aligned}Y_n(T_i) + Y_p(T_i) &= \frac{n_n}{n_p + n_n} + \frac{n_p}{n_p + n_n} = 1 \\ &\Rightarrow Y_p(T_i) = 1 - Y_n(T_i)\end{aligned} \tag{32}$$

F.

We can solve equation Boltzmann equations (eq. 9) for weak interactions where i = n, p. Our temperature interval is defined from $T_i = 100 \cdot 10^9\ K$ to $T_f = 0.1 \cdot 10^9\ K$. The decay rates $\Gamma_{n\to p}$ and $\Gamma_{p\to n}$ are found in table 2 of ref [3].

$$\begin{aligned}\Gamma_{n\to p}(T,q) =& \frac{1}{\tau}\Bigg[\int_1^\infty \frac{(x+q)^2(x^2-1)^{1/2}x}{[1-e^{xZ}[1+e^{-(x+q)Z_\nu}]}dx \\ &+ \int_1^\infty \frac{(x+q)^2(x^2-1)^{1/2}x}{[1-e^{-xZ}[1+e^{(x-q)Z_\nu}]}dx\Bigg] \\ \Gamma_{p\to n}(T,q) =& \Gamma_{n\to p}(T,-q)\end{aligned} \tag{33}$$

where $\tau = 1700\ s$ is the free neutron decay time, $q = (m_n - m_p)/m_e = 2.53$, $Z = m_e c^2/k_B T = 5.93/T_9$ and $Z_\nu = m_e c^2/k_B T_\nu = 5.93/T_{9\nu}$. T_9 is defined as $T/10^9$ and T_ν as $T_\nu/10^9$. Neutrino temeprature T_ν can be found in eq. 16 from task b). Initial conditions are given in eq. 30 and eq. 32.
Solutions to Y_n and Y_p and their equilibrium values can be found in fig. 1.

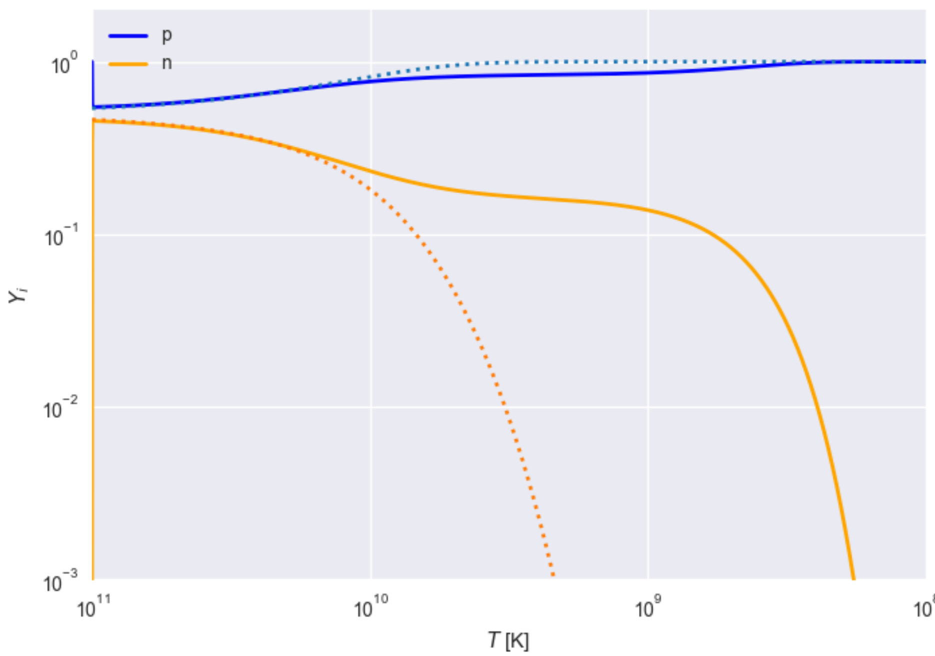

Figure 1. Solutions of Y_n and Y_p from eq. 9 shown in solid lines and equilibrium values from eq. 30 and eq. 32 shown in dotted lines.

G.

The general Boltzmann equation for particle i that interacts with any number of other particles j, both through decays and two-body reactions is given in the task description as

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = \sum_{j\neq i}[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}] + \sum_{ijk}[n_k n_l \gamma_{kl\to ij} - n_i n_j \gamma_{ij\to kl}] \tag{34}$$

We define $\Gamma_{ij\to kl} = n_b\gamma_{ij\to kl} \Rightarrow \gamma_{ij\to kl} = \frac{1}{n_b}\Gamma_{ij\to kl}$, so that we can rewrite eq. 34 with this definition

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = \sum_{j\neq i}[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}] + \sum_{ijk}[\frac{n_k n_l}{n_b}\Gamma_{kl\to ij} - \frac{n_i n_j}{n_b}\Gamma_{ij\to kl}] \tag{35}$$

Inserting eq. 35 into eq. 9 from task A yields

$$
\begin{aligned}
\frac{\mathrm{d}Y_i}{\mathrm{d}(\ln T)} &= -\frac{1}{H}\frac{1}{n_b}\left[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}\right] = -\frac{1}{H}\frac{1}{n_b}\left[\underbrace{\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i}_{\text{Insert } eq.9}\right] \\
&= -\frac{1}{H}\frac{1}{n_b}\left\{\sum_{j\neq i}[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}] + \sum_{ijk}[\frac{n_k n_l}{n_b}\Gamma_{kl\to ij} - \frac{n_i n_j}{n_b}\Gamma_{ij\to kl}]\right\} \\
&= -\frac{1}{H}\left\{\sum_{j\neq i}[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}] + \sum_{ijk}[Y_k Y_l\Gamma_{kl\to ij} - Y_i Y_j\Gamma_{ij\to kl}]\right\}
\end{aligned} \tag{36}
$$

where we used eq. 5 rewrite the number densities to relative number densities.

H.

Deuterium bottleneck occurs when the temperature falls around $T \approx 9 \cdot 10^8\ K$ which is where nuclides heavier than hydrogen are produced. What we have been describing up until this point is what happens before the deuterium bottleneck. We shall now solve Boltzmann equations for $i = n, p, D$ within the same temperature interval and initial conditions as previously. Boltzmann equations for n, p, and D are as follows

$$
\begin{aligned}
\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} &= -\frac{1}{H}\left\{-\lambda_w(n)Y_n + \lambda_w(p)Y_p + \lambda_\gamma(D)Y_D - [pn]Y_nY_p\right\} \\
\frac{\mathrm{d}Y_p}{\mathrm{d}(\ln T)} &= -\frac{1}{H}\left\{-\lambda_w(p)Y_p + \lambda_w(n)Y_n + \lambda_\gamma(D)Y_D - [pn]Y_nY_p\right\} \\
\frac{\mathrm{d}Y_D}{\mathrm{d}(\ln T)} &= -\frac{1}{H}\left\{-\lambda_\gamma(D)Y_D + [pn]Y_nY_p\right\}
\end{aligned} \tag{37}
$$

where $\lambda_w(n) = \Gamma_{n\to p}$, $\lambda_w(p) = \Gamma_{p\to n}$, $\lambda_\gamma(D) = \Gamma_{D\to n} = \Gamma_{D\to p}$, and $[pn] = \Gamma_{np\to D_\gamma}$. The first two decay rates are the same as the previous task (eq. 33). From ref. [3] we find pn and $\lambda_\gamma(D)$.

$$
\begin{aligned}
[pn] &= 2.5 \cdot 10^4\ \rho_b \\
\lambda_\gamma(D) &= 4.68 \cdot 10^9\ [pn]\ \rho_b^{-1}\ T_9^{3/2}\exp(-25.82\ T_9^{-1})
\end{aligned} \tag{38}
$$

where $\rho_b = \Omega_{b0}/\rho_{c0}$ (values can be found in table I) and $T_9 = T/10^9$.

Now, comparing fig. 2 to fig. 1, we see that neutron decays significantly faster, as Deuterium is being produced.

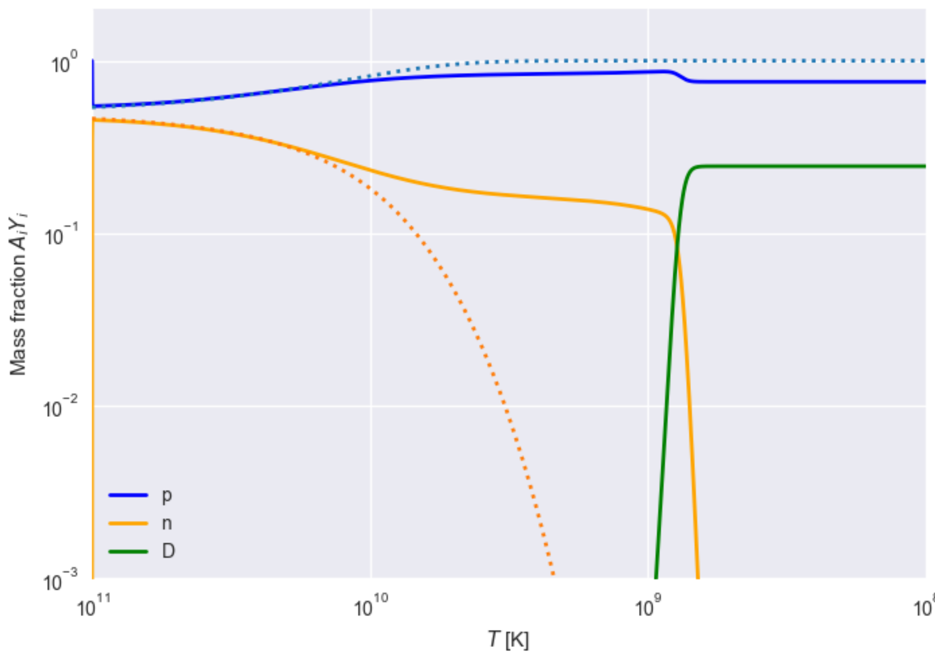

Figure 2. The solution of Y_n, Y_p and $2Y_D$ from eq. 37, shown in solid, as well as the equilibrium values from eq. 30 and eq. 32, shown in dotted lines.

I.

We are now in a position to implement all reactions necessary for accurately computing the abundance of elements up to Li^7. We do this by solving Boltzmann equations (generalized Boltzmann equation can be found in eq. 35) for n, p, D, T, He^3, He^4, and Be^7, using reactions 1)-3) from part a), and 1)-11) + 15)-18) + 20) + 21) from part b of Table 2 in ref. [3]. Results can be found in fig. 3.

J.

We can use BBN to learn more about the content of our universe by comparing our theoretical predictions to observations. The observables we will be using are the abundance fractions of D and Li^7 relative to hydrogen, Y_D/Y_p, and Y_{Li^7}/Y_p, and the mass fraction of He^4, 4Y$_{He^4}$. Remember that: Y$_{He^3}$(today) $= Y_{He^3} + Y_T$, and Y$_{Li^7}$(today) $=$ Y$_{Li^7} + Y_{Be^7}$. The observables we will be comparing our theoretical predictions to are

$$\begin{aligned} Y_D/Y_p &= (2.57 \pm 0.03) \cdot 10^{-5}, \\ 4Y_{He^4} &= 0.254 \pm 0.003, \\ Y_{Li^7}/Y_p &= (1.6 \pm 0.3) \cdot 10^{-10}. \end{aligned} \tag{39}$$

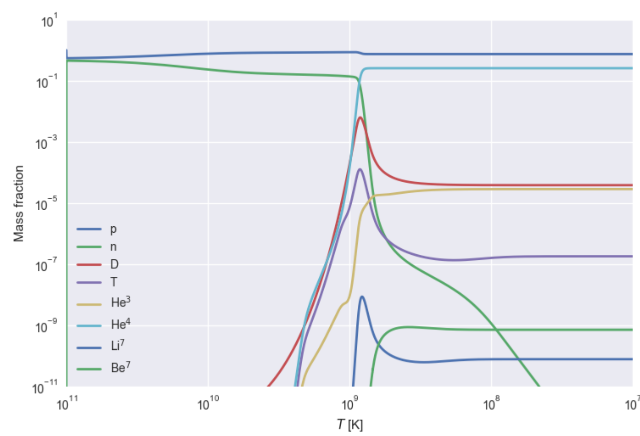

Figure 3. The mass fractions A_iY_i, where A_i is the mass number of particle i (i.e. number of neutrons plus protons).

We compare predictions to observables by computing the relic abundances for different values of $\Omega_{b0} = [0.01, 1]$, and using the chi-squared-method to find the most probable value for Ω_{b0}. Results can be found in fig. 4

We hoped the most probable value for Ω_{b0} to be 0.05, since it is the value we have been using for this model.

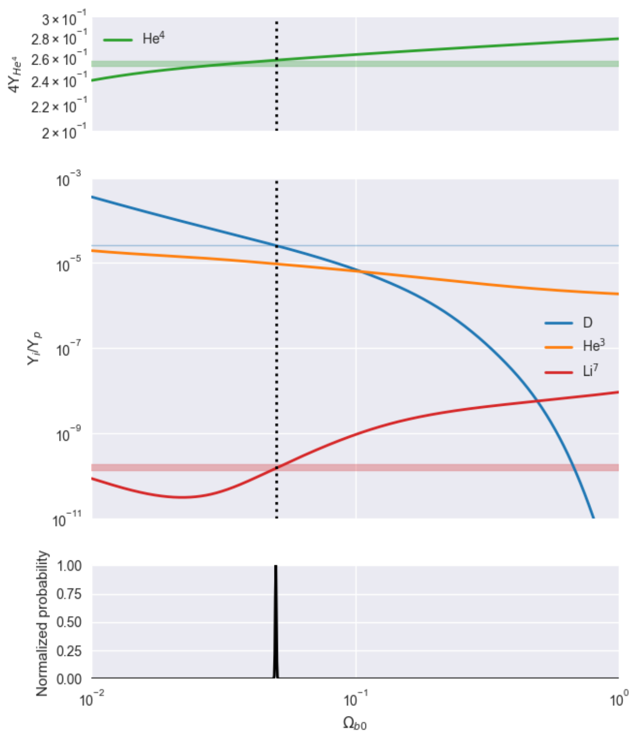

Figure 4. Relic abundance of elements are shown as a function of the baryon density Ω_{b0}, along with measurements eq. 39 (horizontal shaded regions). In the lower plot, the normalized probability is shown. The best-fit value of Ω_{b0} is indicated by the dotted line.

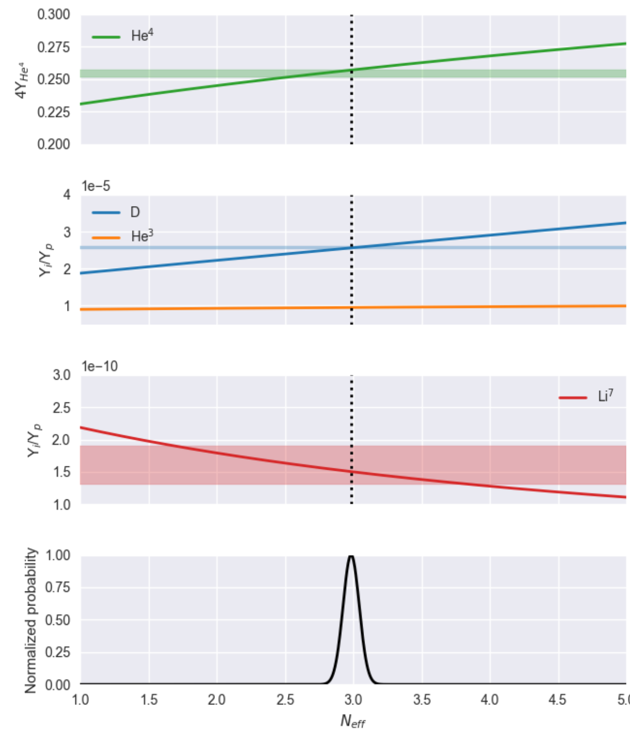

Figure 5. Relic abundance of elements are shown as a function of the effective number of neutrino species N_{eff}, along with measurements eq. 39 (horizontal shaded regions). In the lower plot, the normalized probability is shown. The best-fit value of $N_{\rm eff}$ is indicated by the dotted line.

However, our computations say $\Omega_{b0} \approx 0.03$ is the most probable value. Either our model cannot compute relic abundances accurately, or there is a numerical error in my program. I hope it is the last-mentioned.
The total matter content of the universe is around $\Omega_{m0} = 0.3$, where a significant fraction of this is in the form of some unknown and unseen (dark matter). Our value for Ω_{b0} from BBN tells us that dark matter is non-baryonic matter.

K.

Similarly to task j, we can do the exact same procedure to find the most probable value for the effective number of neutrinos, $N_{\rm eff}$, by comparing theoretical predictions to observables (found in eq. 39).

We find the most probable value for $N_{\rm eff}$ is 3, which is what we expected. This also confirms that there is a numerical error in task j, which lead us to not get the expected value for Ω_{b0}.

II. REFERENCES

[1] Elgarøy, Ø. AST3220 – Cosmology I. [PDF].
[2] Fixsen, D. J. (2009). THE TEMPERATURE OF THE COSMIC MICROWAVE BACKGROUND. The Astrophysical Journal, 707(2), 1. [URL]
[3] Wagoner, R. V., Fowler, W. A., Hoyle, F. (1967). ON THE SYNTHESIS OF ELEMENTS AT VERY HIGH TEMPERATURES. The Astrophysical Journal, 148, 10. [URL]