AST3220 Project 3: Inflation without approximation

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This abstract is abstract.

I. EQUATIONS GIVEN IN PROJECT DESCRIPTION

We are given equations for Planck quantities such as:

- Planck energy

$$E_P^2 = \frac{\hbar c^5}{G} \tag{1}$$

where $\hbar$ is Planck's constant, c is the speed of light, and G is the gravitational constant.

- Planck mass

$$m_P^2 = \frac{\hbar c}{G} \tag{2}$$

- Planck length

$$l_P^2 = \frac{\hbar G}{c^3} \tag{3}$$

Assuming spatial flatness and that the scalar field dominates the energy density, the equations governing the evolution of the scalar field and the scale factor are

$$\ddot{\phi} + 3H\dot{\phi} + \hbar c^3 V'(\phi) = 0 \tag{4}$$

and

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right] \tag{5}$$

We define the quantity

$$H_i \equiv \frac{8\pi G}{3c^2}V(\phi_i) \tag{6}$$

where ϕ_i is the initial value of the field.

II. TASK A

Before solving equations 4 and 5 numerically it is useful to write them in terms of the following dimensionless variables.

- τ, time

$$\tau = H_i t, \quad [\tau] = \left[\frac{1}{s}\right]\cdot[s] = 1 \tag{7}$$

- h, Hubble parameter

$$h = \frac{H}{H_i} \tag{8}$$

This quantity is unitless because H and H_i have the same units.

- ψ, scalar field

$$\psi = \frac{\phi}{E_p} = \frac{[J]}{[J]} = 1 \tag{9}$$

ϕ is measured in units of energy. Thus, ψ is unitless.

- v, potential

$$\begin{aligned} v &= \frac{\hbar c^3}{H_i^2 E_P^2}V \\ [v] &= \frac{\hbar c^3}{\frac{8\pi G}{3c^2}V(\phi_i)\frac{\hbar c^5}{G}}V(\phi) = \frac{3}{8\pi} \end{aligned} \tag{10}$$

where we have used the definition of H_i^2 from equation 6 and the Planck energy defined in equation 1.

III. TASK B

In this task, we want to show the following expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\ln\left(\frac{a}{a_i}\right)\right] = h(\tau) \tag{11}$$

and that eq. 4 and 5 can be rewritten as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 3h\frac{\mathrm{d}\psi}{\mathrm{d}\tau} + \frac{\mathrm{d}v}{\mathrm{d}\psi} = 0 \tag{12}$$

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + v(\psi)\right] \tag{13}$$

A. Showing expression 11

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\ln\left(\frac{a}{a_i}\right)\right] = h(\tau) \tag{14}$$

From the definition in eq. 8 we known that $h = H/H_i$ where $H = \dot{a}/a$ and similarly for H_i. We can therefore begin by multiplying with $\dot{a}/\dot{a}$ inside the parentheses.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\ln\left(\frac{a}{a_i}\cdot\frac{\dot{a}}{\dot{a}}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\ln\left(\underbrace{\frac{a}{\dot{a}}}_{1/H}\cdot\underbrace{\frac{\dot{a}}{a_i}}_{H_i}\right)\right] \tag{15}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\ln\left(\frac{H_i}{H}\right)\right] = \frac{H}{H_i} \tag{16}$$

$$= h(\tau) \tag{17}$$

$$\tag{18}$$

B. Rewriting eq. 4

We begin by writing out the time derivates as differentials

$$\ddot{\phi} + 3H\dot{\phi} + \hbar c^3 V'(\phi) = 0 \tag{19}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + 3H\frac{\mathrm{d}\phi}{\mathrm{d}t} + \hbar c^3\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \tag{20}$$

We arrange the terms in equations 9 and 7 to obtain expressions for t and ϕ

$$\phi = \psi E_P \tag{21}$$

$$t = \frac{\tau}{H_i} \tag{22}$$

which is then inserted into 19

$$\frac{\mathrm{d}^2\psi E_P}{\mathrm{d}\tau^2/H_i^2} + 3H\frac{\mathrm{d}\psi E_P}{\mathrm{d}\tau/H_i} + \hbar c^3\frac{\mathrm{d}V}{\mathrm{d}\psi E_P} = 0 \tag{23}$$

$$E_P H_i^2\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 3H_i H E_P\frac{\mathrm{d}\psi}{\mathrm{d}\tau} + \frac{\hbar c^3}{E_P}\frac{\mathrm{d}V}{\mathrm{d}\psi} = 0 \quad \Big| \cdot \frac{1}{E_P H_i^2} \tag{24}$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 3\underbrace{\frac{H}{H_i}}_{=h(eq.8)}\frac{\mathrm{d}\psi}{\mathrm{d}\tau} + \underbrace{\frac{\hbar c^3}{E_P^2 H_i^2}}_{=v/V(eq.10)}\frac{\mathrm{d}V}{\mathrm{d}\psi} = 0 \tag{25}$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 3h\frac{\mathrm{d}\psi}{\mathrm{d}\tau} + \frac{\mathrm{d}v}{\mathrm{d}\psi} = 0 \tag{26}$$

C. Rewriting eq. 5

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right] \tag{27}$$

We insert the dimensionless variables $t = \tau/H_i$ (eq. 7), $H = hH_i$ (eq. 8) and $\phi = \psi E_P$ (eq. 9).

$$(hH_i)^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\left(\frac{\mathrm{d}\psi E_P}{\mathrm{d}(\tau/H_i)}\right)^2 + V(\phi)\right] \tag{28}$$

$$(hH_i)^2 = \frac{8\pi G}{3c^2}\left[\frac{E_P^2 H_i^2}{2\hbar c^3}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + V(\phi)\right]\Bigg| \cdot \frac{1}{H_i^2} \tag{29}$$

$$h^2 = \frac{8\pi G}{3c^2 H_i^2}\left[\frac{E_P^2 H_i^2}{2\hbar c^3}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + V(\phi)\right] \tag{30}$$

$$= \frac{8\pi G}{3c^2 H_i^2}\frac{\overbrace{E_P^2}^{eq.1} H_i^2}{\hbar c^3}\left[\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + \underbrace{\frac{E_P^2 H_i^2}{\hbar c^3}V(\phi)}_{=v,eq.10}\right] \tag{31}$$

$$= \frac{8\pi \cancel{G}}{3\cancel{c^2}\cancel{H_i^2}}\frac{\cancel{\hbar c^5}\cancel{H_i^2}}{\cancel{G}\cancel{\hbar c^3}}\left[\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + v(\psi)\right] \tag{32}$$

$$= \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + v(\psi)\right] \tag{33}$$

IV. TASK C

The scalar field ψ with potential field $V(\phi)$ is analogous to a ball being dropped down from a hill. If the ball rolls down sufficiently slowly (time derivative somewhat equal to 0), the potential energy can be treated as essentially constant for a significant portion of the way down to the minimum [1]. From equation 13 we see that $d\psi/d\tau \approx 0$, thus the initial value for $d\psi/d\tau$ is irrelevant as long as it does not violate the slow-roll conditions.

V. TASK D

We want to find the initial value for the field that will give 500 e-folds of inflation given the following potential

$$V(\phi) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2 \tag{34}$$

N is the number that measures how many e-foldings are left until inflation ends defined as

$$N(t) = -\frac{8\pi}{E_P^2}\int_t^{t_{\mathrm{end}}}\frac{V}{V'}\dot{\phi}dt = \frac{8\pi}{E_P^2}\int_{\phi_{\mathrm{end}}}^{\phi}\frac{V}{V'}d\phi \tag{35}$$

where ϕ_{end} can be found from the criterion $\epsilon(\phi_{\mathrm{end}}) = 1$. We begin by finding $dV/d\phi$

$$V' = \frac{\partial V}{\partial \phi} = \frac{m^2c^4}{(\hbar c)^3}\phi \tag{36}$$

and inserting into equation 35 for 500 e-folds.

$$500 = \frac{8\pi}{E_P^2}\int_{\phi_{\text{end}}}^{\phi_i} \frac{\frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2}{\frac{m^2c^4}{(\hbar c)^3}\phi} d\phi \tag{37}$$

$$500 = \frac{4\pi}{E_P^2}\int_{\phi_{\text{end}}}^{\phi_i} \phi d\phi \tag{38}$$

$$500 = \frac{4\pi}{E_P^2}\frac{1}{2}[\phi_i^2 - \phi_{\text{end}}^2] \tag{39}$$

$$500 = \frac{2\pi}{E_P^2}[\phi_i^2 - \phi_{\text{end}}^2] \tag{40}$$

The slow-roll parameter ϵ is defined as

$$\epsilon = \frac{E_P^2}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{41}$$

Now we evaluate ϵ for the potential given in eq. 34

$$\begin{aligned}\epsilon &= \frac{E_P^2}{16\pi}\left(\frac{\frac{m^2c^4}{(\hbar c)^3}\phi}{\frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2}\right)^2 \\ &= \frac{E_P^2}{4\pi\phi^2} = \eta\end{aligned} \tag{42}$$

For $\epsilon(\phi_{\text{end}}) = 1$ we have the following expression for ϕ_{end}

$$\epsilon(\phi_{\text{end}}) = 1 \Rightarrow \frac{E_P^2}{4\pi\phi_{\text{end}}^2} = 1 \tag{43}$$

$$\phi_{\text{end}} = \sqrt{\frac{E_P^2}{4\pi}} = \frac{E_P}{2\sqrt{\pi}} \tag{44}$$

Inserting this expression back into equation 40 yields

$$500 = \frac{2\pi}{E_P^2}[\phi_i^2 - \phi_{\text{end}}^2] \tag{45}$$

$$500 = \frac{2\pi}{E_P^2}\left[\phi_i^2 - \frac{E_P^2}{4\pi}\right] \tag{46}$$

$$500 = \frac{2\pi\phi_i^2}{E_P^2} - \frac{1}{2} \tag{47}$$

$$\phi_i^2 = \left(500 + \frac{1}{2}\right)\frac{E_P^2}{2\pi} \tag{48}$$

$$\phi_i = E_P\sqrt{\frac{500 + 1/2}{2\pi}} \tag{49}$$

The initial value for the field that will give 500 e-folds is $\phi_i \approx 8.93E_P$.

VI. TASK E

In this task we wish to solve equations 11, 12 and 13. Before solving the equations we need to rewrite them for easier implementation.

Equation 11 can be simplified in the following way

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\underbrace{\ln\left(\frac{a}{a_i}\right)}_{x}\right] = h(\tau) \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = h\end{gathered} \tag{50}$$

For equation 12, we want to express this in terms of $d^2\psi/d\tau^2$ and substitute $d\psi/d\tau \to y$:

$$\begin{gathered}\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau^2} + 3h\underbrace{\frac{\mathrm{d}\psi}{\mathrm{d}\tau}}_{y} + \frac{\mathrm{d}v}{\mathrm{d}\tau} = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} = -3hy - \frac{\mathrm{d}v}{\mathrm{d}\tau}\end{gathered} \tag{51}$$

Equation 13 can be rewritten using the same substitution

$$\begin{gathered}h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\underbrace{\frac{\mathrm{d}\psi}{\mathrm{d}\tau}}_{y}\right)^2 + v(\psi)\right] \\ h^2 = \frac{8\pi}{3}\left[\frac{1}{2}y^2 + v(\psi)\right]\end{gathered} \tag{52}$$

Lastly, we need an expression for v(ϕ) and its derivative using equation 10 combined with eq. 34 and 6

$$\begin{aligned}v &= \hbar c^3\underbrace{\frac{3c^2 2(\hbar c)^3}{8\pi G m^2c^4\phi_i^2}}_{1/H_i^2}\frac{1}{E_P^2}\underbrace{\frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2}_{V(\phi)} \\ &= \frac{\hbar c^5}{8\pi G E_P^2}\left(\frac{\phi}{\phi_i}\right)^2 = \frac{3}{8\pi}\left(\frac{\phi}{\phi_i}\right)^2\end{aligned} \tag{53}$$

The derivate of $v(\phi)$ is

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}\psi} &= \frac{d}{d\psi}\frac{\hbar c^5}{8\pi G E_P^2}\left(\frac{\phi}{\phi_i}\right)^2 \\ &= \frac{d}{d\psi}\frac{\hbar c^5}{8\pi G\phi_i^2}\underbrace{\frac{\phi^2}{E_P^2}}_{\psi^2} \\ &= \frac{\hbar c^5}{4\pi G\phi_i^2}\psi\end{aligned} \tag{54}$$

where ϕ_i was found in task D. The chosen method for solving the equations is Euler's method. Initial conditions are the following

- $h(0) = H(0)/H_i = 1$
- $\ln(a(0)/a_i) = \ln(a_i/a_i) = 0$

An important relation to remember is

$$\psi = \frac{\phi}{E_P} \tag{55}$$

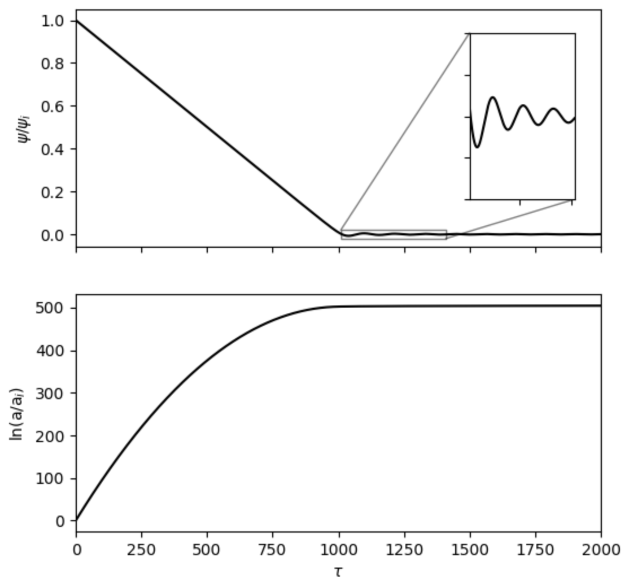

Figure 1. The numerical solution to the scalar field and time evolution of the logarithmic scale factor.

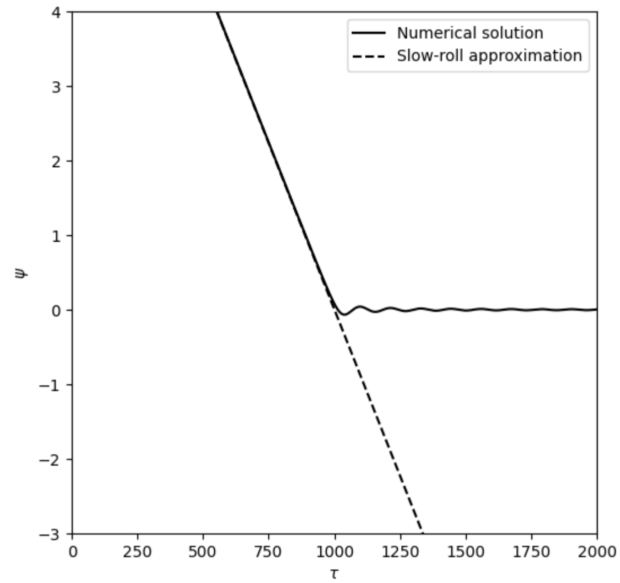

Figure 2. Slow-roll approximation (dashed line) compared to numerical solution (solid line).

which can be used to find ψ_i given ϕ_i.

Based on lectures, we expect ψ to oscillate at the minimum of the potential. The term $3H\dot{\phi}$ in eq. 4 is a friction term, thus the oscillations are damped [1]. Figure VI shows the time evolution of the scalar field ψ and the logarithmic scale factor $\ln(a/a_i)$. Our expectations are confirmed.

VII. TASK F

We want to compare the numerical solution from the previous task with the slow-roll approximation (SRA) given in the lecture notes.

$$\phi(t) = \phi_i - \frac{mc^2 E_P}{\hbar\sqrt{12\pi}} t \tag{56}$$

where H is given by

$$H = \sqrt{\frac{4\pi}{3}} \frac{mc^2}{\hbar E_P} \left(\phi_i - \frac{mc^2 E_P}{\hbar\sqrt{12\pi}} t \right) \tag{57}$$

Rewriting the SRA using $\psi = \psi/E_P$ and $\tau = H_i t$

$$\psi = \psi_i - \frac{\tau}{4\pi\psi_i} \tag{58}$$

Figure 2 shows the slow-roll approximation compared to the numerical solution. The SRA deviates linearly from the numerical solution after $\tau \approx 1000$.

VIII. TASK G

We are interested in plotting the time-evolution of the slow-roll parameter ϵ, i.e. ϵ as a function of τ. We have an expression for ϵ from eq. 42. We implement the expression for ϵ using ϕ calculated in task e, and the result is shown in figure 3. We found N_{tot} = 501.5, which is close to the expected value from SRA predicting 500 e-foldings.

The dashed line shows the ending of inflation at $\epsilon = 1$ where the SRA breaks down. Past this point we see strong oscillations. However, the parameter is no longer relevant.

IX. TASK H

We have that

$$\rho_\phi c^2 = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 + V(\phi) \tag{59}$$

and

$$p_\phi = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 - V(\phi) \tag{60}$$

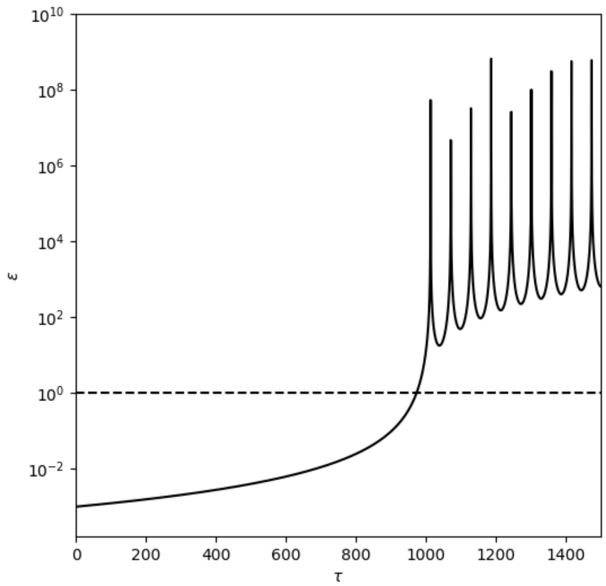

Figure 3. Slow roll parameter ϵ as a function of τ. Dashed line shows the end of inflation at $\epsilon = 1$. The y-axis is logarithmic scaled.

Eq. 60 can be rewritten in terms of the dimensionless variables so

$$p_\phi = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 - V(\phi) \tag{61}$$

$$= \frac{1}{2}\frac{1}{\hbar c^3}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 - V \tag{62}$$

$$= \frac{1}{2}\frac{1}{\hbar c^3}\left(\frac{\mathrm{d}\psi E_P}{\mathrm{d}\tau/H_i}\right)^2 - \frac{H_i^2 E_P^2}{\hbar c^3}v \tag{63}$$

$$= \frac{1}{2}\frac{E_P^2 H_i^2}{\hbar c^3}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 - \frac{H_i^2 E_P^2}{\hbar c^3}v \tag{64}$$

$$= \frac{H_i^2 E_P^2}{\hbar c^3}\left(\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 - v\right) \tag{65}$$

By the same method, we get the following for $\rho_\phi c^2$

$$\rho_\phi c^2 = \frac{H_i^2 E_P^2}{\hbar c^3}\left(\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + v\right) \tag{66}$$

From this we get

$$\frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 + v} \tag{67}$$

X. TASK I

The relation between pressure and density obtained in task h is solved in the same function as task e, i.e. through Euler's method given the initial value for $d\psi/d\tau$ and v.
We expect $d\psi/d\tau$ to be close to -1 in the slow-roll regime before in oscillates between -1 and 1 when the SRA breaks down. Figure 4 shows the relation between pressure and density over time and indeed it confirms our theory.

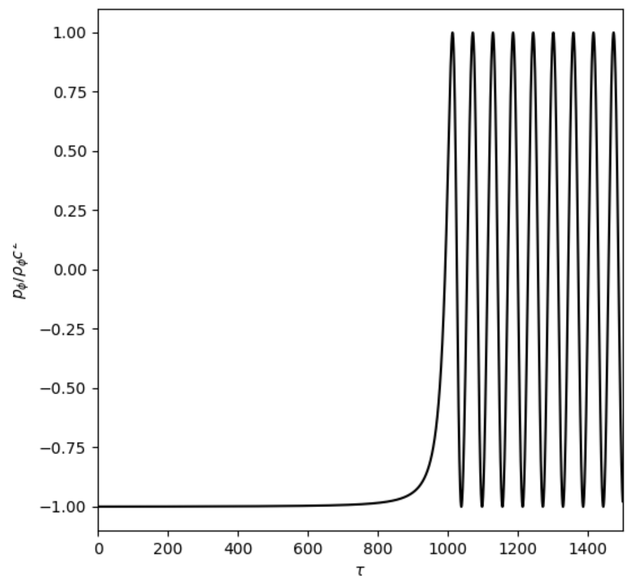

Figure 4. Ratio between pressure and density as a function of time.

XI. TASK J

We are interested in plotting the slow-roll parameters ϵ and η as a function of remaining e-foldings before the end of inflation. From eq. 42 we know that $\epsilon = \eta$, so we only need to plot one. I choose to plot ϵ. In task G we found the total number of e-foldings. The number of e-foldings left is found through this relation

$$N_{\text{left}} = N_{\text{tot}} - \ln(a/a_i) \tag{68}$$

where $\ln(a/a_i)$ sliced so it has the same indices as ϵ when $\epsilon \le 1$, i.e. before the inflation ends. Figure 5 shows the result.

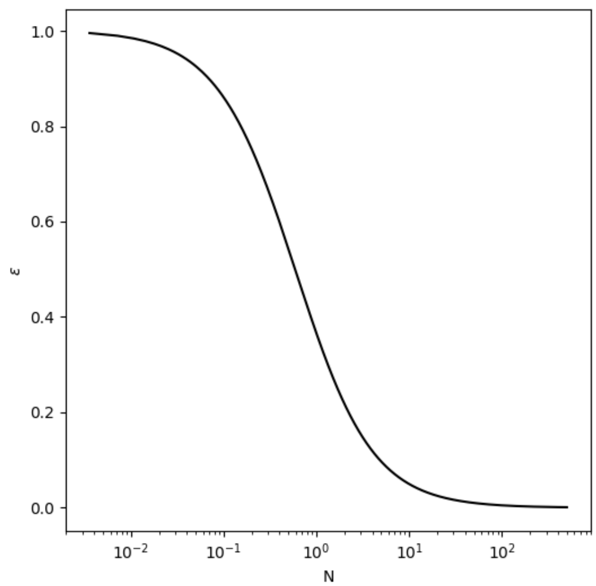

Figure 5. Slow-roll parameter ϵ as functions of N, the number of e-foldings remaining before the end of inflation. The x-axis is logarithmic scaled.

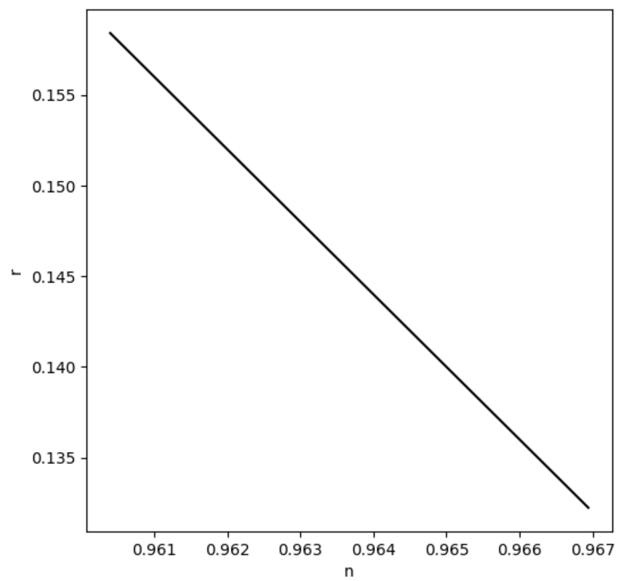

Figure 6. Predictions of the ϕ^2-model of inflation where $N \in [50, 60]$.

XII. TASK K

The tensor-to-scalar ratio r is defined as

$$r = 16\epsilon \tag{69}$$

and the spectral index n is defined as

$$\begin{aligned} n &= 1 - 6\epsilon + 2\underbrace{\eta}_{=\epsilon} \\ \Rightarrow n &= 1 - 4\epsilon \end{aligned} \tag{70}$$

The expression for ϵ is given in eq. 42 where ϕ is found through eq. 35.

$$\epsilon = \frac{E_P^2}{4\pi\left(\frac{NE_P^2}{2\pi} + \phi_{\text{end}}^2\right)} \tag{71}$$

N is varying from 50 to 60. Figure 6 shows the prediction of the ϕ^2-model in the n-r plane.

XIII. TASK L

In the Starobinksy model, the potential is given as

$$V(\phi) = \frac{3M^2M_P^2}{4}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_P}}\right)^2 \tag{72}$$

where $\hbar = c = 1$, so $E_P^2 = \frac{1}{G}$. M_P is the Planck mass now defined as $M_P^2 = \frac{1}{8\pi G} = \frac{E_P^2}{8\pi}$. The slow roll parameter ϵ is defined in eq. 41

$$\epsilon = \frac{E_P^2}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{73}$$

We begin by differentiating V

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{3M^2M_P^2}{4}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_P}}\right)^2 \tag{74}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{3M^2M_P^2}{4}\left(1 - 2e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} + e^{-2\sqrt{\frac{2}{3}}\frac{\phi}{M_P}}\right) \tag{75}$$

$$= \frac{3M^2M_P^2}{4}\left(\frac{2}{M_P}\sqrt{\frac{2}{3}}e^y - \frac{2}{M_P}\sqrt{\frac{2}{3}}e^{2y}\right) \tag{76}$$

$$= \frac{3M^2M_P^{\cancel{2}\,1}}{\cancel{4}\,2}\frac{\cancel{2}}{\cancel{M_P}}\sqrt{\frac{2}{3}}(e^y - e^{2y}) \tag{77}$$

$$= \frac{3}{2}M^2M_P\sqrt{\frac{2}{3}}e^y(1 - e^y) \tag{78}$$

where $y = -\sqrt{\frac{2}{3}}\frac{\phi}{M_P} = -\sqrt{\frac{16\pi}{3}}\psi$. Inserting V and V' into eq. 41

$$\epsilon = \frac{E_P^2}{16\pi}\left(\frac{\frac{3}{2}M^2 M_P\sqrt{\frac{2}{3}}e^y(1-e^y)}{\frac{3M^2M_P^2}{4}(1-e^y)^2}\right)^2 \tag{79}$$

$$= \frac{E_P^2}{16\pi}\left(\frac{2\sqrt{\frac{2}{3}}e^y}{M_P(1-e^y)}\right) \tag{80}$$

$$= \frac{E_P^2}{16\pi}\frac{8}{3M_P^2}\frac{e^{2y}}{(1-e^y)^2} \tag{81}$$

$$= \underbrace{\frac{E_P^2}{8\pi}}_{M_P^2}\frac{4}{3}\frac{e^{2y}}{(1-e^y)^2} \tag{82}$$

$$= \frac{4}{3}\frac{e^{2y}}{(1-e^y)^2} \tag{83}$$

$$\tag{84}$$

The second slow-roll parameter is defined as

$$\eta = \frac{E_P^2}{8\pi}\frac{V''}{V} \tag{85}$$

We found the double derivative of the potential

$$V'' = \frac{\mathrm{d}^2V}{\mathrm{d}\phi^2} = \frac{\mathrm{d}V'}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{3}{2}M^2M_P\sqrt{\frac{2}{3}}e^y(1-e^y) \tag{86}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{3}{2}\sqrt{\frac{2}{3}}M^2M_P(e^y - e^{2y}) \tag{87}$$

$$= \frac{3}{2}\sqrt{\frac{2}{3}}M^2M_P\left(-\sqrt{\frac{2}{3}}\frac{1}{M_P}e^y + \sqrt{\frac{2}{3}}\frac{2}{M_P}e^{2y}\right) \tag{88}$$

$$= \frac{3}{2}\frac{2}{3}M^2M_P\frac{1}{M_P}(-e^y + 2e^{2y}) \tag{89}$$

$$= M^2(-e^y + 2e^{2y}) \tag{90}$$

Inserting V'' into the expression for η yields

$$\eta = \frac{E_P^2}{8\pi}\frac{\cancel{M^2}(-e^y + 2e^{2y})}{\frac{3\cancel{M^2}M_P^2}{4}(1-e^y)^2} \tag{91}$$

$$= \frac{4}{3}\underbrace{\frac{E_P^2}{8\pi}}_{M_P^2}\frac{(-e^y+2e^{2y})}{M_P^2(1-e^y)^2} \tag{92}$$

$$= \frac{4}{3}\frac{(2e^{2y}-e^y)}{(1-e^y)^2} \tag{93}$$

XIV. TASK M

To numerically solve for the Starobinksy model we need an expression for the dimensionless potential v and its derivative $dv/d\psi$. By the definition of v in eq. 10 and the potential in eq. 72 we have

$$\begin{aligned} v &= \frac{\hbar c^3}{H_i^2E_P^2}V(\phi) \\ &= \frac{3\hbar c^5}{8\pi G E_P^2}\frac{V(\psi)}{V(\psi_i)} \end{aligned} \tag{94}$$

Units are switched so that $\hbar = c = 1$ and $E_P^2 = 1/G$ so that

$$\begin{aligned} v &= \frac{3}{8\pi}\frac{V(\psi)}{V(\psi_i)} \\ &= \frac{3}{8\pi}\frac{(1-e^y)^2}{(1-e^{y_i})^2} \end{aligned} \tag{95}$$

where $y \equiv -\sqrt{\frac{16\pi}{3}}\psi$. The derivative of the reduced potential becomes

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}\psi} &= \frac{3}{8\pi}(1-e^{y_i})^{-2}\frac{d}{d\psi}(1-e^y)^2 \\ &= \frac{3}{8\pi}(1-e^{y_i})^{-2}\frac{d}{d\psi}(1-e^{-\sqrt{\frac{16\pi}{3}}\psi})^2 \\ &= \frac{3}{8\pi}(1-e^{y_i})^{-2}\cdot 2(-e^{-y})\left(-\sqrt{\frac{16\pi}{3}}\right)(1-e^y) \\ &= \sqrt{\frac{3}{\pi}}e^y\frac{(1-e^y)}{(1-e^{y_i})^2} \end{aligned} \tag{96}$$

Figure 7 shows the results.

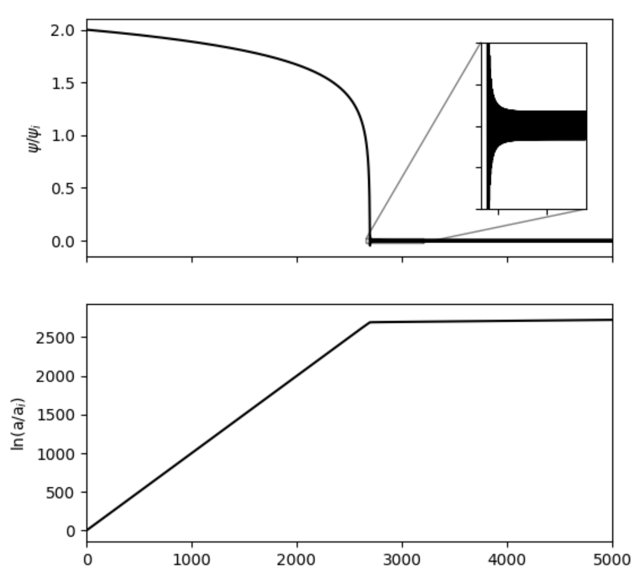

Figure 7. The numerical solution to the scalar field and logarithmic scale factor of the Starobinsky model.

XV. TASK N AND O

The number of e-foldings left till inflation ends is found in the same manner as task j. The slow-roll parameters for this model is defined in eq. 83 and 93. The following approximated values are introduced

$$N \approx \frac{3}{4}e^{-y} \tag{97}$$

$$\epsilon \approx \frac{3}{4N^2} \tag{98}$$

$$\eta \approx -\frac{1}{N} \tag{99}$$

Figure 8 shows the slow-roll parameters and their approximation plotted against number of e-foldings left.

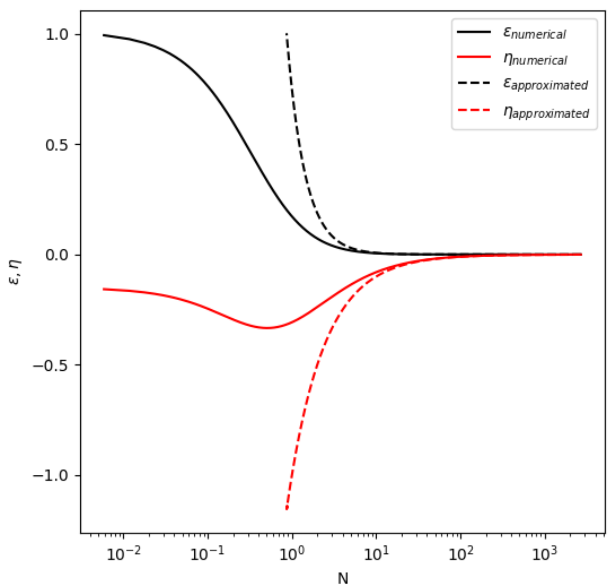

Figure 8. The numerical slow-roll parameters ϵ (solid black) and η (solid red) as a function of N, the number of e-foldings remaining before the end of inflation. The dashed lines are the approximated slow-roll parameters.

As $N \in [50, 60]$ we slice the array for number of e-foldings left and map the corresponding onto the array for ψ. The sliced array are used to calculate ϵ and η, to further calculate n and r. These are the approximated values for the variables

$$n \approx 1 - \frac{2}{N} \tag{100}$$

$$r \approx \frac{12}{N^2} \tag{101}$$

Figure 9 shows the results.

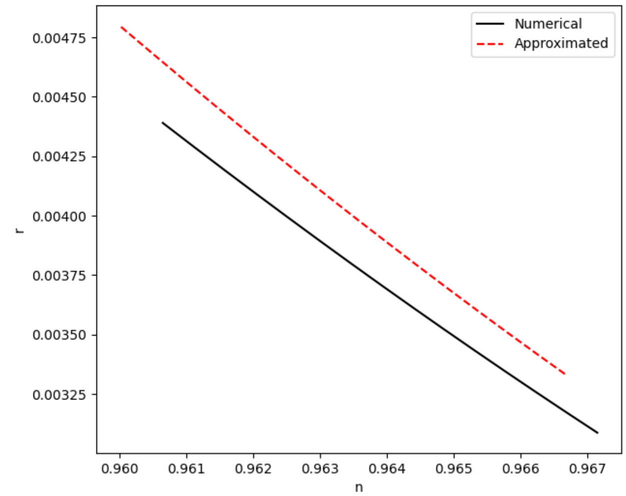

Figure 9. The numerical tensor-to-scalar ratio r plotted against the spectral index n for $N \in [50, 60]$. Black solid line is the numerical results and the red dashed line is the approximated results.

XVI. TASK P

ESA's CMB satellite Planck was Europe's first mission to study the Cosmic Microwave background, launched in 2009 [2]. From the paper "Planck 2018 results. X. Constraints on inflation" [3] they found $n = 0.9649 \pm 0.0042$ and $r < 0.10$. However, ref. [4] combine Planck data with BICEP2/Keck 2015 data and suggest a further improved constraint on r where $r < 0.044$.

Table I. Constraints on the tensor-to-scale-ratio and spectral index.

Model	n	r
ϕ^2	0.9669	0.1322
Starobinsky	0.9606	0.0043

Both models are consistent with the result for n. When it comes to r, the ϕ^2-model deviates rather more from the results than the Starobinsky. Thus, suggesting the Starobinsky model is a valid option for modelling inflation.

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