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Development and implementation of a quantile hedging algorithm for various loss functions

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Introduction and motivation

In the complete market, any European contingent claim has exactly one price V_0. This price allows us to fully hedged payoff of this option by creating a self-financing strategy. Suppose an investor is unwilling to put up the initial amount of capital V_0 required by a perfect hedge the option, but wants to hedge with less capital and is ready to accept some risk. Therefore, it is necessary to perform some optimal partial hedging, which can be achieved with smaller then V_0 amount of capital.
In [1] as a optimal hedging criterion was the minimization the probability that the value of the option at the time of exercise is greater then the value of the self-financing strategy also at the expiry of the option. In this paper, the optimal hedging criterion will be the minimization of the shortfall risk i.e. the expected value of the shortfall weighted by the loss function (the formal definition of this criterion can be found in in section 2).
The aim of the paper consists of two parts. The first is related to the development of quantile hedging methods for various loss functions by supplementing and extending the proofs from [3] to make them clearer. The results of this part, apart from the structure of the optimal hedging, is the determination of the explicit price and quantile hedging formulas of call and put options for the Black-Scholes model, which boils down to hedging the modified call and put options. The second part of the work objective is the implementation of numerical methods to perform optimal quantile hedging for the path-independent option class (also for various loss functions).
In section 1 we describe the mathematical model that we use to value assets and recall the most important facts about the theory of pricing and hedging European contingent claims.
Then, in the sections 2 and 3 we show the existence of a solution to the optimal hedging criterion and give the form of optimal hedging for convex, linear and concave loss functions for the model, which we described in the chapter 1. These sections are based on [3], our aim was to describe the method by which we get the solution to the optimization problem and also to extend the most important proofs from [3] to be clearer.
In the chapter 4 we determined the explicit formulas for the price and hedging modified contingent claims for the call and put option using the theorems from the chapters 2 and 3.
In the last section, we describe basic numerical methods that allow you to determine the form of a modified European contingent claim which solves our optimal security criterion for any option that is path-independent, i.e. the original claim payoff function only depends on the last price of the underlying asset. Moreover, these methods

allow the determination of a hedging portfolio over time for these modified options. In addition, for the purposes of the work, a package was created in **R**, which allows for the valuation of the modified call and put options and determining their hedging strategy for the given model parameters and allows the use of numerical methods described in the chapter 5. The package is available on github in the link: [12].
We assume that the reader has knowledge of probability theory, stochastic processes, statistical hypothesis testing theory (in particular, the Neyman-Pearson lemma) and has a basic knowledge of financial mathematics.

1 Model description and basic definitions of financial mathematics

1.1 Model description

Assume, that at the time period $[0, T]$, there is a non-dividend underlying asset on the market. Let $(\Omega, \mathcal{F}, \mathbf{P})$ means probability space, and $S = (S_t)_{t\in[0,T]}$ with natural filtration $\mathcal{F}_t = \sigma(S_s : s \leq t, s \in [0, T])$ means the process of price of the underlying asset and suppose that it change according to the standard Black-Scholes model with constant volatility $\sigma > 0$. It means, that price process of this asset is a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

with the initial price $S_0 > 0$, where B is a Wiener process under the measure $\mathbf{P}$, and μ is constant. In addition we define risk-free instrument $N = (N_t)_{t\in[0,T]}$ equal:

$$N_t = e^{rt},$$

with $N_0 = 1$, where $r > 0$ is a risk-free rate, which is constant over the time period $[0, T]$.
Using Itô formula, we can describe price process in an explicit form. With that we define discounted price process of the underlying asset $X = (X_t)_{t\in[0,T]}$ as:

$$X_t = \frac{S_t}{N_t} = X_0 \exp\left((\mu - r - 0.5\sigma^2)t + \sigma B_t\right) = X_0 \exp\left((m - 0.5\sigma^2)t + \sigma B_t\right),$$

where $X_0 = S_0$, a $m = \mu - r$.
To price a derivative instrument we need basic knowledge about martingale measure.

Definition 1.1.1. *Process $Z = (Z_t)_{t\in[0,T]}$ we called as martingale, when:*

- *Z_t is measurable under the measure $\mathcal{F}_t$,*
- *Z_t is integrable, $\mathbb{E}[|Z_t|] < \infty$ for every t,*
- *conditional expected value $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$ for every $s \leq t$.*

Definition 1.1.2. *Measure $\mathbf{P}^*$ is called equivalent martingale measure, if it is absolutely continuous with respect to the real measure $\mathbf{P}$ and the discounted price process of the underlying asset $X = (X_t)_{t\in[0,T]}$ is under the measure $\mathbf{P}^*$ a martingale.*

Fact 1. *In the Black-Scholes model there is exactly one equivalent martingale measure $\mathbf{P}^*$.*

Proof. Let's consider the process:

$$B_t^* = B_t - \frac{r-\mu}{\sigma}t$$

According to Girsanov theorem (which we can find in a subsection 7.3 in [5] and in a chapter 15 from [11]) if the process B_t^* is a standard Brownian motion under some measure $\mathbf{P}^*$, then this measure is a martingale measure. Moreover, using Girsanov theorem we have the following Radon-Nikodym derivative form up to time t:

$$\left(\frac{\mathbf{dP}^*}{\mathbf{dP}}\right)_t = \exp\left(-\frac{m}{\sigma}B_t - 0.5\frac{m^2}{\sigma^2}t\right) = \exp\big(0.5t(\frac{m^2}{\sigma^2} - m)\big)\big(\frac{X_t}{X_0}\big)^{-\frac{m}{\sigma^2}}.$$

$\square$

Remark. *In next chapters we will denote Radon-Nikodym derivative $\frac{\mathbf{dP}^*}{\mathbf{dP}}$ as ρ.*

Now let us introduce basic definitions and properties of financial mathematics. These definitions is taken from script [5], chapters 5 and 7.

Definition 1.1.3. *A trading strategy $\bar{\xi} = (\eta_t, \xi_t)_{t\in[0,T]}$ is a predictable process under filtration $(\mathcal{F}_t)_{t\in[0,T]}$.*
The value process $V = (V_t)_{t\in[0,T]}$ associated with a trading strategy $\bar{\xi}$ at the time t is given by linear combination:

$$V_t = \eta_t N_t + \xi_t S_t.$$

The gains process $G = (G_t)_{t\in[0,T]}$ of the portfolio $\bar{\xi}$ over time $[0,t]$ we define as:

$$G_t = \int_0^t \xi_u dS_u + \int_0^t \eta_u dN_u.$$

A trading strategy at the time t means that we have η_t amounts, that grows with risk-free process N_t and ξ_t quantity of underlying asset S_t.

Definition 1.1.4. *We say, that a trading strategy $\bar{\xi}$ is self-financing, when:* $V_t = V_0 + G_t$ *for every* t.
A self-financing strategy has an arbitrage opportunity if its value process satisfied the following conditions:

- $V_0 \leq 0$;
- $V_T \geq 0 \quad \mathbf{P}$ *- almost surely;*
- $\mathbf{P}(V_T > 0) > 0$.

Fact 2. *Let*

$$\bar{V}_t = \frac{V_t}{N_t} = V_0 + \int_0^t \xi_s dX_s.$$

Then, the trading strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ *is self-financial.*

Proof. Increment of process $\bar{V}_t$ is given by:

$$d\bar{V}_t = \xi_t dX_t = \xi_t(\mu - r)X_t dt + \xi_t \sigma X_t dB_t.$$

The Itô formula for function $f(\bar{V}_t, t) = \bar{V}_t e^{rt}$ gives us:

$$dV_t = d(\bar{V}_t e^{rt}) = r\bar{V}_t e^{rt} dt + e^{rt}\xi_t(\mu - r)X_t dt + e^{rt}\xi_t \sigma X_t dB_t,$$

and using the definition of the gains process, we have:

$$dV_t = r(\eta_t N_t + \xi_t S_t)dt + \xi_t(\mu - r)S_t dt + \xi_t \sigma S_t dB_t = \eta_t dN_t + \xi_t dS_t.$$

Integrating on both sides, we get a gains process G_t. □

Remark. *From now on, we will denote the self-financing strategy as* (V_0, ξ).

1.2 Pricing of the European contingent claim

Let us move on to the pricing theory of the European contingent claims:

Definition 1.2.1. *Non-negative measurable function C with respect to $\mathcal{F}_T$ is called a European contingent claim. We will denote discounted value of a European contingent claim as* $H = e^{-rT}C$.

Definition 1.2.2. *A contingent claim C is called attainable (replicable), if there exist a self-financing strategy (V_0, ξ), such that at the maturity time T value process is equal C. In other words:*

$$H = V_0 + \int_0^T \xi_s dX_s.$$

We say that strategy (V_0, ξ) is admissible, if value process V satisfies:

$$V_t \geq 0 \quad \forall t \in [0, T] \quad \mathbf{P} \text{ - almost surely.}$$

Definition 1.2.3. *A market model is said to be complete if every contingent claim is attainable.*

Theorem 1.2.1 (Fundamental theorem of asset pricing)**.** *A market is without arbitrage opportunity if and only if it admits at least one equivalent martingale measure.*

Proof. Proof is in [6] and $Chapter\ VII - 4a$ in the book [9]. □

Theorem 1.2.2 (Second fundamental theorem of asset pricing)**.** *An arbitrage-free market model is complete if and only if there is exactly one equivalent martingale measure.*

Proof. Proof is in [6] and $Chapter\ VII - 4a$ in the book [9]. □

The Black-Scholes model has only one equivalent martingale measure, therefore every contingent claim is attainable. Moreover the value process $(\bar{V}_t)_{t \in [0,T]}$ of the self-financial strategy is $\mathbf{P}^*$-martingale.
Since every European contingent claim is attainable, there exists a self-financial strategy (V_0, ξ) such that $V_T = C$. Therefore for every $t \in [0, T]$, we have:

$$\bar{V}_t = \mathbb{E}^*[\bar{V}_T | \mathcal{F}_t] = \mathbb{E}^*[V_T e^{-rT} | \mathcal{F}_t] = e^{-rT} \mathbb{E}^*[C | \mathcal{F}_t].$$

So, the price of a European contingent claim equals:

$$\Pi(C, t) = e^{-r(T-t)} \mathbb{E}^*[C | \mathcal{F}_t] = V_t.$$

In particular we have: $\Pi(C, 0) = V_0 = \mathbb{E}^*[H]$.

Example 1.2.1. *Let consider European call option with a strike price K, i.e. the payoff of this option is $C = (S_T - K)^+$. Price of this contingent claim at the time t is equal to:*

$$\Pi(C, t) = e^{-r\tau} \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] = e^{-r\tau} \mathbb{E}^*[(S_T - K) \mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t],$$

where $\tau = T - t$. Using the equality $S_T = S_t e^{(r-0.5\sigma)\tau+\sigma(B_T^*-B_t^*)}$ and the fact, that $S_t \in \mathcal{F}_t$, as well as $B_T^* - B_t^*$ are independent of σ-algebra $\mathcal{F}_t$, we have:

$$e^{-r\tau}\mathbb{E}^*[(S_T-K)\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t] = S_t e^{-0.5\sigma^2\tau}\mathbb{E}^*[e^{\sigma(B_T^*-B_t^*)}\mathbf{1}_{\{B_T^*-B_t^*>\frac{\ln(K/S_t)-(r-0.5\sigma^2)\tau}{\sigma}\}}]$$
$$-Ke^{-r\tau}\mathbf{P}^*(B_T^*-B_t^*>\frac{\ln(K/S_t)-(r-0.5\sigma^2)\tau}{\sigma}).$$

$B_T^* - B_t^*$ in the measure $\mathbf{P}^*$ has a normal distribution $N(0,\tau)$, so:

$$\mathbf{P}^*(B_T^*-B_t^*>\frac{\ln(K/S_t)-(r-0.5\sigma^2)\tau}{\sigma}) = \mathbf{P}^*(Z>-\frac{\ln(S_t/K)+(r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}) = N(d_2),$$

where Z is a standard Gaussian random variable, and N is its distribution function, also

$$d_2 = \frac{\ln(S_t/K)+(r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Now let calculate the expected value. We have:

$$\mathbb{E}^*[e^{\sigma(B_T^*-B_t^*)}\mathbf{1}_{\{B_T^*-B_t^*>-d_2\sqrt{\tau}\}}] = \int_{-d_2\sqrt{\tau}}^{\infty}\frac{1}{\sqrt{2\pi\tau}}e^{\sigma x}e^{-\frac{x^2}{2\tau}}dx = \int_{-d_2\sqrt{\tau}}^{\infty}\frac{1}{\sqrt{2\pi\tau}}e^{0.5\sigma^2\tau}e^{-\frac{(x-\sigma\tau)^2}{2\tau}}dx$$
$$= e^{0.5\sigma^2\tau}\mathbf{P}^*(X>-d_2\sqrt{\tau}),$$

where X is a random variable with a $N(\tau\sigma,\tau)$ distribution. In conclusion, the price of the European call option equals:

$$\Pi(C,\tau) = S_t N(d_1) - Ke^{-r\tau}N(d_2),$$

where $d_1 = d_2 + \sigma\sqrt{\tau}$.

Example 1.2.2. *Let us consider European put option with a strike price K, i.e. its payoff equals $C = (K - S_T)^+$. Using the call-put parity, we have:*

$$\Pi(C,t) = \Pi((S_T-K)^+,t) - S_t + Ke^{-r\tau} = Ke^{-r\tau}N(-d_2) - S_t N(-d_1).$$

1.3 Hedging of the European options

In order to protect against the payoff of the European contingent claim, self-financing strategy should be created with an initial price V_0, whose value process at the maturity time T is equal to payoff of that claim. Such operation we called hedging.

Consider square-integrable payoff of a European contingent claim $C \in L^2(\Omega)$ which we can presented in the form of the following stochastic integral:

$$C = \mathbb{E}^*[C] + \int_0^T \zeta_t dB_t^*,$$

where $\zeta = (\zeta_t)_{t\in[0,T]}$ 0,t] is a square-integrable adapted process. Below lemma, will help us find self-financing strategy, leading to hedge European contingent C.

Lemma 1.3.1. *Consider a contingent claim payoff $C \in L^2(\Omega)$ and the process $\zeta = (\zeta_t)_{t\in[0,T]}$. Let ξ_t, η_t be given by:*

$$\xi_t = \frac{e^{-r(T-t)}}{\sigma S_t}\zeta_t,$$

$$\eta_t = \frac{e^{-r(T-t)}\mathbb{E}^*[C|\mathcal{F}_t] - \xi_t S_t}{N_t}.$$

Then the trading strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing and its value process is equal to:

$$V_t = e^{-r(T-t)}\mathbb{E}^*[C|\mathcal{F}_t], \quad 0 \le t \le T.$$

In particular we have: $V_T = C$.

Proof. Proof can be found in [5] as the proof of *Proposition 7.11.* □

In practice, the hedging problem can now be reduced to the computation of the process ζ_t. This computation, called Delta hedging, can be performed by the application of the Itô formula and the Markov property. Below lemma allows us to compute the process ζ in case, the payoff on the contingent claim is of the form $C = \Phi(S_T)$ for some function Φ.

Lemma 1.3.2. *Assume that, Φ is a Lipschitz payoff function. Then the function $C(t,x)$ defined by:*

$$C(t, S_t) = \mathbb{E}^*[\Phi(S_T)|S_t],$$

is in $C^{1,2}([0,T],\mathbb{R})$ and stochastic integral:

$$\Phi(S_T) = \mathbb{E}^*[\Phi(S_T)] + \int_0^T \zeta_t dB_t^*$$

is given by:

$$\zeta_t = \sigma S_t \frac{\partial C}{\partial x}(t, S_t) \quad 0 \le t \le T.$$

In addition we have:

$$\xi_t = e^{-r(T-t)} \frac{\partial C}{\partial x}(t, S_t) \quad 0 \leq t \leq T.$$

Proof. Proof can be found in [5] as the proof of *Proposition 7.12.* $\square$

Example 1.3.1. *In the case of a call option with a strike price K, we have:*

$$\xi_t = \frac{\partial \Pi}{\partial x}(t, S_t) = N(d_1) + S_t \frac{\partial N}{\partial d_1} \frac{\partial d_1}{\partial x} - K e^{-r\tau} \frac{\partial N}{\partial d_2} \frac{\partial d_2}{\partial x} =$$
$$N(d_1) + \frac{1}{\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2} - \frac{K}{S_t \sigma \sqrt{\tau}} e^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_2^2} = N(d_1).$$

Moreover, we have

$$\eta_t = \frac{S_t N(d_1) - K e^{-r\tau} N(d_2) - \xi_t S_t}{e^{rt}} = -K e^{-rT} N(d_2).$$

Example 1.3.2. *In the case of a put option with a strike price K, using call-put parity, we get:*

$$\xi_t = N(d_1) - 1.$$

Moreover, we have

$$\eta_t = \frac{K e^{-r\tau} N(-d_2) + S_t(1 - N(-d_1) - N(d_1))}{e^{rt}} = K e^{-rT} N(-d_2).$$

2 Optimal hedging for a convex function

Let us consider discounted European contingent claim. As we already know, in the case of our model, in order to fully protect against this claim, we need an initial capital of $U_0 = \mathbb{E}^*[H]$.

Suppose that we don't have or we are unwilling to use the initial capital in the amount of arbitrage-free price U_0, and for hedging this option we want to use a smaller capital of $\tilde{V}_0 < U_0$. In [1] the (V_0, ξ) strategy was considered, which minimizes the probability of unsuccessful hedging, i.e. $\mathbf{P}(\bar{V}_T \leq H) = min$ under the constraint, that $V_0 \leq \tilde{V}_0$. This time we want to control the size of the loss $(H - \bar{V}_T)^+$, not only the probability of its occurrence. This approach was proposed in [3], and we will now introduce this concept.

Let us introduce a loss function l which describes the investor's attitude with respect to the shortfall. We assume that l is an increasing convex function defined on $[0, \infty)$, with $l(0) = 0$, moreover which fulfils the condition $\mathbb{E}[l(H)] < \infty$.

Definition 2.0.1. *We define shortfall risk as the expected value of the shortfall weighted by the loss function l*

$$\mathbb{E}[l((H-\bar{V}_T)^+)].$$

Our aim is to find an admissible strategy (V_0, ξ) which, minimize the shortfall risk while, not using more initial capital then $\tilde{V}_0$. Thus we consider the optimization problem:

$$\mathbb{E}[l((H-\bar{V}_T)^+)] = \mathbb{E}[l((H - V_0 - \int_0^T \xi_s dX_s)^+)] = min,$$

under the constraint $V_0 \leq \tilde{V}_0$.

Remark. *In fact, we would prefer minimize $\mathbb{E}[l((C-V_T)^+)]$, i.e. the actual loss weighted by the function l, rather then the shortfall risk $\mathbb{E}[l((H-\bar{V}_T)^+) = \mathbb{E}[l(e^{-rT}(C-V_T)^+)]$. Note, however that in particular, when $l(x)$ is a class of x^p for some $p > 0$ (which we will later discuss), the two problems are equivalent.*

We want to reduce our problem to the search for an element $\tilde{\varphi}$ in the class $\mathcal{R} = \{\varphi : \Omega \to [0,1] \mid \varphi \in \mathcal{F}_T\}$ of "randomized tests" which solves the following optimization problem:

Theorem 2.0.1. *There exists a solution $\tilde{\varphi} \in \mathcal{R}$ to the problem:*

$$\min_{\varphi \in \mathcal{R}} \mathbb{E}[l((1-\varphi)H)]$$

under the constraint $\mathbb{E}^*[\varphi H] \leq \tilde{V}_0$. If l is strictly convex, then any two solutions coincide $\mathbf{P}-p.n$ on $\{H > 0\}$.

To proof this theorem we need the following lemma:

Lemma 2.0.1. *Let (f_n) be a sequence of $[0,\infty)$ valued measurable functions on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. There is a sequence $g_n \in conv(f_n, f_{n+1}, ...)$ such, that $(g_n)_{n\geq 1}$ converges almost surely to a some function $g \in [0, \infty)$. If the convex hull $conv(f_n, n \geq 1)$) is bounded in w L^0 then g jest is finite $\mathbf{P}-a.s.$. If there are $\alpha > 0, \delta > 0$ such that, for all n $\mathbf{P}(f_n > \alpha) > \delta$, then $\mathbf{P}(g > 0) > 0$.*

Proof. The proof can be found in [2] as proof of *Lemma AI.1* □

Proof of the theorem. The proof of this theorem is in [3] as a proof of *Proposition 3.1*, for the sake of clarity of the thesis, it has been rewritten.
Let $\mathcal{R}_0$ consist of those elements of $\mathcal{R}$, that satisfy $\mathbb{E}^*[\varphi H] \leq \hat{V}_0$.

Let φ_n be a minimizing sequence for $\mathbb{E}[l((1-\varphi)H)]$ in $\mathcal{R}_0$. Using the lemma we can choose functions $\tilde{\varphi}_n \in \mathcal{R}_0$ belonging to the convex hull of $conv(\varphi_n, n \geq 1)$ such that $(\tilde{\varphi}_n)$ converges $\mathbf{P} - a.s.$ to some function $\tilde{\varphi} \in \mathcal{R}$.
Since $l(H) \in L^1(\mathbf{P})$ we can use dominated convergence to conclude that

$$\mathbb{E}[l((1-\tilde{\varphi}_n)H)] \to \mathbb{E}[l((1-\tilde{\varphi})H)] = min.$$

On the other hand from the Fatou lemma, we have: $\mathbb{E}^*[\tilde{\varphi}H] \leq \liminf \mathbb{E}^*[\tilde{\varphi}_n H] \leq \tilde{V}_0$, thus $\tilde{\varphi} \in \mathcal{R}_0$.
Let $\tilde{\varphi}$ be a solution. For any $\varphi \in \mathcal{R}_0$ and for $\epsilon \in [0,1]$ we define:

$$\varphi_\epsilon = (1-\epsilon)\tilde{\varphi} + \epsilon\varphi.$$

We have:

$$\mathbb{E}[l((1-\varphi_\epsilon)H)] = \mathbb{E}[l(H - (1-\epsilon)\tilde{\varphi}H - \epsilon\varphi H)].$$

By the convexity of l we get:

$$\mathbb{E}[l((1-\epsilon)(1-\tilde{\varphi})H + \epsilon(1-\varphi)H)] \leq (1-\epsilon)\mathbb{E}[l((1-\tilde{\varphi})H)] + \epsilon\mathbb{E}[l((1-\varphi)H)].$$

If l is strictly convex, then the inequality is strict if

$$\mathbf{P}(\{\varphi \neq \tilde{\varphi}\} \cap \{H > 0\}) > 0.$$

□

Let $\tilde{\varphi}$ be a solution to the problem of the above theorem. Without loss of generality we assume $\tilde{\varphi} = 1$ dla $\{H = 0\}$.
Let us introduce the modified claim $\tilde{H} = \tilde{\varphi}H$, and let us define the process $\tilde{U}$ as

$$\tilde{U}_t = \mathbb{E}^*[\tilde{\varphi}H|\mathcal{F}_t].$$

Since the market model is complete, any contingent claim is attainable, so there is a self-financing strategy $(\tilde{V}_0, \xi)$ that replicates the modified claim $\tilde{H}$ i.e.

$$\mathbb{E}^*[\tilde{H}|\mathcal{F}_t] = \tilde{V}_0 + \int_0^t \xi_s dX_s \quad \forall t \in [0,T].$$

Definition 2.0.2. *For any admissible strategy* (V_0, ξ) *we define the corresponding success ratio as:*

$$\varphi_{(V_0,\xi)} = \mathbf{1}_{\{\bar{V}_T \geq H\}} + \frac{\bar{V}_T}{H}\mathbf{1}_{\{\bar{V}_T < H\}}.$$

Theorem 2.0.2. *The self-financing strategy $(\tilde{V}_0, \xi)$, that hedges the modified contingent claim $\tilde{H} = \tilde{\varphi}H$ solves the optimization problem*

$$\mathbb{E}[l((H - \bar{V}_T)^+)] = \mathbb{E}[l((H - V_0 - \int_0^T \xi_s dX_s)^+)] = min,$$

under the constraint $V_0 \leq \tilde{V}_0$.

Proof. The proof is based on the proof of *Theorem 3.2* in [3], it has been supplemented with additional comments.
Let (V_0, ξ) be any admissible strategy with $V_0 \leq \tilde{V}_0$ and denote by φ the corresponding success ratio. We have:

$$\varphi H = H\mathbf{1}_{\{\bar{V}_T \geq H\}} + \bar{V}_T \mathbf{1}_{\{\bar{V}_T < H\}} = (H \wedge \bar{V}_T),$$

while the shortfall takes the form:

$$(H - \bar{V}_T)^+ = H - (H \wedge \bar{V}_T) = H(1 - \varphi).$$

The value process $(\bar{V}_t)$ is a martingale under the measure $\mathbf{P}^*$ hence, we have:

$$\mathbb{E}^*[\varphi H] = \mathbb{E}^*[(H \wedge \bar{V}_T)] \leq \mathbb{E}^*[\bar{V}_T] = V_0 \leq \tilde{V}_0.$$

Thus, the success ratio satisfies upper bound of the expected value and due to the theorem 2.0.1, we have

$$\mathbb{E}[l((H - \bar{V}_t)^+)] = \mathbb{E}[l(H(1 - \varphi))] \geq \mathbb{E}[l((H(1 - \tilde{\varphi})))],$$

where $\tilde{\varphi}$ is the optimal solution to the problem of the theorem 2.0.1.
The strategy $(\tilde{V}_0, \xi)$ is admissible, because its value process satisfies:

$$\tilde{V}_t = \mathbb{E}^*[\varphi H | \mathcal{F}_t] \geq 0.$$

Moreover, its success ratio satisfies:

$$\varphi_{(\tilde{V}_0, \xi)} H = \tilde{V}_t \wedge H \geq \tilde{\varphi} H \quad \mathbf{P} - a.s. \text{ on } \{H > 0\},$$

because, if $\tilde{V}_T > H$ then $H > \tilde{\varphi}H$, because $\tilde{\varphi}$ has values in the range $[0, 1]$, while, when $\tilde{V}_T \leq H$ then $\tilde{V}_T = \tilde{\varphi}H$ since $\tilde{V}_T$ is the value of the optimal claim decomposition $\tilde{\varphi}H$.
Due to the fact that the function l is increasing and the above inequality, we have:

$$l(H - \tilde{\varphi}H) \geq l(H - \varphi_{(\tilde{V}_0, \xi)} H) \quad \mathbf{P} - a.s. \text{ on } \{H > 0\},$$

and that brings us to the inequality below:

$$\mathbb{E}[l(H - \tilde{\varphi}H)] \geq \mathbb{E}[l(H - \varphi_{(\tilde{V}_0,\xi)}H)].$$

As we know $\tilde{\varphi}$ minimizes the expected value $\mathbb{E}[l(H - \varphi H)]$, therefore $\varphi_{(\tilde{V}_0,\xi)}H = \tilde{\varphi}H$ $\mathbf{P} - a.s.$ on $\{H > 0\}$. Moreover $\varphi_{(\tilde{V}_0,\xi)} = \tilde{\varphi} = 1$ on $\{H = 0\}$, thus the success ratio converges $\mathbf{P} - a.s.$ to $\tilde{\varphi}$. In particular, we have:

$$(H - \tilde{V}_T)^+ = (1 - \tilde{\varphi})H,$$

and because $\tilde{\varphi}$ minimizes $\mathbb{E}[l((1 - \varphi)H)]$ under the constraint $\mathbb{E}^*[\varphi H] \leq \hat{V}_0$, the strategy $(\tilde{V}_0, \xi)$ solves the optimization problem

$$\mathbb{E}[l((H - \bar{V}_T)^+)] = \min,$$

under the constraint $V_0 \leq \tilde{V}_0$. $\square$

Remark. *Note that in the above proof we did not use the convexity of the l function, only that it is increasing. Therefore, this theorem will be apply in the next chapters.*

2.1 Minimizing the expected shortfall

Let us consider the case of a linear loss function $l(x) = x$. Thus, we want to minimize the expected shortfall $\mathbb{E}[(H - \bar{V}_T)^+]$ under the constraint $V_0 \leq \tilde{V}_0$. Theorem 2.0.2 shows that this is equivalent to the optimization problem

$$\min \mathbb{E}[H(1 - \varphi)] = \min(\mathbb{E}[H] - \mathbb{E}[\varphi H]) = \mathbb{E}[H] - \max \mathbb{E}[\varphi H],$$

under the constraint $\mathbb{E}^*[\varphi H] \leq \tilde{V}_0$.
Assume, the drift of the underlying asset is different from the risk-free rate i.e. $\mu \neq r$. Calculation of $\max \mathbb{E}[\varphi H]$ takes the form

$$\mathbb{E}^{\mathbf{Q}}[\varphi] = \max,$$

under the constraint $\mathbb{E}^{\mathbf{Q}^*}[\varphi] \leq \frac{\tilde{V}_0}{\mathbb{E}^*[H]}$, where the measures $\mathbf{Q}$ i $\mathbf{Q}^*$ are defined by Radon-Nikodym derivatives:

$$\frac{\mathbf{dQ}}{\mathbf{dP}} = \frac{H}{\mathbb{E}[H]}, \quad \frac{\mathbf{dQ}^*}{\mathbf{dP}^*} = \frac{H}{\mathbb{E}^*[H]}.$$

The solution to this optimization problem is identified as finding the uniformly most powerful test of the simple hypotheses $\mathbf{Q}^*$ against the simple alternative $\mathbf{Q}$, on the

significance level of $\tilde{V}_0/\mathbb{E}^*[H]$.
The Neyman-Pearson lemma provides a structure of the optimal test $\tilde{\varphi}$. According to this lemma, we have:

$$\varphi(x) = \begin{cases} 1 & \text{when } \frac{\mathbf{dQ}}{\mathbf{dQ*}} > c \\ \\ 0 & \text{when } \frac{\mathbf{dQ}}{\mathbf{dQ*}} < c \end{cases}.$$

The measures $\mathbf{P}, \mathbf{P}^*, \mathbf{Q}, \mathbf{Q}^*$ are absolutely continuous with each other since there are Radon-Nikodym derivatives, thus we have

$$\frac{\mathbf{dQ}}{\mathbf{dQ*}} = \frac{\mathbf{dQ}}{\mathbf{dP}}\frac{\mathbf{dP}}{\mathbf{dP}^*}\frac{\mathbf{dP}^*}{\mathbf{dQ}^*} = \frac{\mathbb{E}^*[H]}{\mathbb{E}[H]}(\rho)^{-1},$$

where $\rho = \frac{\mathbf{dP}^*}{\mathbf{dP}}$. Therefore, when an option is path-independent, the critical set takes the form:

$$\{(\rho)^{-1} > c_1\} = \{X_T^{\frac{m}{\sigma^2}} > c_2\}.$$

To sum up, for $m > 0$ the modified contingent claim takes the form $\tilde{H} = \mathbf{1}_{\{X_T > c\}}H$, and for $m < 0$ we have: $\tilde{H} = \mathbf{1}_{\{X_T < c\}}H$, where constant c we get from the condition:

$$\mathbb{E}^{\mathbf{Q}^*}[\varphi] = \mathbb{E}^*[\varphi H] = \tilde{V}_0.$$

In the case, where $\mu = r$, the real measure $\mathbf{P}$ is itself a martingale measure, and due to that we cannot use the Neyman-Pearson lemma. We want to find $\tilde{\varphi}$ such that

$$\min \mathbb{E}[H(1-\varphi)] = \mathbb{E}[H] - \max \mathbb{E}[\varphi H],$$

under the constraint $\mathbb{E}[\varphi H] \leq \tilde{V}_0$. Obviously, in this case, any modified contingent claim φH that fulfills the condition $\mathbb{E}[\varphi H] = \tilde{V}_0$ is the solution to this problem. In particular, one of these solution is

$$\tilde{\varphi} = \frac{\tilde{V}_0}{\mathbb{E}[H]}.$$

2.2 Structure of the modified claim

Let us assume, the loss function $l \in C^1(0, \infty)$, moreover its derivative l' is strictly increasing with $l'(0+) = 0$ and $l'(\infty) = \infty$. Let $I = (l')^{-1}$ denotes the inverse function to l'. By Theorem 2.0.1 the solution $\tilde{\varphi}$ of our optimization problem exists, and it is unique on $\{H > 0\}$ since l is strictly convex. On $\{H = 0\}$ we set $\tilde{\varphi} = 1$. The following theorem provides the explicit structure of $\tilde{\varphi}$.

Theorem 2.2.1. *The solution* φ *to the optimization problem*

$$\min_{\varphi \in \mathcal{R}} \mathbb{E}[l((1-\varphi)H)],$$

under the constraint $\mathbb{E}^*[\varphi H] \leq \tilde{V}_0$ *is given by:*

$$\tilde{\varphi} = 1 - (\frac{I(c\rho)}{H} \wedge 1) \quad on \quad \{H > 0\},$$

where $\rho = \frac{\mathbf{dP}^*}{\mathbf{dP}}$*, and constant* c *is determined by the condition* $\mathbb{E}^*[\tilde{\varphi} H] = \tilde{V}_0$.

Proof. The proof is based on the proof of [3] (the proof of *Theorem 5.1*), it has been supplemented with additional comments.
We use the method of Karlin [4] in order to reduce the computation of $\tilde{\varphi}$ to an application of the Neyman-Pearson lemma. For $\varphi \in \mathcal{R}$ we define:

$$\varphi_\epsilon = (1-\epsilon)\tilde{\varphi} + \epsilon\varphi.$$

So, we have

$$F_\varphi(\epsilon) = \mathbb{E}[l((1-\varphi_\epsilon)H)] = \mathbb{E}[l(H - (1-\epsilon)\tilde{\varphi}H - \epsilon\varphi)],$$

and since $\tilde{\varphi}$ is optimal, the function $F_\varphi(\epsilon)$ is non decreasing for every $\varphi \in \mathcal{R}$. The derivative of the function F_φ takes the form:

$$F'_\varphi(\epsilon) = \mathbb{E}[l'(H - (1-\epsilon)\tilde{\varphi}H - \epsilon\varphi)(\tilde{\varphi} - \varphi)H].$$

Applying the monotone convergence theorem separately on $\{\varphi > \tilde{\varphi}\}$ and on $\{\varphi < \tilde{\varphi}\}$ we see that the derivative F' exists and satisfies:

$$F'_\varphi(0+) = \lim_{\epsilon \to 0^+} F'_\varphi(\epsilon) = \mathbb{E}[\lim_{\epsilon \to 0^+} l'(H - (1-\epsilon)\tilde{\varphi}H - \epsilon\varphi)(\tilde{\varphi} - \varphi)H] = \mathbb{E}[l'(H(1-\tilde{\varphi}))(\tilde{\varphi} - \varphi)H].$$

The optimality of $\tilde{\varphi}$ means that for any $\varphi \in \mathcal{R}$ the corresponding convex function F_φ on $[0, 1]$ assumes its minimum in $\epsilon = 0$. This is equivalent to $F'_\varphi(0+) \geq 0$ for every $\varphi \in \mathcal{R}$, therefore

$$\mathbb{E}[l'(H(1-\tilde{\varphi}))\tilde{\varphi}H] \geq \mathbb{E}[l'(H(1-\tilde{\varphi}))\varphi H].$$

Let's define the probability measures $\mathbf{Q}, \mathbf{Q}^*$ by the following Radon-Nikodym derivatives:

$$\frac{\mathbf{dQ}}{\mathbf{dP}} = \mathrm{const} l'((1-\tilde{\varphi})H)H, \qquad \frac{\mathbf{dQ}^*}{\mathbf{dP}^*} = \mathrm{const} H.$$

Then the above inequality has the form:

$$\mathbb{E}^{\mathbf{Q}}[\tilde{\varphi}] \geq \mathbb{E}^{\mathbf{Q}}[\varphi] \quad \forall \varphi \in \mathcal{R},$$

while restraint on the price takes the form of:

$$\mathbb{E}^{\mathbf{Q}^*}[\varphi] \leq \text{const}\tilde{V}_0.$$

Therefore, our optimization problem becomes the problem of finding the uniformly most powerful test of the $\mathbf{Q}^*$ hypotheses against the $\mathbf{Q}$ alternative at the significance level of $\text{const}\tilde{V}_0$. Again as in previous chapter, by applying the Neyman-Pearson lemma, we obtain the structure of the optimal test $\tilde{\varphi}$ of the form:

$$\tilde{\varphi}(x) = \begin{cases} 1 & \text{when } \frac{\mathbf{dQ}}{\mathbf{dQ^*}} > c \\ \\ 0 & \text{when } \frac{\mathbf{dQ}}{\mathbf{dQ^*}} < c \end{cases}.$$

The measures $\mathbf{P}, \mathbf{P}^*, \mathbf{Q}, \mathbf{Q}^*$ are absolutely continuous with each other, so we have

$$\frac{\mathbf{dQ}}{\mathbf{dQ^*}} = \frac{\mathbf{dQ}}{\mathbf{dP}} \frac{\mathbf{dP}}{\mathbf{dP^*}} \frac{\mathbf{dP^*}}{\mathbf{dQ^*}} = \text{const } l'((1-\tilde{\varphi})H) \frac{\mathbf{dP}}{\mathbf{dP^*}} = \text{const } l'((1-\tilde{\varphi})H)(\rho)^{-1},$$

where $\rho = \frac{\mathbf{dP^*}}{\mathbf{dP}}$.

So on the set $\{l'((1-\tilde{\varphi})H)(\rho)^{-1} < c\}$ the optimal test $\tilde{\varphi}$ equals 0, while on the set $\{l'((1-\tilde{\varphi})H)(\rho)^{-1} > c\}$ optimal test is equal to 1. Note that if $\tilde{\varphi} = 1$ then $l'((1-\tilde{\varphi})H)(\rho)^{-1} = 0$, because we have $l'(0) = 0$, which is contrary to the test, because $c > 0$. Thus, we have $\tilde{\varphi} < 1$ on $\{H > 0\}$.

In the ordinary Neyman-Pearson situation there is no restriction on the values on the set $\{\frac{\mathbf{dQ}}{\mathbf{dQ^*}} = c\}$ except compliance with the level condition. In our situation however, we have to choose $\tilde{\varphi}$ on this set such, that:

$$(1-\tilde{\varphi})H = I(c\rho),$$

in order to be consistent.

Define:

$$\varphi_c = 1 - (\frac{I(c\rho)}{H} \wedge 1), \quad \text{on} \quad \{H > 0\}.$$

Since the derivative of the loss function is increasing, $I(c\rho)$ increases as c increases. Therefore:

$$\varphi_c H = H - (I(c\rho) \wedge H),$$

goes from H to 0 a.s. as c goes from 0 to ∞. Consequently $\mathbb{E}^*[\varphi_c H]$ goes from $\mathbb{E}^*[H]$ to 0 by dominated convergence theorem. Since the function I is continuous $\mathbb{E}^*[\varphi_c H]$ is continuous in c by dominated convergence. Hence we can find $c \in (0, \infty)$ such that $\mathbb{E}^*[\varphi_c H] = \tilde{V}_0 < \mathbb{E}^*[H]$. For this c we define $\tilde{\varphi} = \varphi_c$.
It remains to check, if $\tilde{\varphi}$ fulfills the compliance conditions. on the set $\{l'((1-\tilde{\varphi})H) = c\rho\}$ optimal test $\tilde{\varphi}$ must satisfy:

$$(1 - \tilde{\varphi})H = (I(c\rho) \wedge H) = I(c\rho).$$

The above equality is satisfied only if:

$$I(c\rho) \leq H \iff c\rho \leq l'(H).$$

We have $(1 - \tilde{\varphi})H \leq H$, and using the properties of the l' and the set structure we get:

$$c\rho = l'((1 - \tilde{\varphi})H) \leq l'(H).$$

On the set: $\{l'((1 - \tilde{\varphi})H) < c\rho\}$ the optimal test should be equal 0, which for the test φ_c is true for $I(c\rho) > H$. The set $\{l'((1 - \tilde{\varphi})H) < c\rho\}$ fulfills this condition. Thus, $\tilde{\varphi}$ is the uniformly most powerful test and is the solution of the optimization problem. □

The modified discounted contingent claim takes the form:

$$\tilde{H} = \tilde{\varphi}H = H - (I(c\rho) \wedge H),$$

where the constant c is determined by the condition $\mathbb{E}^*[\tilde{\varphi}H] = \tilde{V}_0$. According to the theorem 2.0.2 self-financing strategy replicating this modified contingent claim solves the optimization problem:

$$\mathbb{E}[l((H - \bar{V}_T)^+)] = \mathbb{E}[l((H - V_0 - \int_0^T \xi_s dX_s)^+)] = min,$$

under the constraint $V_0 \leq \tilde{V}_0$. Moreover, its shortfall risk is equal to:

$$\mathbb{E}[l((1 - \tilde{\varphi})H)] = \mathbb{E}[l(I(c\rho) \wedge H)] = \mathbb{E}[(l(I(c\rho)) \wedge l(H))].$$

Remark. *Notice that the solution $\tilde{\varphi}$ is a function of H and ρ. If both H and ρ happen to be functions of the final stock price X_T, then $\tilde{\varphi}$ is also a function of X_T.*

Remark. *In the case where $\mu = r$, the martingale measure is $\mathbf{P}$. Then $\rho = 1$, and modified claim takes the form:*

$$\tilde{H} = H - (I(c) \wedge H).$$

2.3 Power loss function

Let us consider the special case, where the loss function is equal to:

$$l(x) = \frac{x^p}{p}, \quad \text{for some } p > 1.$$

Thus, we have $l'(x) = x^{p-1}$, $I(x) = x^{\frac{1}{p-1}}$ and due to theorem 2.2.1, we will obtain optimal hedging by replicating the modified contingent claim:

$$\tilde{H} = \tilde{\varphi} H = H - ((c\rho)^{\frac{1}{p-1}} \wedge H) = H - (c_p(\rho)^{\frac{1}{p-1}} \wedge H),$$

where the constant c_p is determined by the condition $\mathbb{E}^*[\tilde{\varphi} H] = \tilde{V}_0$. The shortfall risk equals:

$$\mathbb{E}[l((1-\tilde{\varphi})H)] = \mathbb{E}[(l(I(c\rho)) \wedge l(H))] = \text{``}\text{``}\frac{1}{p}\mathbb{E}[(c_p^p(\rho)^{\frac{p}{p-1}} \wedge H^p)].$$

Let us now consider the limit $p \to \infty$ corresponding to ever increasing risk aversion with respect to large losses.

Fact 3. *We have the following properties:*

1. *for $p \to \infty$ the modified claim $\tilde{\varphi} H$ converges to $(H - c)^+$ $\mathbf{P} - a.s.$ and in $L^1(\mathbf{P}^*)$, where c is unique constant that satisfies $\mathbb{E}^*[c \wedge H] = \mathbb{E}^*[H] - \tilde{V}_0$;*
2. *if H is a call option with strike price K, then the limit for $p \to \infty$ the modified contingent claim is again a call option at the higher strike $\tilde{K} = K + c$;*
3. *if H is a put option with strike price K, then the limit for $p \to \infty$ the modified contingent claim is again a put option at the lower strike $\tilde{K} = K - c$;*

Proof. The proof $1, 2$ is in [3] as the proof of *Proposition 5.3*, and 3 is an extension of this fact to a put option.

1. If the constant c goes from 0 to ∞, then $(c \wedge H)$ goes from 0 to H, and using dominated convergence theorem $e(c) = \mathbb{E}^*[(c \wedge H)]$ goes from 0 to $\mathbb{E}^*[H]$. The function $e(...)$ is continuous and decreasing for $c \in [0, H]$ hence the constant c is uniquely defined.
   We now show that $\lim_{p\to\infty} c_p = c$. Consider a subsequence c_{p_n} such that $\lim_{n\to\infty} c_{p_n} = \bar{c} \in [0, \infty]$. Then:

   $$c_{p_n}(\rho)^{\frac{1}{p-1}} \wedge H \to \bar{c} \wedge H \quad \mathbf{P} - a.s.,$$

because $(\rho)^{\frac{1}{p-1}} \to 1 \ \ \mathbf{P}-a.s.$.
By dominated convergence we conclude that:

$$\mathbb{E}^*[c_{p_n}(\rho)^{\frac{1}{p-1}} \wedge H] \to \mathbb{E}^*[\bar{c} \wedge H].$$

Hence

$$\mathbb{E}^*[\tilde{\varphi}H] = \mathbb{E}^*[H - (c_{p_n}(\rho)^{\frac{1}{p-1}} \wedge H)],$$

and using the limitation of the initial capital, we get:

$$\mathbb{E}^*[\bar{c} \wedge H] = \mathbb{E}^*[H] - \tilde{V}_0.$$

Therefore $\bar{c} = c$ since c is unique. Applying this argument to lim inf and lim sup we see that $\liminf c_p = c = \limsup c_p$, hence $\lim_{p\to\infty} = c$.

2. We have $(c_p(\rho)^{\frac{1}{p-1}} \wedge H) \to (c \wedge H) \ \ \mathbf{P}-a.s.$. Therefore:

$$\tilde{\varphi}H = H - (c_p(\rho)^{\frac{1}{p-1}} \wedge H) \to H - (c \wedge H) = ((X_T - K)^+ - c)^+ \ \ \mathbf{P}-a.s,$$

where:

$$((X_T - K)^+ - c)^+ = \begin{cases} 0 & \text{then } X_T < K \\ 0 & \text{then } X_T < K + c \\ X_T - K - c & \text{then } X_T \geq K + c \end{cases}$$

Moreover, using the dominated convergence theorem (since $H \in L^1(\mathbf{P}^*)$) we have:

$$\lim_{p\to\infty} \mathbb{E}^*[|(c \wedge H) - (c_p(\rho)^{\frac{1}{p-1}} \wedge H)|] = 0.$$

So with the limit $p \to \infty$, a call option at the strike price K converges $\mathbf{P}-a.s.$ and in $L^1(\mathbf{P}^*)$ to the call option with the strike price $\tilde{K} = K + c$.

3. As with the call option, we have:

$$\tilde{\varphi}H \to (H - c)^+ = (K - c - X_T)^+ \ \ \mathbf{P}-a.s.,$$

and with the limit $p \to \infty$, a put option at the strike price K converges in $L^1(\mathbf{P}^*)$ to the put option with the strike price $\tilde{K} = K - c$ by dominated convergence.
Note that in this case $c \in [0, K)$ since for $c \geq K$, this option is never exercised, so its price is 0.

$\square$

3 Optimal hedging for a concave function

Let us assume that our investor, instead of being a standard risk averse agent, is in fact inclined to take risk. In our model this corresponds to a concave instead of convex loss function.
Hence, let $k : [0, \infty) \to [0, \infty)$ be increasing and strictly concave function with $k(0) = 0$. Our optimization problem is still the same, we want to find self-financing strategy (V_0, ξ) such that:

$$\mathbb{E}[k((H - \bar{V}_T)^+)] = min,$$

under the constraint $V_0 \leq \tilde{V}_0$.
As with the convex function, we want to reduce our problem to searching for an element $\tilde{\varphi} \in \mathcal{R}$.

Theorem 3.0.1. *There exists a solution $\tilde{\varphi} \in \mathcal{R}$ to the problem:*

$$\min_{\varphi \in \mathcal{R}} \mathbb{E}[k((1 - \varphi)H)],$$

under the constraint $\mathbb{E}^*[\varphi H] \leq \tilde{V}_0$. Moreover, the solution $\tilde{\varphi}$ is unique on the set $\{H > 0\}$.

Proof. The existence of the solution $\tilde{\varphi}$ provides the proof of theorem 2.0.1.
Let $\tilde{\varphi}$ be a solution. For any $\varphi \in \mathcal{R}_0$ and for $\epsilon \in [0, 1]$ we define:

$$\varphi_\epsilon = (1 - \epsilon)\tilde{\varphi} + \epsilon\varphi.$$

So we have:

$$\mathbb{E}[k((1 - \varphi_\epsilon)H)] = \mathbb{E}[k((1 - \epsilon)(1 - \tilde{\varphi})H + \epsilon(1 - \varphi)H)].$$

Using the concavity of the k function we get:

$$\mathbb{E}[k((1 - \varphi_\epsilon)H)] \geq (1 - \epsilon)\mathbb{E}[k((1 - \tilde{\varphi})H)] + \epsilon\mathbb{E}[k((1 - \varphi)H)] \geq \mathbb{E}[k((1 - \tilde{\varphi})H)],$$

since $\tilde{\varphi}$ is the solution of the optimization problem. The above inequality is strict for strictly concave k on the set $\{H > 0\}$, thus the $\tilde{\varphi}$ solution is unique on this set. □

Using theorem 2.0.2 both of the above problems are equivalent, thus on $\{H = 0\}$ we set $\tilde{\varphi} = 1$, while on $\{H > 0\}$, the following theorem provides the structure of $\tilde{\varphi}$.

Theorem 3.0.2. *The solution φ of the optimization problem*

$$\min_{\varphi \in \mathcal{R}} \mathbb{E}[k((1-\varphi)H)],$$

under the constraint $\mathbb{E}^*[\varphi H] \leq \tilde{V}_0$ is given by:

$$\tilde{\varphi} = \mathbf{1}_{\{k(H) > cH\rho\}},$$

where $\rho = \frac{\mathbf{dP}^*}{\mathbf{dP}}$, and constant c determined by the condition $\mathbb{E}^*[\tilde{\varphi}H] = \tilde{V}_0$.

Proof. The proof of this theorem is in [3], for the sake of clarity of the thesis, it has been rewritten.
From the concavity of k and from $k(0) = 0$ we have:

$$\mathbb{E}[k((1-\varphi)H + \varphi 0)] \geq \mathbb{E}[k(H)] - \mathbb{E}[\varphi k(H)].$$

If we minimize the lower bound of the above inequality, it will be less then or equal to $\mathbb{E}[k((1-\varphi)H)]$ for every $\varphi \in \mathcal{R}$. Minimizing the lower bound is equivalent to:

$$\max_{\varphi \in \mathcal{R}} \mathbb{E}[\varphi k(H)],$$

under the constraint $\mathbb{E}^*[\varphi H] \leq \tilde{V}_0$. We can find solution to above optimization problem by the Neyman-Pearson lemma. Let us define the probabilistic measures $\mathbf{Q}, \mathbf{Q}^*$ with the following Radon-Nikodym derivatives:

$$\frac{\mathbf{dQ}}{\mathbf{dP}} = \mathrm{const}k(H), \qquad \frac{\mathbf{dQ}^*}{\mathbf{dP}^*} = \mathrm{const}H.$$

Then, our optimization problem becomes the problem of finding the uniformly most powerful test of the $\mathbf{Q}^*$ hypotheses against the $\mathbf{Q}$ alternative at the significance level of $\tilde{V}_0/\mathbb{E}^*[H]$. Using the Neyman-Pearson lemma, we have the following structure of the optimal test $\tilde{\varphi}$:

$$\tilde{\varphi}(x) = \begin{cases} 1 & \text{then } \frac{\mathbf{dQ}}{\mathbf{dQ^*}} > c \\ & \\ 0 & \text{then } \frac{\mathbf{dQ}}{\mathbf{dQ^*}} < c \end{cases},$$

where $\frac{\mathbf{dQ}}{\mathbf{dQ^*}}$ equals:

$$\frac{\mathbf{dQ}}{\mathbf{dQ}^*} = \frac{\mathbf{dQ}}{\mathbf{dP}} \frac{\mathbf{dP}}{\mathbf{dP}^*} \frac{\mathbf{dP}^*}{\mathbf{dQ}^*} = \mathrm{const}\frac{k(H)}{H}(\rho)^{-1},$$

where $\rho = \frac{\mathbf{dP}^*}{\mathbf{dP}}$. Thus, the optimal test is equal to:

$$\tilde{\varphi} = \mathbf{1}_{\{\frac{k(H)}{H}(\rho)^{-1} > c\}} \quad \text{on } \{H > 0\},$$

where constant c is given by the condition $\mathbb{E}^*[\tilde{\varphi}H] = \tilde{V}_0$. Hence, $\tilde{\varphi}$ minimizes the lower bound of the inequality. For such $\tilde{\varphi}$ we have:

$$\mathbb{E}[k((1-\tilde{\varphi})H)] = \mathbb{E}[(1-\tilde{\varphi})k(H)],$$

so $\tilde{\varphi}$ is the solution to the optimization problem. □

Remark. *In the case where $\mu = r$, $\rho = 1$ and the optimal test is as follows:*

$$\tilde{\varphi} = \mathbf{1}_{\{\frac{k(H)}{H} > c\}}.$$

3.1 Power loss function

Now let us consider a concave loss function equal to:

$$k(x) = x^p, \quad \text{for some } p \in (0,1).$$

Then on the set $\{\frac{k(H)}{H}(\rho)^{-1} > c\} \cup \{H = 0\}$, the optimal test takes the form of:

$$\varphi_p = \mathbf{1}_{\{H^{1-p}\rho < a_p\}},$$

where the constant a_p is determined by the initial price condition.
Let us also assume that there is a unique constant a_0 such that $\mathbb{E}^*[\varphi_0 H] = \tilde{V}_0$, where φ_0 denotes the indicator function of the set $\{a_0 > H\rho\}$. With $p \to 0$ we have the following fact:

Fact 4. *For $p \to 0$ the solution φ_p converges to the solution of quantile hedging considered in [1] i.e.*

$$\varphi_p \to \varphi_0 = \mathbf{1}_{\{a_0 > H\rho\}}$$

both $\mathbf{P}-a.s.$ and in $L^1(\mathbf{P}^*)$.

Proof. The proof is in [3] as the proof of *Proposition 5.4*. □

Fact 5. *For $p \to 1$ the solution φ_p converges to the solution φ_1 in the linear case, both $\mathbf{P}-a.s.$ and in $L^1(\mathbf{P}^*)$.*

Proof. Recall that in the case of a linear loss function, the optimal test is:

$$\varphi_1 = \mathbf{1}_{\{(\rho)^{-1}>c\}} = \mathbf{1}_{\{(\rho)<a_1\}},$$

where a_1 is given by $\mathbb{E}^*[\varphi_1 H] = \tilde{V}_0$, and of course it is unique.
Let us consider a convergent subsequence a_{p_n} such that $\lim_{n\to\infty} a_{p_n} \to a^*$. Since $H^{1-p_n} \to 1$, we have:

$$\mathbf{1}_{\{H^{1-p_n}\rho<a_{p_n}\}} \to \mathbf{1}_{\{\rho<a^*\}}, \quad \mathbf{P}-a.s.$$

Moreover, using the dominated convergence theorem, we have:

$$\mathbb{E}^*[H\mathbf{1}_{\{H^{1-p_n}\rho<a_{p_n}\}}] \to \mathbb{E}^*[H\mathbf{1}_{\{\rho<a^*\}}],$$

and this implies $\mathbb{E}^*[H\mathbf{1}_{\{\rho<a^*\}}] = \tilde{V}_0$. From the uniqueness of a_1 it follows that $a^* = a_1$ and applying this argument to the lim sup and lim inf respectively, we see that $\lim_{p\to 1} a_p = a_1$. Consequently

$$\varphi_p = \mathbf{1}_{\{H^{1-p}\rho<a_p\}} \to \mathbf{1}_{\{\rho<a_1\}} = \varphi_1,$$

both $\mathbf{P}-a.s.$ and in $L^1(\mathbf{P}^*)$. □

4 Computations in the Black-Scholes model

Let us now return to the Black-Scholes model that was introduced in the section 1. Recall that in this model, the price of the underlying asset is a geometric Brownian motion, while the Radon-Nikodym derivative $\frac{\mathbf{dP^*}}{\mathbf{dP}}$ is equal to:

$$\rho = \exp\big(0.5t(\frac{m^2}{\sigma^2} - m)\big)\big(\frac{X_T}{X_0}\big)^{-\frac{m}{\sigma^2}},$$

where $m = \mu - r$.

Remark. *By $\tilde{C}$ we will denote accumulated payoff of the modified contingent claim $\tilde{H}$ i.e.:*

$$\tilde{C} = e^{rT}\tilde{H}.$$

4.1 Call option

Consider a call option at strike price K i,e, its payoff at the time T equals $(S_T - K)^+$.

Linear case When the loss function is linear, according to the subsection 2.1 the modified contingent claim for $m > 0$ has the form

$$\tilde{H} = \mathbf{1}_{\{X_T > c_1\}} H = e^{-rT}(S_T - K)^+ \mathbf{1}_{\{S_T > c_2\}}.$$

The price of the modified contingent claim $\tilde{C}$ is:

$$\Pi(\tilde{C}, t) = e^{-r\tau}\mathbb{E}^*[(S_T - K)^+ \mathbf{1}_{\{S_T > c\}} | \mathcal{F}_t],$$

where $\tau = T - t$. Obviously, in the case when $c \leq K$, the above option becomes a normal call option. For the remaining c we have

$$\Pi(\tilde{C}, t) = e^{-r\tau}\mathbb{E}^*[(S_T - c)^+ + (c - K)\mathbf{1}_{\{S_T \geq c\}} | \mathcal{F}_t] = \Pi((S_T - c)^+, t) + e^{-r\tau}(c - K)N(d_2(c, t)),$$

where N is the distribution of the standard normal variable, and

$$d_2(c, t) = \frac{\ln(\frac{S_t}{c}) + (r - 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

The constant c we get from the condition $\Pi(\tilde{C}, 0) = \tilde{V}_0$. According to the subsection 1.3 in order to find the process $\xi = (\xi_t)_{t \in [0,T]}$ we need to calculate the derivative of the price of this claim. So, we have:

$$\xi_t = N(d_1(c, t)) + \frac{c - K}{c\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(c,t)}.$$

Again, using the subsection 2.1 for $m < 0$, the modified claim takes the form $\tilde{H} = e^{-rT}(S_T - K)^+ \mathbf{1}_{\{S_T < c\}}$. For $c \leq K$ the claim is 0. Otherwise, the price of $\tilde{C}$ is:

$$\Pi(\tilde{C}, t) = \mathbb{E}^*[(S_T - K)^+ - (S_T - c)^+ + (K - c)\mathbf{1}_{\{S_T > c\}}].$$

Its value is equal to:

$$\Pi(\tilde{C}, t) = \Pi((S_T - K)^+, t) - \Pi((S_T - c)^+, t) + (K - c)e^{-rT}N(d_2(c, t)),$$

where c is determined by the initial condition $\Pi(\tilde{C}, 0) = \tilde{V}_0$. In this case, ξ is given by:

$$\xi_t = N(d_1(K, t)) - N(d_1(c, t)) + \frac{K - c}{c\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(c,t)}.$$

In the case when $m = 0$, the modified claim is:

$$\tilde{C} = \varphi(S_T - K)^+,$$

and its price is equal to:

$$\Pi(\tilde{C}, t) = \varphi\Pi((S_T - K)^+, t),$$

where $\varphi = \tilde{V}_0 / \Pi((S_T - K)^+, 0)$.

Convex loss function Now consider the convex loss function $l(x) = \frac{x^p}{p}$ for some $p > 1$. From section 2 we know that optimal hedging is obtained by hedging the following claim:

$$\tilde{H} = H - (c_{p_1}(\rho)^{\frac{1}{p-1}} \wedge H) = H - (c_{p_2} X_T^{-\frac{m}{\sigma^2(p-1)}} \wedge H) = e^{-rT}(C - (c_{p_3} S_T^{-\frac{m}{\sigma^2(p-1)}} \wedge C)),$$

where $C = (S_T - K)^+$. This option is not equal 0, when $c_{p_3} S_T^{-\frac{m}{\sigma^2(p-1)}} < C$. So we will examine when this is fulfilled.

For $m > 0$, $c_{p_3} x^{-\frac{m}{\sigma^2(p-1)}}$ is the convex and decreasing function (relative to x), therefore there is at most one point of intersection L with $(x-K)^+$, which is related to the constant c_{p_3} by equation:

$$(L-K)L^{\frac{m}{\sigma^2(p-1)}} = c_{p_3}.$$

Above this point, the claim pays a positive payoff, so the option is:

$$\tilde{H} = e^{-rT}(S_T - K - (L-K)L^{\frac{m}{\sigma^2(p-1)}} S_T^{-\frac{m}{\sigma^2(p-1)}})\mathbf{1}_{\{S_T \geq L\}}.$$

The price of $\tilde{C}$ equals:

$$\Pi(\tilde{C}, t) = e^{-r\tau}\mathbb{E}^*[(S_T - K - (L-K)L^{\frac{m}{\sigma^2(p-1)}} S_T^{-\frac{m}{\sigma^2(p-1)}})\mathbf{1}_{\{S_T > L\}}|\mathcal{F}_t].$$

Using the fact that $S_t \in \mathcal{F}_t$ as well as $B_T^* - B_t^*$ are independent of this σ-algebra we get:

$$\mathbb{E}^*[S_T\mathbf{1}_{\{S_T \geq L\}}|\mathcal{F}_t] = S_t e^{(r-0.5\sigma^2)\tau}\mathbb{E}^*[e^{\sigma(B_T^* - B_t^*)}\mathbf{1}_{\{B_T^* - B_t^* > -\sqrt{\tau}d_2(L,t)\}}]$$

$$= S_t e^{(r-0.5\sigma^2)\tau}\int_{-\sqrt{\tau}d_2(L,t)}^{\infty}\frac{1}{\sqrt{2\pi\tau}}e^{\sigma x}e^{-\frac{x^2}{2\tau}}dx = S_t e^{r\tau}\mathbf{P}^*(Z > -\sqrt{\tau}d_2(L,t)),$$

where Z is a random variable from the distribution $N(\sigma\tau, \tau)$. Ultimately we get:

$$\mathbb{E}^*[S_T\mathbf{1}_{\{S_T \geq L\}}|\mathcal{F}_t] = S_t e^{r\tau}N(d_1(L,t)),$$

where $d_1(L,t) = d_2(L,t) + \sigma\sqrt{\tau}$. It remains to calculate:

$$\mathbb{E}^*[S_T^{-\frac{m}{\sigma^2(p-1)}}\mathbf{1}_{\{S_T > L\}}|\mathcal{F}_t] = S_t^{-\frac{m}{\sigma^2(p-1)}}e^{-\frac{m}{\sigma^2(p-1)}(r-0.5\sigma^2)\tau}\mathbb{E}^*[e^{-\frac{m}{\sigma(p-1)}(B_T^* - B_t^*)}\mathbf{1}_{\{(B_T^* - B_t^* > -\sqrt{\tau}d_2(L,t))\}}]$$

$$= S_t^{-\frac{m}{\sigma^2(p-1)}}e^{-\frac{m}{\sigma^2(p-1)}(r-0.5\sigma^2)\tau}\int_{-\sqrt{\tau}d_2(L,t)}^{\infty}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{m}{\sigma(p-1)}x}e^{-\frac{x^2}{2\tau}}dx.$$

As in the previous case, the integral can be replaced into survival probability of a random variable from a certain normal distribution. Thus, we have

$$S_t^{-\frac{m}{\sigma^2(p-1)}} e^{-\frac{m}{\sigma^2(p-1)}(r-0.5\sigma^2)\tau+0.5\frac{m^2}{\sigma^2(p-1)^2}\tau}\mathbf{P}^*(Z > -\sqrt{\tau}d_2(L,t)),$$

where $Z \sim N(-\frac{m}{\sigma(p-1)}\tau, \tau)$. In summary, the price of the modified contingent claim at the time t is:

$$\begin{aligned}\Pi(\tilde{C},t) = (K-L)(\frac{L}{S_t})^{\frac{m}{\sigma^2(p-1)}} e^{-r\tau} e^{\frac{m}{\sigma^2(p-1)}(0.5\sigma^2-r+0.5\frac{m}{(p-1)})\tau} N(d_2(L,t) - \frac{m\sqrt{\tau}}{\sigma(p-1)}) \\ + S_t N(d_1(L,t)) - K e^{-r\tau} N(d_2(L,t)),\end{aligned}$$

where L is given by the condition $\Pi(\tilde{C},0) = \tilde{V}_0$. By differentiation we obtain the hedging strategy ξ:

$$\xi_t = N(d_1(L,t)) + \frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L,t)} - \frac{K}{L\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L,t)}$$

$$-\frac{m}{\sigma^2(p-1)}(K-L)(\frac{L}{S_t})^{\frac{m}{\sigma^2(p-1)}}\frac{1}{S_t}e^{-r\tau}e^{\frac{m}{\sigma^2(p-1)}(0.5\sigma^2-r+0.5\frac{m}{(p-1)})\tau}N(d_2(L,t) - \frac{m\sqrt{\tau}}{\sigma(p-1)})$$

$$+(K-L)(\frac{L}{S_t})^{\frac{m}{\sigma^2(p-1)}}e^{-r\tau}e^{\frac{m}{\sigma^2(p-1)}(0.5\sigma^2-r+0.5\frac{m}{(p-1)})\tau}\frac{1}{S_t\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5(d_2(L,t)-\frac{m\sqrt{\tau}}{\sigma(p-1)})^2}.$$

For $m < 0$ in the case, when $-\frac{m}{\sigma^2(p-1)} \in (0,1]$, the function $c_{p_3}x^{-\frac{m}{\sigma^2(p-1)}}$ is increasing and concave (linear), therefore there is at most one point of intersection with $(x-K)^+$. The price of this option is the same as in the previous case.

If $-\frac{m}{\sigma^2(p-1)} \in (1,\infty)$, $c_{p_3}x^{-\frac{m}{\sigma^2(p-1)}}$ is increasing and convex, thus it has exactly two points of intersection with $(x-K)^+$ (if it did not have them, the price of the modified claim would be 0). Therefore, the claim takes the form of:

$$\tilde{H} = e^{-rT}(S_T - K - (L_1-K)L_1^{\frac{m}{\sigma^2(p-1)}}S_T^{-\frac{m}{\sigma^2(p-1)}})\mathbf{1}_{\{S_T\in(L_1,L_2)\}} \quad \text{equivalent}$$

$$\tilde{H} = e^{-rT}(S_T - K - (L_2-K)L_2^{\frac{m}{\sigma^2(p-1)}}S_T^{-\frac{m}{\sigma^2(p-1)}})\mathbf{1}_{\{S_T\in(L_1,L_2)\}}.$$

The price $\tilde{C}$ is equal to:

$$(K-L_1)(\frac{L_1}{S_t})^{\frac{m}{\sigma^2(p-1)}}e^{-r\tau}e^k\big(N(-d_2(L_2)+\frac{m\sqrt{\tau}}{\sigma(p-1)})-N(-d_2(L_1)+\frac{m\sqrt{\tau}}{\sigma(p-1)})\big)$$
$$+S_t\big(N(-d_1(L_2))-N(-d_1(L_1))\big)-Ke^{-r\tau}\big(N(-d_2(L_2))-N(-d_2(L_1))\big),$$

where $k=\frac{m}{p-1}\tau(0.5-\frac{r}{\sigma^2}+0.5\frac{m}{\sigma^2(p-1)})$, and $(L_1-K)L_1^{\frac{m}{\sigma^2(p-1)}}=(L_2-K)L_2^{\frac{m}{\sigma^2(p-1)}}$. Calculating the derivative of the expression above, we obtain the form of ξ at the time t:

$$\xi_t=N(-d_1(L_2))-N(-d_1(L_1))-\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}(e^{-0.5d_1^2(L_2)}-e^{-0.5d_1^2(L_1)})$$

$$+\frac{K}{L_2}\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L_2)}-\frac{K}{L_1}\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L_1)}$$

$$-\frac{m}{\sigma^2(p-1)}(K-L_1)(\frac{L_1}{S_t})^{\frac{m}{\sigma^2(p-1)}}\frac{1}{S_t}e^{-r\tau}e^k\big(N(-d_2(L_2)+\frac{m\sqrt{\tau}}{\sigma(p-1)})-N(-d_2(L_1)+\frac{m\sqrt{\tau}}{\sigma(p-1)})\big)$$

$$-(K-L_1)(\frac{L_1}{S_t})^{\frac{m}{\sigma^2(p-1)}}\frac{1}{S_t}e^{-r\tau}e^k\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}\big(e^{-0.5(-d_2(L_2)+\frac{m\sqrt{\tau}}{\sigma(p-1)})^2}-e^{-0.5(-d_2(L_1)+\frac{m\sqrt{\tau}}{\sigma(p-1)})^2}\big).$$

In the case of $m=0$, the modified contingent claim is:

$$\tilde{H}=e^{-rT}((S_T-K)^+-(c_p\wedge(S_T-K)^+))=e^{-rT}(S_T-(c_p+K))^+.$$

The price of this option is equal to the price of the call option with the strike price c_p+K.

Concave loss function Now consider the concave loss function $l(x)=x^p$, for $p\in(0,1)$. From section 3 we have the following modified contingent claim:

$$\tilde{H}=e^{-rT}(S_T-K)^+\mathbf{1}_{\{((S_T-K)^+)^{1-p}<a_pS_T^{\frac{m}{\sigma^2}}\}}.$$

The function $f(x)=((x-K)^+)^{1-p}$ is 0 for $x\le K$ and is increasing and concave for $x>K$.
When $m<0$ then $g(x)=cx^{\frac{m}{\sigma^2}}$ is convex and decreasing, thus it has exactly one

point of intersection L with the function f. Above this point $f(x) > g(x)$, therefore the payoff of the modified contingent claim is 0 for $x \geq L$. To sum up, for $m < 0$ we have:

$$\tilde{H} = e^{-rT}(S_T - K)^+ \mathbf{1}_{\{S_T < L\}},$$

where L fulfills the condition $(L-K)^{1-p} = a_p L^{\frac{m}{\sigma^2}}$. The price of $\tilde{C}$ at the time t is:

$$\Pi(\tilde{C}, t) = \Pi((S_T - K)^+, t) - \Pi((S_T - L)^+, t) + (K - L)e^{-r\tau} N(d_2(L, t)),$$

where $\tau = T - t$, and the constant L is given by the condition $\Pi(\tilde{C}, 0) = \tilde{V}_0$. By differentiation we obtain the hedging strategy:

$$\xi_t = N(d_1(K,t)) - N(d_1(L,t)) + \frac{K-L}{L\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(L,t)}.$$

When $\frac{m}{\sigma^2} \in (0, 1-p]$, the function g is increasing and concave, the exponent in the function g is smaller than the exponent in f, thus again there is at most one point L, where these functions are equal. Above this point, the payoff of the modified option is zero, so both the modified claim and its price are the same as when $m < 0$.
For $\frac{m}{\sigma^2} \in (1-p, 1]$, the function g has a larger exponent than f, thus there are exactly two points L_1 and L_2, where these functions are equal. On the interval $x \in [L_1, L_2]$ we have $f(x) \geq g(x)$, and then payoff $\tilde{H} = 0$. In summary, the modified claim takes the form:

$$\tilde{H} = e^{-rT}(S_T - K)^+ \mathbf{1}_{\{S_T \leq L_1\} \cup \{S_T > L_2\}} = e^{-rT}(S_T - K)^+ (\mathbf{1}_{\{S_T \leq L_1\}} + \mathbf{1}_{\{S_T > L_2\}}),$$

and the constants L_1, L_2 are related to each other by the condition $(L_1 - K)^{1-p} L_1^{-\frac{m}{\sigma^2}} = (L_2 - K)^{1-p} L_2^{-\frac{m}{\sigma^2}}$. The price of the claim is equal to:

$$\begin{aligned}\Pi(\tilde{C}, t) = \Pi((S_T - K)^+, t) - \Pi((S_T - L_1)^+, t) + (K - L_1)e^{-r\tau} N(d_2(L_1, t)) \\ + \Pi((S_T - L_2)^+, t) + e^{-r\tau}(L_2 - K) N(d_2(L_2, t)),\end{aligned}$$

while

$$\begin{aligned}\xi_t = N(d_1(K,t)) - N(d_1(L_1,t)) + N(d_1(L_2,t)) \\ + \frac{K-L_1}{L_1\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(L_1,t)} + \frac{L_2-K}{L_2\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(L_2,t)}.\end{aligned}$$

For $\frac{m}{\sigma^2} \in (1, \infty)$, g is increasing and convex, so there are exactly two points L_1, L_2 in which $f(x) = g(x)$ and the modified contingent claim takes the same form as in the

case, when $\frac{m}{\sigma^2} \in (1-p, 1]$.
When $m = 0$, the modified claim is:

$$\tilde{H} = e^{rT}(S_T - K)^+ \mathbf{1}_{\{((S_T-K)^+)^{1-p} < a_p\}} = e^{-rT}(S_T - K)^+ \mathbf{1}_{\{S_T < L\}}.$$

Therefore, the price of this claim is the same as in the case, when $m < 0$.

4.2 Put option

Consider a put option at strike price K i,e, its payoff at the time T equals $(K - S_T)^+$.

Linear case According to the subsection 2.1 in the case of linear loss function when $m > 0$ the modified conditional claim is:

$$\tilde{H} = e^{-rT}(K - S_T)^+ \mathbf{1}_{\{S_T > c\}},$$

for $c \in [0, K)$, if $c \geq K$ the payoff of the above option is 0. The price of $\tilde{C}$ is equal to:

$$\Pi(\tilde{C}, t) = e^{-r\tau}\mathbb{E}^*[(K - S_T)^+ \mathbf{1}_{\{S_T > c\}} | \mathcal{F}_t] = e^{-r\tau}\mathbb{E}^*[(K - S_T)^+ - (c - S_T)^+ + (c - K)\mathbf{1}_{\{S_T < c\}} | \mathcal{F}_t].$$

After calculating the conditional expected value, we get:

$$\Pi(\tilde{C}, t) = \Pi((K - S_T)^+, t) - \Pi((c - S_T)^+, t) + (c - K)e^{-r\tau}N(-d_2(c, t)),$$

where the constant c is determined by the initial condition $\Pi(\tilde{C}, 0) = \tilde{V}_0$. By differentiation we get the ξ process of the form:

$$\xi_t = N(d_1(K, t)) - N(d_1(c, t)) - \frac{c - K}{c\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(c,t)}.$$

For $m < 0$ the payoff of the modified claim take the form:

$$\tilde{C} = (K - S_T)^+ \mathbf{1}_{\{S_T < c\}}.$$

Therefore, the option price at the time t is equal to:

$$\begin{aligned}\Pi(\tilde{C}, t) = e^{-r\tau}\mathbb{E}^*[(K - S_T)^+ \mathbf{1}_{\{S_T > c\}} | \mathcal{F}_t] &= e^{-r\tau}\mathbb{E}^*[(c - S_T)^+ + (K - c)\mathbf{1}_{\{S_T < c\}} | \mathcal{F}_t] \\ &= \Pi((c - S_T)^+, t) + (K - c)e^{-r\tau}N(-d_2(c, t)),\end{aligned}$$

when $c < K$. Otherwise, the payoff of this option is equal to the payoff of a standard put option. The constant c we get from the condition $\Pi(\tilde{C}, 0) = \tilde{V}_0$, while for hedging we need:

$$\xi_t = N(d_1(c,t)) - 1 - \frac{K-c}{c\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(c,t)}$$

the underlying asset in the portfolio. For $m = 0$ the modified claim is: $\tilde{C} = \varphi(K - S_T)^+$, and its price at the time t is equal to

$$\Pi(\tilde{C}, t) = \varphi \Pi((K - S_T)^+, t),$$

where $\varphi = V_0 / \Pi((K - S_T)^+, 0)$.

Convex loss function As with a call option, consider the convex loss option $l(x) = \frac{x^p}{p}$ for $p > 1$. From subsection 2.3, we have the following modified contingent claim:

$$\tilde{H} = e^{-rT}((K - S_T)^+ - (c_p S_T^{-\frac{m}{\sigma^2(p-1)}} \wedge (K - S_T)^+)).$$

For $-\frac{m}{\sigma^2(p-1)} \in (0,1]$ the function $g(x) = cx^{-\frac{m}{\sigma^2(p-1)}}$ is increasing and concave (linear for $-\frac{m}{\sigma^2(p-1)} = 1$), so it has exactly one point of intersection L with $f(x) = (K - x)^+$. Above this point, the claim payoff $\tilde{H}$ is 0, so we have:

$$\tilde{H} = e^{-rT}\Big((K - S_T) - (K - L) L^{\frac{m}{\sigma^2(p-1)}} S_T^{-\frac{m}{\sigma^2(p-1)}}\Big) \mathbf{1}_{\{S_T < L\}}.$$

Using the calculations for the call option, we get the price of the $\tilde{C}$ at the time t equal to:

$$\begin{aligned}\Pi(\tilde{C}, t) = (L - K)(\frac{L}{S_t})^{\frac{m}{\sigma^2(p-1)}} e^{-r\tau} e^{\frac{m}{p-1}\tau(0.5 - \frac{r}{\sigma^2} + 0.5\frac{m}{\sigma^2(p-1)})} N(-d_2(L,t) + \frac{m\sqrt{\tau}}{\sigma(p-1)}) \\ + K e^{-r\tau} N(-d_2(L,t)) - S_t N(-d_1(L,t)),\end{aligned}$$

where L we determine from the condition $\Pi(\tilde{C}, 0) = \tilde{V}_0$, while the process ξ is given as:

$$\xi_t = \frac{1}{\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(L,t)} - N(-d_1(L,t)) - \frac{K}{L\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5 d_1^2(L,t)}$$

$$-\frac{m}{\sigma^2(p-1)}(L-K)(\frac{L}{S_T})^{\frac{m}{\sigma^2(p-1)}} \frac{1}{S_t} e^{-r\tau} e^{\frac{m}{\sigma^2(p-1)}(0.5\sigma^2 - r + 0.5\frac{m}{(p-1)})\tau} N(-d_2(L,t) + \frac{m\sqrt{\tau}}{\sigma(p-1)})$$

$$-(L-K)(\frac{L}{S_T})^{\frac{m}{\sigma^2(p-1)}} e^{-r\tau} e^{\frac{m}{\sigma^2(p-1)}(0.5\sigma^2 - r + 0.5\frac{m}{(p-1)})\tau} \frac{1}{S_t\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-0.5(-d_2(L,t) + \frac{m\sqrt{\tau}}{\sigma(p-1)})^2}.$$

For $-\frac{m}{\sigma^2(p-1)} \in (1, \infty)$, $g(x) = cx^{-\frac{m}{\sigma^2(p-1)}}$ is increasing and convex, hence it has exactly one point of intersection L with $f(x) = (K-x)^+$. The modified claim and its price at time t have the same form as in the previous case.
For $m > 0$, the function $g(x)$ is decreasing and convex, thus it has at most two points of intersection L_1, L_2 with f such that $0 \leq L_1 < L_2 \leq K$ and are related to each other by the condition $(K-L_1)L_1^{\frac{m}{\sigma^2(p-1)}} = (K-L_2)L_2^{\frac{m}{\sigma^2(p-1)}}$. Hence, the conditional claim takes the form:

$$\tilde{H} = e^{-rT}(K - S_T - (K-L_1)L_1^{\frac{m}{\sigma^2(p-1)}} S_T^{-\frac{m}{\sigma^2(p-1)}})\mathbf{1}_{\{S_T \in (L_1, L_2)\}} \quad \text{equivalent}$$

$$\tilde{H} = e^{-rT}(K - S_T - (K-L_2)L_2^{\frac{m}{\sigma^2(p-1)}} S_T^{-\frac{m}{\sigma^2(p-1)}})\mathbf{1}_{\{S_T \in (L_1, L_2)\}}.$$

By calculating the conditional expected value of $\tilde{C}$ we get the option price equal to:

$$(L_1 - K)(\frac{L_1}{S_t})^{\frac{m}{\sigma^2(p-1)}} e^{-r\tau} e^k \big(N(-d_2(L_2,t) + \frac{m\sqrt{\tau}}{\sigma(p-1)}) - N(-d_2(L_1,t) + \frac{m\sqrt{\tau}}{\sigma(p-1)})\big)$$

$$-S_t\big(N(-d_1(L_2,t)) - N(-d_1(L_1,t))\big) + Ke^{-r\tau}\big(N(-d_2(L_2,t)) - N(-d_2(L_1,t))\big),$$

where $k = \frac{m}{p-1}\tau(0.5 - \frac{r}{\sigma^2} + 0.5\frac{m}{\sigma^2(p-1)})$.
By differentiation of the above expression, we obtain the form of the process ξ:

$$\xi_t = N(-d_1(L_1)) - N(-d_1(L_2)) + \frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}(e^{-0.5d_1^2(L_2)} - e^{-0.5d_1^2(L_1)})$$

$$-\frac{K}{L_2}\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L_2)} + \frac{K}{L_1}\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L_1)}$$

$$-\frac{m}{\sigma^2(p-1)}(L_1 - K)(\frac{L_1}{S_t})^{\frac{m}{\sigma^2(p-1)}}\frac{1}{S_t}e^{-r\tau}e^k\big(N(-d_2(L_2) + \frac{m\sqrt{\tau}}{\sigma(p-1)}) - N(-d_2(L_1) + \frac{m\sqrt{\tau}}{\sigma(p-1)})\big)$$

$$-(L_1 - K)(\frac{L_1}{S_t})^{\frac{m}{\sigma^2(p-1)}}\frac{1}{S_t}e^{-r\tau}e^k\frac{1}{\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}\big(e^{-0.5(-d_2(L_2)+\frac{m\sqrt{\tau}}{\sigma(p-1)})^2} - e^{-0.5(-d_2(L_1)+\frac{m\sqrt{\tau}}{\sigma(p-1)})^2}\big).$$

For $m = 0$, the modified claim is:

$$\tilde{H} = e^{-rT}((K - S_T)^+ - (c_p \wedge (K - S_T)^+)) = e^{-rT}(K - c_p - S_T)^+,$$

while the option price $\tilde{C}$ is equal to the price of the put option at the strike price $K - c_p$.

Concave loss function Now, consider the concave loss function $l(x) = x^p$, for $p \in (0,1)$. From subsection 3.1, the modified contingent claim for optimal hedging takes the form:

$$\tilde{H} = e^{-rT}(K - S_T)^+\mathbf{1}_{\{((K-S_T)^+)^{1-p} < a_p S_T^{\frac{m}{\sigma^2}}\}}.$$

The function $f(x) = ((K - x)^+)^{1-p}$ is equal 0 for $x \geq K$ and is decreasing and concave for $x < K$.
For $\frac{m}{\sigma^2} \in (0,1]$, $g(x) = cx^{\frac{m}{\sigma^2}}$ is increasing and concave, thus it has exactly one point $L \in (0,K]$ such that $f(x) = g(x)$. For $x \in (0,L)$ we have $g(x) < f(x)$, thus the option takes the form:

$$\tilde{H} = e^{-rT}(K - S_T)^+\mathbf{1}_{\{S_T > L\}}.$$

At the time t the price of $\tilde{C}$ is:

$$\Pi(\tilde{C},t) = e^{-r\tau}\mathbb{E}^*[(K - S_T)^+\mathbf{1}_{\{S_T>L\}}|\mathcal{F}_t] = e^{-r\tau}\mathbb{E}^*[(K - S_T)^+ - (L - S_T)^+ + (L - K)\mathbf{1}_{\{S_T<L\}}|\mathcal{F}_t]$$

$$= \Pi((K - S_T)^+, t) - \Pi((L - S_T)^+, t) + (L - K)e^{-r\tau}N(-d_2(L,t)),$$

where the constant L is given by the option price condition at time 0. By differentiation we obtain the formula for ξ process, needed to hedge the option:

$$\xi_t = N(d_1(K,t)) - N(d_1(L,t)) - \frac{L-K}{L\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L,t)}.$$

For $\frac{m}{\sigma^2} \in (1,\infty)$, $g(x) = cx^{\frac{m}{\sigma^2}}$ is increasing and convex, thus there is exactly one point $L \in (0,K]$ such that $f(x) = g(x)$. For $x \in (0,L)$, we have $g(x) < f(x)$, therefore both the payoff of the modified claim $\tilde{C}$ and its price take the same form as in the previous case.
For $m < 0$ the function $g(x)$ is decreasing and convex, so there are at most two points L_1, L_2 for which the functions $g(x), f(x)$ are equal. For $x \in [L_1, L_2]$ we have $f(x) \geq g(x)$ and modified claim is then 0. This means that in this case, the option takes the form:

$$\tilde{H} = (S_T - K)^+ \mathbf{1}_{\{S_T \leq L_1\} \cup \{S_T > L_2\}},$$

and the constants L_1, L_2 are related to each other by the condition $(K-L_1)^{1-p}L_1^{-\frac{m}{\sigma^2}} = (K-L_2)^{1-p}L_2^{-\frac{m}{\sigma^2}}$. The price of the option $\tilde{C}$ at the time t is equal to:

$$\begin{aligned}\Pi(\tilde{C},t) = \Pi((K-S_T)^+,t) - \Pi((L_2-S_T)^+,t) + (L_2-K)e^{-r\tau}N(-d_2(L_2,t))\\ +\Pi((L_1-S_T)^+,t) + (K-L_1)e^{-r\tau}N(-d_2(L_1,t)),\end{aligned}$$

while ξ is given by:

$$\begin{aligned}\xi_t = N(d_1(K,t)) - N(d_1(L_2,t)) - \frac{L_2-K}{L_2\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L_2,t)}\\ +N(d_1(L_1,t)) - 1 - \frac{K-L_1}{L_1\sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-0.5d_1^2(L_1,t)}.\end{aligned}$$

For $m = 0$, the modified claim $\tilde{C}$ equals:

$$\tilde{C} = (K-S_T)^+\mathbf{1}_{\{((K-S_T)^+)^{1-p}<a_p\}} = (K-S_T)^+\mathbf{1}_{\{S_T>L\}}.$$

The price of the above claim at t takes the same form as the option price for $\frac{m}{\sigma^2} \in (0,1]$, and L we determine by the option price condition at time 0.

4.3 Example

In order to illustrate the result of hedging a modified claim, consider an underlying asset with the initial price $S_0 = 100$, the drift $\mu = 0.1$, volatility $\sigma = 0.3$ and a

one-year call option with an exercise price of $K = 100$. Let assume that the risk-free rate is $r = 0.05$. Under the Black-Scholes model, the price of this option is 14.2313. Suppose, we have the initial capital of the 90% call option price, to hedge this option i.e. $\tilde{V}_0 = 12.8081$.
We will minimize the shortfall risk, for power loss function with the exponents equals $0.01, 0.5, 1, 2, 5, 100$, using the formulas derived in subsection 4.1.
First, by using numerical methods of determining the roots of a function (including bisection method, the description of which can be found in chapter 3.1 of [10]) we set the constant from the condition to the initial price of the modified contingent claim. Let us recall that this constant for any loss function is included in a indicator of the modified option payoff moreover, for a convex loss function, this constant is included in the payoff itself. So, we have:

- For $p = 0.01$ the modified option has a constant equal to $L = 196.8459$;
- For $p = 0.5$ the modified option has a constants equal to $L_1 = 196.8459$ and $L_2 = 115651.7$;
- For $p = 1$ the modified option has a constant equal to $c = 116.2422$;
- For $p = 2$ the modified option has a constant equal to $L = 103.4746$;
- For $p = 5$ the modified option has a constant equal to $L = 103.1746$;
- For $p = 100$ the modified option has a constant equal to $L = 103.0797$.

In the case when $p = 0.5$, we have two constants, one of them was determined from the Newton algorithm, which was discussed in chapter 3.3 in [10].

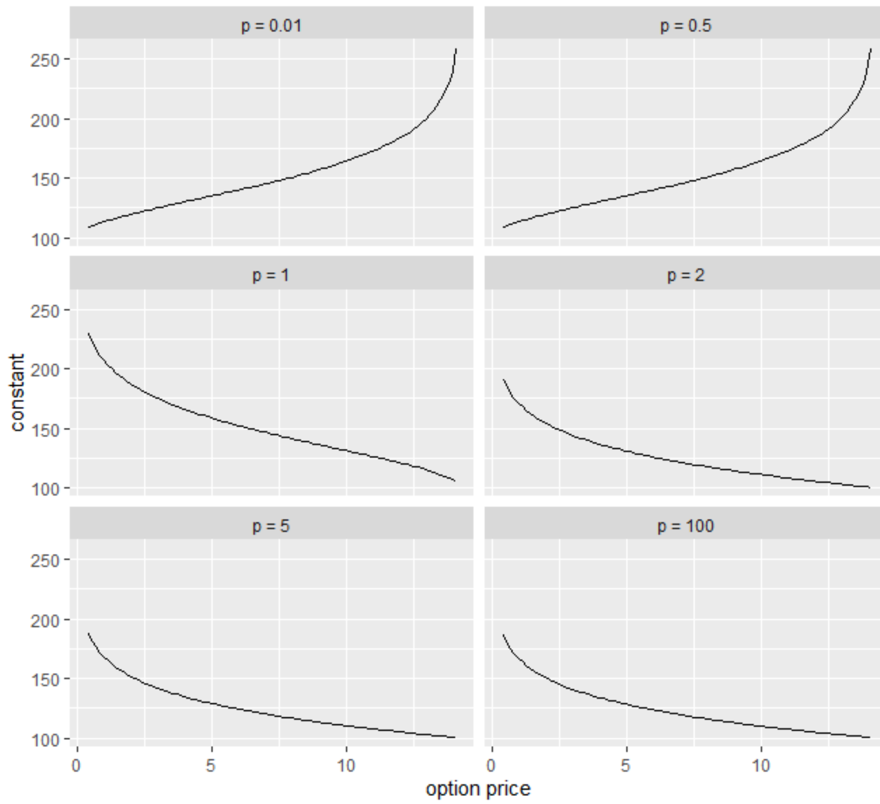

Figure 1: The determined constants depending on the option price.

The figure 1 shows a graph of the determined constant depending on the initial capital. As it could be deduced from the formulas for the option price, in the case of the concave function, as the initial capital increases, the constant increases, while for $p \geq 1$ it decreases. When $p = 0.5$, the constants L_1 i L_2 converges to:

$$x = -\frac{\frac{m}{\sigma^2}K}{1 - p - \frac{m}{\sigma^2}},$$

when the price converges to the call option price. The chart of dependence of the constant L_2 on the price is not shown in the above figure due to the large values of the constant for small option prices, so no changes can be seen for larger option prices.
The modified claims payoffs compared to the standard call option are as follows:

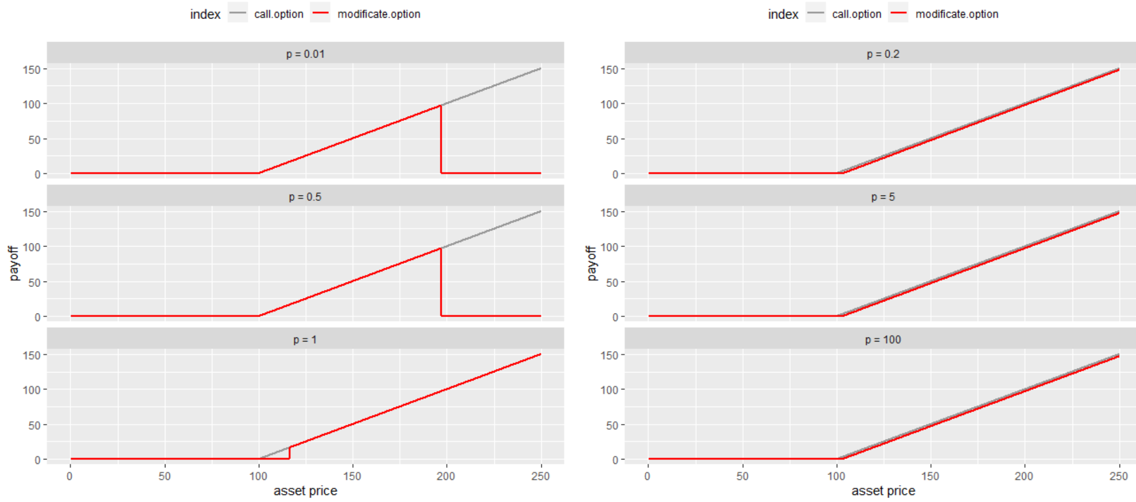

Figure 2: The payoffs for $p = 0.05$, $p = 0.5$, $p = 1$, $p = 2$, $p = 5$, $p = 100$.

In the case when $p = 0.01$, the call option is not hedged at all for high stock prices, for $p = 0.01$ it is similar, with the difference that for stock prices greater than $115,651.7$ we again hedge the entire call option by hedging the modified option. For the linear loss function, the call option is not hedged on the range $[K, 116.2422]$.
For convex loss functions outside the $[0, K]$ interval, the hedging of the modified claim is never sufficient to hedge a call option, but the losses from the exercise of the standard option are quite small.
In order to visualize the results, we simulated the trajectories from the future price of the underlying asset by discretisation of the stochastic differential equation that describes the change in the asset price i.e.

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

So we get a recursive formula for the next stock prices as:

$$S_{t_{i+1}} = S_{t_i} \exp\Big((\mu - 0.5\sigma^2)\delta t + \sigma Z_i\Big),$$

for $i = 0, ..., (\delta t)^{-1}$, where Z_i are i.i.d random variables from $N(0, \delta t)$, and $t_{i+1} - t_i = \delta t$, while by δt we denote the division of time, in our case we assume, that it is $\frac{1}{250}$ per year.
We simulated 10000 trajectories in this way. The first 1000 of them looks like this:

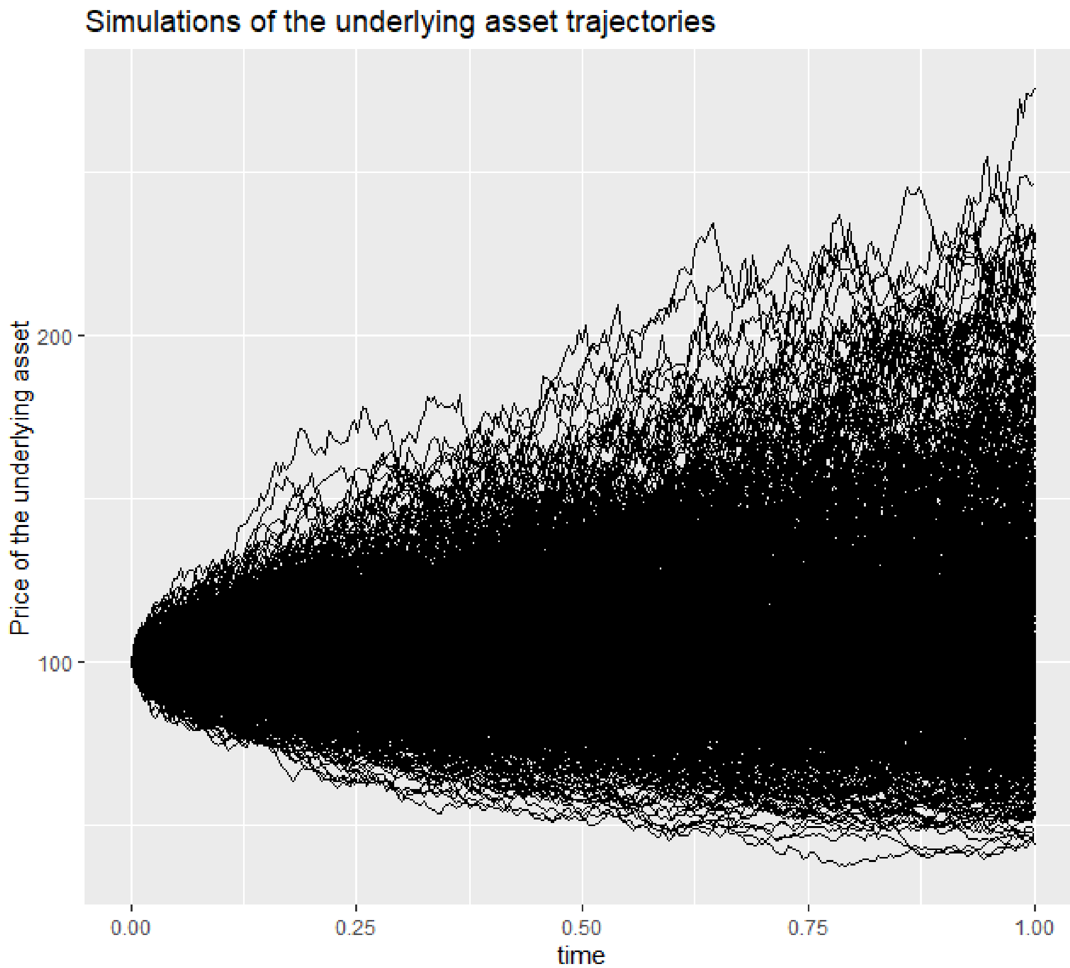

Figure 3: Trajectories of the first 1000 asset price simulations.

Next, we examined the gains/losses of the portfolio, in the case when we hedge the call option as standard i.e. using the full initial capital, as well as using modified claims. The portfolio's gains/losses histograms are as follows:

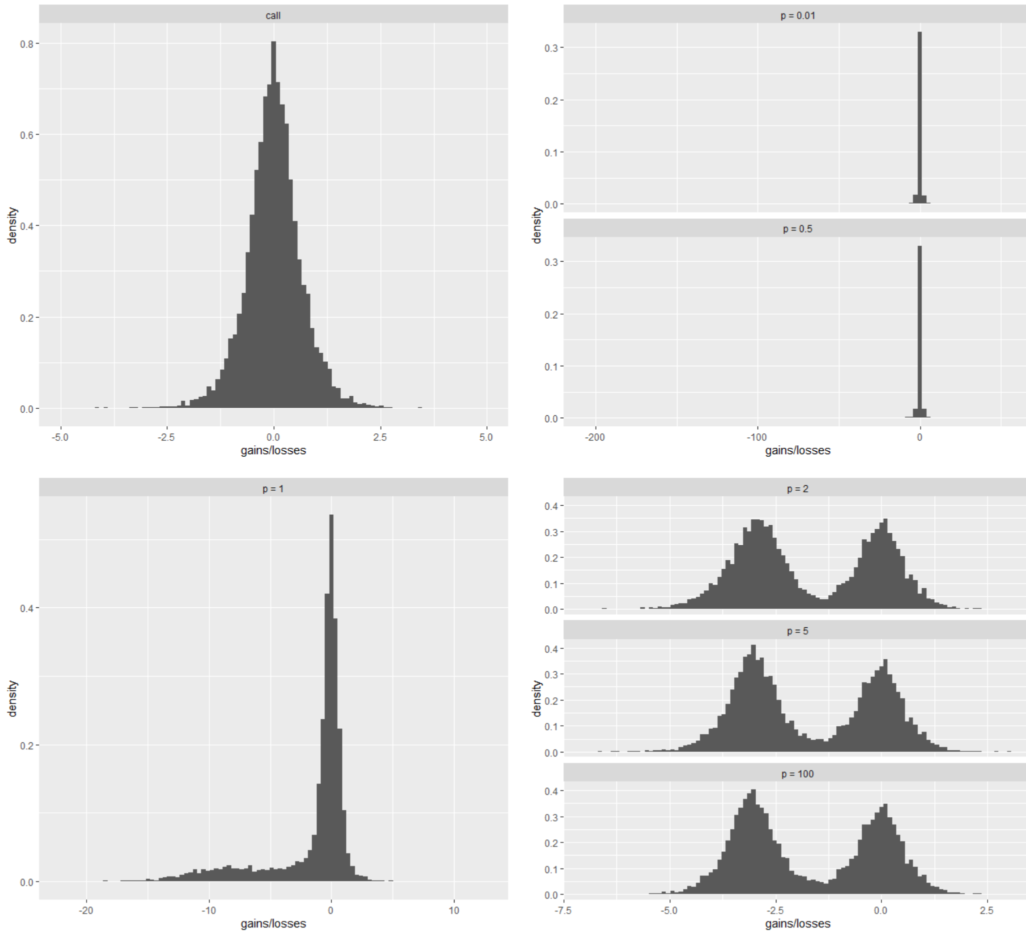

Figure 4: Portfolio's gains/losses histograms for full hedging of the call option and quantile hedge with 90% initial capital for the power loss function with the exponents $p = 0.01$ i $p = 0.5$, $p = 1$ i $p = 2$, $p = 5$ i $p = 100$.

Of course, in the case of a standard hedging, any profits or losses are close to 0, and losses or gains are due to the discretisation of the geometric Brownian motion. In the case of the concave loss functions, we are often hedged, but sometimes the modified option does not provide replication of the call option and this causes large losses (no losses are visible on the chart because they occur very rarely, in this case, when the share price is greater than 196.8459).
For the linear loss function it is similarly to the concave case, except that that the losses are smaller, but they are more frequent.
For convex options, we often have small losses, which again results directly from the

payoff - when we hedge by minimizing the convex loss function, then only for the share price lower then the strike price we are fully hedged.

Statistics	$p = 0.01$	$p = 0.5$	$p = 1$	$p = 2$	$p = 5$	$p = 100$	call
minimum	-204.7759	-204.7759	-22.0559	-6.9707	-6.6311	-6.5226	-4.1842
1 quartile	-0.4578	-0.4578	-1.2426	-3.0292	3.0907	-3.1193	-0.3636
median	-0.0089	-0.0089	-0.2447	-2.1298	-2.1691	-2.1771	0.00742
average	-2.4025	-2.4025	-1.5496	-1.6686	-1.6817	-1.6861	0.0058
3 quartile	0.4006	0.4006	0.2401	-0.0897	-0.0888	-0.0887	0.3688
maximum	55.7709	55.7709	12.4435	2.9966	2.9948	2.9944	3.4814

Table 1: Basic portfolio profit and loss statistics - comparison of basic statistics for different exponents and basic gains / losses statistics for full hedging of the call option.

As we can see in the table 1 the basic statistics when hedging a modified option that minimizes the concave loss function are the same, as both the price of these modified options and the amount of assets needed to hedge for the powers of $p = 0.01$ and $p = 0.5$ are approximately the same. Recall that in the case when $p = 0.5$, the price of the modified option is equal to:

$$\begin{aligned}\Pi(\tilde{C}, t) = \Pi((S_T - K)^+, t) - \Pi((S_T - L_1)^+, t) + (K - L_1)e^{-r\tau}N(d_2(L_1, t)) \\ + \Pi((S_T - L_2)^+, t) + e^{-r\tau}(L_2 - K)N(d_2(L_2, t)).\end{aligned}$$

When L_2 is large (as in our case, where $L_2 = 115651.7$) the components associated with it in the above equation are small, hence the price of this option is equal to the price of the option in the case when $p = 0.01$. The same is true for the ξ process. In a convex case, for different exponents, there are slight differences in the basic statistics.

5 Implementation and numerical analysis

In this section, we will deal with the implementation and verification of how algorithms work, allowing for numerical valuation and hedging the modified conditional claims for more general options.

In the case where the contingent claim is a call or put option, and we are using the Black-Scholes model, in section 4 we have set out explicit formulas for the price and the process ξ which allow to price and hedge these modified options at any time. In the case of other European options that we want to optimally hedge, we would again have to use the conditional expected value to value it theoretically, which may be problematic for some options. To avoid this problem, we use the numerical methods that allow us to estimate the price of modified claims and the ξ process over time. In this chapter we will describe the Monte Carlo method used for valuation of any European option at the time 0 as well as finite difference method for dynamically hedging a path independent option. Both these methods as well as all the formulas derived in the chapter 4 have been implemented in the R package, which is available in [12].

5.1 Monte Carlo simulations

Let us return to the generalized formulas for the modified conditional claims for the Black-Scholes model. Recall that for any discounted path independent option with the payoff H, we have the following modified contingent claims:

- For the concave function i.e. $p < 1$:

$$\tilde{H} = \mathbf{1}_{\{H^{1-p}\rho < a_p\}} H = e^{-rT} \mathbf{1}_{\{C^{1-p} S_T^{-\frac{m}{\sigma^2}} < a\}} C,$$

  where C is the option payoff we want to optimally hedge, while a is a constant.

- In the case of the linear loss function, we have:

$$\tilde{H} = \begin{cases} \mathbf{1}_{\{S_T > c\}} e^{-rT} C & \text{when } m > 0 \\ \mathbf{1}_{\{S_T < c\}} e^{-rT} C & \text{when } m < 0 \end{cases},$$

  while $\tilde{H} = ce^{-rT}C$, when $m = 0$, where c is a constant.

- For the convex loss function i.e. $p > 1$ we get:

$$\tilde{H} = H - (c_p \rho^{\frac{1}{p-1}} \wedge H) = e^{-rT}(C - (cS_T^{-\frac{m}{\sigma^2(p-1)}} \wedge C)),$$

  where c is a constant.

Of course, the prices of the above modified claims are monotonic due to this constant. In the case of the concave loss function, as the constant a increases the price of the $\tilde{C} = e^{rT}\tilde{H}$ increases, in the linear case, for $m \leq 0$ the option price increases with the increase of c and decreases when $m > 0$. For the convex loss function, when c increases, the price of the option decreases. Therefore, in all cases, there are exactly

one constant such the price of the modified contingent claim is $\tilde{V}_0$.
To find these constants we will use the Monte Carlo simulations. Theoretical considerations for Monte Carlo methods can be found in the script [7].
For any option C, we want to estimate the expected value:

$$\Pi(C,t) = e^{-rT}\mathbb{E}^*[C].$$

Let $C_1, ..., C_N$ be the payoffs that have been determined from successive implementations of the price process trajectory $S = (S_t)_{t\ in[0,T]}$ in the martingale measure $\mathbf{P}^*$, described in subsection 1.2. The trajectory simulation algorithm of the underlying asset is described in section 4.3, in this case to get the trajectory in the measure $\mathbf{P}^*$, instead of μ is the risk-free rate r. Then:

$$\bar{\Pi} = e^{-rT}\frac{1}{n}\sum_{i=1}^{N} C_i$$

is an estimate of the option price at the time 0.

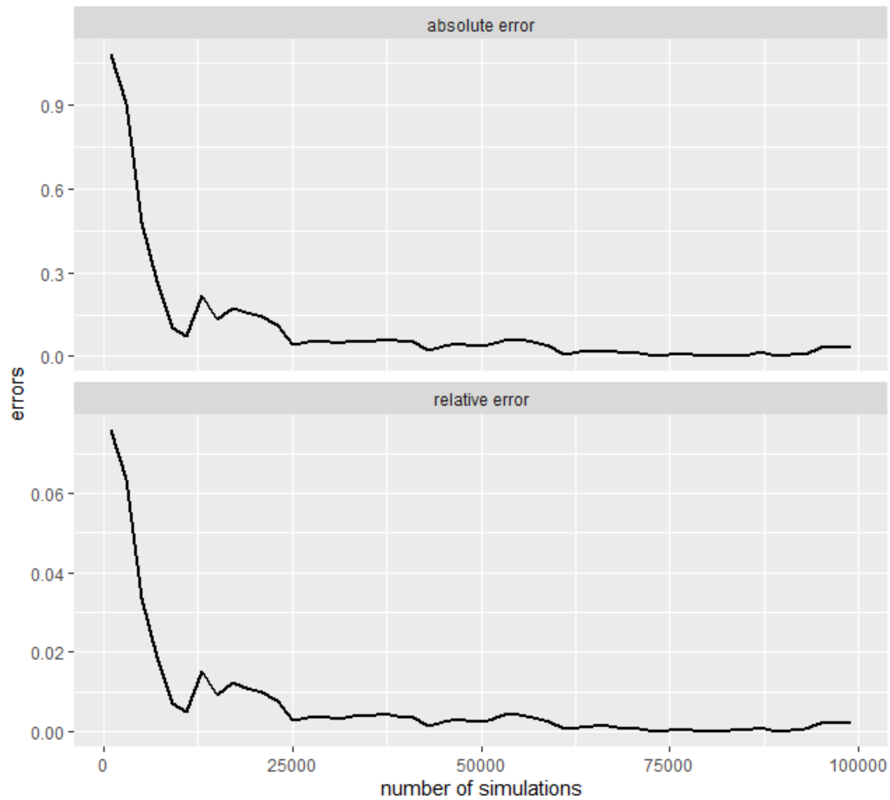

Figure 5: Graphs of errors in the valuation of call options depending on the number of simulations.

The chart 5 shows the relative and absolute error in the valuation of the call option price in respect to the number of simulated trajectories of the underlying asset. The relative and absolute errors compare the results obtained by the formula for the call option price to the result of the estimation from a given number of trajectory simulations.
From the charts 5 we can see that as the number of simulations increases, the error in estimating the call option with the Monte Carlo method decreases. Already for 25000 simulations, this error is low enough.
In the case of the modified option $\tilde{C}$, its payoff depends of the certain constant, which we will determine from the initial condition (i.e that the price of this claim is equal to $\tilde{V}_0$). We will do this using numerical methods (such as the bisection method), which as function values will take the option price determined by Monte Carlo methods.

Example In order to compare the result of the numerical methods, we will check the values of the determined constants determined by the Monte Carlo method with the constants calculated using the formulas from the subsection 4.1.
Consider the underlying asset and the call option with the same parameters as for the subsection 4.3. Again, consider the loss function with the exponents $p = 0.01, 0.5, 1, 2, 5, 100$.

method	$p = 0.01$	$p = 0.5$	$p = 1$	$p = 2$	$p = 5$	$p = 100$
theoretical values	4.91706	0.52303	116.242	45.7365	6.0443	3.1609
Monte Carlo	4.91411	0.52294	116.334	45.9113	6.0663	3.1741

Table 2: Comparison of constants determined by Monte Carlo method and by theoretical solutions.

The table 2 shows the constants determined by the Monte Carlo method (with the number of simulations equal to 100000), as described above, and the constants determined by the bisection method using analytical formulas. In the case of concave and convex loss functions, after determining the constant L, the constant appearing in the general formula for the modified claim was calculated using the formulas for concave and convex loss functions, respectively:

$$a = (L - K)^{1-p} L^{-\frac{m}{\sigma^2}}, \quad c = (L - K) L^{\frac{m}{\sigma^2 (p-1)}}.$$

As we can see in the table above, the constants from the Monte Carlo method estimate quite well the values from the theoretical approach. Of course, for better accuracy, the number of simulations can be increased. It is also worth increasing it in the case of a more complicated original option, or much greater volatility in the model.

5.2 Finite difference algorithm

Thanks to the method described in the previous chapter, for any option, we get the exact form of the payoff from the modified claim. In order to perform an optimal hedging, we need to know the price of the modified option over time (and not only at time 0) and the ξ process over the lifetime of the option. The Monte Carlo method fails in this case.
Consider path-independent options. In this case, one of the effective methods of pricing an option and its hedging is the finite difference method.

Let the share price be equal to $S = i\delta S$ for $i = 0, ..., I$, where δS is the step size of the stock price, while I is such number for which $I\delta S$ approximates the share price at infinity.
Let $t = T - k\delta t$, for $k = 0, ..., J$ will be the time steps, where J is such that $T - J\delta t = 0$. For the share price $i\delta S$ and time $T - k\delta t$ let us denote the option price as:

$$V_i^k = V(i\delta S, T - k\delta t) = \Pi(C, i\delta S, T - k\delta t).$$

Hence, we have $(I + 1)(J + 1)$ option values that can be represented in the form of the grid.

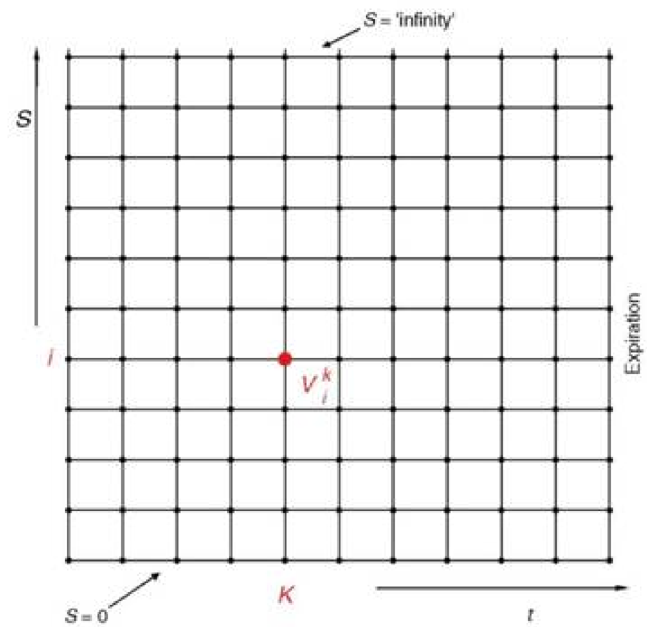

Figure 6: https://ebrary.net/7100/business_finance/what_finite-difference_method

This approach which we will describe below is known as *explicit finite difference*, more about this method and its modifications can be found in the chapter 77 in [8]. Knowing the initial conditions (payoffs at time T) we want to recursively set the next option prices.
Let the contingent claim payoff C is the form of $C = \Phi(S_T)$. Consider the generalized Black-Scholes equation that our model, described in chapter 1 satisfies.

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V = 0.$$

Using the Taylor series we have:

$$V_{i+1}^k = V_i^k + \frac{\partial V}{\partial S}\delta S + \frac{(\delta S)^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{(\delta S)^3}{3!}\frac{\partial^3 V}{\partial S^3} + O((\delta S)^4),$$

and

$$V_{i-1}^k = V_i^k - \frac{\partial V}{\partial S}\delta S + \frac{(\delta S)^2}{2}\frac{\partial^2 V}{\partial S^2} - \frac{(\delta S)^3}{3!}\frac{\partial^3 V}{\partial S^3} + O((\delta S)^4).$$

Subtracting V_{i-1}^k from V_{i+1}^k we get the approximate value of the derivative $\frac{\partial V}{\partial S}$:

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S},$$

with the error of approximation of the order $O((\delta S)^2)$. Similarly, adding both sides we get

$$V_{i+1}^k + V_{i-1}^k = 2V_i^k + (\delta S)^2\frac{\partial^2 V}{\partial S^2} + O((\delta S)^4).$$

So we have the following approximation of the second derivative:

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{(\delta S)^2},$$

with the error $O((\delta S)^2)$.
Likewise, we can determine $\frac{\partial V}{\partial t}$. The approximate value of this derivative is as follows:

$$\frac{\partial V}{\partial t} \approx \frac{V_i^k - V_i^{k+1}}{\delta t},$$

with the error $O(\delta t)$.
In the case of our model, we have: $a(S,t) = 0.5\sigma^2 S^2$, $b(S,t) = rS$, $c(S,t) = -r$. Summarizing, we obtain the generalized Black-Scholes equation in numerical form:

$$\frac{V_i^k - V_i^{k+1}}{\delta t} + a_i^k\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{(\delta S)^2} + b_i^k\frac{V_{i+1}^k - V_{i-1}^k}{2\delta S} + c_i^k V_i^k = O(\delta t, (\delta S)^2),$$

where $a_i^k = 0.5\sigma^2 i\delta S$, $b_i^k = i\delta S$, $c_i^k = -r$.
Since we know the exact values of the options at time T, we want to go backward all the way to time 0, when executing the algorithm. Thus we want to determine the value of V_i^{k+1}. By transforming this numerical equation, we get:

$$V_i^{k+1} = \left(\frac{\delta t}{(\delta S)^2}a_i^k - \frac{\delta t}{2\delta S}b_i^k\right)V_{i-1}^k + \left(1 - 2\frac{\delta t}{(\delta S)^2}a_i^k - \delta t c_i^k\right)V_i^k + \left(\frac{\delta t}{(\delta S)^2}a_i^k + \frac{\delta t}{2\delta S}b_i^k\right)V_{i+1}^k.$$

Initial conditions and stability We have to start the finite difference algorithm from the initial conditions. Of course, in our case it is

$$V_i^0 = \Phi(i\delta S).$$

With the algorithm, we cannot calculate V_0^k and V_I^k for $k > 0$, therefore we have to determine these values in a different way. In our model, when $S = 0$ then $a(S,t) = b(S,t) = 0$, thus the Black-Scholes equation takes the form:

$$\frac{\partial V}{\partial t}(0,t) - rV(0,t) = 0$$

which in numerical form we write as:

$$V_0^{k+1} = (1 - r\delta t)V_0^k.$$

For V_I^k we have two approaches The first, more theoretical, is taken from the chapter in [8]. When the option has a payoff that is at most linear in the underlying for large values of S then $\frac{\partial^2 V}{\partial S^2}(S,t) \to 0$. The finite-difference representation of this is:

$$V_I^k = 2V_{I-1}^k - V_{I-2}^k.$$

The second approach we have implemented in the package is slightly different. It assumes that the parameters I and δS are selected in such a way that the option price to behaves "stable" for the share price $I\delta S$ i.e. for this asset value, nothing unexpected happens with the option's price and the price change over time δt is the same as the change for the asset price of $(I-1)\delta S$. In numerical form, we can write it as:

$$V_I^k = V_I^{k-1} + V_{I-1}^k - V_{I-1}^{k-1}.$$

For the stability of the algorithm [8] in the chapter 77 gives the following restrictions for δt and δS:

$$\delta t \leq \frac{1}{\sigma^2 I^2}, \qquad \delta S \leq \frac{2a}{|b|} = \sigma^2 S.$$

In our model restriction on δS does not make much difference in practice unless the volatility is very small. However, that is a serious limitation on the size of the time step δt. For the proofs of these limitations for stability purposes, and for more information on the finite difference method, see [8], chapter 77.

The number of shares that is required at any given time to hedge a contingent claim for $i = 1, ...I-1$ we determine from the approximate value of the derivative $\frac{\partial V}{\partial S}$, i.e.

$$\frac{\partial V}{\partial S} = \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S},$$

with the error $O((\delta S)^2)$, while for $i = 0, I$ we have successively:

$$\frac{\partial V}{\partial S} = \frac{V_1^k - V_0^k}{\delta S}, \quad \frac{\partial V}{\partial S} = \frac{V_I^k - V_{I-1}^k}{\delta S},$$

with the errors $O(\delta S)$.

Example Again, consider an example of a call option with the strike price $K = 100$ on the underlying asset, the parameters of which are described in subsection 4.3. We again consider concave, linear, and convex loss functions with exponents $p = 0.5$, $p = 1$, $p = 2$. Let us set the number of steps in the stock price to $I = 400$ and the step sizes $\delta S = 1, \delta t = 6.8966e - 05$.
The absolute error in the calculation of the modified option with the parameter $p = 0.5$ is given in the heat map below:

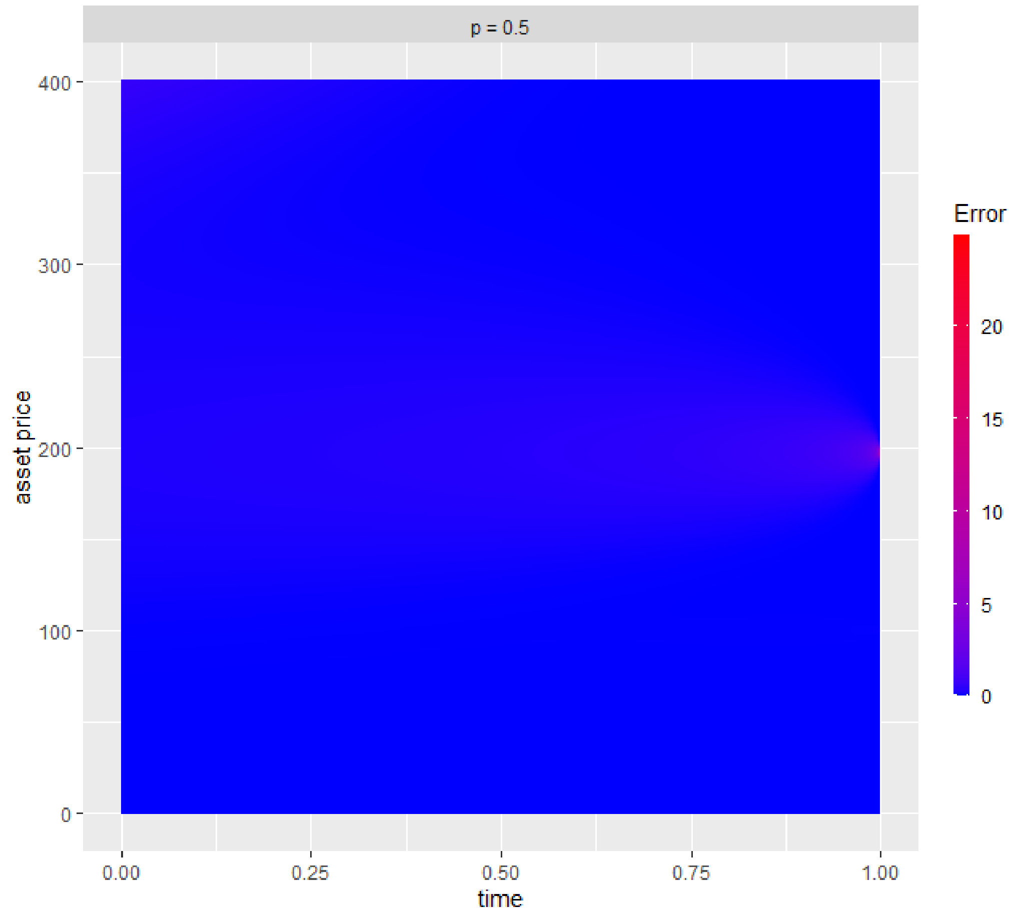

Figure 7: The absolute error in the calculation of the modified option with the parameter p = 0.5

As we can see, the error is quite big. After analyzing the exact results, it turns out that large errors between the theoretical price of the modified claim and the one determined by the finite difference algorithm appear in two places. The first one concerns high asset prices. As a result of selecting too low the maximum price on the grid, the theoretical values of the option price when the time is close to 0 are greater than the finite difference value.
The second, much bigger problem is the initial steps of the algorithm. Due to quite large steps in share prices and the specificity of the option, which is discontinuous at two points, in the finite difference method, there are no these discontinuity points

on the grid, which causes huge errors. Moreover, the value

$$a_p = (L - K)L^{-\frac{m}{\sigma^2}},$$

where L is the constant determined by Newton's method from theoretical option price, it slightly differs from the constant determined by Monte-Carlo method, which also causes an error.

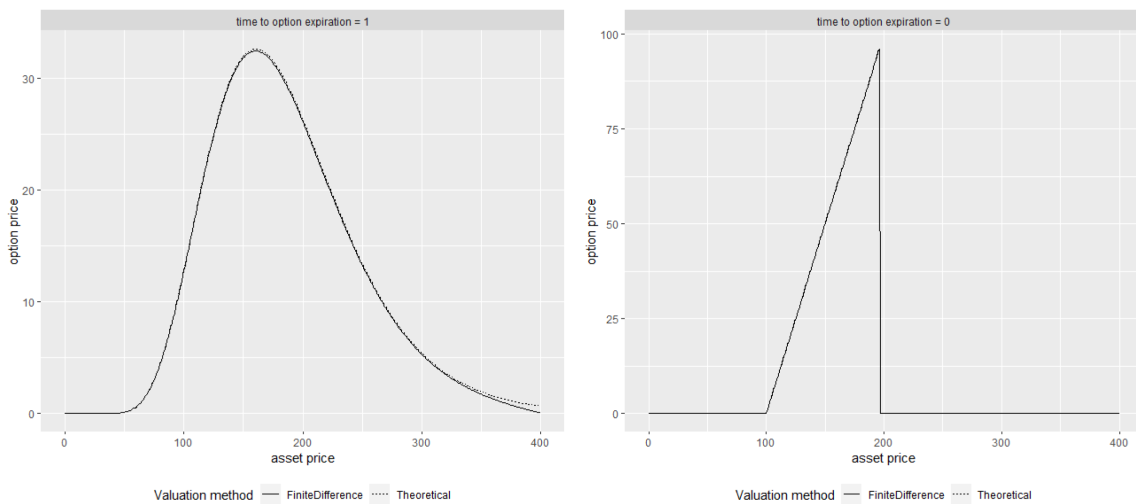

Figure 8: Comparison charts of option pricing using formulas and the finite difference method for the price at the time of 0 and 1.

As we can see in the payoff chart, the theoretically determined payoffs coincide with those determined using numerical methods. If we look at the option price at 0, we can see a slight difference in the option prices, especially when the stock price is large.

In order to correct errors, the one constant L_1 is needed to set, for which there is $i \in 0, ..., I$, such that $L_1 = i\delta S$ and does not change the initial capital too much, then calculate the constant $a = (L_1 - K)^{-\frac{m}{\sigma^2}}$, with which we calculate the payoff of the modified claim using the finite difference method.

In our case, setting the constant at 197, the option price (calculated via MC method) changes from 12.8081 to 12.8457, so it is a 0.2936% change in the option price.

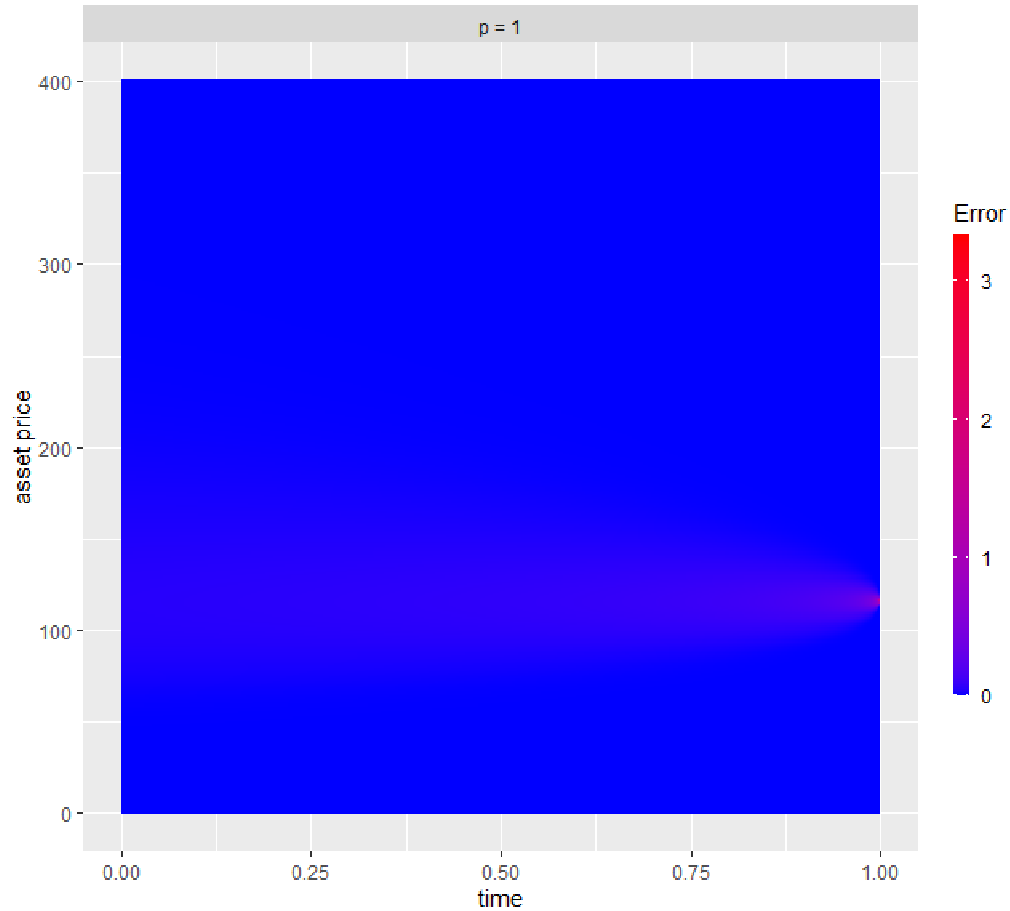

Figure 9: The absolute error in the calculation of the modified option with the parameter p = 1.

The same is true for the linear loss function. The payoff of this modified option is discontinuous due to the share price and the constants are slightly different from each other due to the error in calculating the constant using Monte Carlo methods. Again we can reduce these errors by slightly changing the constants for the modified option so that one of the asset prices on the grid equals the specified constant.

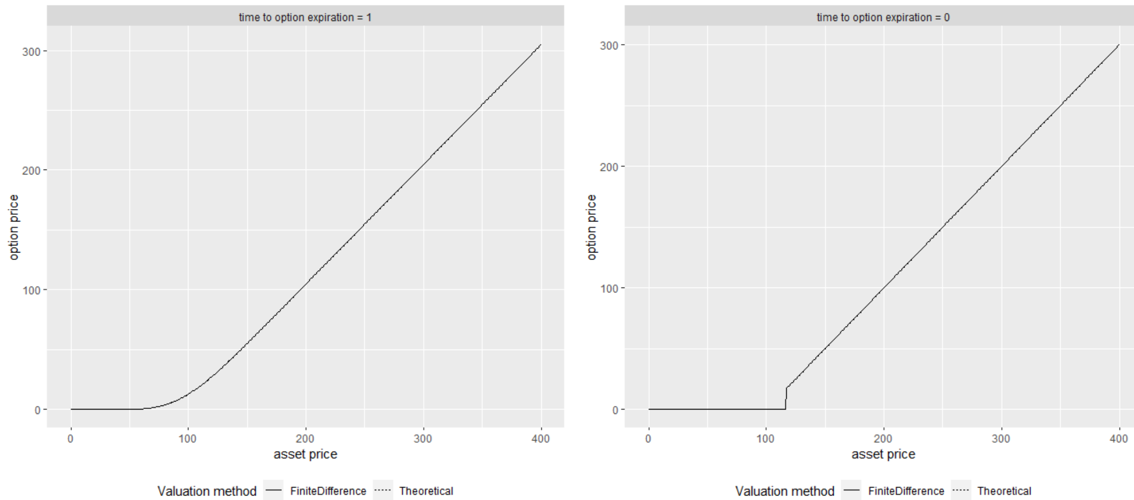

Figure 10: Comparison charts of option pricing using formulas and the finite difference method for the price at the time of 0 and 1.

If we look at the option price depending on the stock price for the time 0 and 1, we are not able to see the differences between the theoretical price and the one determined by algorithms.

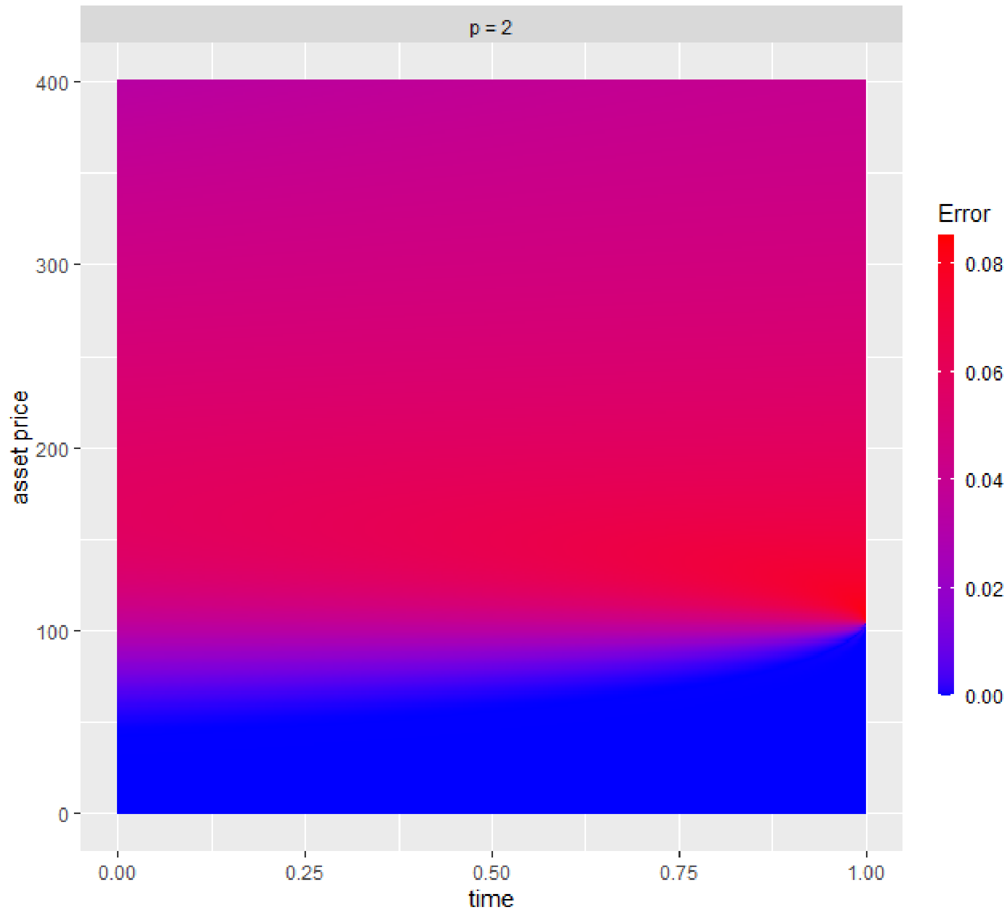

Figure 11: The absolute error in the calculation of the modified option with the parameter p = 2.

In this option, the errors are small, and the largest ones results from the numerical errors of the Monte Carlo method in the constant estimation.

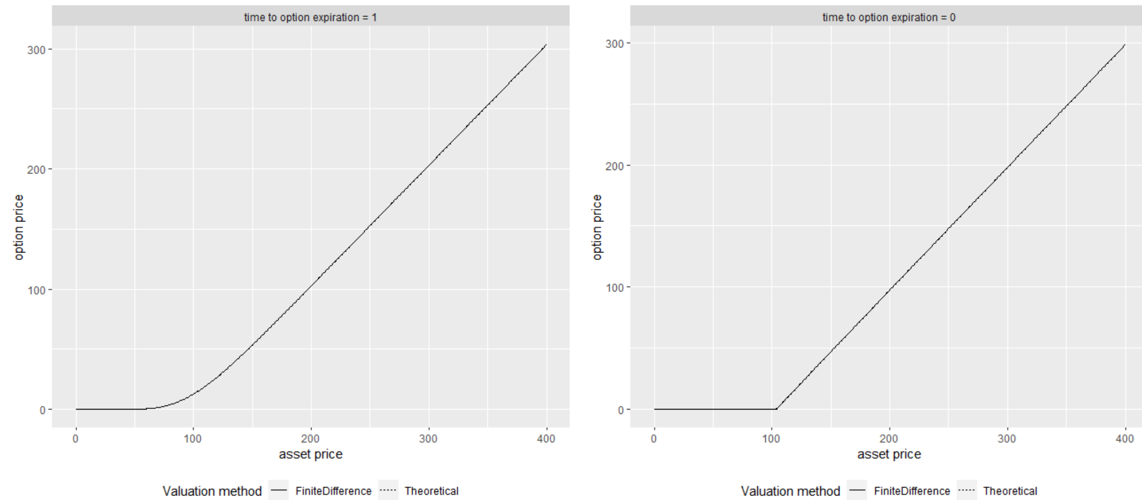

Figure 12: Comparison charts of option pricing using formulas and the finite difference method for the price at the time of 0 and 1.

As in the linear case, there is no difference between the theoretical option price and the one calculated numerically for the time 0 and 1.

5.3 Simulation results

In order to check how all the algorithms work, similarly to the chapter 4.3 we will simulate 10000 trajectories of the underlying asset and then compare the simulation results made in 4.3, which were made thanks to the derived formulas, with the results that we will obtain using the algorithms from chapter 5.
We will do it as follows:

- for 90% of the standard call option price with the strike price $K = 100$, we will determine the constants for the concave, linear and convex loss function using the Monte Carlo method described in 5.1;
- then we will execute the finite difference algorithm for the modified payoff and estimate the number of shares needed to hedge this claim;
- for each trajectory, we'll create a hedging portfolio using the values from the finite difference grid;
- finally, we'll subtract the value of the call payoff from the value of the portfolio at the time T.

In the event, that a the given asset price is between two share values from the finite difference grid, we will use a simple linear interpolation to evaluate the option and the amount of the asset to be hedged:

$$y = y_1 + a(x - x_1), \text{ where } a = \frac{y_1 - y_2}{x_1 - x_2},$$

and $x \in [x_1, x_2]$, while y_1, y_2 are the values of the function from, respectively x_1, x_2.

The portfolio gains/losses histograms are as follows:

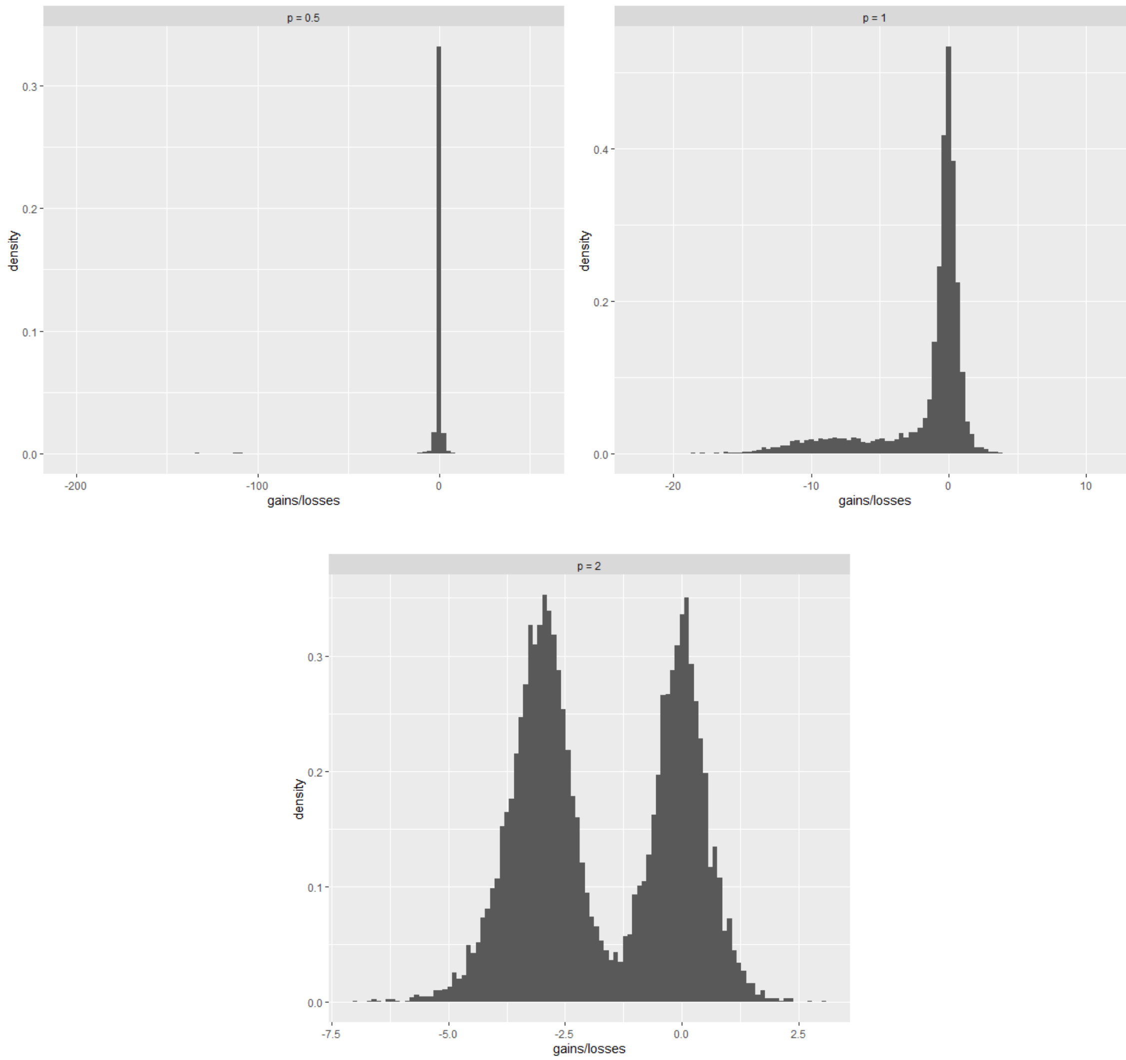

Figure 13: Portfolio profit and loss histograms for $p = 0.5$, $p = 1$ and $p = 2$.

The portfolio's basic profit and loss statistics are as follows:

Basic statistics	theory $p = 0.5$	estimation $p = 0.5$	theory $p = 1$	estimation $p = 1$	theory $p = 2$	estimation $p = 2$
minimum	-204.776	-204.788	-22.056	-22.217	-6.9707	-7.041
1 quartile	-0.4578	-0.4592	-1.2426	-1.2857	-3.0292	-3.1037
median	-0.0089	-0.0101	-0.2447	-0.2543	-2.1298	-2.1906
average	-2.4025	-2.4399	-1.5496	-1.5943	-1.6686	-1.7078
3 quartile	0.4006	0.398	0.2401	0.2384	-0.0897	-0.0904
maximum	55.7709	53.805	12.4435	11.8695	2.9966	2.9914

Table 3: Basic portfolio profit and loss statistics - comparison of theoretical statistics and estimation with numerical methods.

As can be seen in the table 3 the basic statistics calculated on the basis of the portfolios that hedged as per numerical methods in all three cases are very similar to the portfolios that hedged the call option according to the analytical formulas. We can conclude from this, that the numerical methods work quite good, and can be successfully applied to other path independent options, although (especially for finite difference) we should be careful when using them, especially when the payoff of the original option is discontinuous or we are using a concave or linear loss function.

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