MAT327
Introduction to Topology

Class Lecture Notes

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May 7, 2025

Last Updated:

July 25, 2025

Primary source material:

Class Lectures

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Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

III Algebraic Topology

8 Path Homotopy

Lec 19 - Jul 23 (Week 11)

ALGTOP RAAAHHHHHHHHHHHHH

Definition 8.1

given γ_0, γ_1 from x to y, a **path homotopy** from γ_0 to γ_1 is a cts $F : [0,1]^2 \to X$ such that the following holds:

$$F(s,0) = \gamma_0(s) \text{ and } F(s,1) = \gamma_1 \qquad F(0,t) = x \text{ and } F(1,t) = y$$

we say γ_0 is **path homotopic** to γ_1, or $\gamma_0 \simeq_p \gamma_1$, if a pathtopy exists

[pathtopy is certainly a. choice.] note that its important for F to be cts; this is strictly stronger than requiring F to be cts in each coordinate (beloved $xy/x^2 + y^2$).

Example 8.2

Source: Primary Source Material

$A \subseteq \mathbb{R}^n$ cvx $\implies$ any two paths w/ same endpoints are homotopic: for fixed s, take $F(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$. check that this is a pathtopy.

we can generalize pathtopy to deformations of general cts functions.

Definition 8.3

sps $f, g : X \to Y$ cts. a **homotopy** from f to g is a cts $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x) \qquad F(x,1) = g(x)$$

we write $f \simeq g$ in this case.

clearly, both $\simeq, \simeq_p$ are equiv rels. note transitivity uses pasting lemma [technically. its like a single pt tho].

9 The Fundamental Group

[yeah this should go here tbh]

Definition 9.1

for a path γ_0 from x to y and γ_1 from y to z, define:

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s) & s \in [0, 1/2] \\ \gamma_1(2s-1) & s \in [1/2, 1] \end{cases}$$

this induces an operation $*$ on the set of equivalence classes.

Proposition 9.2

$*$ is well-defined on equivalence classes.

Proof.

Source: Primary Source Material

fix $\gamma_0 \simeq \gamma_0'$ and $\gamma_1 \simeq \gamma_1'$. let $F, G : [0,1]^2 \to X$ be pathtopies from γ_i to γ_i' respectively. consider:

$$H(s,t) = \begin{cases} F(2s, t) & (s,t) \in [0, 1/2] \times [0,1] \\ G(2s-1, t) & (s,t) \in [1/2, 1] \times [0,1] \end{cases}$$

it is easy to see that H is then a homotopy from $\gamma_0 * \gamma_1$ to $\gamma_0' * \gamma_1'$. ■

Definition 9.3

a **loop** is a path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$. we say that γ is a loop at x_0 if $\gamma(0) = \gamma(1) = x_0$.

given a fixed x_0, we denote by $\pi_1(X, x_0)$ the set of all pathtopy equiv classes of loops at x_0.

we define by $e_x : [0,1] \to X$ and $\overline{\gamma} : [0,1] \to X$ as:

$$e_x(s) = x \qquad \overline{\gamma}(s) = \gamma(1-s)$$

these are the "constant" and "inverse" paths respectively.

Definition 9.4

given γ, let $\varphi : [0,1] \to [0,1]$ be cts with $\varphi(0) = 0$ and $\varphi(1) = 1$. we call $\gamma \circ \varphi$ a **reparametrization** of γ.

Lemma 9.5

$\gamma \simeq_p \gamma \circ \varphi$

Proof. Source: Primary Source Material

$F(s,t) = \gamma((1-t)s + t\varphi(s))$ is a pathtopy. ∎

okay the rest is just proving that $\pi_1(X, x_0)$ is a grp under $*$. uhh the pfs look kinda annoying to write so im just not gonna. uses the reparametrization tho

Lec 20 - Jul 25 (Week 11)

from last time: given X and a basept $x_0 \in X$, we associated the group $\pi_1(X, x_0)$ to (X, x_0) called the fundamental grp of X w basept x_0.

π_1 is also known as a **functor**.

$$\begin{aligned} \text{Topological space} &\xrightarrow{\pi_1} \text{Group} \\ \text{Continuous map} &\longrightarrow \text{Homomorphism} \\ \text{Homeomorphism} &\longrightarrow \text{Isomorphism} \end{aligned}$$

today we will prove this! whatever that means. in the meantime: did you know the torus has fundamental group $\mathbb{Z}$?

Q: what happens to $\pi_1(X, x_0)$ if we change the basept?

Proposition 9.6

fix $x_0, x_1 \in X$. let α be a path from x_0 to x_1. define $\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\widehat{\alpha}([\gamma]) = [\overline{\alpha} * \gamma * \alpha]$$

then $\widehat{\alpha}$ is well-defined, and $\widehat{\alpha}$ is an isomorphism.

Proof.

Source: Primary Source Material

well-definedness is an exercise. show:

$$\gamma_0 \simeq_p \gamma_1 \implies \overline{\alpha} * \gamma_0 * \alpha \simeq_p \overline{\alpha} * \gamma_1 * \alpha$$

it is a homomorphism because:

$$\begin{aligned}
\widehat{\alpha}([\gamma_0] * [\gamma_1]) = \widehat{\alpha}([\gamma_0 * \gamma_1]) &= [\overline{\alpha} * \gamma_0 * \gamma_1 * \alpha] \\
&= [\overline{\alpha} * \gamma_0 * e_{x_0} * \gamma_1 * \alpha] \\
&= [\overline{\alpha} * \gamma_0 * \alpha * \overline{\alpha} * \gamma_1 * \alpha] \\
&= [\overline{\alpha} * \gamma_0 * \alpha] * [\overline{\alpha} * \gamma_1 * \alpha] \\
&= \widehat{\alpha}([\gamma_0]) * \widehat{\alpha}([\gamma_1])
\end{aligned}$$

it is bijective because:

$$\widehat{\alpha} \circ \widehat{\overline{\alpha}}([\gamma]) = \widehat{\alpha}([\alpha * \gamma * \overline{\alpha}]) = [\overline{\alpha} * \alpha * \gamma * \overline{\alpha} * \alpha] = [\gamma]$$

$\widehat{\overline{\alpha}} \circ \widehat{\alpha}$ is similarly id. ∎

Corollary 9.7

X pathconn then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ for all x_0, x_1. in this case, fundgrp does not depend on basept, and we can denote it $\pi_1(X)$.

Definition 9.8

X is **simply connected** iff X pathconn and fundgrp is trivial.

for instance, any cvx subset of $\mathbb{R}^n$ is simply conn.

notation: we write $\varphi : (X, x_0) \to (Y, y_0)$ to mean that φ cts, $\varphi(x_0) = x_1$.

any map $\varphi : (X, x_0) \to (X, x_1)$ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

this is known as the **induced map** of φ. exercise: check this is well-defined, that is $\gamma_0 \simeq_p \gamma_1 \implies \varphi \circ \gamma_0 \simeq_p \varphi \circ \gamma_1$. idea: if $F : [0,1]^2 \to X$ is path homotopy from γ_0 to γ_1, then $G = \varphi \circ F$ is a path homotopy from $\varphi \circ \gamma_0$ to $\varphi \circ \gamma_1$. (check this!)

Proposition 9.9

let $\varphi : (X, x_0) \to (Y, y_0)$. then the induced map is a homomorphism.

Proof.

Source: Primary Source Material

we check $\varphi_*([\gamma_0] * [\gamma_1]) = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1])$.

$$\begin{aligned} \varphi_*([\gamma_0] * [\gamma_1]) &= \varphi_*([\gamma_0 * \gamma_1]) = [\varphi \circ (\gamma_0 * \gamma_1)] \\ &{\color{red}=} [(\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)] = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1]) \end{aligned}$$

to see red equality, note that $\varphi \circ (\gamma_0 * \gamma_1) = (\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)$. check this! $\blacksquare$

some properties:

(i) if $\varphi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (Z, z_0)$, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(ii) if $\iota : (X, x_0) \to (X, x_0)$ is id, then ι_* is id.

(iii) if $\varphi : (X, x_0) \to (Y, y_0)$ is homeo, then φ_* is iso.

proofs:

(i) $(\varphi \circ \psi)_*([\gamma]) = [\varphi \circ \psi \circ \gamma] = \varphi_*([\psi \circ \gamma]) = \varphi_*(\psi_*([\gamma]))$

(ii) $\iota_*([\gamma]) = [\iota \circ \gamma] = [\gamma]$

(iii) by (i) and (ii), $\varphi_* \circ (\varphi^{-1})_* = \iota_*$ and $(\varphi^{-1})_* \circ \varphi_* = \iota_*$. this also shows $(\varphi_*)^{-1} = (\varphi^{-1})_*$.

summary: given the following:

$$(X, x_0) \xrightarrow{\quad\varphi\quad} (Y, y_0)$$

we can apply π_1 to transform this into:

$$\pi_1(X, x_0) \xrightarrow{\quad\varphi_*\quad} \pi_1(Y, y_0)$$