# MAT388
# Introduction to Knot Theory

## Class Lecture Notes


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*Class Lectures*

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# Contents

# Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

# I Introduction and Basics

## 1 Preliminaries

Lec 1 - Sep 2 (Week 1)

main question of the course: how many ways can we embed a circle into $\mathbb{R}^3$? we'll consider this up to ambient diffeo/homeo, or up to isotopy (comes later).

for example, we claim that there is no homeo $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(U) = T$, from the unknot to the trefoil.

this gives a distinction between knotted and unknotted, but non-trivial knots can also be distinct.

[theres an example here, but i dont have my stylus :)]

so we need some way to identify different knots, such as a particular property. what can we use, how do we think about these distinctions? note that there are infinitely many distinct, but we're really asking about the underlying structure.

**Definition 1.1**

a **knot diagram** is an *immersion* (to be defined) of the circle $S^1 \hookrightarrow \mathbb{R}^2$, so that:

- All non-injective points are 2:1 and (self-)transverse
- Add over/under-crossing data at all double points

[again, an example, but my stylus....]

**Definition 1.2**

an **immersion** is a non-vanishing (full rank) derivative at all points. [precisely, it is a differentiable map whose pushforward is inj]

**Proposition 1.3**

given any $C^\infty$ (or just $C^1$) embedding $K : S^1 \hookrightarrow \mathbb{R}^3$, a "generic" linear projection of $K(S^1)$ gives a knot diagram.

conversely, any knot diagram defines a knot $K(S^1) \subseteq \mathbb{R}^3$, which is unique up to "isotopy".

**Proof.**

Source: Primary Source Material

(sketch)(very sketchy)

consider $K(S^1) \subseteq \mathbb{R}^3$; it has a tangent vector everywhere. consider $\{\text{directions}\} \subseteq S^2$. this is a smooth map $S^1 \to S^2$; in particular, it is not surjective by Sard's theorem. as a consequence, the complement of the image is full measure (open + dense).

pick a pt not in the image and project in this direction. the tangent vector to $K$ is thus never parallel, so the projection is an immersion.

other issues: possibly self-tangent, possibly $n : 1$. but, none of these are generic (i.e. do a dimension count, apply Sard's)

for the converse: let $x, y$ coords be the coords in the projection. the indeterminacy is $z(\theta)$, since $(x(\theta), y(\theta)) \in \mathbb{R}^2$ are determined by the diagram.

crossings are then double pts $\left\{\theta_1^+, \theta_1^-\right\}, \left\{\theta_2^+, \theta_2^-\right\}$ as pairs of pts in $S^1$, say up to $k$ pts. (necessarily finite since they are necessarily isolated -¿ cpt -¿ finite, or sth)

then choice is(of?) a function $z : S^1 \to \mathbb{R}$ such that $z(\theta_j^+) > z(\theta_j^-)$ - this represents which strand is above the other, in a sense.

the space of all functions $z$ is then convex and thus connected:

$$z_t(\theta) = tz(\theta) + (1-t)\widetilde{z}(\theta)$$

is a valid choice of $z$. so there are many possible choices, but all can be interpolated, thus are isotopic(?). ∎

**Definition 1.4**

a **tri-colouring** of a knot diagram is a choice of colour in {RGB} for each "arc" in the diagram, such that at each crossing, either *one* or *three* colours are used[meet]. it is required to use all three colours.

for instance, the trefoil can very easily be tri-coloured, as well as the "$6_1$ knot", or more generally a ($k$-th) "twist" knot. as a non-example, the "figure-8" knot and unknot have no tri-colouring.

it seems like this only depends on the diagram, but in fact it only depends on a knot up to smooth isotopy. why? what?

(non-clarifying proof/explanation) we use a black box: reidemeister's theorem

two diagrams present isotopic knots in $\mathbb{R}^3$ iff they differ by a finite sequence of these moves:

- R1 - twisting a strand
- R2 - moving two non-intersecting strands atop each other
- R3 - moving a strand behind a crossing if it is "under both strands"

for example: [see diagram 1]

pf of tri-colourability: note that for each move, if there is a chosen tri-colouring before the move, there is a unique tri-colouring after.

ok, but what's actually happening? real answer: consider the group homomorphisms $\pi_1(\mathbb{R}^3 \setminus K(S^1)) \to S_3$.

sps $K \subseteq \mathbb{R}^3$ is a knot (abusing notation). if $K$ isotopic to $\widetilde{K}$, or if there is a homeo/diffeo $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi(K) = \widetilde{K}$, then $\mathbb{R}^3 \setminus K \simeq \mathbb{R}^3 \setminus \widetilde{K}$. thus, we can equivalently as questions about the complements; are these the same? what about $\pi_1(\mathbb{R}^3 \setminus K)$? this will give us our first powerful invariant for knots.

note for $K = U$ the unknot, we can draw a knot "parallel" to the unknot on the inside, and this is homotopic to $U$ with $\pi_1(\mathbb{R}^3 \setminus K) \simeq \mathbb{Z}$. but for the trefoil, this may not be the case - a parallel curve may not be homotopic to $T$.

this seems to distinguish $U$ and $T$, maybe, but how to define anything?

lets look at $U \subseteq \mathbb{R}^3$. let $x_0$ be your eyeball. if $f : [0,1] \to \mathbb{R}^3 \setminus U$ never passes behind the knot, then taking all the light rays from $f(\theta)$ to $x_0$ defines a homotopy $H$, so $[f] = [e]$. otherwise, if it passes through $U$, then we get a winding number, so $\pi_1(\mathbb{R}^3 \setminus U) \simeq \mathbb{Z}$.

for a more complicated diagram, we have a group elem for each *arc* of the diagram. however, given a loop around some arc, we can slide it "along" the arc as long as we don't accidentally cross over any other arcs.

so we'd have the following relation: [see diagram 2]

this is known as the **wirtinger presentation** - we will see the proof next time.

Lec 2 - Sep 4 (Week 2)

on quercus - notes for $C^\infty$ topology.

coming up, we want to discuss van kampen's theorem - to do this, we need to discuss free products/quotients, or "amalgamated free products".

**Definition 1.5**

let $G_1, G_2, H$ be groups and $f_i : H \to G_i$ two group homos. we define the free product $G_1 *_H G_2$ in two ways:

- the "correct" defn coming from category thy says that for any maps $g_i : G_i \to K$ s.t. $g_1 \circ f_1 = g_2 \circ f_2$, then there exists a unique $\varphi : G_1 *_H G_2 \to K$ such that the maps "factor", that is:

$$g_j = \varphi \circ \iota_j$$

  where $\iota_j$ are the inclusion maps [see diagram 3]

- the "useful" defn says that if $G_1$ has the presentation

$$G_1 = \langle \gamma_1 \cdots \gamma_k | r_1 \cdots r_\ell \rangle$$

  with $r_j = 1$ in $G_1$ and $G_1 = \text{Free}(\{\gamma_j\}) / \{r_j = 1\}$, and similarly

$$G_2 = \langle \delta_1 \cdots \delta_n | s_1 \cdots s_m \rangle$$

then:

$$G_1 *_H G_2 = \left\langle \gamma_1 \cdots \gamma_k \delta_1 \cdots \delta_n \middle| \begin{array}{c} r_1 \cdots r_\ell \\ s_1 \cdots s_m \\ f_1(a_1) \cdot f_2(a_1)^{-1} \\ f_1(a_2) \cdot f_2(a_2)^{-1} \\ \vdots \end{array} \right\rangle$$

$$H = \langle a_1 \cdots a_\alpha | t_1 \cdots t_\beta \rangle$$

since $F_1(t_i) = 1$ in $G_1$.

lmao.

**Theorem 1.6: Van Kampen's Theorem**

Let $U_1 \cup U_2 = X$ for open pconn $U_i$ s.t. $U_1 \cap U_2$ is connected, $x_0 \in U_1 \cap U_2$.

Then $\pi_1(X, x_0) \simeq \pi_1(U_1, x_0) *_H \pi_1(U_2, x_0)$, where $H = \pi_1(U_1 \cap U_2, x_0)$.

proof sketch:

let $f : [0,1] \to X$ where $f \in \Omega(X, x_0)$. we want to find finitely many $t_1, \ldots, t_m \in [0,1]$ such that $f|_{[t_j, t_{j+1}]}$ is entirely in $U_1$ or $U_2$ [see diagram 4]. note that these $t_j$ are different than the $t_i$ in [the defn].

take paths between $f(t_j)$ and $x_0$ inside $U_1 \cap U_2$. then by homotopy, $[f]$ is in the smallest subgroup containing $\pi_1(U_1)$ and $\pi_1(U_2)$.

for any $[\gamma] \in \pi_1(U_1 \cap U_2)$, we see the same thing in $\pi_1(X)$. [see diagram 5] thus, we have a surjection $\pi_1(U_1) *_H \pi_1(U_2) \twoheadrightarrow \pi_1(X)$, where $H = \pi_1(U_1 \cap U_2)$. it remains to show that the kernel is 0 - left as "exercise".

this defn is used for an example below:

**Definition 1.7**

the **genus** $g$ **handle body** is the bounded 3D region inside the surface as shown in [diagram 6] with examples.

we show some examples of applications(?).

- $\pi_1(S^1) \simeq \mathbb{Z}$, not proved with van kampen but a different way.
- $\pi_1(X \times Y) \simeq \pi_1(X) \oplus \pi_1(Y)$. pf: any $f : I \longrightarrow X \times Y$ is $f_1 : I \longrightarrow X$ and $f_2 : I \longrightarrow Y$, with $f = (f_1, f_2)$ - same for homotopies.
- $\pi_1(B^n) = \{e\}$. pf: homotope any loop to 0 along radial lines.
- $\pi_1(S^1 \times D^2) \simeq \mathbb{Z}$ - note this is the solid torus.
- $\pi_1(H_g) = \text{Free}_g$. pf: for $g = 2$, denote each handle by $U_i$, then $U_1 \cap U_2 \simeq B^3$. by VK, $\pi_1(H_g) \simeq \mathbb{Z} *_{\{e\}} \mathbb{Z} \simeq \mathbb{Z} * \mathbb{Z} \simeq F_2$. then for $g > 2$, induct. see [diagram 7].
- $\pi_1(S^3) \simeq \{e\}$. pf: $\pi_1(S^3) \simeq \mathbb{Z} *_H \mathbb{Z}$, where $H = \mathbb{Z} \oplus \mathbb{Z}$. note we have that $\mathbb{Z} \oplus \mathbb{Z} \simeq \pi_1(S^1 \times S^1 \times (-\varepsilon, \varepsilon))$ - see below

recall that $S^n \setminus \{x_0\} \simeq \mathbb{R}^n$ by stereographic projection. we claim $S^3 = H_1 \cup_{S^1 \times S^1} H_1$. we'll prove this in two different ways (maam there are 5 minutes left). [not an adjoint space, just notation - unioning "by their boundaries" or "over their boundaries" or sth. actually it may have been an adjoint space]

take $S^3 \subseteq \mathbb{C}^2$. then

$$S^3 = \left\{ |z_1|^2 + |z_2|^2 = 1 \right\} = \left\{ |z_1|^2 \leq \frac{1}{2}, |z_2|^2 = 1 - |z_1|^2 \right\}$$

note that in the second set, the first portion gives $D^2$ and the second gives $S^1$, so we have $D^2 \times S^1$. unioning this with the same set with swapped coordinates gives the result.

for the second way, see diagram 8.