MAT327
Introduction to Topology

Class Lecture Notes

Notes by:

Emerald (Emmy) Gu

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Prof. Luciano Salvetti Martinez

Contents

Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

III Algebraic Topology

8 Path Homotopy

Lec 19 - Jul 23 (Week 11)

ALGTOP RAAAHHHHHHHHHHHHH

Definition 8.1

given γ_0, γ_1 from x to y, a **path homotopy** from γ_0 to γ_1 is a cts $F : [0,1]^2 \to X$ such that the following holds:

$$F(s,0) = \gamma_0(s) \text{ and } F(s,1) = \gamma_1(s) \qquad F(0,t) = x \text{ and } F(1,t) = y$$

we say γ_0 is **path homotopic** to γ_1, or $\gamma_0 \simeq_p \gamma_1$, if a pathtopy exists

[pathtopy is certainly a. choice.] note that its important for F to be cts; this is strictly stronger than requiring F to be cts in each coordinate (beloved $xy/x^2 + y^2$).

Example 8.2 Source: Primary Source Material

$A \subseteq \mathbb{R}^n$ cvx $\implies$ any two paths w/ same endpoints are homotopic: for fixed s, take $F(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$. check that this is a pathtopy.

we can generalize pathtopy to deformations of general cts functions.

Definition 8.3

sps $f, g : X \to Y$ cts. a **homotopy** from f to g is a cts $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x) \qquad F(x,1) = g(x)$$

we write $f \simeq g$ in this case.

clearly, both $\simeq, \simeq_p$ are equiv rels. note transitivity uses pasting lemma [technically. its like a single pt tho].

9 The Fundamental Group

[yeah this should go here tbh]

Definition 9.1

for a path γ_0 from x to y and γ_1 from y to z, define:

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s) & s \in [0, 1/2] \\ \gamma_1(2s-1) & s \in [1/2, 1] \end{cases}$$

this induces an operation $*$ on the set of equivalence classes.

Proposition 9.2

$*$ is well-defined on equivalence classes.

Proof. Source: Primary Source Material

fix $\gamma_0 \simeq \gamma_0'$ and $\gamma_1 \simeq \gamma_1'$. let $F, G : [0,1]^2 \to X$ be pathtopies from γ_i to γ_i' respectively. consider:

$$H(s,t) = \begin{cases} F(2s, t) & (s,t) \in [0, 1/2] \times [0,1] \\ G(2s-1, t) & (s,t) \in [1/2, 1] \times [0,1] \end{cases}$$

it is easy to see that H is then a homotopy from $\gamma_0 * \gamma_1$ to $\gamma_0' * \gamma_1'$. ■

Definition 9.3

a **loop** is a path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$. we say that γ is a loop at x_0 if $\gamma(0) = \gamma(1) = x_0$.

given a fixed x_0, we denote by $\pi_1(X, x_0)$ the set of all pathtopy equiv classes of loops at x_0.

we define by $e_x : [0,1] \to X$ and $\overline{\gamma} : [0,1] \to X$ as:

$$e_x(s) = x \qquad \overline{\gamma}(s) = \gamma(1-s)$$

these are the "constant" and "inverse" paths respectively.

Definition 9.4

given γ, let $\varphi : [0,1] \to [0,1]$ be cts with $\varphi(0) = 0$ and $\varphi(1) = 1$. we call $\gamma \circ \varphi$ a **reparametrization** of γ.

Lemma 9.5

$\gamma \simeq_p \gamma \circ \varphi$

Proof. Source: Primary Source Material

$F(s,t) = \gamma((1-t)s + t\varphi(s))$ is a pathtopy. ∎

okay the rest is just proving that $\pi_1(X, x_0)$ is a grp under $*$. uhh the pfs look kinda annoying to write so im just not gonna. uses the reparametrization tho

Lec 20 - Jul 25 (Week 11)

from last time: given X and a basept $x_0 \in X$, we associated the group $\pi_1(X, x_0)$ to (X, x_0) called the fundamental grp of X w basept x_0.

π_1 is also known as a **functor**.

$$\begin{aligned} \text{Topological space} &\xrightarrow{\pi_1} \text{Group} \\ \text{Continuous map} &\longrightarrow \text{Homomorphism} \\ \text{Homeomorphism} &\longrightarrow \text{Isomorphism} \end{aligned}$$

today we will prove this! whatever that means. in the meantime: did you know the torus has fundamental group $\mathbb{Z}$?

Q: what happens to $\pi_1(X, x_0)$ if we change the basept?

Proposition 9.6

fix $x_0, x_1 \in X$. let α be a path from x_0 to x_1. define $\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\widehat{\alpha}([\gamma]) = [\overline{\alpha} * \gamma * \alpha]$$

then $\widehat{\alpha}$ is well-defined, and $\widehat{\alpha}$ is an isomorphism.

Proof. Source: Primary Source Material

well-definedness is an exercise. show:

$$\gamma_0 \simeq_p \gamma_1 \implies \overline{\alpha} * \gamma_0 * \alpha \simeq_p \overline{\alpha} * \gamma_1 * \alpha$$

it is a homomorphism because:

$$\begin{aligned} \widehat{\alpha}([\gamma_0] * [\gamma_1]) = \widehat{\alpha}([\gamma_0 * \gamma_1]) &= [\overline{\alpha} * \gamma_0 * \gamma_1 * \alpha] \\ &= [\overline{\alpha} * \gamma_0 * e_{x_0} * \gamma_1 * \alpha] \\ &= [\overline{\alpha} * \gamma_0 * \alpha * \overline{\alpha} * \gamma_1 * \alpha] \\ &= [\overline{\alpha} * \gamma_0 * \alpha] * [\overline{\alpha} * \gamma_1 * \alpha] \\ &= \widehat{\alpha}([\gamma_0]) * \widehat{\alpha}([\gamma_1]) \end{aligned}$$

it is bijective because:

$$\widehat{\alpha} \circ \widehat{\overline{\alpha}}([\gamma]) = \widehat{\alpha}([\alpha * \gamma * \overline{\alpha}]) = [\overline{\alpha} * \alpha * \gamma * \overline{\alpha} * \alpha] = [\gamma]$$

$\widehat{\overline{\alpha}} \circ \widehat{\alpha}$ is similarly id. ∎

Corollary 9.7

X pathconn then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ for all x_0, x_1. in this case, fundgrp does not depend on basept, and we can denote it $\pi_1(X)$.

Definition 9.8

X is **simply connected** iff X pathconn and fundgrp is trivial.

for instance, any cvx subset of $\mathbb{R}^n$ is simply conn.

notation: we write $\varphi : (X, x_0) \to (Y, y_0)$ to mean that φ cts, $\varphi(x_0) = x_1$.

any map $\varphi : (X, x_0) \to (X, x_1)$ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

this is known as the **induced map** of φ. exercise: check this is well-defined, that is $\gamma_0 \simeq_p \gamma_1 \implies \varphi \circ \gamma_0 \simeq_p \varphi \circ \gamma_1$. idea: if $F : [0, 1]^2 \to X$ is path homotopy from γ_0 to γ_1, then $G = \varphi \circ F$ is a path homotopy from $\varphi \circ \gamma_0$ to $\varphi \circ \gamma_1$. (check this!)

Proposition 9.9

let $\varphi : (X, x_0) \to (Y, y_0)$. then the induced map is a homomorphism.

Proof. Source: Primary Source Material

we check $\varphi_*([\gamma_0] * [\gamma_1]) = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1])$.

$$\begin{aligned} \varphi_*([\gamma_0] * [\gamma_1]) &= \varphi_*([\gamma_0 * \gamma_1]) = [\varphi \circ (\gamma_0 * \gamma_1)] \\ &= [(\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)] = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1]) \end{aligned}$$

to see red equality, note that $\varphi \circ (\gamma_0 * \gamma_1) = (\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)$. check this! ■

some properties:

(i) if $\varphi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (Z, z_0)$, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(ii) if $\iota : (X, x_0) \to (X, x_0)$ is id, then ι_* is id.

(iii) if $\varphi : (X, x_0) \to (Y, y_0)$ is homeo, then φ_* is iso.

proofs:

(i) $(\varphi \circ \psi)_*([\gamma]) = [\varphi \circ \psi \circ \gamma] = \varphi_*([\psi \circ \gamma]) = \varphi_*(\psi_*([\gamma]))$

(ii) $\iota_*([\gamma]) = [\iota \circ \gamma] = [\gamma]$

(iii) by (i) and (ii), $\varphi_* \circ (\varphi^{-1})_* = \iota_*$ and $(\varphi^{-1})_* \circ \varphi_* = \iota_*$. this also shows $(\varphi_*)^{-1} = (\varphi^{-1})_*$.

summary: given the following:

$$(X, x_0) \xrightarrow{\quad\varphi\quad} (Y, y_0)$$

we can apply π_1 to transform this into:

$$\pi_1(X, x_0) \xrightarrow{\quad\varphi_*\quad} \pi_1(Y, y_0)$$

10 Covering Spaces

Lec 21 - Jul 30 (Week 12)

today's mission: prove $\pi_1(S^1) = \mathbb{Z}$. we notate $I = [0, 1]$.

proof sketch: let $\omega_n(s) = (\cos(2\pi sn), \sin(2\pi sn))$. this is a loop in S^1 at $x_0 = (1, 0)$ that does n revolutions. moves ccw if $n > 0$, cw if $n < 0$. we want to show any loop in S^1 is pathtopic to some ω_n.

idea: show every $\gamma : I \to S^1$ can be "uniquely lifted" to a path $\widetilde{\gamma} : I \to \mathbb{R}$ from 0 to some n. embed $\mathbb{R} \hookrightarrow \mathbb{R}^3$ as the "helix":

$$s \to (\cos(2\pi s), \sin(2\pi s), s)$$

let $P : \mathbb{R} \to S^1$ be the projection of the helix onto the xy-plane. we want to show two things:

(a) for any loop $\gamma : I \to S^1$, there is a unique $\widetilde{\gamma} : I \to \mathbb{R}$ starting at 0 s.t.:
   [commutative diagram]

(b) for every pathtopy $F : I^2 \to S^1$ s.t. $F(0,0) = x_0$, there is a unique pathtopy $\widetilde{F} : I^2 \to \mathbb{R}$ s.t. $\widetilde{F}(0,0) = 0$ and:

[commutative diagram]

Definition 10.1

given spaces X, E, we say $P : E \to X$ is a **covering map** if for every $x_0 \in X$ there is an open $x_0 \in U$ which is "evenly covered by P", i.e.:

$$P^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} V_\alpha$$

is a union of pairwise disjoint open sets in E such that the map $P|_{V_\alpha} : V_\alpha \to U$ is a homomorphism.

in this context, we say E is a **covering space** of X. each V_α is also known as a **sheet** or **slice**.

some examples:

- $\mathrm{id} : X \to X$ is a cvring space (1-sheeted)
- $P : \mathbb{R} \to S^1$ given by $P(s) = (\cos(2\pi s), \sin(2\pi s))$ (countably many sheets)
- $P_n : S^1 \to S^1$ given by $P_n(z) = z^n$, where $S^1 \subseteq \mathbb{C}$ (n-sheeted)

properties of covering maps:

- $\forall\, x \in X$, $P^{-1}(x)$ is a discrete subspace of E. that is, every $e \in P^{-1}(x)$ has an open $V \subseteq E$ s.t. $V \cap P^{-1}(x) = \{e\}$
- cvring maps are open maps (exercise)
- cvring maps are **local homeos**. that is, for all $e \in E$, there is an open $e \in V \subseteq E$ s.t. $P|_V : V \to P(V)$ is a homeo

Definition 10.2

let $P : E \to X$ be a cvring map, $f : X \to Y$ be cts. a **lifting of f** is a cts $\widetilde{f} : Y \to E$ s.t. $f = P \circ \widetilde{f}$

[diagram]

Lemma 10.3: Path-lifting property

let $P : E \to X$ be a cvring map and $x = P(e)$. if $\gamma : I \to X$ is a path starting at x, then there is a unique lifting to a path $\widetilde{\gamma} : I \to E$ starting at e.

Proof.

Source: Primary Source Material

step 1: find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\gamma([t_{i-1}, t_i])$ is contained in some evenly covered open U_i.

for all $t \in I$, $\gamma(t) \in X$. then $\exists$ open nbhd U_t of $\gamma(t)$ s.t. each U_t evenly cvred:

$$I = \bigcup_{t \in I} \gamma^{-1}(U)$$

by lebesgue number lemma, let $\delta > 0$ s.t. for any $\operatorname{diam}(A) < \delta$, there is some $t \in I$ s.t. $A \subseteq \gamma^{-1}(U_t)$. choose a partition $P = \{t_0, \ldots, t_n\}$ s.t. $\|P\| = \max |t_i - t_{i-1}| < \delta$. then we have that $[t_{i-1}, t_i] \subseteq \gamma^{-1}(U)$ for some open U evenly cvred.

step 2: we prove existence of $\widetilde{\gamma}$. we construct $\widetilde{\gamma}$ inductively on each subinterval of the partition.

$\widetilde{\gamma}(0) = e_0$. sps $\widetilde{\gamma}$ defined on $[0, t_{i-1}]$. we extend to $[0, t_i]$ by defining $\widetilde{\gamma}$ on the next subinterval $[t_{i-1}, t_i]$.

by step 1, there is an open U evenly covered s.t. $\gamma([t_{i-1}, t_i]) \subseteq U$. notice that $\widetilde{\gamma}(t_{i-1}) \in P^{-1}(U)$ since:

$$P \circ \widetilde{\gamma}(t_{i-1}) = \gamma(t_{i-1}) \in U$$

then $\widetilde{\gamma}(t_{i-1}) \in V_\alpha$ where V_α is a sheet of $P^{-1}(U)$.

recall $P|_{V_\alpha} : V_\alpha \to U$ is homeo. define:

$$\widetilde{\gamma}(s) = \left(P^{-1}|_{V_\alpha}\right)(\gamma(s)) \qquad s \in [t_{i-1}, t_i]$$

note $\widetilde{\gamma}|_{[0,t_i]}$ cts by pasting lemma, and $\gamma = P \circ \widetilde{\gamma}$.

step 3: we prove uniqueness of $\widetilde{\gamma}$.

sps $\widehat{\gamma}$ is another lifting of γ with $\widehat{\gamma}(0) = e_0$. we show $\widehat{\gamma}(s) = \widetilde{\gamma}(s)$ for all $s \in [t_{i-1}, t_i]$ inductively.

sps $\widehat{\gamma}|_{[0,t_{i-1}]} = \widetilde{\gamma}|_{[0,t_{i-1}]}$. note $\widehat{\gamma}([t_{i-1}, t_i])$ conn and $\widehat{\gamma}(t_{i-1}) \in V_\alpha$ where V_α is the sheet we used to define $\widetilde{\gamma}$ on $[t_{i-1}, t_i]$ in step 2.

it then follows that $\widehat{\gamma}([t_{i-1}, t_i]) \subseteq V_\alpha$ by conn. since $\gamma = P \circ \widehat{\gamma}$:

$$\widehat{\gamma}(s) = \left(P^{-1}|_{V_\alpha}\right)(\gamma(s)) = \widetilde{\gamma}(s)$$

for all $s \in [t_{i-1}, t_i]$.

■

Lemma 10.4: Path-homotopy lifting property

sps $P : E \to X$ cvring map, $x = P(e)$. if $F : I^2 \to X$ is a pathtopy with $F(0,0) = x$, then there is a unique lifting $\widetilde{F} : I^2 \to E$ which is a **pathtopy** in E s.t. $\widetilde{F}(0,0) = e$.

pf: very similar to the one above. not writing allat

Corollary 10.5

let $P : E \to X$ be a cvring map and $x = P(e)$. if $\gamma_0 \simeq_p \gamma_1$, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$.

Theorem 10.6

$\pi_1(S^1) = \mathbb{Z}$.

dont rly wanna put this in a box

let $x_0 = (1,0) \in S^1$ and $P : \mathbb{R} \to S^1$ be the cvring map given by:

$$P(s) = (\cos(2\pi s), \sin(2\pi s))$$

given $[\gamma] \in \pi_1(S^1, x_0)$, let $\widetilde{\gamma} : I \to \mathbb{R}$ be the unique lifting of γ s.t. $\widetilde{\gamma}(0) = 0$. define $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ given by:

$$\varphi([\gamma]) = \widetilde{\gamma}(1) \in P^{-1}(x_0) = \mathbb{Z}$$

first, we show φ is well-defined.

by prev crll, if $\gamma_0 \simeq_p \gamma_1$, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$. in particular, $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.

next, φ is surjective.

fix $n \in \mathbb{Z}$. consider $\omega_n(s) = (\cos(2\pi sn), \sin(2\pi sn))$. then $\varphi([\omega_n]) = \widetilde{\omega_n}(1) = n$.

next, φ is injective.

sps $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$ s.t. $\varphi([\gamma_0]) = \varphi([\gamma_1])$. note $\widetilde{\gamma_0}, \widetilde{\gamma_1} : I \to \mathbb{R}$ are paths in $\mathbb{R}$. since $\mathbb{R}$ cvx, then $\widetilde{\gamma_0} \simeq_p \widetilde{\gamma_1}$ via pathtopy $\widetilde{F} : I^2 \to \mathbb{R}$. then, $P \circ F : I^2 \to S^1$ is a pathtopy from $P \circ \widetilde{\gamma_0} = \gamma_0$ to $P \circ \widetilde{\gamma_1} = \gamma_1$, so $[\gamma_0] = [\gamma_1]$.

finally, φ is homo.

given $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$, let:

$$\widetilde{\gamma_0}(1) = n \qquad \widetilde{\gamma_1}(1) = m$$

then $\gamma_0 \simeq_p \omega_n$ and $\gamma_1 \simeq_p \omega_m$. we show $\varphi([\omega_n] * [\omega_m]) = n + m$.

let $\widetilde{\omega_n}$ be the lifting starting at 0 and ending at n. let $\widetilde{\omega_m}$ be the lifting starting at n. then, we have that:

$$\varphi([\omega_n] * [\omega_m]) = (\widetilde{\omega_n} * \widetilde{\omega_m})(1) = n + m$$

note: check that $\widetilde{\omega_n} * \widetilde{\omega_m}$ is indeed a lifting of $\omega_n * \omega_m$.

11 Retractions

Lec 22 - Aug 6 (Week 13)

last time, we showed that $\pi_1(S^1) = \mathbb{Z}$. today we examine some applications.

Definition 11.1

let $A \subseteq X$. we say A is [a] **retract** of X if there is a cts $r : X \to A$ s.t. $r(a) = a$ for all $a \in A$. we call the map r a **retraction**.

Proposition 11.2

if A a retract of X, then the homo given as

$$\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$$

induced by the inclusion $\iota : A \to X$ is injective.

Proof. Source: Primary Source Material

let $r : X \to A$ be a retraction. note $r \circ \iota : A \to A$ is the identity, so $r_* \circ \iota_*$ is the trivial homo. since r_* is a left-inv of ι_*, we are then done. ■

Example 11.3 Source: Primary Source Material

S^1 is *not* a retract of $D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$.

by the prev thm, $\iota_* : \pi_1(S^1) \to \pi_1(D^2)$ would be inj, but:

$$\pi_1(S^1) = \mathbb{Z} \qquad \pi_1(D^2) = \{e\}$$

so this is not possible.

Example 11.4 Source: Primary Source Material

S^1 is a retract of the "figure 8" space (i.e. $S^1 \vee S^1$).

[label each copy of S^1 as A, B resp. and the base pt as x_0.] then the map r given by

$$r(x) = \begin{cases} x & x \in A \\ x_0 & x \in B \end{cases}$$

is a retraction.

Example 11.5 Source: Primary Source Material

$S^1 \vee S^1$ is *not* a retract of $D^2 \vee D^2$.

by contra, sps $r : D^2 \vee D^2 \to S^1 \vee S^1$ is a retraction. then:

$$D^2 \hookrightarrow D^2 \vee D^2 \longrightarrow S^1 \vee S^1 \longrightarrow S^1$$

would be a retraction $D^2 \to S^1$, a contradiction.

Definition 11.6

let $A \subseteq X$. we say A is a **deformation retract** if $\mathrm{id} : X \to X$ is homotopic to a retraction via $F : X \times I \to X$ s.t. $F(a,t) = a$ for all $t \in I$ and $a \in A$.

the homotopy F is called a **deformation retraction**.

Example 11.7 Source: Primary Source Material

S^1 is a deform retract of $\mathbb{R}^2 \setminus \{0\}$.

take $F : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\}$ as:

$$F(x,t) = (1-t)x + \frac{tx}{\|x\|}$$

this is a deform retraction.

Example 11.8 Source: Primary Source Material

consider $X = \mathbb{R}^3 \setminus \{\lambda e_3\}$, or $\mathbb{R}^3$ without the z-axis. then, $\mathbb{R}^2 \setminus \{0\}$ is a deform retract of X.

take $F((x,y,z),t) = (x,y,(1-t)z)$.

Example 11.9 Source: Primary Source Material

let X be $\mathbb{R}^2$ minus two pts. then $S^1 \vee S^1$ is a deform retract of X.

[u can just visualize this one tbh.]

Proposition 11.10

if A deform retract of X and $a_0 \in A$, then the homo $\iota_* : \pi_1(A,a_0) \to \pi_1(X,a_0)$ induced by $\iota : A \to X$ is iso.

Proof. Source: Primary Source Material

it suffices to show ι_* surj.

fix $F : X \times I \to X$ deform retraction of X onto A, and $[\gamma] \in \pi_1(X, a_0)$. consider $G : I \times I \to X$ as:

$$G(s,t) = F(\gamma(s), t)$$

note G is a pathtopy from γ to some loop α in A:

$$G(0,t) = F(\gamma(0), t) = F(a_0, t) = a_0$$

for all $t \in I$. then:

$$\iota_*([\alpha]) = [\alpha] = [\gamma]$$

■

Corollary 11.11

$\pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}$.

ok lets prove brower fixed pt (for D^2) using algtop. step one:

Definition 11.12

a cts $f : X \to Y$ is **nullhomotopic** if $f \simeq e_{x_0}$, i.e. f homotopic to a constant map.

Proposition 11.13

let $h : S^1 \to X$. tfae:

(i) h nulltopic

(ii) there exists cts ext $k : D^2 \to X$ of h

(iii) h_* is trivial homo

Proof.

Source: Primary Source Material

(i) $\implies$ (ii)

sps $h \simeq e_{x_0}$. let $F : S^1 \times I \to X$ be homotopy from h to e_{x_0}.

note $D^2 \cong (S^1 \times I)/(S^1 \times \{1\})$. consider $p : S^1 \times I \to D^2$ given as $p(x,t) = (1-t)x$. this is a qmap.

F constant on $S^1 \times \{1\}$. by properties of quotients, there exists some cts $k : D^2 \to X$ s.t. $F = k \circ p$. then for $x \in S^1$, $k(x) = F(x,0) = h(x)$.

(ii) $\implies$ (iii)

sps $k : D^2 \to X$ cts ext of h. let $\iota : S^1 \to D^2$ be inclusion. note $h = k \circ \iota$.

then $h_* = k_* \circ \iota_*$. note:

$$\iota_* : \pi_1(S^1) \to \pi_1(D^2) \qquad \pi_1(S^1) = \mathbb{Z} \qquad \pi_1(D^2) = \{e\}$$

thus ι_* trivial, so h_* trivial.

(iii) $\implies$ (i)

sps h_* trivial. note S^2 is a quotient of I, since $S^1 = I/\{0,1\}$ with:

$$x_0 = (1,0) \qquad p(s) = (\cos(2\pi s), \sin(2\pi s))$$

let $[p] \in \pi_1(S^1, x_0)$ and $h_*[p] = [e_{h(x_0)}]$. note $h_*([p]) = [h \circ p] =: [f]$.

fix pathtopy $F : I^2 \to X$ from f to $e_{h(x_0)}$. let $q(s,t) = (p(s), t)$; this is qmap from I^2 to $S^1 \times I$.

since F pathtopy, it is constant on pts identified by q. then, there exists some cts $G : S^1 \times I \to X$ s.t. $F = G \circ q$. to check homotopy, note that:

$$G(x,0) = F(s,0) = f(s) = h(p(s)) = h(x)$$

Example 11.14

Source: Primary Source Material

$\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$, the inclusion map, is not nulltopic since $\iota_* : \mathbb{Z} \to \mathbb{Z}$ is the id, so nontrivial. for the same reason, $\mathrm{id} : S^1 \to S^1$ is not nulltopic.

ok, back to fixed points. step two: vector fields on D^2.. what.

Definition 11.15

a **vector field** on D^2 is a cts $\mathcal{V} : D^2 \to \mathbb{R}^2$. we say $\mathcal{V}$ is **non-vanishing** if we have that $\mathcal{V}(x) \neq 0$ for all $x \in D^2$.

Proposition 11.16

if $\mathcal{V} : D^2 \to \mathbb{R}^2 \setminus \{0\}$ non-vanishing vecfield, then:

(1) there exists $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$ for some $\alpha < 0$. that is, $\mathcal{V}(x)$ points directly inwards

(2) there exists $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$ for some $\alpha > 0$, that is, $\mathcal{V}(x)$ points directly outwards

Proof.

Source: Primary Source Material

(2) follows from applying (1) to $-\mathcal{V}$.

by contradiction, sps no $\mathcal{V}(x)$ point directly inwards. consider the map given by $h = \mathcal{V}|_{S^1} : S^1 \to \mathbb{R}^2 \setminus \{0\}$. then $\mathcal{V}$ is a cts ext of h; by prev. prop, h is nulltopic. we claim h homotopic to inclusion $\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$.

consider $F(s,t) = (1-t)h(s) + ts$. we check $F : S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$, i.e., $F(s,t) \neq 0$. indeed, sps $F(s,t) = 0$. then $(1-t)h(s) = -ts$. so $\mathcal{V}(s) = h(s) = -ts/(1-t)$, so $\mathcal{V}(s)$ points inwards, contradiction.

but $\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$ not nulltopic, so contradiction (again). ■

Theorem 11.17: Brouwer's Fixed Point Theorem (for the 2D disc)

if $f : D^2 \to D^2$ cts, then there is $x \in D^2$ s.t. $f(x) = x$.

Proof.

Source: Primary Source Material

by contra, sps $f(x) \neq x$ for all $x \in X$. let $\mathcal{V}(x) = f(x) - x$, so $\mathcal{V}$ is non-vanishing vecfield. by prev thm, there is $\alpha > 0$ and $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$.

but $f(x) = (\alpha + 1)x \notin D^2$, a contradiction. ∎

is brouwer fixed pt true for D^n? yes [duh], but it requires more advanced algtop [x doubt]: homotopy theory and homotopy groups $\pi_n(X)$.