# MAT327
# Introduction to Topology

## Class Lecture Notes

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# Contents

# Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

# I  Algebraic Topology

## 1  Path Homotopy

Lec 19 - Jul 23 (Week 11)

ALGTOP RAAAHHHHHHHHHHHHH

> **Definition 1.1**
>
> given $\gamma_0, \gamma_1$ from $x$ to $y$, a **path homotopy** from $\gamma_0$ to $\gamma_1$ is a cts $F : [0,1]^2 \to X$ such that the following holds:
>
> $$F(s,0) = \gamma_0(s) \text{ and } F(s,1) = \gamma_1 \qquad F(0,t) = x \text{ and } F(1,t) = y$$
>
> we say $\gamma_0$ is **path homotopic** to $\gamma_1$, or $\gamma_0 \simeq_p \gamma_1$, if a pathtopy exists

[pathtopy is certainly a. choice.] note that its important for $F$ to be cts; this is strictly stronger than requiring $F$ to be cts in each coordinate (beloved $xy/x^2 + y^2$).

Source: Primary Source Material

> **Example 1.2**
>
> $A \subseteq \mathbb{R}^n$ cvx $\implies$ any two paths w/ same endpoints are homotopic: for fixed $s$, take $F(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$. check that this is a pathtopy.

we can generalize pathtopy to deformations of general cts functions.

> **Definition 1.3**
>
> sps $f, g : X \to Y$ cts. a **homotopy** from $f$ to $g$ is a cts $F : X \times [0,1] \to Y$ with
>
> $$F(x,0) = f(x) \qquad F(x,1) = g(x)$$
>
> we write $f \simeq g$ in this case.

clearly, both $\simeq, \simeq_p$ are equiv rels. note transitivity uses pasting lemma [technically. its like a single pt tho].

# 2　The Fundamental Group

[yeah this should go here tbh]

## Definition 2.1

for a path $\gamma_0$ from $x$ to $y$ and $\gamma_1$ from $y$ to $z$, define:

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s) & s \in [0, 1/2] \\ \gamma_1(2s - 1) & s \in [1/2, 1] \end{cases}$$

this induces an operation $*$ on the set of equivalence classes.

## Proposition 2.2

$*$ is well-defined on equivalence classes.

## Proof.

Source: Primary Source Material

fix $\gamma_0 \simeq \gamma_0'$ and $\gamma_1 \simeq \gamma_1'$. let $F, G : [0,1]^2 \to X$ be pathtopies from $\gamma_i$ to $\gamma_i'$ respectively. consider:

$$H(s,t) = \begin{cases} F(2s, t) & (s, t) \in [0, 1/2] \times [0, 1] \\ G(2s - 1, t) & (s, t) \in [1/2, 1] \times [0, 1] \end{cases}$$

it is easy to see that $H$ is then a homotopy from $\gamma_0 * \gamma_1$ to $\gamma_0' * \gamma_1'$. ∎

## Definition 2.3

a **loop** is a path $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1)$. we say that $\gamma$ is a loop at $x_0$ if $\gamma(0) = \gamma(1) = x_0$.

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given a fixed $x_0$, we denote by $\pi_1(X, x_0)$ the set of all pathtopy equiv classes of loops at $x_0$.

we define by $e_x : [0,1] \to X$ and $\overline{\gamma} : [0,1] \to X$ as:

$$e_x(s) = x \qquad \overline{\gamma}(s) = \gamma(1-s)$$

these are the "constant" and "inverse" paths respectively.

### Definition 2.4

given $\gamma$, let $\varphi : [0,1] \to [0,1]$ be cts with $\varphi(0) = 0$ and $\varphi(1) = 1$. we call $\gamma \circ \varphi$ a **reparametrization** of $\gamma$.

### Lemma 2.5

$\gamma \simeq_p \gamma \circ \varphi$

### Proof.

Source: Primary Source Material

$F(s,t) = \gamma((1-t)s + t\varphi(s))$ is a pathtopy. ∎

okay the rest is just proving that $\pi_1(X, x_0)$ is a grp under $*$. uhh the pfs look kinda annoying to write so im just not gonna. uses the reparametrization tho

Lec 20 - Jul 25 (Week 11)

from last time: given $X$ and a basept $x_0 \in X$, we associated the group $\pi_1(X, x_0)$ to $(X, x_0)$ called the fundamental grp of $X$ w basept $x_0$.

$\pi_1$ is also known as a **functor**.

$$\text{Topological space} \xrightarrow{\pi_1} \text{Group}$$
$$\text{Continuous map} \longrightarrow \text{Homomorphism}$$
$$\text{Homeomorphism} \longrightarrow \text{Isomorphism}$$

today we will prove this! whatever that means. in the meantime: did you know the torus has fundamental group $\mathbb{Z}$?

Q: what happens to $\pi_1(X, x_0)$ if we change the basept?

**Proposition 2.6**

fix $x_0, x_1 \in X$. let $\alpha$ be a path from $x_0$ to $x_1$. define $\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\widehat{\alpha}([\gamma]) = [\overline{\alpha} * \gamma * \alpha]$$

then $\widehat{\alpha}$ is well-defined, and $\widehat{\alpha}$ is an isomorphism.

**Proof.**

well-definedness is an exercise. show:

$$\gamma_0 \simeq_p \gamma_1 \implies \overline{\alpha} * \gamma_0 * \alpha \simeq_p \overline{\alpha} * \gamma_1 * \alpha$$

it is a homomorphism because:

$$\begin{aligned}
\widehat{\alpha}([\gamma_0] * [\gamma_1]) = \widehat{\alpha}([\gamma_0 * \gamma_1]) &= [\overline{\alpha} * \gamma_0 * \gamma_1 * \alpha] \\
&= [\overline{\alpha} * \gamma_0 * e_{x_0} * \gamma_1 * \alpha] \\
&= [\overline{\alpha} * \gamma_0 * \alpha * \overline{\alpha} * \gamma_1 * \alpha] \\
&= [\overline{\alpha} * \gamma_0 * \alpha] * [\overline{\alpha} * \gamma_1 * \alpha] \\
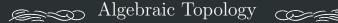&= \widehat{\alpha}([\gamma_0]) * \widehat{\alpha}([\gamma_1])
\end{aligned}$$

it is bijective because:

$$\widehat{\alpha} \circ \widehat{\overline{\alpha}}([\gamma]) = \widehat{\alpha}([\alpha * \gamma * \overline{\alpha}]) = [\overline{\alpha} * \alpha * \gamma * \overline{\alpha} * \alpha] = [\gamma]$$

$\widehat{\overline{\alpha}} \circ \widehat{\alpha}$ is similarly id. ∎

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## Corollary 2.7

$X$ pathconn then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ for all $x_0, x_1$. in this case, fundgrp does not depend on basept, and we can denote it $\pi_1(X)$.

## Definition 2.8

$X$ is **simply connected** iff $X$ pathconn and fundgrp is trivial.

for instance, any cvx subset of $\mathbb{R}^n$ is simply conn.

notation: we write $\varphi : (X, x_0) \to (Y, y_0)$ to mean that $\varphi$ cts, $\varphi(x_0) = x_1$.

any map $\varphi : (X, x_0) \to (X, x_1)$ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

this is known as the **induced map** of $\varphi$. exercise: check this is well-defined, that is $\gamma_0 \simeq_p \gamma_1 \implies \varphi \circ \gamma_0 \simeq_p \varphi \circ \gamma_1$. idea: if $F : [0,1]^2 \to X$ is path homotopy from $\gamma_0$ to $\gamma_1$, then $G = \varphi \circ F$ is a path homotopy from $\varphi \circ \gamma_0$ to $\varphi \circ \gamma_1$. (check this!)

## Proposition 2.9

let $\varphi : (X, x_0) \to (Y, y_0)$. then the induced map is a homomorphism.

## Proof.

we check $\varphi_*([\gamma_0] * [\gamma_1]) = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1])$.

$$\varphi_*([\gamma_0] * [\gamma_1]) = \varphi_*([\gamma_0 * \gamma_1]) = [\varphi \circ (\gamma_0 * \gamma_1)]$$
$$= [(\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)] = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1])$$

to see red equality, note that $\varphi \circ (\gamma_0 * \gamma_1) = (\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)$. check this! $\blacksquare$

some properties:

(i) if $\varphi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (Z, z_0)$, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(ii) if $\iota : (X, x_0) \to (X, x_0)$ is id, then $\iota_*$ is id.

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(iii) if $\varphi : (X, x_0) \to (Y, y_0)$ is homeo, then $\varphi_*$ is iso.

proofs:

(i) $(\varphi \circ \psi)_*([\gamma]) = [\varphi \circ \psi \circ \gamma] = \varphi_*([\psi \circ \gamma]) = \varphi_*(\psi_*([\gamma]))$

(ii) $\iota_*([\gamma]) = [\iota \circ \gamma] = [\gamma]$

(iii) by $(i)$ and $(ii)$, $\varphi_* \circ (\varphi^{-1})_* = \iota_*$ and $(\varphi^{-1})_* \circ \varphi_* = \iota_*$. this also shows $(\varphi_*)^{-1} = (\varphi^{-1})_*$.

summary: given the following:

$$(X, x_0) \xrightarrow{\quad \varphi \quad} (Y, y_0)$$

we can apply $\pi_1$ to transform this into:

$$\pi_1(X, x_0) \xrightarrow{\quad \varphi_* \quad} \pi_1(Y, y_0)$$

# 3   Covering Spaces

<div align="right">Lec 21 - Jul 30 (Week 12)</div>

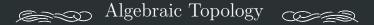today's mission: prove $\pi_1(S^1) = \mathbb{Z}$. we notate $I = [0, 1]$.

proof sketch: let $\omega_n(s) = (\cos(2\pi s n), \sin(2\pi s n))$. this is a loop in $S^1$ at $x_0 = (1, 0)$ that does $n$ revolutions. moves ccw if $n > 0$, cw if $n < 0$. we want to show any loop in $S^1$ is pathtopic to some $\omega_n$.

idea: show every $\gamma : I \to S^1$ can be "uniquely lifted" to a path $\widetilde{\gamma} : I \to \mathbb{R}$ from 0 to some $n$. embed $\mathbb{R} \hookrightarrow \mathbb{R}^3$ as the "helix":

$$s \to (\cos(2\pi s), \sin(2\pi s), s)$$

let $P : \mathbb{R} \to S^1$ be the projection of the helix onto the $xy$-plane. we want to show two things:

(a) for any loop $\gamma : I \to S^1$, there is a unique $\widetilde{\gamma} : I \to \mathbb{R}$ starting at 0 s.t.:

[commutative diagram]

(b) for every pathtopy $F : I^2 \to S^1$ s.t. $F(0,0) = x_0$, there is a unique pathtopy $\widetilde{F} : I^2 \to \mathbb{R}$ s.t. $\widetilde{F}(0,0) = 0$ and:

[commutative diagram]

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**Definition 3.1**

given spaces $X, E$, we say $P : E \to X$ is a **covering map** if for every $x_0 \in X$ there is an open $x_0 \in U$ which is "evenly covered by $P$", i.e.:

$$P^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} V_\alpha$$

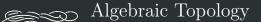is a union of pairwise disjoint open sets in $E$ such that the map $P|_{V_\alpha} : V_\alpha \to U$ is a homomorphism.

in this context, we say $E$ is a **covering space** of $X$. each $V_\alpha$ is also known as a **sheet** or **slice**.

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some examples:

- id $: X \to X$ is a cvring space (1-sheeted)

- $P : \mathbb{R} \to S^1$ given by $P(s) = (\cos(2\pi s), \sin(2\pi s))$ (countably many sheets)

- $P_n : S^1 \to S^1$ given by $P_n(z) = z^n$, where $S^1 \subseteq \mathbb{C}$ ($n$-sheeted)

properties of covering maps:

- $\forall x \in X$, $P^{-1}(x)$ is a discrete subspace of $E$. that is, every $e \in P^{-1}(x)$ has an open $V \subseteq E$ s.t. $V \cap P^{-1}(x) = \{e\}$

- cvring maps are open maps (exercise)

- cvring maps are **local homeos**. that is, for all $e \in E$, there is an open $e \in V \subseteq E$ s.t. $P|_V : V \to P(V)$ is a homeo

## Definition 3.2

let $P : E \to X$ be a cvring map, $f : X \to Y$ be cts. a **lifting of $f$** is a cts $\widetilde{f} : Y \to E$ s.t. $f = P \circ \widetilde{f}$

[diagram]

## Lemma 3.3: Path-lifting property

let $P : E \to X$ be a cvring map and $x = P(e)$. if $\gamma : I \to X$ is a path starting at $x$, then there is a unique lifting to a path $\widetilde{\gamma} : I \to E$ starting at $e$.

## Proof.

step 1: find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\gamma([t_{i-1}, t_i])$ is contained in some evenly covered open $U_i$.

> for all $t \in I$, $\gamma(t) \in X$. then $\exists$ open nbhd $U_t$ of $\gamma(t)$ s.t. each $U_t$ evenly cvred:
> $$I = \bigcup_{t \in I} \gamma^{-1}(U)$$
>
> by lebesgue number lemma, let $\delta > 0$ s.t. for any $\mathrm{diam}(A) < \delta$, there is some $t \in I$ s.t. $A \subseteq \gamma^{-1}(U_t)$. choose a partition $P = \{t_0, \ldots, t_n\}$ s.t. $\|P\| = \max |t_i - t_{i-1}| < \delta$. then we have that $[t_{i-1}, t_i] \subseteq \gamma^{-1}(U)$ for some open $U$ evenly cvred.

step 2: we prove existence of $\widetilde{\gamma}$. we construct $\widetilde{\gamma}$ inductively on each subinterval of the partition.

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$\widetilde{\gamma}(0) = e_0$. sps $\widetilde{\gamma}$ defined on $[0, t_{i-1}]$. we extend to $[0, t_i]$ by defining $\widetilde{\gamma}$ on the next subinterval $[t_{i-1}, t_i]$.

by step 1, there is an open $U$ evenly covered s.t. $\gamma([t_{i-1}, t_i]) \subseteq U$. notice that $\widetilde{\gamma}(t_{i-1}) \in P^{-1}(U)$ since:

$$P \circ \widetilde{\gamma}(t_{i-1}) = \gamma(t_{i-1}) \in U$$

then $\widetilde{\gamma}(t_{i-1}) \in V_\alpha$ where $V_\alpha$ is a sheet of $P^{-1}(U)$.

recall $P|_{V_\alpha} : V_\alpha \to U$ is homeo. define:

$$\widetilde{\gamma}(s) = \left(P^{-1}|_{V_\alpha}\right)(\gamma(s)) \qquad s \in [t_{i-1}, t_i]$$

note $\widetilde{\gamma}|_{[0,t_i]}$ cts by pasting lemma, and $\gamma = P \circ \widetilde{\gamma}$.

step 3: we prove uniqueness of $\widetilde{\gamma}$.

sps $\widehat{\gamma}$ is another lifting of $\gamma$ with $\widehat{\gamma}(0) = e_0$. we show $\widehat{\gamma}(s) = \widetilde{\gamma}(s)$ for all $s \in [t_{i-1}, t_i]$ inductively.

sps $\widehat{\gamma}|_{[0,t_{i-1}]} = \widetilde{\gamma}|_{[0,t_{i-1}]}$. note $\widehat{\gamma}([t_{i-1}, t_i])$ conn and $\widehat{\gamma}(t_{i-1}) \in V_\alpha$ where $V_\alpha$ is the sheet we used to define $\widetilde{\gamma}$ on $[t_{i-1}, t_i]$ in step 2.

it then follows that $\widehat{\gamma}([t_{i-1}, t_i]) \subseteq V_\alpha$ by conn. since $\gamma = P \circ \widehat{\gamma}$:

$$\widehat{\gamma}(s) = \left(P^{-1}|_{V_\alpha}\right)(\gamma(s)) = \widetilde{\gamma}(s)$$

for all $s \in [t_{i-1}, t_i]$.

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### Lemma 3.4: Path-homotopy lifting property

sps $P : E \to X$ cvring map, $x = P(e)$. if $F : I^2 \to X$ is a pathtopy with $F(0,0) = x$, then there is a unique lifting $\widetilde{F} : I^2 \to E$ which is a **pathtopy** in $E$ s.t. $\widetilde{F}(0,0) = e$.

pf: very similar to the one above. not writing allat

### Corollary 3.5

let $P : E \to X$ be a cvring map and $x = P(e)$. if $\gamma_0 \simeq_p \gamma_1$, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$.

### Theorem 3.6

$\pi_1(S^1) = \mathbb{Z}$.

dont rly wanna put this in a box

let $x_0 = (1,0) \in S^1$ and $P : \mathbb{R} \to S^1$ be the cvring map given by:

$$P(s) = (\cos(2\pi s), \sin(2\pi s))$$

given $[\gamma] \in \pi_1(S^1, x_0)$, let $\widetilde{\gamma} : I \to \mathbb{R}$ be the unique lifting of $\gamma$ s.t. $\widetilde{\gamma}(0) = 0$. define $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ given by:

$$\varphi([\gamma]) = \widetilde{\gamma}(1) \in P^{-1}(x_0) = \mathbb{Z}$$

first, we show $\varphi$ is well-defined.

⌐ by prev crll, if $\gamma_0 \simeq_p \gamma_1$, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$. in particular, $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$. ⌐

next, $\varphi$ is surjective.

⌐ fix $n \in \mathbb{Z}$. consider $\omega_n(s) = (\cos(2\pi s n), \sin(2\pi s n))$. then $\varphi([\omega_n]) = \widetilde{\omega_n}(1) = n$. ⌐

next, $\varphi$ is injective.

sps $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$ s.t. $\varphi([\gamma_0]) = \varphi([\gamma_1])$. note $\widetilde{\gamma}_0, \widetilde{\gamma}_1 : I \to \mathbb{R}$ are paths in $\mathbb{R}$. since $\mathbb{R}$ cvx, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$ via pathtopy $\widetilde{F} : I^2 \to \mathbb{R}$. then, $P \circ F : I^2 \to S^1$ is a pathtopy from $P \circ \widetilde{\gamma}_0 = \gamma_0$ to $P \circ \widetilde{\gamma}_1 = \gamma_1$, so $[\gamma_0] = [\gamma_1]$.

finally, $\varphi$ is homo.

given $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$, let:

$$\widetilde{\gamma}_0(1) = n \qquad \widetilde{\gamma}_1(1) = m$$

then $\gamma_0 \simeq_p \omega_n$ and $\gamma_1 \simeq_p \omega_m$. we show $\varphi([\omega_n] * [\omega_m]) = n + m$.

let $\widetilde{\omega_n}$ be the lifting starting at $0$ and ending at $n$. let $\widetilde{\omega_m}$ be the lifting starting at $n$. then, we have that:

$$\varphi([\omega_n] * [\omega_m]) = (\widetilde{\omega_n} * \widetilde{\omega_m})(1) = n + m$$

note: check that $\widetilde{\omega_n} * \widetilde{\omega_m}$ is indeed a lifting of $\omega_n * \omega_m$.