MAT388
Introduction to Knot Theory

Class Lecture Notes

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Class Lectures

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Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

I Introduction and Basics

1 Preliminaries

Lec 1 - Sep 2 (Week 1)

main question of the course: how many ways can we embed a circle into $\mathbb{R}^3$? we'll consider this up to ambient diffeo/homeo, or up to isotopy (comes later).

for example, we claim that there is no homeo $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(U) = T$, from the unknot to the trefoil.

this gives a distinction between knotted and unknotted, but non-trivial knots can also be distinct.

[theres an example here, but i dont have my stylus :)]

so we need some way to identify different knots, such as a particular property. what can we use, how do we think about these distinctions? note that there are infinitely many distinct, but we're really asking about the underlying structure.

Definition 1.1

a **knot diagram** is an *immersion* (to be defined) of the circle $S^1 \hookrightarrow \mathbb{R}^2$, so that:

- All non-injective points are 2:1 and (self-)transverse
- Add over/under-crossing data at all double points

[again, an example, but my stylus....]

Definition 1.2

an **immersion** is a non-vanishing (full rank) derivative at all points. [precisely, it is a differentiable map whose pushforward is inj]

Proposition 1.3

given any C^∞ (or just C^1) embedding $K : S^1 \hookrightarrow \mathbb{R}^3$, a "generic" linear projection of $K(S^1)$ gives a knot diagram.

conversely, any knot diagram defines a knot $K(S^1) \subseteq \mathbb{R}^3$, which is unique up to "isotopy".

Source: Primary Source Material

Proof.

(sketch)(very sketchy)

consider $K(S^1) \subseteq \mathbb{R}^3$; it has a tangent vector everywhere. consider $\{\text{directions}\} \subseteq S^2$. this is a smooth map $S^1 \to S^2$; in particular, it is not surjective by Sard's theorem. as a consequence, the complement of the image is full measure (open + dense).

pick a pt not in the image and project in this direction. the tangent vector to K is thus never parallel, so the projection is an immersion.

other issues: possibly self-tangent, possibly $n : 1$. but, none of these are generic (i.e. do a dimension count, apply Sard's)

for the converse: let x, y coords be the coords in the projection. the indeterminacy is $z(\theta)$, since $(x(\theta), y(\theta)) \in \mathbb{R}^2$ are determined by the diagram.

crossings are then double pts $\left\{\theta_1^+, \theta_1^-\right\}, \left\{\theta_2^+, \theta_2^-\right\}$ as pairs of pts in S^1, say up to k pts. (necessarily finite since they are necessarily isolated -¿ cpt -¿ finite, or sth)

then choice is(of?) a function $z : S^1 \to \mathbb{R}$ such that $z(\theta_j^+) > z(\theta_j^-)$ - this represents which strand is above the other, in a sense.

the space of all functions z is then convex and thus connected:

$$z_t(\theta) = tz(\theta) + (1-t)\widetilde{z}(\theta)$$

is a valid choice of z. so there are many possible choices, but all can be interpolated, thus are isotopic(?). ∎

Definition 1.4

a **tri-colouring** of a knot diagram is a choice of colour in {RGB} for each "arc" in the diagram, such that at each crossing, either *one* or *three* colours are used[meet]. it is required to use all three colours.

for instance, the trefoil can very easily be tri-coloured, as well as the "6_1 knot", or more generally a (k-th) "twist" knot. as a non-example, the "figure-8" knot and unknot have no tri-colouring.

it seems like this only depends on the diagram, but in fact it only depends on a knot up to smooth isotopy. why? what?

(non-clarifying proof/explanation) we use a black box: reidemeister's theorem

two diagrams present isotopic knots in $\mathbb{R}^3$ iff they differ by a finite sequence of these moves:

- R1 - twisting a strand
- R2 - moving two non-intersecting strands atop each other
- R3 - moving a strand behind a crossing if it is "under both strands"

for example: [example]

pf of tri-colourability: note that for each move, if there is a chosen tri-colouring before the move, there is a unique tri-colouring after.

ok, but what's actually happening? real answer: consider the group homomorphisms $\pi_1(\mathbb{R}^3 \setminus K(S^1)) \to S_3$.

sps $K \subseteq \mathbb{R}^3$ is a knot (abusing notation). if K isotopic to $\widetilde{K}$, or if there is a homeo/diffeo $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi(K) = \widetilde{K}$, then $\mathbb{R}^3 \setminus K \simeq \mathbb{R}^3 \setminus \widetilde{K}$. thus, we can equivalently as questions about the complements; are these the same? what about $\pi_1(\mathbb{R}^3 \setminus K)$? this will give us our first powerful invariant for knots.

note for $K = U$ the unknot, we can draw a knot "parallel" to the unknot on the inside, and this is homotopic to U with $\pi_1(\mathbb{R}^3 \setminus K) \simeq \mathbb{Z}$. but for the trefoil, this may not be the case - a parallel curve may not be homotopic to T.

this seems to distinguish U and T, maybe, but how to define anything?

lets look at $U \subseteq \mathbb{R}^3$. let x_0 be your eyeball. if $f : [0,1] \to \mathbb{R}^3 \setminus U$ never passes behind the knot, then taking all the light rays from $f(\theta)$ to x_0 defines a homotopy H, so $[f] = [e]$. otherwise, if it passes through U, then we get a winding number, so $\pi_1(\mathbb{R}^3 \setminus U) \simeq \mathbb{Z}$.

for a more complicated diagram, we have a group elem for each *arc* of the diagram. however, given a loop around some arc, we can slide it "along" the arc as long as we don't accidentally cross over any other arcs.

so we'd have the following relation: [more diagrams]

this is known as the **wirtinger presentation** - we will see the proof next time.