
MAT327

Introduction to Topology

Class Lecture Notes

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Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.



I Algebraic Topology

1 Path Homotopy

Lec 19 - Jul 23 (Week 11)

ALGTOP RAAAHHHHHHHHHHHH

Definition 1.1

given γ_0, γ_1 from x to y , a **path homotopy** from γ_0 to γ_1 is a cts $F : [0, 1]^2 \rightarrow X$ such that the following holds:

$$F(s, 0) = \gamma_0(s) \text{ and } F(s, 1) = \gamma_1(s) \quad F(0, t) = x \text{ and } F(1, t) = y$$

we say γ_0 is **path homotopic** to γ_1 , or $\gamma_0 \simeq_p \gamma_1$, if a pathtopy exists

[pathtopy is certainly a. choice.] note that its important for F to be cts; this is strictly stronger than requiring F to be cts in each coordinate (beloved $xy/x^2 + y^2$).

Example 1.2

Source: Primary Source Material

$A \subseteq \mathbb{R}^n$ cvx \implies any two paths w/ same endpoints are homotopic: for fixed s , take $F(s, t) = (1 - t)\gamma_0(s) + t\gamma_1(s)$. check that this is a pathtopy.

we can generalize pathtopy to deformations of general cts functions.

Definition 1.3

sps $f, g : X \rightarrow Y$ cts. a **homotopy** from f to g is a cts $F : X \times [0, 1] \rightarrow Y$ with

$$F(x, 0) = f(x) \quad F(x, 1) = g(x)$$

we write $f \simeq g$ in this case.

clearly, both \simeq, \simeq_p are equiv rels. note transitivity uses pasting lemma [technically. its like a single pt tho].





2 The Fundamental Group

[yeah this should go here tbh]

Definition 2.1

for a path γ_0 from x to y and γ_1 from y to z , define:

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s) & s \in [0, 1/2] \\ \gamma_1(2s - 1) & s \in [1/2, 1] \end{cases}$$

this induces an operation $*$ on the set of equivalence classes.

Proposition 2.2

$*$ is well-defined on equivalence classes.

Proof.

Source: Primary Source Material

fix $\gamma_0 \simeq \gamma'_0$ and $\gamma_1 \simeq \gamma'_1$. let $F, G : [0, 1]^2 \rightarrow X$ be pathtopies from γ_i to γ'_i respectively. consider:

$$H(s, t) = \begin{cases} F(2s, t) & (s, t) \in [0, 1/2] \times [0, 1] \\ G(2s - 1, t) & (s, t) \in [1/2, 1] \times [0, 1] \end{cases}$$

it is easy to see that H is then a homotopy from $\gamma_0 * \gamma_1$ to $\gamma'_0 * \gamma'_1$. ■

Definition 2.3

a **loop** is a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = \gamma(1)$. we say that γ is a loop at x_0 if $\gamma(0) = \gamma(1) = x_0$.



given a fixed x_0 , we denote by $\pi_1(X, x_0)$ the set of all path-homotopy equiv classes of loops at x_0 .

we define by $e_x : [0, 1] \rightarrow X$ and $\bar{\gamma} : [0, 1] \rightarrow X$ as:

$$e_x(s) = x \quad \bar{\gamma}(s) = \gamma(1 - s)$$

these are the “constant” and “inverse” paths respectively.

Definition 2.4

given γ , let $\varphi : [0, 1] \rightarrow [0, 1]$ be cts with $\varphi(0) = 0$ and $\varphi(1) = 1$. we call $\gamma \circ \varphi$ a **reparametrization** of γ .

Lemma 2.5

$$\gamma \simeq_p \gamma \circ \varphi$$

Proof.

Source: Primary Source Material

$F(s, t) = \gamma((1 - t)s + t\varphi(s))$ is a path-homotopy. ■

okay the rest is just proving that $\pi_1(X, x_0)$ is a grp under $*$. uhh the pfs look kinda annoying to write so im just not gonna. uses the reparametrization tho

Lec 20 - Jul 25 (Week 11)

from last time: given X and a basept $x_0 \in X$, we associated the group $\pi_1(X, x_0)$ to (X, x_0) called the fundamental grp of X w basept x_0 .

π_1 is also known as a **functor**.

$$\begin{array}{ccc} \text{Topological space} & \xrightarrow{\pi_1} & \text{Group} \\ \text{Continuous map} & \longrightarrow & \text{Homomorphism} \\ \text{Homeomorphism} & \longrightarrow & \text{Isomorphism} \end{array}$$



today we will prove this! whatever that means. in the meantime: did you know the torus has fundamental group \mathbb{Z} ?

Q: what happens to $\pi_1(X, x_0)$ if we change the basept?

Proposition 2.6

fix $x_0, x_1 \in X$. let α be a path from x_0 to x_1 . define $\hat{\alpha} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ as:

$$\hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

then $\hat{\alpha}$ is well-defined, and $\hat{\alpha}$ is an isomorphism.

Proof.

Source: Primary Source Material

well-definedness is an exercise. show:

$$\gamma_0 \simeq_p \gamma_1 \implies \bar{\alpha} * \gamma_0 * \alpha \simeq_p \bar{\alpha} * \gamma_1 * \alpha$$

it is a homomorphism because:

$$\begin{aligned} \hat{\alpha}([\gamma_0] * [\gamma_1]) &= \hat{\alpha}([\gamma_0 * \gamma_1]) = [\bar{\alpha} * \gamma_0 * \gamma_1 * \alpha] \\ &= [\bar{\alpha} * \gamma_0 * e_{x_0} * \gamma_1 * \alpha] \\ &= [\bar{\alpha} * \gamma_0 * \alpha * \bar{\alpha} * \gamma_1 * \alpha] \\ &= [\bar{\alpha} * \gamma_0 * \alpha] * [\bar{\alpha} * \gamma_1 * \alpha] \\ &= \hat{\alpha}([\gamma_0]) * \hat{\alpha}([\gamma_1]) \end{aligned}$$

it is bijective because:

$$\hat{\alpha} \circ \hat{\bar{\alpha}}([\gamma]) = \hat{\alpha}([\alpha * \gamma * \bar{\alpha}]) = [\bar{\alpha} * \alpha * \gamma * \bar{\alpha} * \alpha] = [\gamma]$$

$\hat{\bar{\alpha}} \circ \hat{\alpha}$ is similarly id. ■



Corollary 2.7

X pathconn then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ for all x_0, x_1 . in this case, fundgrp does not depend on basept, and we can denote it $\pi_1(X)$.

Definition 2.8

X is **simply connected** iff X pathconn and fundgrp is trivial.

for instance, any cvx subset of \mathbb{R}^n is simply conn.

notation: we write $\varphi : (X, x_0) \rightarrow (Y, y_0)$ to mean that φ cts, $\varphi(x_0) = y_0$.

any map $\varphi : (X, x_0) \rightarrow (Y, y_0)$ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ as:

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

this is known as the **induced map** of φ . exercise: check this is well-defined, that is $\gamma_0 \simeq_p \gamma_1 \implies \varphi \circ \gamma_0 \simeq_p \varphi \circ \gamma_1$. idea: if $F : [0, 1]^2 \rightarrow X$ is path homotopy from γ_0 to γ_1 , then $G = \varphi \circ F$ is a path homotopy from $\varphi \circ \gamma_0$ to $\varphi \circ \gamma_1$. (check this!)

Proposition 2.9

let $\varphi : (X, x_0) \rightarrow (Y, y_0)$. then the induced map is a homomorphism.

Proof.

Source: Primary Source Material

we check $\varphi_*([\gamma_0] * [\gamma_1]) = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1])$.

$$\begin{aligned} \varphi_*([\gamma_0] * [\gamma_1]) &= \varphi_*([\gamma_0 * \gamma_1]) = [\varphi \circ (\gamma_0 * \gamma_1)] \\ &= [(\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)] = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1]) \end{aligned}$$

to see red equality, note that $\varphi \circ (\gamma_0 * \gamma_1) = (\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)$. check this! ■

some properties:

- (i) if $\varphi : (X, x_0) \rightarrow (Y, y_0)$ and $\psi : (Y, y_0) \rightarrow (Z, z_0)$, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
- (ii) if $\iota : (X, x_0) \rightarrow (X, x_0)$ is id, then ι_* is id.





(iii) if $\varphi : (X, x_0) \rightarrow (Y, y_0)$ is homeo, then φ_* is iso.

proofs:

$$(i) \quad (\varphi \circ \psi)_*([\gamma]) = [\varphi \circ \psi \circ \gamma] = \varphi_*([\psi \circ \gamma]) = \varphi_*(\psi_*([\gamma]))$$

$$(ii) \quad \iota_*([\gamma]) = [\iota \circ \gamma] = [\gamma]$$

(iii) by (i) and (ii), $\varphi_* \circ (\varphi^{-1})_* = \iota_*$ and $(\varphi^{-1})_* \circ \varphi_* = \iota_*$. this also shows $(\varphi_*)^{-1} = (\varphi^{-1})_*$.

summary: given the following:

$$(X, x_0) \xrightarrow{\varphi} (Y, y_0)$$

we can apply π_1 to transform this into:

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, y_0)$$

3 Covering Spaces

Lec 21 - Jul 30 (Week 12)

today's mission: prove $\pi_1(S^1) = \mathbb{Z}$. we notate $I = [0, 1]$.

proof sketch: let $\omega_n(s) = (\cos(2\pi sn), \sin(2\pi sn))$. this is a loop in S^1 at $x_0 = (1, 0)$ that does n revolutions. moves ccw if $n > 0$, cw if $n < 0$. we want to show any loop in S^1 is pathtopic to some ω_n .

idea: show every $\gamma : I \rightarrow S^1$ can be “uniquely lifted” to a path $\tilde{\gamma} : I \rightarrow \mathbb{R}$ from 0 to some n . embed $\mathbb{R} \hookrightarrow \mathbb{R}^3$ as the “helix”:

$$s \rightarrow (\cos(2\pi s), \sin(2\pi s), s)$$

let $P : \mathbb{R} \rightarrow S^1$ be the projection of the helix onto the xy -plane. we want to show two things:

(a) for any loop $\gamma : I \rightarrow S^1$, there is a unique $\tilde{\gamma} : I \rightarrow \mathbb{R}$ starting at 0 s.t.:

[commutative diagram]





- (b) for every path $F : I^2 \rightarrow S^1$ s.t. $F(0,0) = x_0$, there is a unique path $\tilde{F} : I^2 \rightarrow \mathbb{R}$ s.t. $\tilde{F}(0,0) = 0$ and:
[commutative diagram]

Definition 3.1

given spaces X, E , we say $P : E \rightarrow X$ is a **covering map** if for every $x_0 \in X$ there is an open $U \ni x_0$ which is “evenly covered by P ”, i.e.:

$$P^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} V_\alpha$$

is a union of pairwise disjoint open sets in E such that the map $P|_{V_\alpha} : V_\alpha \rightarrow U$ is a homeomorphism.

in this context, we say E is a **covering space** of X . each V_α is also known as a **sheet** or **slice**.

some examples:

- $\text{id} : X \rightarrow X$ is a covering space (1-sheeted)
- $P : \mathbb{R} \rightarrow S^1$ given by $P(s) = (\cos(2\pi s), \sin(2\pi s))$ (countably many sheets)
- $P_n : S^1 \rightarrow S^1$ given by $P_n(z) = z^n$, where $S^1 \subseteq \mathbb{C}$ (n -sheeted)

properties of covering maps:

- $\forall x \in X$, $P^{-1}(x)$ is a discrete subspace of E . that is, every $e \in P^{-1}(x)$ has an open $V \subseteq E$ s.t. $V \cap P^{-1}(x) = \{e\}$
- covering maps are open maps (exercise)
- covering maps are **local homeos**. that is, for all $e \in E$, there is an open $V \ni e \subseteq E$ s.t. $P|_V : V \rightarrow P(V)$ is a homeo





Definition 3.2

let $P : E \rightarrow X$ be a coving map, $f : X \rightarrow Y$ be cts. a **lifting of f** is a cts $\tilde{f} : Y \rightarrow E$ s.t. $f = P \circ \tilde{f}$

[diagram]

Lemma 3.3: Path-lifting property

let $P : E \rightarrow X$ be a coving map and $x = P(e)$. if $\gamma : I \rightarrow X$ is a path starting at x , then there is a unique lifting to a path $\tilde{\gamma} : I \rightarrow E$ starting at e .

Proof.

Source: Primary Source Material

step 1: find a partition $0 = t_0 < t_1 < \dots < t_n = 1$ s.t. $\gamma([t_{i-1}, t_i])$ is contained in some evenly covered open U_i .

┌ for all $t \in I$, $\gamma(t) \in X$. then \exists open nbhd U_t of $\gamma(t)$ s.t. each U_t evenly cvred:

$$I = \bigcup_{t \in I} \gamma^{-1}(U_t)$$

by lebesgue number lemma, let $\delta > 0$ s.t. for any $\text{diam}(A) < \delta$, there is some $t \in I$ s.t. $A \subseteq \gamma^{-1}(U_t)$. choose a partition $P = \{t_0, \dots, t_n\}$ s.t. $\|P\| = \max |t_i - t_{i-1}| < \delta$. then we have that $[t_{i-1}, t_i] \subseteq \gamma^{-1}(U)$ for some open U evenly cvred.

step 2: we prove existence of $\tilde{\gamma}$. we construct $\tilde{\gamma}$ inductively on each subinterval of the partition.



┌ $\tilde{\gamma}(0) = e_0$. sps $\tilde{\gamma}$ defined on $[0, t_{i-1}]$. we extend to $[0, t_i]$ by defining $\tilde{\gamma}$ on the next subinterval $[t_{i-1}, t_i]$.

by step 1, there is an open U evenly covered s.t. $\gamma([t_{i-1}, t_i]) \subseteq U$. notice that $\tilde{\gamma}(t_{i-1}) \in P^{-1}(U)$ since:

$$P \circ \tilde{\gamma}(t_{i-1}) = \gamma(t_{i-1}) \in U$$

then $\tilde{\gamma}(t_{i-1}) \in V_\alpha$ where V_α is a sheet of $P^{-1}(U)$.

recall $P|_{V_\alpha} : V_\alpha \rightarrow U$ is homeo. define:

$$\tilde{\gamma}(s) = (P^{-1}|_{V_\alpha})(\gamma(s)) \quad s \in [t_{i-1}, t_i]$$

note $\tilde{\gamma}|_{[0, t_i]}$ cts by pasting lemma, and $\gamma = P \circ \tilde{\gamma}$. ┐

step 3: we prove uniqueness of $\tilde{\gamma}$.

┌ sps $\hat{\gamma}$ is another lifting of γ with $\hat{\gamma}(0) = e_0$. we show $\hat{\gamma}(s) = \tilde{\gamma}(s)$ for all $s \in [t_{i-1}, t_i]$ inductively.

sps $\hat{\gamma}|_{[0, t_{i-1}]} = \tilde{\gamma}|_{[0, t_{i-1}]}$. note $\hat{\gamma}([t_{i-1}, t_i])$ conn and $\hat{\gamma}(t_{i-1}) \in V_\alpha$ where V_α is the sheet we used to define $\tilde{\gamma}$ on $[t_{i-1}, t_i]$ in step 2.

it then follows that $\hat{\gamma}([t_{i-1}, t_i]) \subseteq V_\alpha$ by conn. since $\gamma = P \circ \hat{\gamma}$:

$$\hat{\gamma}(s) = (P^{-1}|_{V_\alpha})(\gamma(s)) = \tilde{\gamma}(s)$$

for all $s \in [t_{i-1}, t_i]$. ┐





Lemma 3.4: Path-homotopy lifting property

spcs $P : E \rightarrow X$ coving map, $x = P(e)$. if $F : I^2 \rightarrow X$ is a pathmap with $F(0,0) = x$, then there is a unique lifting $\tilde{F} : I^2 \rightarrow E$ which is a **pathmap** in E s.t. $\tilde{F}(0,0) = e$.

pf: very similar to the one above. not writing allat

Corollary 3.5

let $P : E \rightarrow X$ be a coving map and $x = P(e)$. if $\gamma_0 \simeq_p \gamma_1$, then $\tilde{\gamma}_0 \simeq_p \tilde{\gamma}_1$.

Theorem 3.6

$\pi_1(S^1) = \mathbb{Z}$.

dont rly wanna put this in a box

let $x_0 = (1,0) \in S^1$ and $P : \mathbb{R} \rightarrow S^1$ be the coving map given by:

$$P(s) = (\cos(2\pi s), \sin(2\pi s))$$

given $[\gamma] \in \pi_1(S^1, x_0)$, let $\tilde{\gamma} : I \rightarrow \mathbb{R}$ be the unique lifting of γ s.t. $\tilde{\gamma}(0) = 0$. define $\varphi : \pi_1(S^1, x_0) \rightarrow \mathbb{Z}$ given by:

$$\varphi([\gamma]) = \tilde{\gamma}(1) \in P^{-1}(x_0) = \mathbb{Z}$$

first, we show φ is well-defined.

by prev crll, if $\gamma_0 \simeq_p \gamma_1$, then $\tilde{\gamma}_0 \simeq_p \tilde{\gamma}_1$. in particular, $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.

next, φ is surjective.

fix $n \in \mathbb{Z}$. consider $\omega_n(s) = (\cos(2\pi sn), \sin(2\pi sn))$. then $\varphi([\omega_n]) = \tilde{\omega}_n(1) = n$.

next, φ is injective.





┌ sps $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$ s.t. $\varphi([\gamma_0]) = \varphi([\gamma_1])$. note $\tilde{\gamma}_0, \tilde{\gamma}_1 : I \rightarrow \mathbb{R}$ are paths in \mathbb{R} . since \mathbb{R} cvx, then $\tilde{\gamma}_0 \simeq_p \tilde{\gamma}_1$ via pathtopy $\tilde{F} : I^2 \rightarrow \mathbb{R}$. then, $P \circ F : I^2 \rightarrow S^1$ is a pathtopy from $P \circ \tilde{\gamma}_0 = \gamma_0$ to $P \circ \tilde{\gamma}_1 = \gamma_1$, so $[\gamma_0] = [\gamma_1]$. ┐

finally, φ is homo.

┌ given $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$, let:

$$\tilde{\gamma}_0(1) = n \quad \tilde{\gamma}_1(1) = m$$

then $\gamma_0 \simeq_p \omega_n$ and $\gamma_1 \simeq_p \omega_m$. we show $\varphi([\omega_n] * [\omega_m]) = n + m$.

let $\widetilde{\omega}_n$ be the lifting starting at 0 and ending at n . let $\widetilde{\omega}_m$ be the lifting starting at n . then, we have that:

$$\varphi([\omega_n] * [\omega_m]) = (\widetilde{\omega}_n * \widetilde{\omega}_m)(1) = n + m$$

note: check that $\widetilde{\omega}_n * \widetilde{\omega}_m$ is indeed a lifting of $\omega_n * \omega_m$. ┐

4 Retractions

Lec 22 - Aug 6 (Week 13)

last time, we showed that $\pi_1(S^1) = \mathbb{Z}$. today we examine some applications.

Definition 4.1

let $A \subseteq X$. we say A is [a] **retract** of X if there is a cts $r : X \rightarrow A$ s.t. $r(a) = a$ for all $a \in A$. we call the map r a **retraction**.

Proposition 4.2

if A a retract of X , then the homo given as

$$\iota_* : \pi_1(A, a_0) \rightarrow \pi_1(X, a_0)$$

induced by the inclusion $\iota : A \rightarrow X$ is injective.



**Proof.**

Source: Primary Source Material

let $r : X \rightarrow A$ be a retraction. note $r \circ \iota : A \rightarrow A$ is the identity, so $r_* \circ \iota_*$ is the trivial homo. since r_* is a left-inv of ι_* , we are then done. \blacksquare

Example 4.3

Source: Primary Source Material

S^1 is *not* a retract of $D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$.

by the prev thm, $\iota_* : \pi_1(S^1) \rightarrow \pi_1(D^2)$ would be inj, but:

$$\pi_1(S^1) = \mathbb{Z} \quad \pi_1(D^2) = \{e\}$$

so this is not possible.

Example 4.4

Source: Primary Source Material

S^1 is a retract of the “figure 8” space (i.e. $S^1 \vee S^1$).

[label each copy of S^1 as A, B resp. and the base pt as x_0 .] then the map r given by

$$r(x) = \begin{cases} x & x \in A \\ x_0 & x \in B \end{cases}$$

is a retraction.

Example 4.5

Source: Primary Source Material

$S^1 \vee S^1$ is *not* a retract of $D^2 \vee D^2$.

by contra, sps $r : D^2 \vee D^2 \rightarrow S^1 \vee S^1$ is a retraction. then:

$$D^2 \hookrightarrow D^2 \vee D^2 \longrightarrow S^1 \vee S^1 \longrightarrow S^1$$

would be a retraction $D^2 \rightarrow S^1$, a contradiction.



Definition 4.6

let $A \subseteq X$. we say A is a **deformation retract** if $\text{id} : X \rightarrow X$ is homotopic to a retraction via $F : X \times I \rightarrow X$ s.t. $F(a, t) = a$ for all $t \in I$ and $a \in A$.
the homotopy F is called a **deformation retraction**.

Example 4.7

Source: Primary Source Material

S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$.

take $F : \mathbb{R}^2 \setminus \{0\} \times I \rightarrow \mathbb{R}^2 \setminus \{0\}$ as:

$$F(x, t) = (1 - t)x + \frac{tx}{\|x\|}$$

this is a deformation retraction.

Example 4.8

Source: Primary Source Material

consider $X = \mathbb{R}^3 \setminus \{\lambda e_3\}$, or \mathbb{R}^3 without the z -axis. then, $\mathbb{R}^2 \setminus \{0\}$ is a deformation retract of X .

take $F((x, y, z), t) = (x, y, (1 - t)z)$.

Example 4.9

Source: Primary Source Material

let X be \mathbb{R}^2 minus two pts. then $S^1 \vee S^1$ is a deformation retract of X .

[u can just visualize this one tbh.]

Proposition 4.10

if A is a deformation retract of X and $a_0 \in A$, then the homomorphism $\iota_* : \pi_1(A, a_0) \rightarrow \pi_1(X, a_0)$ induced by $\iota : A \rightarrow X$ is iso.

**Proof.**

Source: Primary Source Material

it suffices to show ι_* surj.

fix $F : X \times I \rightarrow X$ deform retraction of X onto A , and $[\gamma] \in \pi_1(X, a_0)$. consider $G : I \times I \rightarrow X$ as:

$$G(s, t) = F(\gamma(s), t)$$

note G is a pathtopy from γ to some loop α in A :

$$G(0, t) = F(\gamma(0), t) = F(a_0, t) = a_0$$

for all $t \in I$. then:

$$\iota_*([\alpha]) = [\alpha] = [\gamma]$$

■

Corollary 4.11

$$\pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}.$$

ok lets prove brower fixed pt (for D^2) using algtop. step one:

Definition 4.12

a cts $f : X \rightarrow Y$ is **nullhomotopic** if $f \simeq e_{x_0}$, i.e. f homotopic to a constant map.

Proposition 4.13

let $h : S^1 \rightarrow X$. tfae:

- (i) h nulltopic
- (ii) there exists cts ext $k : D^2 \rightarrow X$ of h
- (iii) h_* is trivial homo



**Proof.**

Source: Primary Source Material

(i) \implies (ii)

┌

sps $h \simeq e_{x_0}$. let $F : S^1 \times I \rightarrow X$ be homotopy from h to e_{x_0} .

note $D^2 \cong (S^1 \times I)/(S^1 \times \{1\})$. consider $p : S^1 \times I \rightarrow D^2$ given as $p(x, t) = (1 - t)x$. this is a qmap.

F constant on $S^1 \times \{1\}$. by properties of quotients, there exists some cts $k : D^2 \rightarrow X$ s.t. $F = k \circ p$. then for $x \in S^1$, $k(x) = F(x, 0) = h(x)$.

└

(ii) \implies (iii)

┌

sps $k : D^2 \rightarrow X$ cts ext of h . let $\iota : S^1 \rightarrow D^2$ be inclusion. note $h = k \circ \iota$.

then $h_* = k_* \circ \iota_*$. note:

$$\iota_* : \pi_1(S^1) \rightarrow \pi_1(D^2) \quad \pi_1(S^1) = \mathbb{Z} \quad \pi_1(D^2) = \{e\}$$

└

(iii) \implies (i)

┌

sps h_* trivial. note S^2 is a quotient of I , since $S^1 = I/\{0, 1\}$ with:

$$x_0 = (1, 0) \quad p(s) = (\cos(2\pi s), \sin(2\pi s))$$

let $[p] \in \pi_1(S^1, x_0)$ and $h_*[p] = [e_{h(x_0)}]$. note $h_*([p]) = [h \circ p] =: [f]$.

fix pathcopy $F : I^2 \rightarrow X$ from f to $e_{h(x_0)}$. let $q(s, t) = (p(s), t)$; this is qmap from I^2 to $S^1 \times I$.

since F pathcopy, it is constant on pts identified by q . then, there exists some cts $G : S^1 \times I \rightarrow X$ s.t. $F = G \circ q$. to check homotopy, note that:

$$G(x, 0) = F(s, 0) = f(s) = h(p(s)) = h(x)$$

└

■



**Example 4.14**

Source: Primary Source Material

$\iota : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$, the inclusion map, is not nulltopic since $\iota_* : \mathbb{Z} \rightarrow \mathbb{Z}$ is the id, so nontrivial. for the same reason, $\text{id} : S^1 \rightarrow S^1$ is not nulltopic.

ok, back to fixed points. step two: vector fields on D^2 .. what.

Definition 4.15

a **vector field** on D^2 is a cts $\mathcal{V} : D^2 \rightarrow \mathbb{R}^2$. we say \mathcal{V} is **non-vanishing** if we have that $\mathcal{V}(x) \neq 0$ for all $x \in D^2$.

Proposition 4.16

if $\mathcal{V} : D^2 \rightarrow \mathbb{R}^2 \setminus \{0\}$ non-vanishing vecfield, then:

- (1) there exists $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$ for some $\alpha < 0$. that is, $\mathcal{V}(x)$ points directly inwards
- (2) there exists $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$ for some $\alpha > 0$, that is, $\mathcal{V}(x)$ points directly outwards

Proof.

Source: Primary Source Material

(2) follows from applying (1) to $-\mathcal{V}$.

by contradiction, sps no $\mathcal{V}(x)$ point directly inwards. consider the map given by $h = \mathcal{V}|_{S^1} : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$. then \mathcal{V} is a cts ext of h ; by prev. prop, h is nulltopic. we claim h homotopic to inclusion $\iota : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$.

consider $F(s, t) = (1 - t)h(s) + ts$. we check $F : S^1 \times I \rightarrow \mathbb{R}^2 \setminus \{0\}$, i.e., $F(s, t) \neq 0$. indeed, sps $F(s, t) = 0$. then $(1 - t)h(s) = -ts$. so $\mathcal{V}(s) = h(s) = -ts/(1 - t)$, so $\mathcal{V}(s)$ points inwards, contradiction.

but $\iota : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$ not nulltopic, so contradiction (again). ■




Theorem 4.17: Brouwer's Fixed Point Theorem (for the 2D disc)

if $f : D^2 \rightarrow D^2$ cts, then there is $x \in D^2$ s.t. $f(x) = x$.

Proof.

Source: Primary Source Material

by contra, sps $f(x) \neq x$ for all $x \in X$. let $\mathcal{V}(x) = f(x) - x$, so \mathcal{V} is non-vanishing vecfield. by prev thm, there is $\alpha > 0$ and $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$. , but $f(x) = (\alpha + 1)x \notin D^2$, a contradiction. ■

is brouwer fixed pt true for D^n ? yes [duh], but it requires more advanced algtop [x doubt]: homotopy theory and homotopy groups $\pi_n(X)$.

Lec 23 - Aug 8 (Week 13)

we now move on to borsuk ulam.

Definition 4.18

fix $x \in S^n$. we say $-x$ is the **antipode** of x .

a map $f : S^n \rightarrow S^n$ is **antipode-preserving** if for all x , $f(-x) = -f(x)$.

Proposition 4.19

if $h : S^1 \rightarrow S^1$ cts and antipode preserving, then h *not* nulltopic.

Proof.

Source: Primary Source Material

consider the cvrng map $q : S^1 \rightarrow S^1$ given by:

$$q(\cos \theta, \sin \theta) = (\cos 2\theta, \sin 2\theta) \quad q(z) = z^2$$

note $q(z) = q(-z)$ and is a qmap. consider $q \circ h : S^1 \rightarrow S^1$. this has the property:

$$(q \circ h)(-z) = q(-h(z)) = q(h(z)) = (q \circ h)(z)$$

note there exists cts k s.t. $q \circ h = k \circ q$. let $x_0 = (1, 0)$. we claim $k_* : \pi_1(S^1, x_0) \rightarrow \pi_1(S^1, x_0)$ is *not* trivial.





┌ consider the path $\tilde{\gamma}$ from x_0 to $-x_0$. then $\gamma = q \circ \tilde{\gamma}$ is a loop at x_0 , and:

$$k_*([\gamma]) = [k \circ \gamma] = [k \circ q \circ \tilde{\gamma}] = [q \circ h \circ \tilde{\gamma}]$$

it suffices to show $[q \circ h \circ \tilde{\gamma}] \neq [e_{x_0}]$.

sps otw, i.e. $q \circ h \circ \tilde{\gamma} \simeq_p e_{x_0}$. note $h \circ \tilde{\gamma}$ is the lifting of $q \circ h \tilde{\gamma}$ starting at x_0 . thus, the pathcopy lifts to a pathcopy from $h \circ \tilde{\gamma}$ to e_{x_0} . but:

$$h \circ \tilde{\gamma}(1) = h(-x_0) = -h(x_0) = -x_0 \neq x_0.$$

note we can assume $h(x_0) = x_0$. otw, consider the rotation $\rho : S^1 \rightarrow S^1$ with $\rho(h(x_0)) = x_0$. apply the result to $\rho \circ h$. thus $\rho \circ h$ not nulltopic, so h not nulltopic.

now, we claim $h_* : \pi_1(S^1, x_0) \rightarrow \pi_1(S^1, x_0)$ is not trivial.

┌ we know $k_* : \mathbb{Z} \rightarrow \mathbb{Z}$ nontrivial, thus must be inj. also, $q_* : \mathbb{Z} \rightarrow \mathbb{Z}$ given as $q_*(a) = 2a$, so q_* inj. therefore, $q_* \circ h_* = k_* \circ q_*$ must be inj. in particular, this means h_* cannot be trivial.

Proposition 4.20

there is no cts antipode-preserving map $g : S^2 \rightarrow S^1$.

Proof.

Source: Primary Source Material

sps by contradiction g is such a map. let $E \subseteq S^2$ be an equator of S^2 . consider $h := g|_E : S^1 \rightarrow S^1$. then h is cts and antipode-preserving.

by the prev thm, h is not nulltopic. however, h has cts ext to the upper hemisphere of S^2 , which is homeo to D^2 , a contradiction.



**Theorem 4.21: Borsuk-Ulam**

if $f : S^2 \rightarrow \mathbb{R}^2$ cts, there exists $x \in S^2$ s.t. $f(x) = f(-x)$.

Proof.

Source: Primary Source Material

spw otw, i.e. $f(x) \neq f(-x)$ for all x . consider $g : S^2 \rightarrow S^1$ given by:

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

then, notice that:

$$g(-x) = \frac{f(-x) - f(x)}{\|f(x) - f(-x)\|} = -g(x)$$

this contradicts the prev thm. ■

notice that it is not true for the torus - simply project onto \mathbb{R}^2 .

