MAT354
Complex Analysis

Class Lecture Notes

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Class Lectures

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Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

I Introduction

1 Preliminaries

Lec 1 - Sep 3 (Week 1)

i think sth abt the discriminant?

cardanos soln to simplified cubics $x^3 + cx + d = 0$ (circa 1535):

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}$$

tartaglia (circa 1539) said: let $x = t - \frac{b}{3a}$ in $ax^3 + bx^2 + cx + d = 0$. this yields a cubic $At^3 + Bt + C = 0$, letting us use cardanos soln.

but this means each cubic has exactly one real solution - that doesnt make sense! for instance, the roots of $x^3 - 3x = 0$ are $x = 0, \pm\sqrt{3}$.

lets try cardanos formula for $x^3 - 3x = 0$: given $c = -3, d = 0$, we have:

$$\begin{aligned} x &= \sqrt[3]{0 + \sqrt{0 + \frac{(-3)^3}{27}}} + \sqrt[3]{0 - \sqrt{0 + \frac{(-3)^3}{27}}} \\ &= \sqrt[3]{\sqrt{-1}} + \sqrt[3]{-\sqrt{-1}} \\ &= \sqrt[3]{\sqrt{-1}} - \sqrt[3]{\sqrt{-1}} \\ &= \sqrt[3]{i} - \sqrt[3]{i} \end{aligned}$$

note that i has 3 cubic roots - $x^3 - i = 0$ has 3 complex roots, given by $-i$, $\sqrt{\frac{3}{2} + \frac{1}{2}i}$, and $-\sqrt{\frac{3}{2} + \frac{1}{2}i}$.

i lwk not paying attn lol

complex analysis is *NOT* a generalization of real analysis! for instance, consider:

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \qquad f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

note f is differentiable at $x = 0$ - this can be seen using the limit defn. however, $f''(x)$ is clearly not defined at $x = 0$.

this doesnt happen in complex functions - differentiable functions have differentiable derivatives!

recall in real analysis, $f : \mathbb{R} \to \mathbb{R}$ is diff'ble at x_0 if there is L such that:

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} = L \qquad 0 < |x - x_0| < \delta \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon$$

a complex $F : \mathbb{C} \to \mathbb{C}$ is diff'ble at $z = z_0$ if there is L such that:

$$0 < |z - z_0| < \delta \implies \left| \frac{F(z) - F(z_0)}{z - z_0} - L \right| < \varepsilon$$

whats different? well... in $\mathbb{C}$, the "absolute value" is in fact a norm, and that means we approach within a *ball*, rather than only from two directions. as a result, this is a *strictly stronger condition* than real differentiability.

wasnt looking. notice $a := \mathfrak{Re}\, z = \frac{z+\overline{z}}{2}$ and $b := \mathfrak{Im}\, z = \frac{z-\overline{z}}{2i}$.

Example 1.1

Source: Primary Source Material

write equation of a line $ax + by = c$ with $a^2 + b^2 \neq 0$ in terms of z and $\overline{z}$:

$$a\left(\frac{z+\overline{z}}{2}\right) + b\left(\frac{z-\overline{z}}{2i}\right) = c \implies \left(\frac{a}{2} + \frac{b}{2i}\right) z + \left(\frac{a}{2} - \frac{b}{2i}\right) \overline{z} = c$$

still not looking

consider:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

it can be shown that $f^{(n)}(0) = 0$ for all n, so the taylor series at $x = 0$ is 0. yet the taylor series of f does not approach f on a nbhd of 0.

thus, a real function can be C^∞ but still not analytic (ie no taylor series exact expansion), but this is not the case for complex functions - differentiable functions are always analytic.

some properties:

$$\frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2} \qquad |z| = |\overline{z}| \qquad -|z| \leq \mathfrak{Re}\, z \leq |z| \qquad -|z| \leq \mathfrak{Im}\, z \leq |z|$$