

# Embedded Contact Homology of Prequantization Bundles

Jo Nelson & Morgan Weiler

Rice University

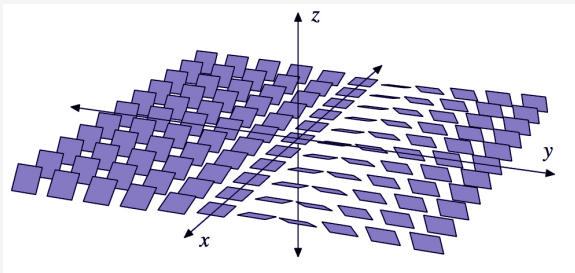
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<https://math.rice.edu/~jkn3/WHVSS-slides.pdf>

# Contact structures

## Definition

A **contact structure** is a maximally nonintegrable hyperplane field.

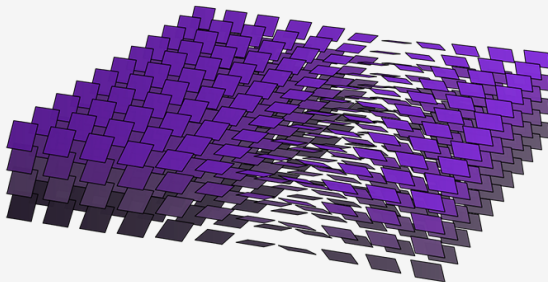


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# Contact structures

## Definition

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$$\xi = \ker(dz - ydx)$$

The kernel of a 1-form  $\lambda$  on  $Y^{2n-1}$  is a contact structure whenever

- $\lambda \wedge (d\lambda)^{n-1}$  is a volume form  $\Leftrightarrow d\lambda|_{\xi}$  is nondegenerate.

## Definition

The Reeb vector field  $R$  on  $(Y, \lambda)$  is uniquely determined by

- $\lambda(R) = 1$ ,
- $d\lambda(R, \cdot) = 0$ .

The **Reeb flow**,  $\varphi_t : Y \rightarrow Y$  is defined by  $\frac{d}{dt}\varphi_t(x) = R(\varphi_t(x))$ .

A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \quad (0.1)$$

and is **embedded** whenever (0.1) is injective.

Given an embedded **Reeb orbit**  $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow Y$ ,  
the linearized flow along  $\gamma$  defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \rightarrow (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$  is called the **linearized return map**.

If 1 is not an eigenvalue of  $d\varphi_T$  then  $\gamma$  is **nondegenerate**.

Nondegenerate orbits are either **elliptic** or **hyperbolic** according to whether  $d\varphi_T$  has eigenvalues on  $S^1$  or real eigenvalues.

$\lambda$  is **nondegenerate** if all Reeb orbits associated to  $\lambda$  are nondegenerate.

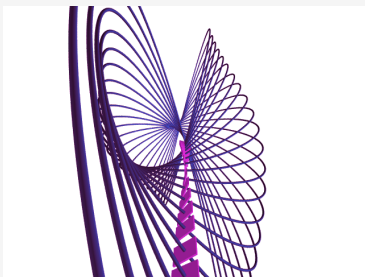
# Reeb orbits on $S^3$

$$S^3 := \{(u, v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \lambda = \frac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv).$$

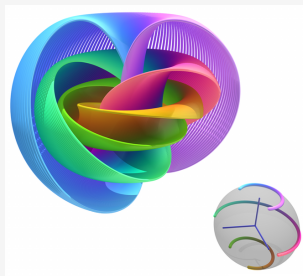
The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu \frac{\partial}{\partial u} - i\bar{u} \frac{\partial}{\partial \bar{u}} + iv \frac{\partial}{\partial v} - i\bar{v} \frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is  $\varphi_t(u, v) = (e^{it}u, e^{it}v)$ .



Patrick Massot



Niles Johnson,  $S^3/S^1 = S^2$

# The Hopf Fibration



**Niles Johnson**

<http://www.nilesjohnson.net>

## Theorem (Boothby-Wang construction '58)

*Let  $(\Sigma_g, \omega)$  be a Riemann surface and  $e$  a negative class in  $H_2(\Sigma_g; \mathbb{Z})$ . Let  $p : Y \rightarrow \Sigma_g$  be the principal  $S^1$ -bundle with Euler class  $e$ . Then there is a connection 1-form  $\lambda$  on  $Y$  whose Reeb vector field  $R$  is tangent to the  $S^1$ -action.*

- $(Y, \lambda)$  is the **prequantization bundle** over  $(\Sigma_g, \omega)$ .
- The Reeb orbits of  $R$  are the  $S^1$ -fibers of this bundle.
- The Reeb orbits of  $R$  are degenerate.
- $d\lambda = p^*\omega$
- $p_*\xi = T\Sigma_g$

# Perturbed Reeb dynamics of prequantization bundles

Use a Morse-Smale  $H : \Sigma \rightarrow \mathbb{R}$ ,  $|H|_{C^2} < 1$  to perturb  $\lambda$ :

$$\lambda_\varepsilon := (1 + \varepsilon \mathfrak{p}^* H) \lambda$$

The perturbed Reeb vector field is

$$R_\varepsilon = \frac{R}{1 + \varepsilon \mathfrak{p}^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon \mathfrak{p}^* H)^2}$$

where  $\tilde{X}_H$  is the horizontal lift of  $X_H$  to  $\xi$ . If  $p \in \text{Crit}(H)$  then  $X_H(p) = 0$ .

The **action** of a closed orbit  $\gamma$  is  $\mathcal{A}(\gamma) := \int_\gamma \lambda_\varepsilon$ .

Fix  $L > 0$ .  $\exists \varepsilon > 0$  such that if  $\gamma$  is an orbit of  $R_\varepsilon$  and

- if  $\mathcal{A}(\gamma) < L$  then  $\gamma$  is nondegenerate and projects to  $p \in \text{Crit}(H)$ ;
- if  $\mathcal{A}(\gamma) > L$  then  $\gamma$  loops around the tori above the orbits of  $X_H$ , or is a larger iterate of a fiber above  $p \in \text{Crit}(H)$ .

# Fiber orbits of prequantization bundles

Recall

$$R_\epsilon = \frac{R}{1 + \epsilon \mathfrak{p}^* H} + \frac{\epsilon \tilde{X}_H}{(1 + \epsilon \mathfrak{p}^* H)^2}$$

Denote the  $k$ -fold cover projecting to  $p \in \text{Crit}(H)$  by  $\gamma_p^k$ .

We have

$$CZ_\tau(\gamma_p^k) = RS_\tau(\text{fiber}^k) - \frac{\dim(\Sigma)}{2} + \text{ind}_p(H).$$

Using the constant trivialization of  $\xi = \mathfrak{p}^* T\Sigma$ ,  $RS_\tau(\text{fiber}^k) = 0$ .

Thus

$$CZ_\tau(\gamma_p^k) = \text{ind}_p(H) - 1.$$

# Fiber orbits of prequantization bundles

Recall

$$CZ_\tau(\gamma_p^k) = \text{ind}_p(H) - 1$$

Only positive hyperbolic orbits have even  $CZ$ .

If  $\text{ind}_p(H) = 1$  then  $\gamma_p$  is positive hyperbolic.

Since  $\mathfrak{p}$  is a bundle, all linearized return maps are close to  $Id$ .

Hence no negative hyperbolic orbits.

If  $\text{ind}_p(H) = 0, 2$  then  $\gamma_p$  is elliptic.

Assume  $H$  is perfect. Denote

- the index zero elliptic orbit by  $e_-$
- the index two elliptic orbit by  $e_+$ ,
- the hyperbolic orbits by  $h_1, \dots, h_{2g}$ .

**Embedded contact homology (ECH)** is a Floer theory for closed  $(Y^3, \lambda)$  and  $\Gamma \in H_1(Y; \mathbb{Z})$ .

For nondegenerate  $\lambda$ , the chain complex  $ECC_*(Y, \lambda, \Gamma, J)$  is generated as a  $\mathbb{Z}_2$  vector space by **orbit sets**  $\alpha = \{(\alpha_i, m_i)\}$ , which are finite sets for which:

- $\alpha_i$  is an embedded Reeb orbit
- $m_i \in \mathbb{Z}_{>0}$
- $\sum_i m_i [\alpha_i] = \Gamma$
- If  $\alpha_i$  is hyperbolic,  $m_i = 1$ .

The **grading**  $*$  comes from the relative **ECH index**  $I(\alpha, \beta)$ , a combination of  $c_1(\ker \lambda)$ ,  $CZ(\alpha_i^k)$ ,  $CZ(\beta_j^k)$ , and the relative self-intersection.

# Almost complex structures and $\partial^{ECH}$

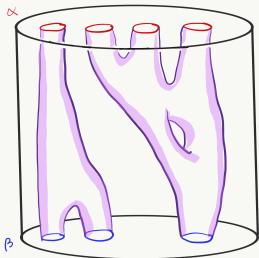
A  $\lambda$ -**compatible almost complex structure** is a complex structure  $J$  on  $T(\mathbb{R} \times Y)$ , for which:

- $J$  is  $\mathbb{R}$ -invariant
- $J\xi = \xi$ , positively with respect to  $d\lambda$
- $J(\partial_s) = R$ , where  $s$  denotes the  $\mathbb{R}$  coordinate

$\langle \partial^{ECH} \alpha, \beta \rangle$  counts **currents**, disjoint unions of  **$J$ -holomorphic curves**

$$u : (\dot{\Sigma}, j) \rightarrow (\mathbb{R} \times Y, J), \quad du \circ j = J \circ du$$

which are asymptotically cylindrical to orbit sets  $\alpha$  and  $\beta$  at  $\pm\infty$ .



*For generic  $J$ ,  
ECH index one yields  
somewhere injective.*

-Hutchings' Haiku

# Embedded contact homology differential $\partial^{ECH}$

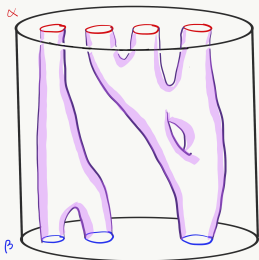
Theorem (Hutchings-Taubes '09)

$(\partial^{ECH})^2 = 0$ , so  $(ECC_*(Y, \lambda, \Gamma, J), \partial^{ECH})$  is a chain complex.

Theorem (Taubes, Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda)

*The homology depends only on  $(Y, \ker \lambda, \Gamma)$ .*

We denote the homology by  $ECH_*(Y, \ker \lambda, \Gamma)$ .



*Dee squared is zero;  
obstruction bundle gluing  
is complicated.*

-Hutchings-Taubes' Haiku

## Theorem (Nelson-Weiler, 90%)

Let  $(Y, \xi = \ker \lambda)$  be a prequantization bundle over  $(\Sigma_g, \omega)$ . Then

$$\bigoplus_{\Gamma \in H_1(Y; \mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$$

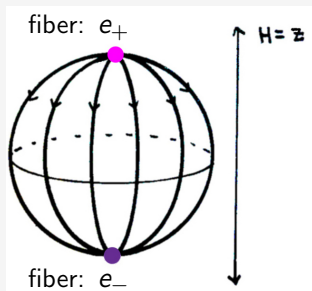
Inspired by the 2011 PhD thesis of Farris.

- 1 The critical points of a perfect  $H$  form a basis for  $H_*(\Sigma_g; \mathbb{Z}_2)$ . The generators of  $ECH$  are of the form  $e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$  where  $m_i = 0, 1$ , so correspond to a basis for  $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$ .
- 2 We will prove  $\partial^{ECH}$  only counts cylinders corresponding to Morse flows on  $\Sigma_g$ , therefore  $\partial^{ECH}(e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+})$  is a sum over all ways to apply  $\partial^{Morse}$  to  $h_i$  or  $e_+$ .

# Our favorite fibration on $S^3$

## Example $(S^3, \lambda)$

The ECH of  $S^3$  is the  $\mathbb{Z}_2$ -vector space generated by terms  $e_-^{m-} e_+^{m+}$ , where  $|e_-| = 2, |e_+| = 4$ . Note that  $*$  is not the grading on  $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$ , since  $|e_-^2| = 6$ .



The fibers above the critical points of the height function on  $S^2$  represent  $e_{\pm}$ .

We have  $\partial^{ECH} = 0$  because  $\partial^{Morse} = 0$ .

# Lens spaces $L(k, 1)$

$L(k, 1)$  is the total space of the prequantization bundle with Euler number  $-k$  on  $S^2$ .

Corollary (Nelson-Weiler, 95%)

*With its prequantization contact structure  $\xi_k$ ,*

$$ECH_*(L(k, 1), \xi_k, \Gamma) \cong \begin{cases} \mathbb{Z}_2 & \text{if } * \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{else} \end{cases}$$

*for all  $\Gamma \in H_1(L(k, 1); \mathbb{Z})$ .*

# Finer points of the isomorphism

Fix a negative Euler class  $e$ . For  $\Gamma \in \{0, \dots, -e - 1\}$ ,

$$ECH_*(Y, \xi, \Gamma) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Lambda^{\Gamma - ne} (H_*(\Sigma_g; \mathbb{Z}_2))$$

## Proposition (Nelson-Weiler)

Let  $\alpha = e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$  and let  $\beta = e_-^{n_-} h_1^{n_1} \cdots h_{2g}^{n_{2g}} e_+^{n_+}$ .

Let  $N = n_- + n_+ + \sum_j n_j$  and  $m = \frac{(m_- + m_+ + \sum_i m_i) - N}{-e}$ . Then

$$I(\alpha, \beta) = (2 - 2g)m - m^2 e + 2mN + m_+ - m_- - n_+ + n_-$$

Using this formula, we obtain

$$I(e_+^{N+e}, e_-^N) = 2g - 2$$

# $ECH_*(Y, \xi, 0)$ for $g = 2, e = -1$

Recall  $I(e_+^{N+e}, e_-^N) = 2g - 2$ . Set  $*(\alpha) = I(\alpha, \emptyset)$ .

	$* = -2$	$* = -1$	$* = 0$	$* = 1$	$* = 2$	$* = 3$	$* = 4$
$\Lambda^0$			$\emptyset$				
$\Lambda^1$	$e_-$	$h_i$	$e_+$				
$\Lambda^2$	$e_-^2$	$e_- h_i$	$e_- e_+$	$h_i e_+$	$e_+^2$		
			$h_i h_j$				
$\Lambda^3$			$e_-^3$	$e_-^2 h_i$	$e_-^2 e_+$	$e_- h_i e_+$	$e_- e_+^2 \dots$
				$e_- h_i h_j$	$h_i h_j h_k$	$h_i h_j e_+$	
$\Lambda^4$							$e_-^4 \dots$

## Theorem (Nelson-Weiler, 90%)

Let  $(Y, \xi = \ker \lambda)$  be a prequantization bundle over  $(\Sigma_g, \omega)$ . Then

$$\bigoplus_{\Gamma \in H_1(Y; \mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$$

- 1 There exists  $\varepsilon > 0$  so that the generators of  $ECC_*^L(Y, \lambda_\varepsilon, J)$  consist solely of orbits which are fibers over critical points.
- 2 Prove that  $\partial^{ECH, L}$  only counts cylinders which are the union of fibers over Morse flow lines in  $\Sigma$ .
- 3 Finish with a direct limit argument, sending  $\varepsilon \rightarrow 0$  and  $L \rightarrow \infty$ , in addition to the isomorphism with Seiberg-Witten.

# Pseudoholomorphic Cylinders

- Pseudoholomorphic cylinders correspond to Floer trajectories on  $\Sigma_g$  (Moreno, Siefring)
- Floer trajectories on  $\Sigma_g$  correspond to Morse flows (Floer, Salamon-Zehnder)
- Cylinder counts permit use of fiberwise  $S^1$ -invariant  $J$ , even for multiply covered curves, by automatic transversality (Wendl)

## Theorem (N. 2017)

*The cylindrical contact homology chain complex of a prequantization bundle over  $\Sigma_g$  is generated by infinitely many copies of the Morse complex of  $\Sigma_g$ , and on each copy the cylindrical differential agrees with the Morse differential.*

# Higher genus curve counting difficulties (Farris)

- Can count cylinders using the complex structure  $J_{\Sigma_g} = \mathfrak{p}^* j_{\Sigma_g}$ , the  $S^1$ -invariant lift of  $j_{\Sigma_g}$ .

(YAY! Automatic transversality!)

- $J_{\Sigma_g}$ -holomorphic cylinders correspond to Morse trajectories on  $\Sigma_g$ .

- **Cannot use**  $J_{\Sigma_g}$  for higher genus curves!

$J_{\Sigma_g}$  cannot be independently perturbed at the intersection points of  $\pi_Y u$  with a given  $S^1$ -orbit by an  $S^1$ -invariant perturbation.

(YIKES!  $J_{\Sigma_g}$  is not typically regular!)

- There will always be a regular  $J$  for moduli spaces of higher genus curves, but we cannot assume  $J$  is  $S^1$ -invariant.

(CURVE COUNTING NO LONGER OBVIOUS...)

# Domain dependent almost complex structures (Farris)

**Forsake  $J_{\Sigma_g}$  for an  $S^1$ -invariant domain dependent perturbation,**

$$\{J_{\Sigma_g}^z\}_{z \in \dot{\Sigma}}$$

- Akin to time-dependence in Hamiltonian Floer theory.
- Implicit function theorem relates counts of nearby moduli spaces

**Higher genus curves and multiply covered cylinders do not contribute to  $\partial^{ECH}$**

- Transversality guarantees index 1 holomorphic curves do not exist unless they are fixed by the  $S^1$ -action.
- Otherwise the curve lives in a moduli space of dimension  $\geq 2$ .
- But  $\langle \partial^{ECH} \alpha, \beta \rangle$  only counts curves where  $\pi_Y \circ u$  is isolated.
- So we only count cylinders projecting to Floer trajectories.

**Remaining issue** (modulo direct limits):

- Hutchings set up ECH with a domain independent  $J$ ...

# One parameter family of complex structures

Consider  $\{\mathfrak{J}_t\}_{t \in [0,1]}$  a family of  $S^1$ -invariant domain dependent almost complex structures in  $\mathbb{R} \times Y$ ,

$$\mathfrak{J}_0 := \{J_{\Sigma_g}^Z\}_{z \in \dot{\Sigma}}$$

$$\mathfrak{J}_1 := J \in \mathcal{J}^{reg}(Y, \lambda).$$

## Lemma

*For generic paths, the moduli space  $\mathcal{M}_t = \mathcal{M}(\alpha, \beta, \mathfrak{J}_t)$  is cut out transversely save for a discrete number of times  $t_0, \dots, t_\ell \in (0, 1)$ .*

*For each such  $t_i$ ,  $\partial^{ECH}$  can change either by*

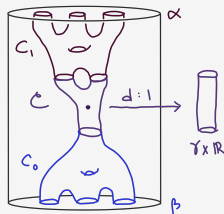
- *creation/destruction of a pair of oppositely signed curves;*
- *an “ECH handleslide.”*

*In either case, the homology is unaffected. PHEW!*

# Handleslides do not impact curve counts

At a handleslide  $t_i$ ,  $\{C_k \mid \text{ind}_{\text{Fred}}(C_k) = 1\}$  breaks into a building with:

- an index 1 curve  $C_1$  at top (or bottom)
- branched covers  $\mathcal{C}$  of  $\gamma \times \mathbb{R}$  with  $\text{ind}_{\text{Fred}}(\mathcal{C}) = 0$
- an index 0 curve  $C_0$  at bottom (or top)



*Branched covers cannot appear as the top-most or bottom-most level.*  
(Hutchings - N '16, Cristofaro-Gardiner - Hutchings - Zhang)  
*Hooray! We can invoke obstruction bundle gluing...*

$$\#\mathcal{M}(\alpha, \beta, \mathfrak{J}_{t_i+\epsilon}) = \#\mathcal{M}(\alpha, \gamma, \mathfrak{J}_{t_i-\epsilon}) + \#G(C_1, C_0) \cdot \#\mathcal{M}(\gamma, \beta, \mathfrak{J}_{t_i}),$$

OBG gives a combinatorial formula for  $\#G \in \mathbb{Z}$ , based on  
the partitions at  $-\infty$  ends of  $C_1$ ,  
the partitions at  $+\infty$  ends of  $C_0$ .

No need to explicitly compute  $\#G$  as inductively  $\#\mathcal{M}(\gamma, \beta, \mathfrak{J}_{t_i}) = 0!$

# Filtrations and computations

There is no geometric Morse-Bott ECH.

Denote by  $ECH_*^L(Y, \lambda_\varepsilon, \Gamma)$  the homology of the chain complex of ECH generators with action  $\leq L$ . (It's independent of  $J$ .)

Hutchings-Taubes '13: Cobordism and inclusion maps give us

$$\begin{array}{ccc} ECH_*^L(Y, \lambda_\varepsilon, \Gamma) & \longrightarrow & ECH_*^L(Y, \lambda_{\varepsilon'}, \Gamma) \\ \downarrow & & \downarrow \\ ECH_*^{L'}(Y, \lambda_\varepsilon, \Gamma) & \longrightarrow & ECH_*^{L'}(Y, \lambda_{\varepsilon'}, \Gamma) \end{array} \quad \begin{array}{l} \text{which commute,} \\ \text{for } \varepsilon' < \varepsilon, L' > L. \end{array}$$

We can now compute

$$\lim_{\varepsilon \rightarrow 0, L \rightarrow \infty} ECH_*^L(Y, \lambda_\varepsilon, \Gamma, J) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2). \quad (0.2)$$

That the LHS of (0.2) is  $ECH_*(Y, \xi, \Gamma)$  uses a similar filtration on Seiberg-Witten Floer homology from Hutchings-Taubes.

There is a degree  $-2$  map

$$U : ECC_*(Y, \xi, \Gamma) \rightarrow ECC_{*-2}(Y, \xi, \Gamma)$$

which counts  $J$ -holomorphic curves passing through a base point.

$U$  is equivalent to the  $U$  maps on Seiberg-Witten and Heegaard Floer homologies.

In the case of prequantization bundles, we expect  $U$  to count index 2 Morse flow lines and sections of the  $\mathbb{C}^*$  bundle  $\mathbb{R} \times Y \rightarrow \Sigma_g$ .

$U$  is Useful:

- Find index 2 holomorphic curves, since  $U$  is an invariant;
- **ECH capacities**, which obstruct symplectic embeddings;
- Proving stabilization results.

From Seiberg-Witten and Heegaard Floer homologies we know  $U$  is an isomorphism if  $*$  is large enough. Therefore:

**Theorem (Nelson-Weiler, 90%)**

*If  $e = -1$  and  $g > 1$ , then for  $*$  large enough,*

$$ECH_*(Y, \xi) \cong \mathbb{Z}_2^{2^{2g}-1}.$$

*and  $U$  is an isomorphism.*

We expect to prove this theorem entirely in ECH once we can compute the  $U$  map.

### Proposition (Colin-Honda '13)

*If  $\phi \in \text{Mod}(\Sigma_g)$  is periodic, then  $(Y, \xi)$  is supported by an open book decomposition with page  $\Sigma_g$  and monodromy  $\phi$  and is a Seifert fiber space over the orbifold  $\Sigma_g/\phi$ . There is a contact form for  $\xi$  whose Reeb vector field is tangent to the fibers.*

We will generalize our prequantization methods to circle bundles over orbifolds to understand the dynamics of symplectomorphisms which are freely homotopic to  $\phi$ , extending the Calabi invariant bounds in Weiler's thesis to genus 0 open books.

