Mathematics for Computer Science(30470023) Spring 2020

Lecture 8 — April 13, 2020

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1 Overview

Last time we came across the up-down permutation problem, and got the exponential generating function. To compute the Taylor expansion of it, this class we introduce a strong tool "Cauchy's Residue Theorem" (and for the preparations, also a few other concepts). Then, we apply this theorem to solve two examples, one of which is the Taylor expansion of exponential generating function of the up-down permutation sequence.

2 Review: Counting the number of up-down permutation

Consider "exponential generating function" which we obtained last week. We define b_n as follows:

$$b_n = \frac{a_n}{n!}.$$

Then,

$$B(x) = \sum_{n \geq 0} b_n x^n = \tan(x).$$

All we need to do is to find out the Taylor expansion of $B(x)$.

3 Complex analysis

First, we need to introduce the concept of Pole singularity.

Definition 3.1 (Pole singularity). For a function $f(z)$, if it can be expressed as follows:

$$f(z) = \sum_{-l \leq m < 0} c_m (z - z_0)^m + \sum_{n \geq 0} c_n (z - z_0)^n,$$

where

$$\sum_{n \geq 0} c_n (z - z_0)^n$$

converges when

$$0 < |z - z_0| < \epsilon. \qquad \diamondsuit$$

This form is called Pole singularity. Here c_{-1} is called residue.

Definition 3.2 (simple pole). $f(z)$ is called simple pole at $z = z_0$ if $l = 1$, i.e.

$$f(z) = \frac{c_{-1}}{z - z_0} + \sum_{n \geq 0} c_n (z - z_0)^n$$

◇

Theorem 3.3 (Calculate the residue value). If we know f has pole singularity at z_0, then the residue of f at z_0 can be calculated by:

$$c_{-1} = \lim_{z \to z_0} (z - z_0) f(z).$$

If the limit exists and is finite, we know that this is a simple pole, and we get the residue value. ◇

Example 3.4 (Can't apply Thm3.3). Consider the following function: for $z_0 \in \mathbb{C}$,

$$f(z) = \frac{1}{(z - z_0)^3} - \frac{5}{(z - z_0)^2} + \frac{7}{z - z_0} + \sum_{n \geq 0} c_n (z - z_0)^n.$$

It has the residue on z_0, but we cannot get it from the theorem above because

$$\lim_{z \to z_0} (z - z_0) f(z) \to \infty.$$

When it comes to a not simple pole case, we have to compute the residue by expanding the function.

Example 3.5 (Calculate directly). Consider the function

$$f(z) = \frac{e^z}{(z - i)^2}.$$

We want to calculate its residue directly. Let

$$z = z_0 + \Delta.$$

Thus

$$f(z) = \frac{e^{z_0 + \Delta}}{\Delta^2} = e^i \frac{1 + \Delta + \frac{\Delta^2}{2!} + \cdots}{\Delta^2}$$

and

$$f(z) = e^i \frac{1}{\Delta^2} + e^i \frac{1}{\Delta} + e^i (\frac{1}{2!} + \frac{1}{3!} \Delta + \cdots).$$

Here we get

$$c_{-1} = e^i,$$

that is the residue of f at $z = i$.
We don't really care the other values, but need to write them out in order to determine the residue.

Next, we introduce two observations about (complex) integration.

Fact 3.6.

$$\left| \int_{\mathbb{P}} f(z) \mathrm{d}z \right| \leq \max\{|f(z)| : z \in \mathbb{P}\} \cdot \text{length of curve } \mathbb{P}.$$

◇

Proof. For any partition $z_1, z_2, ..., z_n,\ z_i \in P$ of the curve P,

$$\left|\sum_{i=1}^{n-1} f(z_i)(z_{i+1} - z_i)\right| \leq \max\{|f(z)| : z \in \mathbb{P}\} \cdot \text{length of curve } \mathbb{P}$$

Let $n \to \infty$, and we get the result. □

Example 3.7 (Homework last week). Calculate the following function, where $\mathbb{P}$ is the unit counter-clockwise circle around $(0, 0)$.

$$g(k) = \int_{\mathbb{P}} z^k \mathrm{d}z = ?$$

The result is

$$\begin{cases} g(k) = 0 & k \geq 0 \\ g(k) = 2\pi i & k = -1 \\ g(k) = 0 & k < -1. \end{cases} \tag{1}$$

And we noticed that this integration has something to do with the residue of $f(z)$. That is true, and is proved by Cauchy. Before showing that theorem, we have to make one more definition.

Definition 3.8 (Simple Curve). A curve is <u>simple</u> if the curve does not cross itself, except possibly at end points when it is a closed curve. ♢

Theorem 3.9 (Cauchy's Residue Theorem). Let $\mathbb{P}$ be a simple closed curve (counter-clockwise). Let f be differntiable everywhere inside (and on) the curve, except for a finite number of pole sigularities: z_j with residue r_j. Then the integration has the following property:

$$\frac{1}{2\pi i} \oint_{\mathbb{P}} f(z) \mathrm{d}z = \sum_{j=1}^{m} r_j.$$

♢

So if the residues sum to 0 in the areas between $\mathbb{P}$ and $\mathbb{P}'$ (which is another simple closed counter-clockwise curve), then

$$\oint_{\mathbb{P}} f(z) \mathrm{d}z = \oint_{\mathbb{P}'} f(z) \mathrm{d}z.$$

Example 3.10. Calculate

$$\int_{-\infty}^{+\infty} \frac{1}{1 + x^4} \mathrm{d}x$$

using Cauchy's Residue Theorem. Let $\mathbb{P}$ be a (big) semi-circle on the upper half complex plane with radius R, and $\mathbb{P}'$ be the upper circle, we have

$$\int_{-R}^{+R} \frac{1}{1 + x^4} \mathrm{d}x + \int_{\mathbb{P}'} \frac{1}{1 + x^4} \mathrm{d}x = 2\pi i \sum r_j.$$

Now, what is the singularity? That question is easy ,there are only 4 on the plane. 2 of them are on the upper half plane.

$$z^4 = -1$$

$$z = e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{-i\pi}{4}}, e^{\frac{-3i\pi}{4}}$$

Using L'Hospital's rule, we get:

$$r_1 = \lim_{z \to z_1} \frac{z - z_1}{1 + z^4} = \frac{1}{4z_1^3},$$

and

$$r_2 = \lim_{z \to z_2} \frac{z - z_2}{1 + z^4} = \frac{1}{4z_2^3}.$$

Let $\mathbb{P}'$ is the upper circle, then when $R \to \infty$,

$$\left| \int_{\mathbb{P}'} \frac{1}{1 + x^4} \mathrm{d}x \right| \leq max\{ \left| \frac{1}{1 + x^4} \right| | x \in \mathbb{P}' \} \cdot (length\ of\ P') \leq \frac{\pi R}{R^4} \to 0.$$

So we get

$$\oint_{\mathbb{P}} \frac{1}{1 + z^4} \mathrm{d}z = 2\pi i \frac{1}{4} (\frac{1}{z_1^3} + \frac{1}{z_2^3}) = \frac{\pi}{\sqrt{2}}.$$

Let $R \to \infty$ and we get the answer:

$$\int_{-\infty}^{+\infty} \frac{1}{1 + x^4} \mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

4 Taylor expansion of $\tan(x)$

We first need to determine $\tan(z)$, z on the complex plane, at which z does $\tan(z)$ has singularity. We perform some transformation to make it simpler.

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1}{i}(1 - \frac{2e^{-iz}}{e^{iz} + e^{-iz}}) = \frac{1}{i}(1 - \frac{2}{e^{2iz} + 1})$$

We only need to find out the roots of $e^{2iz} + 1 = 0$. They turn out to be:

$$2z = (2m + 1)\pi, m \in \mathbb{Z}.$$

Then we define

$$z_m = (m + \frac{1}{2})\pi, m \in \mathbb{Z}.$$

Now, we can get to solve the original problem. Our plan is shown as follows. We first introduce a new function:

Observation 4.1. Let $f(z) = \tan(z)$, then $\frac{f(z)}{z^{n+1}}$ have poles in:

$$\begin{cases} z_* = 0, \\ z_j = (j + \frac{1}{2})\pi. \end{cases} \tag{2}$$

(One more than $\tan(z)$.) ◇

By Cauchy's Residue Theorem, we have

$$\frac{1}{2\pi i}\oint_{\mathbb{P}}\frac{f(z)}{z^{n+1}}\mathrm{d}z=\sum r_i.$$

$f(z)$ has a Tayloy expansion near 0:

$$f=\sum_{k\geq 0} b_k x^k.$$

So that we have:

$$\frac{1}{2\pi i}\oint_{\mathbb{P}:\text{near }0}\frac{f(z)}{z^{n+1}}\mathrm{d}z=b_n.$$

Then we have

$$\frac{1}{2\pi i}\oint_{\mathbb{P}}\frac{f(z)}{z^{n+1}}\mathrm{d}z=b_n+\sum\frac{\text{residue of }\tan(z)\text{ at }z_j}{z_j^{n+1}}.$$

Next, we solve it by two steps.
1 Determine residue of $\tan(z)$ at z_j

Claim 4.2. Residue of $\tan(z)$ at z_j is -1. ♢

Proof. Homework. □

2 Prove $|\tan(z)|=O(1)$ for z at $\mathbb{P}$. The curve $\mathbb{P}$ is a symmetric square that does not come near the singularities:

$$z=\pm m\pi+iy, y\in[-m\pi,+m\pi], z=x\pm im\pi, x\in[-m\pi,+m\pi].$$

Fact 4.3.

$$|\tan(z)|<5$$

for any $z\in\mathbb{P}$. $\mathbb{P}$ is the square curve defined above. ♢

Proof. Homework. □

Then when $R_m\to\infty$,

$$\left|\frac{1}{2\pi i}\oint_{\mathbb{P}}\frac{f(z)}{z^{n+1}}\mathrm{d}z\right|<\frac{5}{2\pi}\frac{2\pi R_m}{R_m^{n+1}}\to 0.$$

As a result,

$$b_n=-\sum_m\frac{(-1)}{z_m^{n+1}}=\sum_m\frac{1}{((m+\frac{1}{2})\pi)^{n+1}}.$$

Then we make some calculation to see if this result make sense. First, we noticed that when $n\in 2\mathbb{Z}$,

$$b_n=0.$$

That is good, for it is consistent with $\tan(z)$ being an odd function.
When $n\in 2\mathbb{Z}+1$, we have

$$b_n=2\sum_{m\geq 0}\frac{1}{((m+\frac{1}{2})\pi)^{n+1}}.$$

The summation is hard to compute, however, we can get calculate it by brute force when n is small, and we can get a asymptotical form when n is large.

Theorem 4.4 (The formula of b_n). For $n \in 2\mathbb{Z}+1$,

$$b_n = 2(\frac{2}{\pi})^{n+1}(\frac{1}{1^{n+1}} + \frac{1}{3^{n+1}} + \cdots). \qquad \diamondsuit$$

When $n \to \infty$,

$$b_n \approx 2(\frac{2}{\pi})^{n+1}.$$

We have an explicit series for any b_n, For small n, one can manually derive the summation. So the number of up-down permutation is

$$a_n = n!b_n \approx 2(\frac{2}{\pi})^{n+1}n!.$$

Observation 4.5. We have two different ways to compute a_n, and when n is small, we will get some interesting results. By Taylor expansion:

$$a_1 = b_1 = 1.$$

By Thm 4.6,

$$b_1 = 2(\frac{2}{\pi})^2(\frac{1}{1^2} + \frac{1}{3^2} + \cdots).$$

Combine these two, we get:

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}.$$

We can use this to derive that:

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6}. \qquad \diamondsuit$$