Mathematics for Computer Science (30470023) Spring 2020

Lecture 3 — Mar 2, 2020

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1 Overview

In this lecture we discribe the solution to the Online Auction Problem. The surprising result is that it is the best to skip first $\frac{1}{\mathrm{e}}$ offers. Then we introduce the concept of conditional probability, and also the distributive principle as well as the chain rule. Finally we introduce the expectation for a random variable.

2 The Online Auction Problem (Secretary's Problem/Beauty contest)

In last lecture we introduced the Online Auction Problem:

Problem 2.1. *A music concert ticket is selling. There is a stream of n offers with prices $x_1, x_2, \cdots, x_n$, for what strategy can we get the best price with highest probability?*

Remark 2.2. The strategy is non-regrettable, means that we cannot go back to the previous offers we rejected.

2.1 Formalizing

We can simply assume that $x_1, x_2, \cdots, x_n$ is a permutation of $1, 2, \cdots, n$. The probability space $\mathbb{P} = (\mathcal{U}, p)$ is denoted by

- $\mathcal{U}$ is the set of all permutations of $\{1, 2, \cdots, n\}$;
- For all $u \in \mathcal{U}$, $p(u) = \dfrac{1}{n!}$;
- T is the event of accepting the best offer (i.e. the price n).

For integer $k (1 \leq k < n)$, we denote strategy k as

1. Skip the first k offer;
2. For $j > k$, take the first x_j, such that $x_j > \max\{x_1, x_2, \cdots, x_k\}$.

Then the probability of getting the best price $p_{n,k} = \frac{|T|}{|\mathcal{U}|} = \frac{|T|}{n!}$. To calculate it, we should first obtain the properties of the permutations where we can reach the best price.

A permutation $u = (x_1, x_2, \cdots, x_n)$ where $x_j = n$ is in T iff

1. $j > k$;
2. $\max\{x_1, x_2, \cdots, x_{j-1}\} = \max\{x_1, x_2, \cdots, x_k\}$.

To continue the calculation, we first introduce two basic principles.

2.2 Two basic principles

Theorem 2.3 (Addition principle). If $S = S_1 \cup S_2 \cup \cdots \cup S_m$ is a disjoint union, then

$$|S| = \sum_{i=1}^{m} |S_i|.$$

Remark 2.4. As a simple application for addition principle, we have $\Pr(S) \le \sum_{i=1}^{m} \Pr(S_i)$, the equality holds iff the S_i are disjoint.

Theorem 2.5 (Multiplication principle). If S has items characterized by $s \in S, s = (i_1, i_2, \cdots, i_l)$, and $1 \le i_j \le c_j$ for all $1 \le j \le l$, then

$$|S| = c_1 c_2 \cdots c_l.$$

2.3 The solution of the Online Auction Problem

Lemma 2.6. Denote T_j as the set of permutations in T that $x_j = n$, then

$$|T_j| = n! \cdot \frac{1}{n} \cdot \frac{k}{j-1}.$$

We will prove to this lemma later.

Applying the lemma, we get that

$$|T| = \sum_{j>k} |T_j| = n! \cdot \frac{k}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n}\right),$$

so

$$\begin{aligned} p_{n,k} &= \frac{k}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n}\right) \\ &\approx \frac{k}{n}(\ln n - \ln k) \\ &= \frac{k}{n} \ln \frac{n}{k} \end{aligned}$$

by derivation we get that when $k = \frac{1}{\mathrm{e}} n$ the probability reaches the maximum $\frac{1}{\mathrm{e}}$.

3 Conditional Probability

Definition 3.1. Let V, W be two events such that $\Pr(W) \neq 0$. The conditional probability of V (conditioned on W) is defined by

$$\Pr(V|W) = \frac{\Pr(V \cap W)}{\Pr(W)}.$$

Remark 3.2.
1. $V \cap W$ represents the event that both V and W happen simultaneously;
2. The conditional probability can be seen as a restricted probability space $\mathbb{P}' = (W, p')$ with $p'(u) = \dfrac{p(u)}{\Pr(W)}$;
3. In particular, define $\Pr(V|W) = 0$ if $\Pr(W) = 0$.

Theorem 3.3 (Distributive principle). Let $W_1, W_2, \cdots, W_m$ be disjoint events, and their union is $W = W_1 \cup W_2 \cup \cdots \cup W_m$, then for arbitrary event T, we have

$$\Pr(T|W) = \frac{\sum_{i=1}^{m} \Pr(W_i)\Pr(T|W_i)}{\Pr(W)}.$$

Proof. Since $W_1, W_2, \cdots, W_m$ are disjoint, we get that $T \cap W_1, T \cap W_2, \cdots, T \cap W_m$ are disjoint and their union is $T \cap W$. So by the addition principle,

$$\begin{aligned}
\sum_{i=1}^{m} \Pr(W_i)\Pr(T|W_i) &= \sum_{i=1}^{m} \Pr(T \cap W_i) \\
&= \Pr(T \cap W) \\
&= \Pr(T|W)\Pr(W).
\end{aligned}$$
$\square$

This distributive principle can be applied to the Online Auction Problem.

Example 3.4 (Proof of Lemma 2.6). In the Online Auction Problem, we denote $W_j (j \geq k+1)$ to be the set of permutations that $x_j = n$, then $\mathcal{U} = W_{k+1} \cup \cdots \cup W_n$, and $\Pr(W_j) = \dfrac{1}{n}$. Since $\mathrm{Pt}(T|W_j)$ is the probability that we can get the best price in the case of $x_j = n$, i.e. the maximum of $x_1, x_2, \cdots, x_{j-1}$ appears in $x_1, x_2, \cdots, x_k$, so $\mathrm{Pt}(T|W_j) = \dfrac{k}{j-1}$. This gives a brief proof to Lemma 2.6.

Theorem 3.5 (Chain Rule). Let $V_1, V_2, \cdots, V_k$ be events that not forced to be disjoint, and the event that they happen simutaneously is $T = V_1 \cap V_2 \cap \cdots \cap V_k$, then

$$\Pr(T) = \Pr(V_1)\Pr(V_2|V_1)\Pr(V_3|V_1 \cap V_2) \cdots \Pr(V_k|V_1 \cap V_2 \cap \cdots \cap V_{k-1})$$

Remark 3.6. The chain rule is a generalization of the "multiplication principle".

Example 3.7. In birthday paradox, denote $\bar{T}$ to be the event that everyone in class has a unique birthday, and V_j is the student #1, #2, $\cdots$, #j have different birthdays, so $T = V_1 \cap V_2 \cap \cdots \cap V_k$, and

$$\Pr(V_j | V_1 \cap V_2 \cap \cdot \cap V_{j-1}) = 1 - \frac{j-1}{n}.$$

Appling the Chain Rule we get

$$\Pr(T) = \Pr(V_1)\Pr(V_2|V_1) \cdots \Pr(V_k|V_1 \cap V_2 \cap \cdots \cap V_{k-1}) = \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right).$$

4 Expectation

Definition 4.1 (Expectation). For a probability space $\mathbb{P} = (\mathcal{U}, p)$, a random variable X over $\mathcal{U}$ is a real-valued mapping $X : \mathcal{U} \to (-\infty, +\infty)$.

The expectation of X is defined by

$$\mathbb{E}(X) = \sum_{u \in U} X(u)p(u).$$

Theorem 4.2 (linear of expectation). If random variable $X = X_1 + X_2$, then

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2).$$

Proof. by the definition of expctation,

$$\mathbb{E}(X) = \sum_{u \in U} p(u)X(u) = \sum_{u \in U} p(u)(X_1(u) + X_2(u)) = \mathbb{E}(X_1) + \mathbb{E}(X_2).$$

Example 4.3. Throw n coins independently, each coin has bias b, i.e.

$$\begin{cases} \Pr(X_i = 1) = b \\ \Pr(X_i = 0) = 1 - b \end{cases}.$$

Denote X to be the number of 1s you throw. Then $\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n) = bn$.

Example 4.4. Take a random permutation u of $\{1, 2, \cdots, n\}$, look at its cycle representation, denote by $X(u)$ the number of cycles in its cycle representation.

To calculate the expectation of $X(u)$, we can write the random variable $X(u)$ to be some easily calculated random variables. Denote $X_i = \frac{1}{l}$ be the random variables of i, where l is the length of the cycle that i is on. Then $X = \sum_{i=1}^{n} X_i$.

So $\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i)$. The rest of the problem is solved in homework.

Example 4.5. We play a game on a rooted tree. In every step we randomly pick a node v and delete v and all its descendents, repeat the step until the tree is empty. Denote by X the number of steps we will make before tree is empty. The result is also shown in homework.