Mathematics for Computer Science (30470023) Spring 2020

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1 Overview

In this lecture, we further explore the power of generating functions. By multiplying two generating functions, we can easily apply convolution on series, thus handling more complicated recursion relations. We use this method to solve two classical combinatorial counting problems − counting the methods to parenthesize an expression, and counting the number of "up-down permutations". Finally, to evaluate the taylor series of some functions, there is a brief introduction to some core concepts of complex analysis.

2 Reviewing generating functions

For a series $\{a_i\}_{i\geq 0}$, define its generating function as a formal power series

$$A(x) = \sum_{i\geq 0} a_i x_i. \tag{1}$$

Let $A(x)$ and $B(x)$ be the generating function of two series $\{a_i\}_{i\geq 0}$ and $\{b_i\}_{b\geq 0}$. We can correspond some transformations on the series to the transformation on generating functions:

Transformation	Series	Generating function
Linear transformation	$\{a_i + c \cdot b_i\}_{i\geq 0}$	$A(x) + c \cdot B(x)$
Subscript shift	$\{a_{i+k}\}_{i\geq 0}$	$x^k A(x)$
Convolution	$\left\{\sum_{j=0}^{i} a_j b_{i-j}\right\}_{i\geq 0}$	$A(x) \cdot B(x)$

Binomial theorem provides fundamental rules to calculate the generating functions:

Theorem 2.1 (Binomial Theorem). If $n \geq 0$ is an integer, then it is possible to expand the n^{th} power of $x + y$ as

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-1}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n. \tag{2}$$

$\diamondsuit$

Theorem 2.2 (Newton's Generalized Binomial Theorem). For an arbitrary number r, define $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} = \frac{(r)_k}{k!}$, where $(\cdot)_k$ is the Pochhammer symbol. Then if x, y are real numbers with $|x| > |y|$ and r is any complex number, we have

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \tag{3}$$

$$= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2))}{3!} x^{r-3} y^3 + \cdots \tag{4}$$

♢

3 Parenthesizing an expression

Example 3.1 (Parenthesizing an expression). Assume there are n terms, $x_1, x_2, \cdots, x_n$, and we need to find the sum of them. For the reason that the computer can only add two numbers at a time, we need to parenthesize the expression $x_1 + x_2 + \cdots + x_n$ at first.

For example, if $n = 5$, $((x_1 + x_2) + (x_3 + (x_4 + x_5)))$ is a feasible method to parenthesize the expression $x_1 + x_2 + x_3 + x_4 + x_5$.

Actually, if there are n terms, the computer needs $n - 1$ additions to get the total sum. Hence, every parenthesized expression should have $n - 1$ parenthesis. Define a_n is the number of ways to parenthesize an expression with n terms. Then we have

$$a_n = \frac{1}{n} \binom{2n-2}{n-2}. \tag{5}$$

Proof. In order to get a recursive sequence, we consider the last addition operation, which corresponds to the largest parenthesis. Assume the last addition operation is between $(x_1 + x_2 + \cdots + x_k)$ and $(x_{k+1} + x_{k+2} + \cdots + x_n)$. Then the last addition operation is $((x_1 + x_2 + \cdots + x_k) + (x_{k+1} + x_{k+2} + \cdots + x_n))$.

$\forall k \in \{1, 2, \cdots, n-1\}$, there are a_k ways to parenthesize $x_1 + x_2 + \cdots + x_k$ and a_{n-k} ways to parenthesize $x_{k+1} + x_{k+2} + \cdots + x_n$. Therefore, if the last addition operation is between $(x_1 + x_2 + \cdots + x_k)$ and $(x_{k+1} + x_{k+2} + \cdots + x_n)$, there are $a_k a_{n-k}$ ways to parenthesize $x_1 + x_2 + \cdots + x_n$ in total. To sum up, for $n \geq 2$, we have

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k}. \tag{6}$$

It is easy to find that $a_1 = 1, a_2 = 1, a_3 = 2$. We can verify that the recursion is correct when $n = 2, 3$. Define $a_0 = 0$, then when $n \geq 2$, the recursion can be written as

$$a_n = \sum_{k=0}^{n} a_k a_{n-k}. \tag{7}$$

Define the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$, then we have the following equation

$$\begin{aligned}
A(x) - a_0 - a_1 x &= \sum_{n\geq 2}^{\infty} a_n x^n & (8)\\
&= \sum_{n\geq 2}^{\infty} \sum_{k=0}^{n} a_k a_{n-k} x^k x^{n-k} & (9)\\
&= (a_0 + a_1 x + a_2 x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots) & (10)\\
&= A(x)^2. & (11)
\end{aligned}$$

Then we have $A(x)^2 - A(x) + x = 0$, so $A(x) = \frac{1 \pm \sqrt{1-4x}}{2}$. For the reason that $A(0) = a_0 = 0$, we get a closed form of $A(x)$ as

$$\begin{aligned}
A(x) &= \frac{1 - (1-4x)^{\frac{1}{2}}}{2} & (12)\\
&= \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}} & (13)\\
&= \frac{1}{2} - \frac{1}{2} \sum_{n\geq 0}^{\infty} (-4x)^n \binom{\frac{1}{2}}{n}. & (14)
\end{aligned}$$

Here, we have used the Newton's Generalized Binomial Theorem to expand $(1-4x)^{\frac{1}{2}}$.

Therefore, we have can compute a_n as

$$\begin{aligned}
a_n &= -\frac{1}{2}\binom{\frac{1}{2}}{n}(-4)^n & (15)\\
&= -\frac{1}{2}(-4)^n \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!} & (16)\\
&= -\frac{1}{2}(-4)^n \frac{1}{2^n} \frac{(-1)^{n-1}}{n!}(2n-3)!! & (17)\\
&= -\frac{1}{2}(-4)^n \frac{(-1)^{n-1}}{2^{2n-1}} \frac{1}{n} \binom{2n-2}{n-1} & (18)\\
&= \frac{1}{n}\binom{2n-2}{n-1}. & (19)
\end{aligned}$$

$\square$

Remark 3.2. In combinatorial mathematics, the **Calatan numbers** are defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \quad \text{for } n \geq 1. \tag{20}$$

In our example, $a_n = C_{n-1}$. Actually, the Calatan numbers and their generalized forms are the solution of many counting numbers.

4 Up-down permutations

Lemma 4.1.

$$\tan x = x + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!} + o(x^{2n+1}) \quad \text{for } |x| < \frac{\pi}{2}. \tag{21}$$

where B_{2n} is the $2n^{th}$ Bernoulli Number. ♢

Proof. By definition,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n. \tag{22}$$

Therefore,

$$\frac{x}{e^x-1} + \frac{x}{2} = \frac{x}{2}\frac{e^x+1}{e^x-1} = \frac{x}{2}\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{x}{2}\coth(\frac{x}{2}). \tag{23}$$

Replace x with $2x$. Then we have

$$x\coth(x) = 1 + \sum_{n=0}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} x^{2n}. \tag{24}$$

For the reason that $x\cot(x) = ix\coth(ix)$, we can get

$$\cot(x) = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}(-1)^{n-1}B_{2n}}{(2n)!} x^{2n-1}. \tag{25}$$

For the reason that $\tan(x) = \cot(x) - 2\cot(2x)$, we can get

$$\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^{2n}-1)2^{2n}B_{2n}}{(2n)!} x^{2n-1}. \tag{26}$$

□

Example 4.2 ("UP-DOWN" permutations). Suppose n is an odd positive number. We define a permutation σ of $\{1, 2, \cdots, n\}$ as an "UP-DOWN" permutation if and only if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots < \sigma(n-1) > \sigma(n)$. Define a_n as the number of "UP-DOWN" permutations on $\{1, 2, \cdots, n\}$. Then we have

$$a_n = \frac{(-1)^{\frac{n-1}{2}}(2^{n+1}-1)2^{n+1}}{n+1} B_{n+1}. \tag{27}$$

Proof. In order to find a recursion of a_n, we consider the index of n. Suppose $\sigma(k+1) = n$. Therefore, we can see the permutation as two permutations. One is on $(\sigma(1), \sigma(2), \cdots, \sigma(k))$ and another is on $(\sigma(k+2), \sigma(k+3), \cdots, \sigma(n))$. Both permutations are "UP-DOWN" permutations.

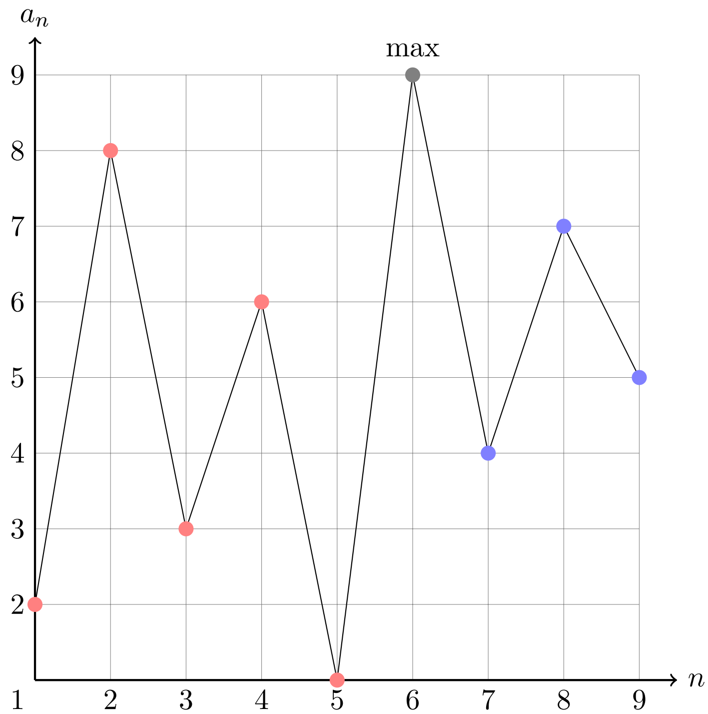

Figure 1: Visualization of an up-down sequence

For example, in figure 1, the index of $n = 9$ is 6 and $k = 5$. One permutation is on $(2, 8, 3, 6, 5)$ and another is on $(4, 7, 5)$. They are both "UP-DOWN" permutations.

There are $\binom{n-1}{k}$ ways to allocate the numbers in the two permutations. There are a_k "UP-DOWN" permutations on $(\sigma(1), \sigma(2), \cdots, \sigma(k))$ and a_{n-k-1} permutations on $(\sigma(k+2), \sigma(k+3), \cdots, \sigma(n))$. Therefore, for $\forall k \in \{1, 3, 5, \cdots, n-2\}$, if $\sigma(k+1) = n$, there are $\binom{n-1}{k} a_k a_{n-1-k}$ permutations. Hence, for $n \geq 3$, we have

$$a_n = \sum_{k \in \{1,3,5,\cdots,n-2\}} \binom{n-1}{k} a_k a_{n-1-k} \tag{28}$$

$$= \sum_{k \in \{1,3,5,\cdots,n-2\}} \frac{(n-1)!}{k!(n-k-1)!} a_k a_{n-1-k} \tag{29}$$

$$= (n-1)! \sum_{k \in \{1,3,5,\cdots,n-2\}} \frac{a_k}{k!} \frac{a_{n-k-1}}{(n-k-1)!}. \tag{30}$$

Also, it is easy to see that $a_1 = 1, a_3 = 2$. Define $b_k = \frac{a_k}{k!}$ and $B(x) = \sum_{n=1,3,\cdots} b_n x^n$, then we have

$$
\begin{aligned}
nb_n &= \sum_{k\in\{1,3,5,\cdots,n-2\}} b_k b_{n-1-k} && (31)\\
\Longrightarrow \sum_{n=3,5,\cdots} nb_n x^n &= x \sum_{n=3,5,\cdots} \sum_{k\in\{1,3,5,\cdots,n-2\}} b_k b_{n-1-k} x^k x^{n-1-k} && (32)\\
\Longrightarrow & x(B'(x) - b_1) = xB(x)^2 \qquad (\text{because } B'(x) = \sum_{n=1,3,\cdots} nb_n x^{n-1}) && (33)\\
\Longrightarrow & B'(x) = 1 + B(x)^2 \qquad (\text{because } b_1 = 1) && (34)\\
\Longrightarrow & \int \frac{dy}{1+y^2} = \int dx \qquad (\text{suppose } y = B(x)) && (35)\\
\Longrightarrow & B(x) = y = \tan(x + \theta_0) && (36)
\end{aligned}
$$

where θ_0 is a constant.

For the reason that $B(0) = 0$, we can take $\theta_0 = 0$. Hence, according to lemma 4.1, we have

$$
\begin{aligned}
B(x) &= \tan(x) && (37)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2^{2k}-1)2^{2k}B_{2k}}{(2k)!} x^{2k-1}. && (38)
\end{aligned}
$$

Therefore, we can get b_{2k-1} as

$$\frac{(-1)^{k-1}(2^{2k}-1)2^{2k}B_{2k}}{(2k)!}. \tag{39}$$

Because $a_k = k!b_k$ for $\forall k \in \mathbb{Z}^+$, we can finally get a_n as

$$
\begin{aligned}
a_n &= \frac{(-1)^{k-1}(2^{2k}-1)2^{2k}(2k-1)!B_{2k}}{(2k)!} && (40)\\
&= \frac{(-1)^{k-1}(2^{2k}-1)2^{2k-1}}{k} B_{2k} && (41)\\
&= \frac{(-1)^{\frac{n-1}{2}}(2^{n+1}-1)2^{n+1}}{n+1} B_{n+1}. && (42)
\end{aligned}
$$

where $n = 2k-1, k = 1, 2, \cdots$. □

Remark 4.3. "UP-DOWN" permutations can also be called as **Alternating permutations**. The determination of a_n in our example is called **Andre's problem**. The numbers a_n are known as **Euler numbers**.

5 Complex analysis roller coaster

Definition 5.1 (Complex function). A *complex function* is a function from a subset of $\mathbb{C}$ to $\mathbb{C}$. ♢

Definition 5.2 (Continouity). A complex function f is continuous at z_0, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $z \in \mathbb{C}$

$$|z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \epsilon. \tag{43}$$

♢

5.1 Differentiation of complex function

Definition 5.3 (Differentiation). The differentiation of a complex function f at z_0 is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{44}$$

f is differentiable at z_0 if the differentiation of $f'(z_0)$ exists. ♢

Example 5.4. f defined as $x + iy \mapsto x + iy^2$ (x, y are real numbers) is not differentiable at 0. Because

$$\lim_{x \to 0} \frac{f(x)}{x} = 1 \tag{45}$$

but

$$\lim_{y \to 0} \frac{f(yi)}{yi} = 0 \tag{46}$$

Example 5.5. f defined as $z \mapsto z^3$ is differentiable at each point because

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (3z_0^2 + 3z(z - z_0) + (z - z_0)^2) = 3z_0^2. \tag{47}$$

Definition 5.6 (Holomorphic function). A complex function f defined on a open set S is holomorphic if f is differentiable at each point in S. ♢

Theorem 5.7 (Holomorphic implies infinitely differentiable). If f is holomorphic in S, f is infinitely differentiable in S. In other words, for any $z \in S$ and $n \geq 0$, $f^{(n)}(x)$ exists. ♢

Theorem 5.8 (Holomorphic implies analytic). If f is holomorphic in S, for any $z_0 \in S$, f can be expanded as a power series

$$\sum_{i \geq 0} a_i (z - z_0)^i \tag{48}$$

that converges to $f(z)$ in some open disk centered at z_0. We call this power series the Taylor series of f at z_0. Particularly, if S is a open disk centered at z_0, the Taylor series converges to $f(z)$ at any point $z \in S$. ♢

Example 5.9. All familiar Taylor series still holds in complex cases. For example, if we choose a branch of the multivalued function $\ln z$,

$$\ln(1 + z) = \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} z^i \tag{49}$$

for $|z| < 1$.

Definition 5.10 (Meromorphic function). A complex function f is meromorphic in an open set S if f is holomorphic in S except for a set of isolated points[1]. $\diamondsuit$

Definition 5.11 (Pole). The poles of a meromorphic function f is the zeros of $1/f$. $\diamondsuit$

Example 5.12. $1+i$ is the pole for meromorphic function $f(z) = \dfrac{1}{z-(1+i)}$.

Definition 5.13 (Pole Singularity). z_0 is a pole of meromorphic function f. f has a pole singularity at $z = z_0$, if for some $\epsilon > 0$,

$$f(z) = \sum_{1 \le j \le m} \frac{c_j}{(z-z_0)^j} + \sum_{n \ge 0} b_n (z-z_0)^m \tag{50}$$

for all z satisfying $0 < |z - z_0| < \epsilon$, where m, c_j and b_n are constants and $c_m \neq 0$. m is called the order of the pole z_0, c_1 is called the residue of f at z_0. A pole is called *simple*, if its order equals to 1. $\diamondsuit$

5.2 Integration of complex function

Definition 5.14 (Integration). C is a curve on complex plane and u, v is the start point and end point of C. Let $u = z_0, z_1, z_2, \ldots, z_n = v$ be a series of successive points on C. When

$$\max_{1 \le i \le n} |z_i - z_{i-1}| \to 0, \tag{51}$$

the limit of

$$\sum_{i=1}^{n} f(z_i)(z_i - z_{i-1}) \tag{52}$$

is called the integration of f on C, denoted by

$$\int_C f(z)\,\mathrm{d}z. \tag{53}$$

$\diamondsuit$

The integration depends on the direction of C. If we swap the start point and the end point, the integration is multiplied by -1. If the start point coincides with the end point, the integration is called *contour integration*.

Computation of integration If C can be described as a parametric curve $\beta(t)$ $(a \le t \le b)$ with differentiable β, and f can be decomposed as $x + yi \mapsto u(x,y) + iv(x,y)$, then

$$\int_C f(z)\,\mathrm{d}z = \int_a^b \beta'(t) f(\beta(t))\,\mathrm{d}t. \tag{54}$$

[1] A set of points isolated, if there exists no point z such that every neighbor of z contains infinite number of points in this set.