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Topological mesh operators

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ABSTRACT

In this paper we introduce an unified framework for topological manipulation on triangulated 2-manifolds with or without boundary. We show that there are two kinds of primitive operators on the underlying meshes: operators that change the topological characteristic of the mesh and operators that just modify its combinatorial structure. We present such operators and demonstrate that they provide a complete and coherent set of elementary operations for mesh construction and editing.

1. Introduction

Triangulated meshes constitute one of the fundamental representations for objects in Computer Graphics and Geometric Modeling. They describe the spatial support where attributes of the objects are defined, such as geometry and texture. Moreover, current graphics hardware are optimized for such representations.

Although other representations, such as point sets, are becoming increasingly popular in recent years, polygonal representations are still prevalent and necessary in one way or another. The main reason is that meshes describe in a convenient piecewise manner the *global* space, intrinsic to the object. Point sets, on the other hand, provide only a *local* description. Indeed, the generation of polygonal meshes from point data is an active area of research.

Two-dimensional surfaces are, arguably, the most common type of object in computer graphics. Moreover, we are often interested in non-degenerate surfaces, i.e. 2D manifolds, since they allow efficient multiresolution representations, compact data structures and simple geometric approximations. These objects are best represented by *combinatorial structures* such as simplicial meshes.

Contributions. In this paper we investigate operators to build, unbuild and modify combinatorial two-dimensional manifolds with or without boundary.

In particular, we introduce an unified framework for primitive operations on combinatorial 2-manifolds with or without boundary. This mathematical framework is based on the integration of two fundamental theories: the Handlebody theory and the Stellar theory.

We further define a complete and sufficient set of operators to change the combinatorial structure, as well as, the topological characteristic of a polygonal mesh. This computational framework is substantiated by the main theorems of the Handlebody and Stellar theories. These new operators form a complete and coherent set and they not depend on the space where the surface is embedded.

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We finally propose a concise application program interface (API) for the implementation of these operators, give examples of prototype applications and point out how the framework could be incorporated with advantages in previously known algorithms in Geometric Modeling and Computer Graphics.

Paper outline. Section 2 introduces some concepts of combinatorial topology. Section 3 describes previous and related works. Section 4 presents the Handlebody and Stellar theories. Section 5 proposes the complete set of operators for surface modeling. Section 6 presents a suitable API for the proposed framework. Section 7 describes example applications improved by the use of these operators. Finally, Section 8 concludes this work by giving some final remarks and suggestion for future work.

2. Fundamental concepts

In this section, we present some fundamental concepts of combinatorial topology that will be used on this work.

2.1. Basic topological concepts

A *simplex* σ^p of dimension p (p-*simplex*, for short) is the convex hull of $p+1$ points $\{v_0, \ldots, v_p\}$, $v_i \in \mathbb{R}^m$, in general position, i.e., the vectors $v_1 - v_0, v_2 - v_0, \ldots, v_p - v_0$ are linearly independent. The points $v_0, \ldots, v_p$ are called the *vertices* of σ. A *face* of σ is the convex span of some of the vertices of σ and therefore is also a simplex. The simplices of dimensions 2 and 1 will be called, respectively, *triangles* and *edges*. If τ is a face of a simplex σ, then τ is said to be incident to σ. The *boundary* of a p-simplex σ, denoted by $\partial\sigma$, is the collection of all of its faces except σ itself. Two k-simplices σ and $\rho \in K$ are *adjacent* when $\sigma \cap \rho \neq \emptyset$, and *independent* otherwise. The *valence* or *degree* of a vertex $v \in K$ is the number of edges which have v as a vertex, and is denoted by $\deg(v)$.

A *simplicial complex* K is a finite set of simplices containing all their subsimplices such that if ρ and σ belong to K, then either ρ and σ meet at a subsimplex τ, or ρ and σ are independent. A simplicial complex K is *connected* if it cannot be represented as a union of two non-empty disjoint subcomplexes. A *component* of a complex K is a connected subcomplex that it is not contained in a larger connected subcomplex of K.

The *underlying polyhedron* $|K| \subset \mathbb{R}^m$ corresponds to the union of the simplices in K. A *triangle mesh* is the underlying polyhedron of a 2-dimensional simplicial complex.

The *join* $\sigma \star \tau$ of independent simplices σ and τ is the simplex whose vertices are those of both σ and τ. The *join* of complexes K and L, written $K \star L$, is $\{\sigma \star \tau\colon\ \sigma \in K, \tau \in L\}$ if $\sigma \in K$ and $\tau \in L$, σ and τ are independent. Consider a simplicial complex K and $\sigma \in K$. The local neighborhood of σ is described by the following elements:

- The *open star* of σ is
$$\operatorname{star}(\sigma, K) = \{\tau \in K\colon\ \sigma \text{ is a face of } \tau\}.$$
- The *star* of σ is
$$\overline{\operatorname{star}}(\sigma, K) = \big\{\tau \in K\colon\ \tau \text{ is a face of an element of } \operatorname{star}(\sigma, K)\big\}.$$
- The *link* of σ is
$$\operatorname{link}(\sigma, K) = \{\tau \in K\colon\ \tau \text{ and } \sigma \text{ are independent and } \sigma \star \tau \in K\}.$$

Definition 1 *(Combinatorial surface).* A simplicial complex S, $|S| \subset \mathbb{R}^m$, is a *combinatorial surface* if every edge in S is bounding either one or two triangles and if the link of a vertex in S is homeomorphic either to an interval or to a circle.

The edges in a combinatorial surface S incident to only one face are called *boundary edges*. Vertices incident to boundary edges are called *boundary vertices*. The subcomplex of S of those boundary simplices forms the *boundary* of S and is denoted by ∂S. The boundary of a combinatorial surface is a collection of closed curves. The edges and vertices that are not on the boundary are called, respectively, *interior edges*and *interior vertices*.

A combinatorial surface is *orientable* when it is possible to choose a coherent orientation for all of its simplices, where coherent means that two adjacent triangles induce opposite orientations on their common interior edge. The set of faces, edges and vertices of a surface S will be denoted, respectively, by $F(S)$, $E(S)$ and $V(S)$.

2.2. The Euler characteristic of surfaces

The topological setting applied to boundary representation of solids (Baumgart, 1975) has traditionally been the Euler–Poincaré theory, dated from the turn of the XIXth century (Poincaré, 1893). A very important theorem from this theory is the classification theorem for surfaces, that says:

Theorem 2. *Any connected oriented combinatorial surface with boundary is homeomorphic to either a sphere or a connected sum of $g > 0$ tori, in any case with some finite number of disks removed. No two of these surfaces are homeomorphic.*

The *Euler characteristic of a connected combinatorial surface* S, denoted by $\chi(S)$ with f faces, e edges and v vertices is defined as

$$\chi(S) = v - e + f.$$

Poincaré (1893) proved a very important topological invariant for a oriented combinatorial surface with boundary. It says that any homeomorphic oriented combinatorial surface has the same Euler characteristic

$$\chi(S) = v - e + f = 2(s - g) - b,$$

where s is the number of surface connected components, g is the number of genus on the surface, and b is the number of boundary curves components. The equation above is called the Euler–Poincaré formula.

3. Related works

The representation of a surface by a polygonal mesh is usually made of two parts: the connectivity and the geometry. The connectivity defines neighborhood relations within the surface, while the geometry defines the shape embedding in ambient space.

In this paper, we are mainly concerned with the combinatorial structure of a surface. For this reason, we will not address geometric issues extensively here. Nonetheless, we observe that the combinatorial representation of a mesh influence the implementation of geometric operations in non-trivial ways.

Accordingly, we review related work in the area, which fall into three categories: combinatorial data structures; topological operators; geometric and multiresolution operators.

3.1. Combinatorial data structures

The neighborhood relations within a mesh are encoded by a combinatorial graph that indicates incidence and adjacency relationships among vertices, edges and faces. The major issue in terms of topological data structures is the trade-off between the size of the representation, the execution time of queries; and the flexibility to edit the structure. A general overview of data structures for meshes can be found in de Floriani and Hui (2007).

Practically all combinatorial data structures for 2-dimensional manifolds are based on edges or on faces. Edge-based data structures mainly stores, for each edge, its incident vertices and adjacent edges, while face-based data structures stores, for each face, its vertices and the adjacent faces.

The classical *Winged-Edge* (Baumgart, 1975) structure links vertices and faces through the edges, and also includes information about orientation. Several data structures based on edges have then been proposed. One example is the *Quad-Edge* (Guibas and Stolfi, 1985) data structure that represents both the primal and dual graphs of the mesh. An important characteristic of the *Quad-Edge* data structure is that it was defined together with an *Edge Algebra* (see comments in the next subsection). Other very significant example is the *Half-Edge* (Mäntylä, 1988) data structure that decouples the two uses of an edge by adjacent faces and encodes the face orientation by a cycle of half-edges. This structure is very popular in recent implementations (Boissonnat et al., 2002; Botsch et al., 2002). The *Handle-Edge* data structure (Castelo et al., 1992; Lage et al., 2005) extends the *Half-Edge* in order to represent surfaces with boundary. An explicit representation of the surface boundary will play an important role in the implementation of the operators to be presented in this work.

Some examples of face-based data structure are the one proposed by Higashi et al. (1995), which is designed for robust geometric computations, and the *Corner-Table* (Rossignac et al., 2001), which is optimized for memory consumption rather than for structure changes. A recently proposed data structure, which is the counterpart of Quad-Edge, is *Gems* a structure for d-dimensional triangulations, that stores for each top simplex the list of its incident simplices (Montagner and Stolfi, 2006).

All the above data structures were designed to represent only manifold surfaces. Among others, the *Radial-Edge* (Weiler, 1985) and the *Non-manifold Indexed Data Structure with Adjacencies* (*NMIA*) (de Floriani and Hui, 2003) are two examples of data structures that can represent non-manifold objects, as well. In this paper we adopt an edge-based mesh representation. It is similar to the half-edge, but it is enhanced to support manifolds with boundary. Such data structure shows to be very suitable for a simple implementation of the proposed mesh operators. However, the topological operators proposed in our framework can also be implemented using other data structures, such as the Corner-Table.

3.2. Topological operators

Surface modifications are implemented through operators on its mesh representation. These operators can be classified according to their level of abstraction and their functionality.

Euler operators (Mäntylä, 1988) are low level operators for editing a mesh representation of the boundary of a solid. They are based on the Euler–Poincaré theory, which states that the topology of a compact oriented combinatorial surface S with boundary is characterized by its Euler characteristic $\chi(S) = |V| - |E| + |F|$, where $|V|$, $|E|$ and $|F|$ indicate respectively the number of vertices, edges and faces of S. The Euler characteristic classifies compact orientable surfaces according to the Euler formula: $\chi(S) = 2s - 2g - b$, where s is the number of connected components, g is the number of genus (e.g. through

holes, tunnels or handles), and b is the number of boundary curves of the surface. Mäntylä proved that *Euler operators* form a complete set of modeling primitives for manifold solids (Mäntylä, 1988). That is, every topologically valid polyhedron can be constructed from an initial polyhedron by a finite sequence of *Euler operators*. There are two groups of such operations: the *make* group and the *kill* group. The main disadvantage of the *Euler operators* is that, in the process of editing a mesh with these atomic operations, some intermediate results may not represent valid solids. Moreover, the *Euler operator* that generates a genus, assumes that the 2-manifold being operated is the boundary of a solid in $\mathbb{R}^3$. Therefore, *Euler operators* are usually encapsulated into higher level operators.

Quad-Edge operators are low-level operators based on the *Edge Algebra* defined by Guibas and Stolfi (1985). Their main advantage is conciseness, and they further show that only two atomic operations are sufficient for the construction and modification of arbitrary topological graphs embedded in two-dimensional manifolds. The *Gems* data structure has also an simple algebra associated with it. The *Gem Algebra* is based on just two topological operators: *create* and *splice*. We remark that these operators are conceptually equivalent to our handle operators.

The operators proposed in this paper work at a higher-level than the above ones, and as such, could be defined either in terms of the *Euler* or *Quad-Edge* operators — although this is not necessary. Here we have chosen to define them directly, as atomic operations, since we believe that they provide the right level of abstraction. Moreover, these operators have the advantage of always maintaining a valid mesh.

3.3. Multiresolution operators

Because of their importance in applications, many high level operators have been proposed to change the resolution of a mesh. These operators can be used for mesh simplification or mesh refinement. The meshes they operate on can have regular or irregular connectivity.

Multiresolution operators for regular meshes are usually associated with simplification and subdivision algorithms. In this area, the classical operators are the *quadrisection* for faces (Catmull and Clark, 1978; Loop, 1987) and vertices (Doo and Sabin, 1978) (e.g. *primal* and *dual* refinement). The drawback of these operators is that they cannot be used for adaptive refinement without compromising the regularity of the mesh. Recently, two new schemes, $\sqrt{3}$ subdivision (Kobbelt, 2000) and $\sqrt{2}$ subdivision (Velho and Zorin, 2001), introduced operators that are suitable for adaptive refinement. These schemes employ trisection and bisection operators, respectively.

The most popular multiresolution operators for irregular meshes are the *edge collapse* and its inverse, the *edge split*. Hoppe et al. (1993) proved that these two operators can be used to transform between any two equivalent simplicial complexes, if respecting the link condition (Edelsbrunner, 2002). Although *edge collapse* was designed originally in connection with progressive meshes (Hoppe, 1996), it has also been extensively used in many mesh simplification methods (Garland and Heckbert, 1997).

The operators proposed in this paper have more expressive power than the multiresolution geometric operators discussed above and can be used to implement them.

3.4. Geometric operators

As we mentioned before, geometric operations are not the focus of this paper. Nonetheless, we would like to briefly discuss their relationship with topological operators. Some geometric operations, such as warping deformations, are defined only in terms of point-wise information of a shape embedded in the ambient space. Therefore, these operators are independent of the mesh structure, once the geometry of the point is given.

Other geometric operators, such as the *umbrella* operator (Taubin, 1995) used in Laplace smoothing, depend on the local geometry of the surface. There are also operators that associate geometric quantities with elements of the mesh, for example differential properties (Desbrun et al., 2000). These two types of operators need information about the neighborhood of a topological entity, and, thus, they rely on queries about the mesh structure.

The data structure proposed in this paper supports efficient mesh queries and can be augmented with geometric attributes associated with different topological elements. Thus, it is suitable for the implementation of geometric operators.

3.5. Overview

In this paper we introduce a complete and minimal set of high-level operators that can be used to change both mesh connectivity and topology. These operators are based on two main theories of combinatorial topology: namely the *Handlebody* and *Stellar* theories. They consist of the following atomic operations for:

- Building, unbuilding and changing the topology of a mesh:
  create(v0, v1, v2)/destroy(f) — generates/eliminates a connected component defined by the triangle $(v0, v1, v2)$/face f, respectively;
  glue(e0, e1)/unglue(e) — joins/splices two pieces of the mesh boundary defined by the pair of edges $e0$, $e1$/edge e, respectively.
- Modifying the connectivity and resolution of a mesh:

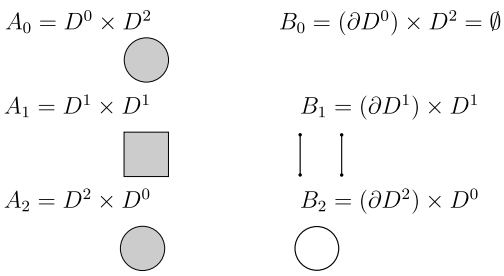


Fig. 1. 2D handles: $H_0 = (A_0, B_0)$; $H_1 = (A_1, B_1)$; $H_2 = (A_2, B_2)$.

flip(e) − alters the structure of a mesh region defined by two adjacent triangles by swapping their common edge e.
split(σ)/flip(σ) − refines/simplifies the mesh by respectively subdividing/refining either a triangle $\sigma = f$ or an edge $\sigma = e$.

As we will show in the rest of the paper, these operators can naturally express the algorithms used in geometric modeling and computer graphics applications.

4. Mathematical framework

In this section, we lay out the fundamental concepts of our framework for mesh operations. We distinguish between two kinds of operators on meshes: the ones that change the topology of the mesh, and the one that just alter its combinatorial structure.

Operators that change the mesh topology globally are based on the Handlebody theory, while operators that alter locally the combinatorial structure of the mesh are based on the Stellar theory. Both apply on combinatorial surfaces.

4.1. Handlebody theory

The Handlebody theory (Milnor, 1963) refines the Euler–Poincaré theory by bringing several new topological invariants for n-dimensional manifolds. The fundamental problem of Handlebody theory is to study the topological changes generated by handle attachments to a manifold with boundary.

In the surface case, three types of handles are to be defined and they will be distinguished by an index λ that varies from 0 to 2. Here, D^i denotes the i-dimensional disk and ∂P denotes the boundary of a set P.

Definition 3. For 2-dimensional manifolds, a handle of index λ, denoted by H_λ, is a pair of topological spaces (A_λ, B_λ) such that $B_\lambda \subset A_\lambda$, $A_\lambda = D^\lambda \times D^{2-\lambda}$ and $B_\lambda = \partial D^\lambda \times D^{2-\lambda}$.

According to this definition, one can observe that: 1) the set A_0 is a 2-disk and B_0 is the empty space; 2) the set A_1 is a square and B_1 is defined to be two of its opposite sides and 3) the set A_2 is a 2-disk and B_2 is its boundary (see Fig. 1). Observe that B_λ is naturally identified with a subset of the boundary of A_λ, i.e. there is a natural homeomorphism between B_λ and a subset of ∂A_λ.

A handle $H_\lambda = (A_\lambda, B_\lambda)$ is attached to a surface S by identifying B_λ with a subset I of ∂S and glueing $A_\lambda \supset B_\lambda$ to S along I.

The next theorem is the main mathematical tool in which the Handlebody theory is based.

Theorem 4 *(Handlebody decomposition). For every orientable surface S there is a finite sequence of surfaces $\{S_i\}$, $i = 0..N$, such that $S_0 = \emptyset$, $S_N = S$ and the surface S_i is obtained by attaching a handle $H_\lambda = (A_\lambda, B_\lambda)$ to the boundary of S_{i-1}. This sequence is called a handlebody decomposition of S.*

Fig. 2 illustrates the handlebody decomposition of a torus, $S_4 = (((S_0 + H_0) + H_1) + H_1) + H_2$.

When a handle $H_\lambda = (A_\lambda, B_\lambda)$ is attached to the boundary of S_{i-1} to obtain S_i, a topological change is generated and such change depends *only* on the index λ.

Theorem 5. *If S_i is obtained by attaching the handle H_λ to S_{i-1}, then $\chi(S_i) = \chi(S_{i-1}) + (-1)^\lambda$.*

As a consequence, the Euler characteristic of a surface S provided with a handlebody decomposition $\{S_i\}$, $i = 0..N$, is

$$\chi(S) = |H_0| - |H_1| + |H_2|$$

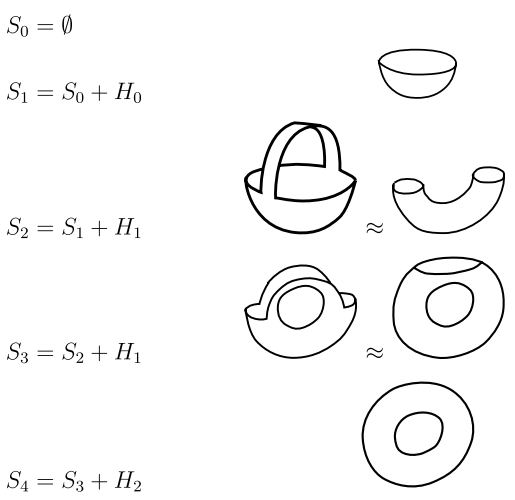


Fig. 2. Handlebody decomposition of a torus, $S_4 = (((S_0 + H_0) + H_1) + H_1) + H_2$.

where $|H_k|$, $k \in \{0, 1, 2\}$, corresponds to the number of handles of type k in $\{S_i\}$. For example, in the handlebody decomposition of the torus in Fig. 2, there are one handle H_0, two handles H_1, and one handle H_2. The formula above is, then, verified, since the Euler characteristic of a torus is zero. This is a topological invariant introduced by the Handlebody theory.

Handles can be attached to an orientable surface with boundary in such a way to preserve its orientability, i.e., the identification of B_λ into ∂S preserves the orientation of S. If one starts with an orientable surface, then after attaching a handle coherently the surface is again orientable.

We observe that if we keep track of the number of connected components and the number of boundary curves, we can easily calculate the number of genus on the surface and classify it whenever it is necessary. We will now present how to count those two numbers by studying the topological changes caused by a handle attachment that preserves the orientability.

0-*Handle.* The topological change generated by a handle attachment of index 0 is a creation of a new surface component (see S_1 in Fig. 2). This handle attachment increases the Euler characteristic by one.

1-*Handle.* When the handle H_1 is coherently attached to a surface S_i, three situations can occur:

(1) The set A_1 is attached to disjoint intervals on the same boundary curve component. In this case, the topological change is the inclusion of a new boundary curve component in the surface (see S_2 in Fig. 2).
(2) The set A_1 is attached to intervals on different boundary curve components of the same surface component. The topological change is here characterized by the creation of a new genus on the surface. In addition, the number of boundary curve components decreases (see S_3 in Fig. 2).
(3) The set A_1 is attached to intervals on different surface components. Here, a boundary curve component and a surface component is removed.

In these three situations, when a handle H_1 is attached coherently to S_{i-1} to obtain S_i, we have $\chi(S_i) = \chi(S_{i-1}) - 1$. Observe that, all of them alter the number of boundary curves. Moreover, the last one also changes the number of connected components on the surface.

2-*Handle.* Handles of index 2 close a boundary curve component (see S_4 in Fig. 2).

Concluding, there are three types of handles and five different situations in which they can be attached to a boundary surface.

4.2. Stellar theory

In the previous section, we saw how to change the topology of a manifold. Now, we will see how to manipulate the structure of a combinatorial surface *without* modifying its topology, which is the main point of Stellar theory (Alexander, 1930; Newman, 1926; Pachner, 1991; Lickorish, 1999).

As we have seen in Section 2.1, the link and the star of a simplex σ provide a combinatorial description of the neighborhood of σ. We can use them to define certain changes in a triangle mesh, without modifying essentially (i.e., "topologically") that neighborhood. That is, we do not want to change the topology of the realization of the surface in $\mathbb{R}^3$. The *stellar operations* provide a such change. They comprise *bistellar moves* and *stellar subdivision*:

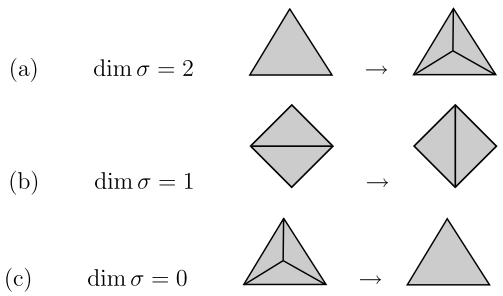


Fig. 3. Two-dimensional bistellar moves.

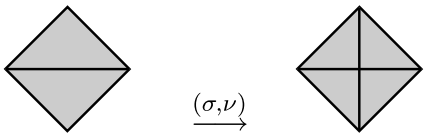


Fig. 4. Two-dimensional stellar subdivision on interior edges.

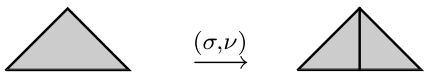


Fig. 5. Two-dimensional stellar subdivision on boundary edges.

Definition 6. Let K be an n-dimensional simplicial complex. Take an r-simplex $\sigma \in K$, and an $(n-r)$-simplex $\tau \notin K$, such that $\text{link}(\sigma, K) = \partial\tau$. Then, the operation $\kappa(\sigma, \tau)$, called *bistellar move*, consists of changing K by removing $\sigma \star \partial\tau$ and inserting $\partial\sigma \star \tau$.

The bistellar moves are atomic operations that make local changes to the neighborhood of an simplex, while maintaining the integrity of its combinatorial structure. In the case of combinatorial surfaces, there are three types of bistellar moves, for $\dim\sigma = 2, 1, 0$, called 2-move, 1-move, and 0-move. They are shown in Fig. 3.

The fundamental result of the Stellar theory is given by the following theorem:

Theorem 7. *(See Newman, 1926; Pachner, 1991.) Two combinatorial surfaces are piecewise linearly homeomorphic if and only if they are bistellar equivalent.*

The above result guarantees that bistellar moves can change any triangulation of a closed piecewise linear manifold to any other. A version of this theorem for manifolds with boundary uses all stellar operations, including stellar subdivision (Pachner, 1991).

Definition 8. Let K be a 2-dimensional simplicial complex, take an r-simplex $\sigma \in K$ and a vertex ν in the interior of σ. The operation (σ, ν) removes $\overline{\text{star}}(\sigma, K)$ and replaces it with $\nu \star \partial\sigma \star \text{link}(\sigma, K)$. Such operation is called a *stellar subdivision* and its inverse $(\sigma, \nu)^{-1}$ is called a *stellar weld*.

Note that some of the stellar subdivision and welds are also stellar moves as for example $\kappa(\sigma, \nu)$ and $\kappa(\nu, \sigma)$ for $\dim\sigma = 2$ (see the top and bottom rows of Fig. 3).

In dimension 2, this new operation is the stellar subdivision on edges, called 1-split. It is shown in Fig. 4 the interior edge case and in Fig. 5 the boundary edge case.

Stellar subdivision is a very powerful concept and it is the cornerstone of Stellar theory. Here, we will only mention some results of the stellar subdivision theory (Alexander, 1930).

Proposition 9. *Any stellar move, $\kappa(\sigma, \tau)$, is the composition of a stellar subdivision and a weld, namely $(\tau, \nu)^{-1}(\sigma, \nu)$.*

This result can be easily seen through an example, shown in Fig. 6.

Proposition 10. *Any stellar operation can be decomposed into a finite sequence of elementary stellar operations on* edges.

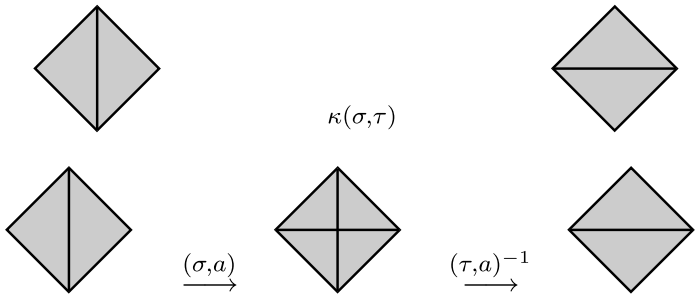


Fig. 6. A bistellar move on an edge can be decomposed into a subdivision and an weld.

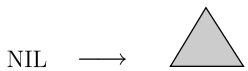


Fig. 7. Handle operator of type 0 (triangle creation).

This result is even stronger than the previous one. It basically allows us to restate the main theorem of Stellar theory only in terms of operations on edges. We will use this result to state a simple framework for mesh creation and edition which uses only operations on edges, making it particularly suited for implementation on classical data structures.

5. Computational framework

The purpose of this section is to introduce a new set of topological operators based on the concepts of Handlebody and Stellar theories. This set includes operators for building/unbuilding meshes and to change the structure and resolution of a mesh.

We remark that, although the Handlebody theory can be applied to general combinatorial manifolds, the Stellar theory is restricted to simplicial complexes. Therefore, from now on, we will focus on triangular meshes. This is not a limitation, since any manifold surface can be triangulated and, in practice, triangular meshes are a common choice in applications.

5.1. Handle operators

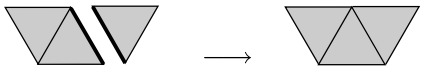
5.1.1. Building Handle operators

The Handlebody theory presented in Section 4.1 studies the topological changes in a surface caused by a handle attachment. There are three types of handles to build a handlebody decomposition of a surface. From a combinatorial point of view, we define three types of operators to represent the handle attachments:

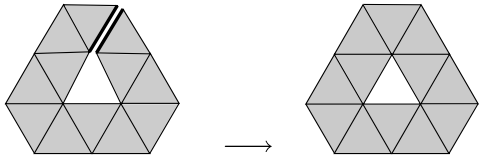
- Handle operator of type 0 – This operator creates a new combinatorial surface component with only one triangle (see Fig. 7).
- Handle operator of type 1 – The purpose of this operator is to identify two given boundary edges *with no vertices in common*. There are three situations for this group:
  Case (a): the boundary edges are on different surfaces. In this case the operator attaches the surfaces and removes one boundary curve (see Fig. 8(a)).
  Case (b): the given boundary edges are incident to the same boundary curve. The operator splits the boundary curve into two different components (see Fig. 8(b)).
  Case (c): the boundary edges are on different boundary curves on a surface component. It creates a new genus in the surface and reduce in one the number of boundary curve components of the surface (see Fig. 8(c)).
- Handle operator of type 2 – This operator identifies two given boundary edges *with two vertices in common*. The operator closes one boundary curve component and transform those boundary vertices into two interior vertices (see Fig. 9).

According to the definitions above, we observe that if a Handle operator of type λ is applied to a combinatorial surface S_1 to obtain S_2, then $\chi(S_2) = \chi(S_1) + (-1)^{\lambda}$. This is a direct consequence of Theorem 5.

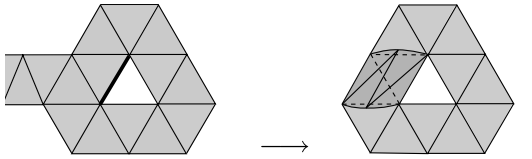
One can observe that the Handle operators of type 1 and type 2 identify two boundary edges to make an interior edge. The first is applied when the edges have no vertices in common, and the second when the edges have two vertices in common. Thus, there is one missing case to consider: when the boundary edges have one vertex in common. So, it is suitable to define the Zip operator, which identifies two boundary edges with one vertex in common. This operator removes one edge and one vertex, then it doesn't change the Euler characteristic of the surface. Its main purpose is to close the vertex link (see Fig. 10). In fact, such operator can be derived from the building Handle operators together with their inverse. However, it is very convenient to have a direct implementation of it.



(a) Boundary edges belong to different surfaces

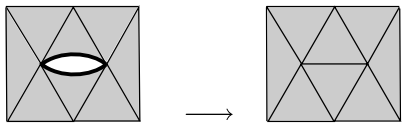


(b) Boundary edges belong to the same boundary curve of a surface



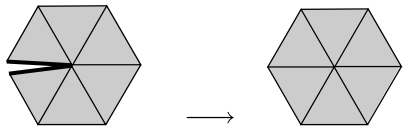
(c) Boundary edges belong to different boundary curves of a surface

Fig. 8. Handle operator of type 1 (joining boundaries).



Boundary edges have two vertices in common

Fig. 9. Handle operator of type 2 (closing boundaries).



Boundary edges have one vertex in common

Fig. 10. Zip operator.

5.1.2. Unbuilding Handle operators

There is an inverse operator for each building Handle operator presented. The topological changes caused by their inverse operation are now described.

The unbuilding Handle operator of index zero destroys a triangle. Unbuilding Handle operators of index 1 and index 2 split an interior edge into two boundary edges. There are five cases to consider when splitting an interior edge. Such cases are distinguished according to the number of boundary vertices incident to the interior edge that will be operated, which could be 2, 1 or 0. The unbuilding Handle operator of type 1 is used when the incident vertices to the interior edge are both in the surface boundary. The unbuilding Handle operator of type 2 is applied when the incident vertices of the interior edge are on the interior of the surface. In the last case, when the interior edge has one vertex in the boundary, one should use the inverse Zip operator.

The topological changes caused by an unbuilding Handle operator of index 1 when applied to a given interior edge e, depend on the answer to the following question:

Are the boundary vertices incident to e on different boundary curve components?

If the answer is affirmative then the unbuilding Handle operator will remove one boundary curve component (see the unbuilding operation in Fig. 8(b)). In contrary, the second question has to be answered.

Are those vertices on the same boundary curve component?

When the vertices are incident to the same boundary curve, the unbuilding operation not only will add a new boundary curve component to the surface but also it will either decrease the genus (see unbuilding of Fig. 8(c)) or disconnect the surface (see unbuilding of Fig. 8(a)).

Unbuilding operator of index 2 duplicates an interior edge with zero incident boundary vertices. The topological change in this situation is an addition of a new boundary curve to the surface.

The inverse Zip operator (the unzip operator) is applied when the interior edge e has one incident vertex on the boundary. It simply duplicates an interior edge and transforms an interior vertex into a boundary vertex.

5.1.3. Algebraic properties

A surface S is said to be valid if it satisfies the definition of a combinatorial surface. So the Euler–Poincaré formula is a necessary condition for the validity of the model.

$$\chi(S) = v - e + f = 2(s - g) - b,$$

where $v, e, f, b, g,$ and s represent, respectively, the number of vertices, the number of edges, the number of faces, the number of boundary curves, the number of genus and the number of connected components on the surface.

Similarly to Braid et al. (1980), we consider a six-dimensional space $\mathbb{E}^6$, whose axes are: v, e, f, b, g, s. The canonical basis $\mathbb{E}^6$ is denoted by $\mathcal{B}$. This consideration allows us to say that the Euler–Poincaré formula can be rewritten in such a way to represent a hyperplane (five-dimensional subspace) EP on this vector space:

$$v - e + f - 2(s - g) + b = 0.$$

It is suitable to define a new basis for this six-dimensional space $\mathbb{E}^6$ in such a way that a combination of Handle operators could be explicitly represented as a vector on the EP hyperplane. This new basis, called the *Handle operator* basis and denoted by $\mathcal{H}$, is defined in the following way:

- Handle operator of type 0 $\leftrightarrow (1, 0, 0, 0, 0, 0)_{\mathcal{H}}$.
- Handle operator of type 1 (a): boundary edges on different surface components $\leftrightarrow (0, 1, 0, 0, 0, 0)_{\mathcal{H}}$.
- Handle operator of type 1 (b): boundary edges on the same boundary component $\leftrightarrow (0, 0, 1, 0, 0, 0)_{\mathcal{H}}$.
- Handle operator of type 1 (c): boundary edges on the same surface component but on different boundary curves $\leftrightarrow (0, 0, 0, 1, 0, 0)_{\mathcal{H}}$.
- Handle operator of type 2 $\leftrightarrow (0, 0, 0, 0, 1, 0)_{\mathcal{H}}$.

In order to complete the six-dimensional space we can use the perpendicular vector of the Euler–Poincaré hyperplane, which coordinate on the Handle operators basis is $(0, 0, 0, 0, 0, 1)_{\mathcal{H}}$, and on the canonical basis on the Euclidean space is $(1, -1, 1, 1, 2, -2)_{\mathcal{B}}$.

Now, we would like to change the basis from $\mathcal{H}$ to the canonical basis $\mathcal{B}$ of $\mathbb{E}^6$, whose axis are $v, e, f, \partial s, g, s$. In this way, we could obtain the number of cell elements after the application of a combination of Handle operators, or in the inverse way we could obtain the number of Handle operators to be applied to build a surface with a given number of cell elements.

To find the solution of this change of basis problem we need to relate the vectors of the Handle operators basis with the vectors of the basis $\mathcal{B}$. Those relations are expressed as follows:

- Handle operator of type 0: $(1, 0, 0, 0, 0, 0)_{\mathcal{H}} \leftrightarrow (3, 3, 1, 1, 0, 1)_{\mathcal{B}}$.
- Handle operator of type 1 (a): $(0, 1, 0, 0, 0, 0)_{\mathcal{H}} \leftrightarrow (-2, -1, 0, -1, 0, -1)_{\mathcal{B}}$.
- Handle operator of type 1 (b): $(0, 0, 1, 0, 0, 0)_{\mathcal{H}} \leftrightarrow (-2, -1, 0, 1, 0, 0)_{\mathcal{B}}$.
- Handle operator of type 1 (c): $(0, 0, 0, 1, 0, 0)_{\mathcal{H}} \leftrightarrow (-2, -1, 0, -1, 1, 0)_{\mathcal{B}}$.
- Handle operator of type 2: $(0, 0, 0, 0, 1, 0)_{\mathcal{H}} \leftrightarrow (0, -1, 0, -1, 0, 0)_{\mathcal{B}}$.

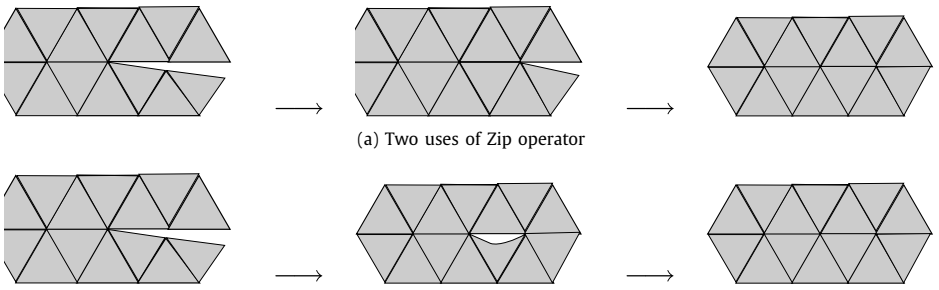
We can now introduce the *transition matrix* Λ as a change of basis matrix from the Handle operators' base to the base on the Euclidean space.

$$\Lambda = \begin{pmatrix} 3 & -2 & -2 & -2 & 0 & 1 \\ 3 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

So with this matrix we can compute in advance the characteristic of the surface after a combination of the Handle operators by just a matrix–vector multiplication:

$$[v']_{\mathcal{B}} = \Lambda[v]_{\mathcal{H}}.$$

For example, a vector $(2, 1, 0, 0, 0, 0)_{\mathcal{H}}$ says that two Handle operators of type 0 and one Handle operator of type 1 (a) has been applied. Multiplying this vector by Λ we obtain $(4, 5, 2, 1, 0, 1)_{\mathcal{B}}$, which means that the resulted surface has 4 vertices, 5 edges, 2 faces, 1 boundary curve, 0 genus, and 1 connected component.



(a) Two uses of Zip operator

(b) One use of Handle operator of type 1(b) and one of type 2

Fig. 11. Obtaining the Zip operator by the use of Handle operators.

In order to compute the number of Handle operators needed to build a surface with boundary, we can use the inverse matrix of Λ:

$$\Lambda^{-1} = \frac{1}{12}\begin{pmatrix} -1 & 1 & 11 & -1 & -2 & 2 \\ -3 & 3 & 9 & -3 & -6 & -6 \\ -2 & -4 & 10 & 4 & -4 & 4 \\ -2 & 2 & -2 & -2 & 8 & 4 \\ 3 & -9 & 15 & -3 & -6 & 6 \\ 1 & -1 & 1 & 1 & 2 & -2 \end{pmatrix}.$$

In order to illustrate the use of Λ^{-1}, suppose now that we would like to build a tetrahedron that has 4 vertices, 6 edges, 4 faces, no boundary curves and no genus, and 1 connected component. Thus, such surface is represented by the vector $(4,6,4,0,0,1)_{\mathcal{B}}$. If we multiply such vector by Λ^{-1} we obtain the vector $(4,3,1,0,2,0)_{\mathcal{H}}$. This means that one way to obtain a tetrahedron is to apply 4 Handle operators of type 0, 3 Handle operator of type 1(a), 1 Handle operator of type 1 (b) and 2 Handle operators of type 2. It is important to observe that this procedure has to be exercised with care since the unbuilding operators count as -1.

Using this algebraic properties, we could also verify that the Zip operator can be obtained by the application of consecutive Handle operations. Fig. 11 shows two possible ways to zip two boundary edges:

- One is by the two uses of the Zip operator. The transition caused by two Zip operators on the surface is $(-2,-2,0,0,0,0)_{\mathcal{B}}$, i.e. the resulted surface has -2 vertices and -2 edges than the original one (see Fig. 11(a)).
- An other is by the use of a Handle operators of type 1 (b) and one of type 2. To obtain such transition by the use of Handle operators, we multiply this vector by the matrix Λ^{-1} whose result is $(0,0,1,0,1,0)_{\mathcal{H}}$, which is the desired result (see Fig. 11(b)).

5.1.4. Remarks on Handle operators

With the set of Handle operators presented above one can build and unbuild all kinds of orientable combinatorial surfaces with or without boundary. The unbuilding Handle operators shall be used to perform cut operations on the surface, while the building Handle operators shall be used to make paste operations. More formally, we could proclaim the following results:

Theorem 11. *If $\mathcal{S}$ is a connected and oriented combinatorial surface with or without boundary, then there is a finite sequence of unbuilding Handle operators that can completely remove $\mathcal{S}$.*

Sketch of the proof. The procedure to destroy the surface is the following: For every interior edge e on $\mathcal{S}$, apply one of the unbuilding Handle operators (including the inverse of the Zip operator). Observe that in Section 5.1.2 we study all the possible cases to apply such operators to an interior edge, and that is always possible to do that. After that, the resulted surface $\mathcal{S}'$ has $|F(\mathcal{S})|$ surface components, where each component has only one face. Then, we can apply the unbuilding Handle operator of type 0 to each one destroying all of them. □

If we apply this sequence of operators in the other way round, we obtain the handlebody decomposition of the surface.

Corollary 12. *Every orientable combinatorial surface with or without boundary can be created with a finite sequence of building Handle operators.*

Observe that all Handle operators presented in Sections 5.1.1 and 5.1.2, except the Handle operator of type 0 and its inverse, are applied to edges. The building operators identify two boundary edges to make an interior edge and the unbuilding

operators split an interior edge to build two boundary edges. In those sections we investigate all the possibilities to paste that could occur. In all the cases the resulted surface is a valid one.

Theorem 13. *Handle operators cannot generate invalid combinatorial surface.*

To conclude, from now on we will call the Handle operators presented in this section as *low-level* Handle operators. This is because in Section 6 we will present a suitable API that uses them in a higher level.

5.2. Stellar operators

The Stellar theory presented in Section 4.2 studies structural modifications to the neighborhood of a simplex that do not alter the topology. These modifications are the stellar moves, stellar subdivision and welds. They can be used to change the connectivity and the resolution of a mesh.

We classify the Stellar operators in terms of their effect in the number of faces, $|F|$, in the mesh. Accordingly, there are three groups of operators:

- isolevel;
- refinement; and
- simplification.

5.2.1. Isolevel Stellar operators

The isolevel operators keep the resolution of the mesh at the same level. Thus, they do not change $|F|$. The operator in this group is the bistellar 1-move, also called 1-flip (or edge flip). It simply exchanges two existing triangles by two new triangles. This operator is shown in Fig. 3(b).

The edge flip is a very powerful operator for changing the combinatorics of the mesh structure without altering its resolution or topology. For this reason, it is the basis of many computational geometry algorithms.

5.2.2. Refinement Stellar operators

The refinement operators increase $|F|$, and thus the resolution of the mesh. The operators in this group are the 2-split (face split), and 1-split (edge split).

The face split replaces one existing triangle with three new triangles, and thus, it increases $|F|$ by 2. This operator is shown in Fig. 3(a).

The edge split has two cases. When the edge is an internal edge, the edge split replaces two existing triangles sharing that edge with four new triangles. When the edge is a boundary edge, it replaces one existing triangle with two new triangles. This operator increases $|F|$, by 1 or 2, depending of whether the edge belongs to the boundary or not. Figs. 4 and 5 show the 1-split of an internal and a boundary edge.

5.2.3. Simplification Stellar operators

The simplification operators are the inverse of the refinement operators. The inverse of the face split is the face weld, and the inverse of the edge split is the edge weld, In the case of simplification of an element in the interior of the mesh, the face weld replaces three faces incident in a vertex with one face removing that vertex, and the edge weld replaces the four faces incident in a vertex with two faces such that the vertex is substituted by one of the two possible edges that gives a triangulation of the region defined by its link.

Observe that weld operations $(\sigma, v)^{-1}$ are specified through a vertex v, whose star defines the neighborhood to be changed.

5.2.4. Remarks on Stellar operators

At this point it is appropriate to note that Stellar operators can be used as primitives to define other multiresolution operators.

For example, edge collapse and its inverse, vertex split, can be decomposed into a sequence of elementary stellar operations. This is a natural consequence of Theorem 7. More specifically, the edge collapse is given by a composition of edge flips and a final edge weld, while the vertex split is given by an edge split composed wit a sequence of edge flips. This is shown in Fig. 12.

We remark that stellar operations are more flexible in general. In the case of edge collapse/vertex split, it is easy to see that there are many possible sequences of edge flips leading to the final edge weld. Therefore, those edges flips can be chosen in such a way that the quality of the mesh is improved, for example, those with bad aspect ratios (Vieira et al., 2003). The geometric result of this flips could generate singularities on the geometric polyhedron, in order to avoid that the well known link condition has to be verified (Edelsbrunner, 2002).

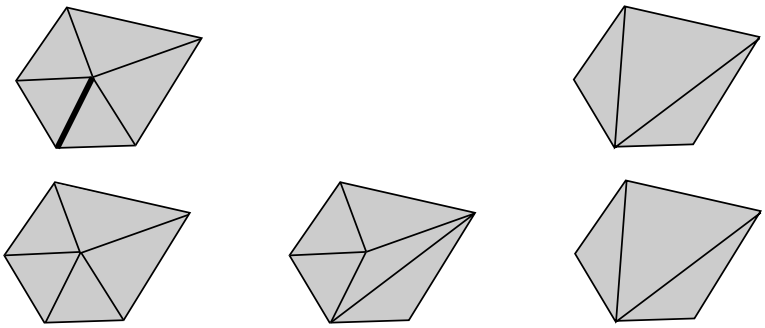


Fig. 12. Decomposition of an edge collapse (top) into an edge swap followed by an edge weld (bottom).

6. Implementation framework

In this section we propose an application program interface (API) for a mesh library based on the Handle and Stellar operators. To implement such API, we adopt an edge-based data structure similar to the half-edge that supports manifolds with boundary, although the operators can be defined on any data structure representing the connectivity of the mesh. Such data structure shows to be very suitable for a simple implementation of the proposed mesh operators. Its detailed description is on the appendix. All the code and examples are available on the web (Velho et al., 2005),

The API consists of the set of queries, Handle operators, Stellar operators and an additional set of higher-level derived operators.

6.1. Queries

The mesh operators need answers of queries and navigation on the mesh structure. The main useful queries are: c = link(s); and c = star(s). Note that they can take as arguments a simplex s of dimension 0 (vertex), 1 (edge) or 2 (face). In our implementation, we use only the vertex star, which returns an adjacency iterator object c, called *circulator* (Mehlhorn et al., 1997). Another useful query, that is directly derived from the star of a vertex is the function degree(v), which is $|\text{star}(v)|$.

We also have the basic operators of the edge algebra (Guibas and Stolfi, 1985): v = org(e) (origin vertex v of a half edge e); f = left(e) (face f to the left of a half edge e); h = sym(e) (symmetric half edge h); and n = lnext(e) (next half edge n on left face). These functions are trivially computed from edge-based data structures.

In order to have an efficient implementation of the Handle operators, it is important to have an explicit representation of the boundary components. Since with that it is possible to identify whether two boundary edges are on the same boundary component or not. For that reason, we also have the query: is_boundary(s), that returns *true* if the simplex s belongs to the mesh boundary.

6.2. Handle operators

The building and unbuilding Handle operators allow cutting and pasting on the surface. They are: f = create(v0, v1, v2) (creates a new triangular face f); destroy(f) (destroys an existing face); glue(e0, e1) ("identifies" two boundary edges to make one interior edge), and unglue(e) (splits one interior edge to make two boundary edges).

The glue(e0, e1) operator internally decides whether to use a Handle operator of type 1, a Handle operator of type 2 or a Zip operator. This decision is done in constant time using our data structure, by simply identifying the boundary components of e0 and e1 and counting how many vertices in common e0 and e1 have. The low level implementation of the Handle operator of types 1 and 2 and the Zip operator has constant time complexity, using a union-find data structure for the boundary.

The unglue(e) operator internally decides whether to use an unbuilding Handle operator of type 1 or type 2 or the unzip operator. The complexity in the worst case of this decision is done in linear time on the number of faces. Given an interior edge we first have count how many interior vertices are incident to it. If it has two incident interior vertices, we have to apply the unbuilding Handle operator of type 2. In the case it has only one incident interior vertex, we have to apply the unzip operator. Both of them have constant time complexity. Otherwise, we have to apply the unbuilding Handle operator of type 1, whose complexity is linear in the number of faces, since we have to identify whether the surface component will be subdivided.

6.3. Stellar operators

The Stellar operators allow changing the resolution and structure of the mesh. They are: flip(e) (swaps the edge e); split(e) (bisects the edge e and its incident faces); split(f) (trisects the face f); weld(v) (inverse of the split operators, which applies to edges and faces).

Note that flip is only defined for internal edges. In our C++ implementation, split is defined using overload of operators. weld deduces the type operation from the star of the vertex v.

Using our data structure, we can affirm that the operators flip(e), split(e), split(f), and weld(v) have constant time complexity.

6.4. Derived, higher-level operators

Although the Handle and Stellar operators described in the previous subsections form a complete set of atomic operators to manipulate the combinatorial structure of a mesh, it may be convenient to define some derived mesh operators that are instrumental in common applications.

These operators can be constructed by a composition of basic Handle or Stellar operators. They encapsulate abstract higher-level operations on a mesh.

Here we will define the operators attach, detach, and remove, that will be useful in the next section.

The operator attach(e, v) augments the mesh by adding to its boundary a new triangle defined by some boundary edge e, and a new vertex v, as shown in pseudo-code below.

```
procedure attach(Edge e, Vertex v)
  Require: e ∈ boundary
  f ← create(org(e), org(sym(e)), v)
  glue(e, f.edge[0])
```

The operator detach(f) is the inverse of attach, it shrinks the mesh by deleting a triangle from the mesh. It can be a triangle on the boundary or an internal triangle.

```
procedure detach(Face f)
  for e ∈ f do
    if not is_boundary(e) then
      unglue(e)
    destroy(f)
```

The operator remove(v) decreases the resolution of the mesh by eliminating one arbitrary vertex v. In order to be able to apply the stellar weld operator, it first needs to perform some edge swaps to make the degree D of the vertex compatible with the desired operation (i.e., in the case of an internal vertex, degree $D = 4$ for edge weld or degree $D = 3$ for face weld, and in the case of a boundary vertex, degree $D = 3$ for edge weld). Note that the selection of edges to be swapped may take into account the aspect ratio of resulting triangles.

```
procedure remove(Vertex v, int D)
  while degree(v) > D do
    e ← select_edge(star(v))
    flip(e)
  weld(v)
```

We could also define an operator insert(v), which would be the inverse of remove. However, this type of operator not only increases the mesh resolution, but usually also changes the mesh geometry (for example, in the context of mesh subdivision). Because of this dependency, we prefer not defining it here.

7. Applications and examples

Mesh operators embody the fundamental transformations for combinatorial manifolds. Applications that adopt meshes as a surface representation can greatly benefit from our operators, because they provide the correct level of abstraction for algorithm design and guarantee that the representation is always valid.

In this section we discuss how our framework fits into graphics applications. Below we give examples of the several algorithms for geometric modeling that employed some of the concepts presented in this paper. We also describe how these prototypical applications can fully exploit our mesh operators.

7.1. Mesh construction

Mesh construction is perhaps the most basic geometric modeling application. In this context, advancing front algorithms constitute a flexible and principled way to create a mesh representation. This type of algorithm starts with a seed triangle and grows the surface by gluing new triangles to the surface boundary. The Handle operators, create and glue, allow a very robust and concise implementation of this algorithm (Medeiros et al., 2003).

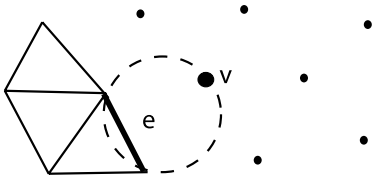


Fig. 13. Advancing front with ball pivoting.

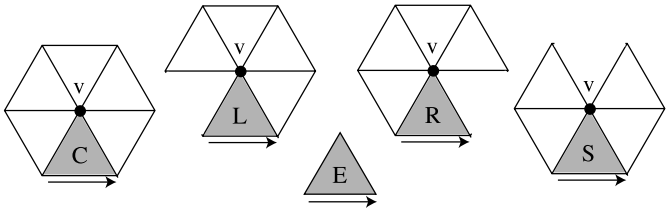


Fig. 14. Edgebreaker CLERS moves: C create, L left, E end, R right, S split.

Ball-pivoting (Bernardini et al., 1999) reconstructs a polygonal surface from point samples using an advancing front algorithm. The name of this method comes from the geometric criterium to select new points to be added to the mesh, which uses a ball of radius α such that the generated mesh is a subset of the Delaunay triangulation. See Fig. 13.

The ball-pivoting method is very suited to the reconstruction of uniform point samplings of a surface, as for example in the case of 3D scanning. Algorithm 1 shows the pseudo-code of the method.

Algorithm 1 Ball Pivoting

```
while points to process do
    while (e ← candidate edge) ≠ ∅ do
        v ← ball_pivot(e)
        attach(e, v)
    if (v0, v1, v2) ← find_seed then
        f ← create(v0, v1, v2)
        new_front(f)
```

Besides the Ball Pivoting, there are several other mesh construction methods that are based on the advancing front algorithm (Schreiner et al., 2006; Scheidegger et al., 2005). These methods differ mainly in two aspects: the type of surface definition (i.e., parametric, implicit, points, etc); and the geometric criterium to get sample points.

7.2. Mesh encoding and compression

Once a mesh has been constructed we are left with the problem of storing and transmitting it. In other words, we need an external representation that encodes the geometry and connectivity of the mesh.

Additionally, since meshes give only a piecewise linear surface approximation, they are often very big, which lead to the need for compression.

Among the different strategies to compress the connectivity of meshes, many of the successful approaches are based on G. Taubin and J. Rossignac's *topological surgery* (Taubin and Rossignac, 1998). The Edgebreaker scheme (Rossignac, 1999) is one example. This kind of algorithm cut the surface along a set of edges during the encode step. Therefore, they could be naturally expressed in terms of unbuilding Handle operators and the unzip. As an important consequence, the Handle operators together with the Zip operator are the natural ones to reconstruct the surface during the decoding process.

The Edgebreaker compression represents mesh connectivity as a dual graph of the triangle mesh that has been cut. For compact surfaces homeomorphic to the sphere this graph is a tree, and the technique guarantees a compression of less than 2 bits per triangle.

The encoding algorithm visits each triangle of the mesh in a depth-first order using five types of moves, called C, L, E, R, and S. Each triangle is labeled with the code indicating the way it was traversed. Fig. 14 shows the $CLERS$ moves.

The resulting $CLERS$ string is a compact encoding of the mesh connectivity.

Algorithm 2 shows the pseudo-code of the Edgebreaker encoding procedure. It uses the concept of a *gate*, i.e., the current edge for traversing the dual graph. The algorithm starts with an initial gate e, and it writes the opposite vertex v and performs a default move to the right (code C). Note that, subsequently the algorithm has four more options to continue at a gate: move to the left (code L); move to the right (code R); move both to left and right (code S) and no move (code E).

In all these cases, the mesh has already been cut, such that the vertex opposite to the gate was previously visited and stored.

Algorithm 2 EdgeBreaker(e)

```
repeat
    v ← org(lprev(e))
    if v not visited then
        write geometry(v); mark v as visited
        output C; e ← lprev(e)
    else if right face visited then
        if left face visited then
            output E; return
        else
            output R; e ← lnext(e)
    else
        if left face visited then
            output L; e ← lprev(e)
        else
            output S; EdgeBreaker(lprev(e)); e ← lnext(e)
until true
```

We remark that this encoding process is equivalent to unbuilding the mesh. In fact, this could be accomplished by using the operation detach after visiting each triangle.

The decoding of a $CLERS$ string builds the mesh essentially by the reverse of the encoding process. The method Spirale Reversi (Isenburg and Snoeyink, 2001), shown in Algorithm 3 does exactly that. It reads the string backwards while constructing the mesh.

Observe that the structure of this algorithm is very similar to the one for Advancing Front mesh construction.

Algorithm 3 Spirale Reversi

```
e ← initial edge
while c ← read code do
    v ← get_vertex(e, c)
    if c = C|L|R|S then
        attach(e, v)
        switch (c)
        case C: e ← lprev(e)
        case R: e ← lnext(e)
        case L: e ← lprev(e)
        case S: pop(lprev(e))
                e ← lnext(e)
    else // c = E
        push(e)
        e = (get_vertex(), get_vertex())
        create(v, org(e), dest(e))
```

Edgebreaker can be further extended to encode and decode surfaces with genus (see Lopes et al., 2002). The operators of our framework prove to be useful in the design and analysis of algorithms based on *topological surgery*.

7.3. Mesh refinement and subdivision surfaces

Classical modeling techniques employ polynomial or rational patches, such as B-splines or NURBS, in the geometric design of smooth surfaces. In these applications a path is represented by a control polygon.

Subdivision surfaces generalize spline patches to non-regular meshes. In this setting, the surface is the limit of applying a subdivision scheme to a control polygon. The subdivision scheme is defined by topological rules for mesh refinement and geometric rules for vertex smoothing.

A simple subdivision scheme for triangle meshes is the $\sqrt{3}$ (Kobbelt, 2000). It employs for mesh refinement a combination of the stellar operators face splits and edge flips. The two basic topological rules are shown in Fig. 15.

The $\sqrt{3}$ subdivision algorithm consists in the repeated application of the refinement and smoothing rules to a control mesh. The pseudo-code is given in Algorithm 4, below.

Most refinement methods for subdivision surfaces can be implemented with stellar operators. Velho (2001b) showed that both primal (Catmull and Clark, 1978) and dual (Doo and Sabin, 1978) schemes can be factorized using edge splits. Other schemes, such as the $\sqrt{2}$ subdivision (Velho and Zorin, 2001) also use edge splits.

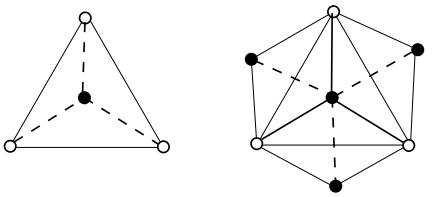


Fig. 15. $\sqrt{3}$ subdivision rules: face split and edge flip.

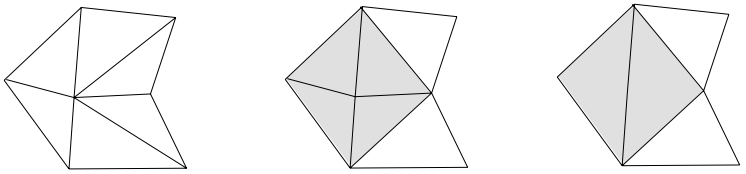


Fig. 16. Four-face cluster simplification: flips, and weld.

Algorithm 4 sqrt3 Subdivide(M)

```
for face f ∈ M do
    v ← split(f)
    smooth_new_vertex(v)
for all old edges e do
    flip(e)
for all old vertices o do
    smooth_old_vertex(o)
```

7.4. Mesh simplification and hierarchical structures

In many applications, a surface is densely sampled and approximated by a triangular mesh. This is the case, for example, of 3D scanning and some scientific simulations.

A consequence of such a process is that these meshes are usually very redundant taking more memory space than necessary for a given approximation accuracy. The solution to this problem is simplification! Mesh simplification algorithms work by eliminating vertices that do not convey relevant geometric information.

Simplification is essentially an optimization problem: we want to compute a mesh with the minimum number of elements such that a surface is approximated with small error.

The full optimization problem is intractable and most methods employ a "greedy" strategy to find suitable local minima. The basic structure of these methods is as follows: mesh vertices are kept in a priority queue according to some error function. Then, simplification is performed by removing vertices with smallest error until the desired mesh size is reached.

In that way, simplification algorithms can also used to generate approximations of the surface at multiple levels of detail and build a hierarchical structure. This is done by globally simplifying independent regions that completely cover the mesh.

One such algorithm is the four-face cluster simplification (Velho, 2001a) that adopts the quadric error metric (Garland and Heckbert, 1997) and divides the mesh into regions of four triangles that are simplified using edge weld, after appropriate edge flips. See Fig. 16.

The method employs the stellar vertex remove operator, and its pseudo-code is shown in Algorithm 5.

Algorithm 5 Stellar Simplification

```
assign quadrics to vertices of M
for all v ∈ M do
    compute error E(v)
for j = 1, N do
    put v ∈ V^j into priority queue Q
    while (v ← pop(Q)) ≠ ∅ do
        if v not marked then
            remove(v)
            locally recompute quadrics and update Q
```

Any simplification method that is based on edge collapse can be implemented using edge flips and edge splits (Velho, 2001a; Vieira et al., 2003). One major advantage of the stellar vertex removal over edge collapse is that it produces meshes

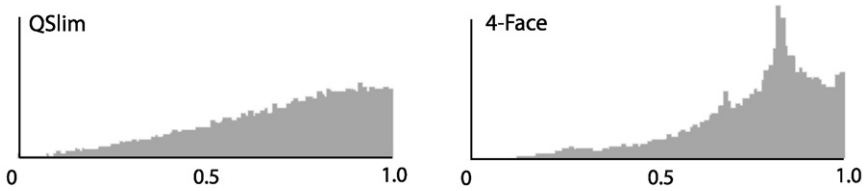


Fig. 17. Histograms of triangle aspect ratio for QSlim and 4-face cluster simplification.

with better quality, since aspect ratio of triangle are taking into account in the optimization. Fig. 17 shows a histogram of a mesh simplified from 80k to 40k triangles by both the QSlim and four-face cluster algorithms.

Furthermore, their locality properties make Stellar operators very suitable for creating multiresolution structures. Progressive meshes (Hoppe, 1996) and binary multi-triangulations (Velho and Gomes, 2000) are examples of hierarchical data structures that can be built with these operators.

7.5. Mesh adaptation

Multiresolution structures constitute the foundation for selecting the appropriate level of detail and adapt a mesh to different situations in modeling and visualization, such as display resolution in view dependent rendering.

However, in most applications the level of detail must vary spatially across the mesh and in time as the conditions change. To resolve this issue, variable resolution adaptation comes into play! The process amounts to coupling an adaptive mechanism on top of a multiresolution structure.

A powerful variable-resolution structure for mesh adaptation is the *semi-regular* 4–8 mesh (Velho, 2004). It is the two-dimensional version of the $n-D$ Restricted Binary Multi-Triangulation (Mello et al., 2002). The 4–8 mesh has the underlying structure of a triangulated quadrangulation (or *tri-quad* mesh). The properties of this structure makes possible to construct a "virtual" multiresolution, while enforcing that adjacent triangles do not differ by more than one level of resolution. Such a restriction guarantees a gradual transition in the mesh adaptation.

Another advantage of the adaptive 4–8 mesh is that, thanks to its regular structure, the hierarchy does not need to be explicitly stored. For this, it is assumed that we are given a base mesh and a function to compute samples of the surface over this base domain.

The dynamic 4–8 adaptation mechanism consists in repeatedly coarsening and refining the mesh while conditions change. For that, two priority queues guide the simplification and subdivision based on some application dependent adaptation function. Note that this is a conservative strategy, since the mesh is first simplified and only then refined. The pseudo-code of the method is show in Algorithm 6.

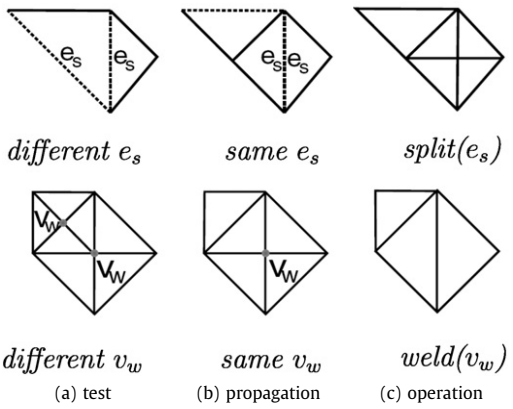
Algorithm 6 Dynamic 4–8 Adaptation

```
read 4–8 base mesh
initialize priority queues Q_r and Q_s
repeat
    change mesh and update queues
    while (v ← pop(Q_s)) ≠ ∅ do
        if priority(v) < T then
            SIMPLIFY(v)
        else break
    while (e ← pop(Q_r)) ≠ ∅ do
        if priority(e) > T then
            REFINE(e)
        else break
until (quit)
```

In order to maintain the mesh invariant that enforces a limit of one-level difference, the concepts of *split edge* and *weld vertex* are employed. The split edge is the internal edge of a tri-quad block. As the mesh is refined, neighbor triangles must have the same split edge to form a block (and therefore, be at the same resolution level). When this is not the case one of the neighbors must be subdivided first, propagating the resolution restriction. The same reasoning applies in the case of simplification for weld vertices. Fig. 18 illustrates the restriction mechanism.

The implementation of the restricted adaptation mechanism for refinement/simplification has a very simple recursive implementation, as shown in Algorithm 7. Note that the overall structure of both procedures is essentially the same, revealing the symmetry of the process.

The adaptive 4–8 mesh can be used in many types of applications where a surface is modified dynamically or even when the surface is static by the computation requirements change over time.



(a) test (b) propagation (c) operation

Fig. 18. Dependency propagation for refinement and simplification.

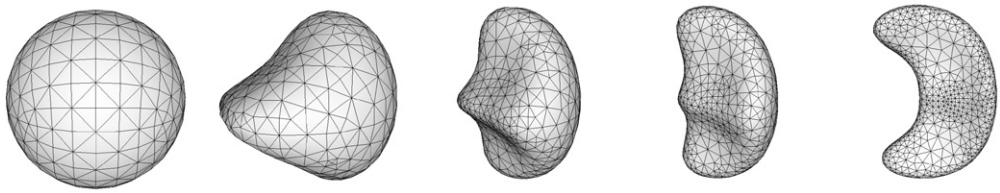


Fig. 19. Adapted meshes generated with a spiraling field at iterations 0, 20, 30, 35, and 63, using 0.012 as time step.

Algorithm 7 Restricted 4–8 Refinement/Simplification

```
procedure REFINE(Edge e)
    for f ∈ star(e) do
        if f.split_edge() ≠ e then
            Refine(f.split_edge())
    split(e)

procedure SIMPLIFY(Vertex w)
    repeat
        v ← max_level_neighbor(w)
        if level(v) > level(w) then
            Simplify(v)
    until (degree(v) ≠ degree(w))
    weld(w)
```

One example of such an application is the approximation of deformable surfaces (de Goes et al., 2006). In that case, may be the result of a physical simulation or some other form of procedural animation. Fig. 19 shows the visualization of a numeric simulation using the level-set method. The surface is the zero-set of a signed distance function representing an interface between two materials. It is a front tracked and evolved by the level-set simulation. For the example in this figure we used a spiraling analytical field from Enright et al. (2005), which advects a sphere of radius 0.15 centered at $(0.35, 0.35, 0.35)$ with velocity:

$$\begin{cases} u(x,y,z) = 2\sin^2(\pi x)\sin(2\pi y)\sin(2\pi z), \\ v(x,y,z) = -\sin(2\pi x)\sin^2(\pi y)\sin(2\pi z), \\ w(x,y,z) = -\sin(2\pi x)\sin(2\pi y)\sin^2(\pi z). \end{cases} \tag{1}$$

Note how the mesh resolution is nicely adapted to the geometric features of the deforming surface. Table 1 gives the times for each stage of the computation. We remark that the adaptation is very efficient and the total time is dominated by the level-set simulation.

Another application that takes advantage of the adaptive 4–8 mesh is the animation of facial expressions (Goldenstein et al., 2005), where the deformation comes from tracking human faces in videos.

Table 1
Time statistics in milliseconds. We measure the initialization time, an average time for deforming and adapting the current mesh, and the total average time.

stage	*Init*	*Deform*	*Adapt*	*Total*
ms	475	3208	562	3770

8. Conclusions

We presented in this paper an unified framework for the representation of combinatorial 2-manifolds with or without boundary. This representation includes two kinds of primitive operators on the underlying meshes: operators that change the topological characteristic of the mesh and operators that just modify its combinatorial structure.

The main characteristics of the proposed framework are:

- The operators don't generate, in any moment, non-manifold objects. Which is the case of several *Euler operators*.
- They are based on two important theories, and they do not depend where the surface is embedded.

We also introduced a new data structure that explicitly represents the boundary curves. This data structure shows to be very useful for the implementation of those operators. A prototype implementation is available online (Velho et al., 2005).

We note that other data structures for mesh representation, such as OpenMesh (Botsch et al., 2002), may use the topological operators proposed in our framework. Also, canonical descriptions for combinatorial manifolds could be constructed using Handle and Stellar operators (for example: Normal Meshes with a non-regular triangulation).

The Handlebody and the Stellar theories apply as well in the three-dimensional context. Thus, the authors pretend extend this work to volumetric meshes.

For the non-manifold extension, one can use the combinatorial stratification of cell complexes proposed by Pesco et al. (2004), to stratify the complex in several manifold parts. And then, use the extension of Handle and Stellar operators for one and three-dimensional manifolds.

Appendix A. Mesh representation

Different data structures can be used to implement the mesh operators defined above. We have chosen an edge-based topological data structure because it gives a good compromise between simplicity and generality.

In our topological data structure, a mesh is a collection of surface components pointers.

```
struct Mesh {
  Container<Surface*> surfaces;
}
```

The surface is structured as $S = (V, E, F, B)$ where V, E, F, B are the collections of vertices, edges, faces and boundary curves respectively. These sets are stored in containers of pointers to the correspondent topological data structures.

```
struct Surface {
  Container<Face*>   faces;
  Container<Edge*>   edges;
  Container<Vertex*>  vertices;
  Container<Edge*>   bndries;
}
```

The face structure stores a pointer to the first half-edge of its outer loop. Here, we assume triangular faces and thus, the face loop contains exactly three edges.

```
struct Face {
  Half_Edge*   he_ref;
  Mesh*  sm_ref;
}
```

An edge is formed by two half-edges. In the case it is representing a boundary edge one of these half-edges points to a null face.

```
struct Edge {
  Half_Edge   he[2];
}
```

The half-edge is the central topological element of the data structure. It stores a pointer to its initial vertex, a pointer to the next half-edge in the face loop, and pointers to the edge and face it belongs to. Note that the mate half-edge can be accessed through the pointer to its parent edge.

```
struct Half_Edge {
  Vertex*    org_ref;
  Half_Edge*    next_ref;
  Face*    f_ref;
  Edge*    e_ref;
}
```

The vertex structure stores one pointer to an incident half-edge. In the case of a boundary vertex, this half-edge is part of the boundary curve. This representation makes it trivial to identify if a vertex is on the boundary or is in the interior of the surface. Also, it is instrumental not only for the implementation of the vertex star iterator, but also for the boundary curve iterator. The vertex structure also holds a pointer to vertex geometry.

```
struct Vertex {
  Half_Edge*    star_i;
  Point*  p;
}
```

The point data structure stores a pointer to the vertex. It represents a "bridge" between geometry and topology. It also stores geometric data of the vertex and can also hold additional data, such as normals, texture and parametric coordinates.

```
struct Point {
  Vertex*    v;
  Data*      d;
}
```

These last two data structures separate the roles of points and vertices which are to represent geometry and topology of the mesh, respectively. This is a robust approach to compare geometric coincidences between vertices. Surfaces reconstruction is a typical example where this is necessary. Indeed the geometric operations acts on sample points (some may belong to the boundary or not) whereas mesh vertices attach them by handle operations whenever new triangles are created. See for example Medeiros et al. (2003).

For a more detailed description of the data structure and to obtain the source code of the Handle and Stellar operators using it see Velho et al. (2005).

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