

Unit 1: Probability models and axioms

MITx Statistics and Data Science

Seung Hyun Sung

11/24/2021

[1] Sample Space and Events

- Random experiment: A mechanism that produces a definite outcome with **uncertainty**
- Sample Space Ω : “List” (Set) of all the possible outcome
 - Mutually exclusive
 - Collectively exhaustive
 - Art: to be at the “right” granularity
- Event: Subset of the sample space

Discrete Sample space

- Use of Sequential Description can be very helpful (Tree Model)
- Finite: Events that can be considered are ‘at least 1 coin face-front’
- Infinite: under a experiment where you throw a coin until the back comes out. The ‘n-th back’ event consideration.

Continuous Sample space

- (x, y) such that $0 \leq x, y \leq 1$
- The probability of each individual point in continuous value is essentially 0
- Then we cannot know its probabilities, hence we use the interval (sets of outcome) assigned as sub-space
- $S = \{x | 0 \leq x \leq \inf\}$

[2] Probability axioms

- **Nonnegativity:** $P(A) \geq 0$, Where A is the event (subset) | Consequences: $P(A) \leq 1$
- **Normalisation:** $P(\Omega) = 1$
- **Additivity:** if $A \cap B = \emptyset$, mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

Properties

- From non-negativity: $P(A) \leq 1$
- From normalization: $P(\emptyset) = 0$
- For A, B, C disjoint: $P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
- For k disjoint events: $P(\{s_1, s_2, \dots, s_k\}) = P(\{s_1\}) + \dots + P(\{s_k\}) = P(s_1) + \dots + P(s_k)$.
- If $A \subset B$, then $P(A) \leq P(B)$.

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where $P(A \cap B) \geq 0$.
- $P(A \cup B) \leq P(A) + P(B)$.
- $P(A \cup B \cup C) = P(A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C)) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$.

[3] Probability calculation steps

- Specify the sample space
- Specify probability law
- Identify an event of interest
- Calculate...

Discrete Uniform Law

- Assume sample space consists of n equally likely elements
- Assume A consists of k elements

$$P(A) = k * \frac{1}{n}$$

Probability calculation

- Uniform probability law: Probability = Area
- Probabilities of each individual point is zero and also the line is zero

Countable Activity: Discrete but infinite sample space

Given $P(n) = \frac{1}{2^n}, n = 1, 2, \dots$,

$$P(\{2, 4, 6, \dots\}) = P(\{2\} \cup \{4\} \cup \dots) = P(2) + P(4) + \dots = \frac{1}{2^2} + \frac{1}{2^4} + \dots = \frac{1}{4} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots\right) = \frac{1}{4} \cdot \frac{1}{1 - 1/4} = \frac{1}{3}$$

Using the formula:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2} \cdot \frac{1}{1 - 1/2} = 1$$

Discussion - Countable additivity

If A_1, A_2, \dots is an infinite **sequence** of **disjoint** events, then $P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$

This axiom only applies for a countable “sequence” of events. This requires that the infinite sets should be discrete rather than continuous.

- A case when it does not apply: The sample space is the 2D plane. For any real number x , let A_x be the subset of the plane that consists of all points of the vertical line through the point $(x, 0)$ i.e. $A_x = \{(x, y) : y \in \mathbb{R}\}$.

$$P\left(\bigcup A_x\right) \neq \sum P(A_x)$$

[4] Mathematical background

Sets: a collection of distinct elements

- For some sets $S_n, n = 1, 2, \dots$,
- $x \in \bigcup_n S_n$ iff $x \in S_n$ for some n ; $x \in \bigcap_n S_n$ iff $x \in S_n$, for all n .

Properties:

- $S \cup T = T \cup S, \quad S \cup (T \cup U) = (S \cup T) \cup U$
- $S \cap (T \cup U) = (S \cap T) \cup (S \cap U), \quad S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$
- $(S^c)^c = S, \quad S \cap S^c = \emptyset$
- $S \cup \Omega = \Omega, \quad S \cap \Omega = S$

De Morgan's Law

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c; \quad \left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

Convergence:

- If $a_i \leq a_{i+1}$, for all i , then either 1. the sequence “converges to ∞ ” 2. the sequence converges to some real number a
- If $|a_i - a| \leq b_i$ for all i and $b_i \rightarrow 0$, then $a_i \rightarrow a$.

Infinite series:

$$\sum_{i=1}^{\infty} a_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

provided limit exists.

- If $a_i \geq 0$, limit exists.
- If terms a_i do not all have the same sign
 - Limit may exist but be different if we sum in a different order
 - Limit exists and independent of order of summation of $\sum_{i=1}^{\infty} |a_i| < \infty$.

Geometric series

$$\sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \dots = \frac{1}{1 - \alpha}, \quad |\alpha| < 1$$

Proof:

Set $S = 1 + \alpha + \alpha^2 + \dots$, Compute $(1 - \alpha)S = 1 - \alpha^{n+1}$. When $n \rightarrow \infty$, $(1 - \alpha)S = 1$.