Unit 1: Probability models and axioms

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[1] Sample Space and Events

- Random experiment: A mechanism that produces a definite outcome with **uncertainty**
- Sample Space Ω: "List" (Set) of all the possible outcome
  - Mutually exclusive
  - Collectively exhaustive
  - Art: to be at the "right" granularity
- Event: Subset of the sample space

Discrete Sample space

- Use of Sequential Description can be very helpful (Tree Model)
- Finite: Events that can be considered are 'at least 1 coin face-front'
- Infinite: under a experiment where you throw a coin until the back comes out. The 'n-th back' event consideration.

Continuous Sample space

- (x, y) such that $0 \leq x, y \leq 1$
- The probability of each individual point in continuous value is essentially 0
- Then we cannot know its probabilities, hence we use the interval (sets of outcome) assigned as sub-space
- $S = \{x | 0 \leq x \leq \inf\}$

[2] Probability axioms

- **Nonnegativity:** $P(A) \geq 0$, Where A is the event (subset) | Consequences: $P(A) \leq 1$
- **Normalisation:** $P(\Omega) = 1$
- **Additivity:** if $A \cap B = \oslash$, mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

Properties

- From non-negativity: $P(A) \leq 1$
- From normalization: $P() = 0$
- For A, B, C disjoint: $P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
- For k disjoint events: $P(\{s_1, s_2, ..., s_k\}) = P(\{s_1\}) + ... + P(\{s_k\}) = P(s_1) + ... + P(s_k)$.
- If $A \subset B$, then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where $P(A \cap B) \geq 0$.
- $P(A \cup B) \leq P(A) + P(B)$.
- $P(A \cup B \cup C) = P(A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C)) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$.

[3] Probability calculation steps

- Specify the sample space
- Specify probability law
- Identify an event of interest
- Calculate...

Discrete Uniform Law

- Assume sample space consists of n equally likely elements
- Assume A consists of k elements

$$P(A) = k * \frac{1}{n}$$

Probability calculation

- Uniform probability law: Probability = Area
- Probabilities of each individual point is zero and also the line is zero

Countable Activity: Discrete but infinite sample space

Given $P(n) = \frac{1}{2^n}, n = 1, 2, ...,$

$$P(\{2, 4, 6, ...\}) = P(\{2\} \cup \{4\} ...) = P(2) + P(4) + ... = \frac{1}{2^2} + \frac{1}{2^4} + ... = \frac{1}{4}(1 + \frac{1}{4} + \frac{1}{4^2} + ...) = \frac{1}{4} \cdot \frac{1}{1 - 1/4} = \frac{1}{3}$$

Using the formula:

$$\sum_{n=1}^{\infty} = \frac{1}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} = \frac{1}{2} \cdot \frac{1}{1 - 1/2} = 1$$

Discussion - Countable additivity

If $A_1, A_2, ...$ is an infinite **sequence** of **disjoint** events, then $P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$

This axiom only applies for a countable "sequence" of events. This requires that the infinite sets should be discrete rather than continuous.

- A case when it does not apply: The sample space is the 2D plane. For any real number x, let A_x be the subset of the plane that consists of all points of the vertical line through the point $(x, 0)$ i.e. $A_x = \{(x, y) : y \in \mathrm{Re}\}$.

$$P(\bigcup A_x) \neq \sum P(A_x)$$

[4] Mathematical background

Sets: a collection of distinct elements

- For some sets $S_n, n = 1, 2, ...,$
- $x \in \bigcup_n S_n$ iff $x \in S_n$ for some n; $x \in \bigcap_n S_n$ iff $x \in S_n$, for all n.

Properties:

- $S \cup T = T \cup S, \quad S \cup (T \cup U) = (S \cup T) \cup U$
- $S \cap (T \cup U) = (S \cap T) \cup (S \cap U), \quad S \cup (T \cap U) = (S \cup T) \cap (S \cap U)$
- $(S^c)^c = S, \quad S \cap S^c =$
- $S \cup \Omega = \Omega, \quad S \cap \Omega = S$

De Morgan's Law

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c; \quad \left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

Convergence:

- If $a_i \leq a_{i+1}$, for all i, then either 1. the sequence "converges to ∞" 2. the sequence converges to some real number a
- If $|a_i - a| \leq b_i$ for all i and $b_i \to 0$, then $a_i \to a$.

Infinite series:

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

provided limit exists.

- If $a_i \geq 0$, limit exists.
- If terms a_i do not all have the same sign
  - Limit may exist but be different if we sum in a different order
  - Limit exists and independent of order of summation of $\sum_{i=1}^{\infty} |a_i| < \infty$.

Geometric series

$$\sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + ... = \frac{1}{1-\alpha}, \quad |\alpha| < 1$$

Proof:

Set $S = 1 + \alpha + \alpha^2 + ...$, Compute $(1-\alpha)S = 1 - \alpha^{n+1}$. When $n \to \infty$, $(1-\alpha)S = 1$.