FMI Layered Standard for DAE (FMI-LS-DAE)

Contents

This layered standard on top of FMI 3.0 defines how to exchange dynamic models in differential algebraic equation form.

Although the document refers to version 3.0 of the FMI standard, everything described in this document also applies to all subsequent minor versions. For further information on compatibility, see section Versioning and Layered Standards in the FMI 3.0 specification.

1. Introduction

1.1. Intent of this Document

A differential-algebraic system of equations (DAE) is a system of equations that either contains differential equations and algebraic equations, or is equivalent to such a system [1].

- Avoid the requirement of index reduction inside of FMUs.
  - This may improve accuracy due to better drift handling.
- Avoid local nonlinear equation solvers inside of FMUs.
  - This may improve accuracy and avoid problems with different local and global error tolerances.
- Preserve the sparseness of DAE systems which is lost for the corresponding reduced ODE systems.
  - This may improve the performance by usage of the sparseness.
- Allow connections between constraint FMUs.
  - Connecting reduced ODE FMUs may lead globally to a non-solvable (singular) system but not for unreduced DAE FMUs.

1.2. How to read this Document

1.3. Overview

2. Common Concepts

Mathematical Notation

DAE representation

There are different forms of DAEs.

For simplicity discontinuous states are currently disregarded.

Fully-Implicit DAE

The generic *fully-implicit DAE*, or *implicit DAE*, formulation:

$$F(\dot{z}(t), z(t), t) = 0$$

where $z(t)$ represents state and additional variables and t is time.

A similar and sometimes more useful representation:

$$F(\dot{x}(t), x(t), y(t), t) = 0$$

with dynamic variables $x(t)$, algebraic variables $y(t)$ and time t.

Note that $x(t)$ is not necessarily the state variables, more on this in section 2.1.1.

Semi-Explicit Index 1 / Hessenberg index 1 DAE

The *semi-explicit index 1*, also called *Hessenberg index 1*, DAE representation

$$\begin{aligned} \dot{x}(t) &= f(x(t),\, y(t),\, t) \\ 0 &= g(x(t),\, y(t),\, t) \end{aligned}$$

with g solvable for y, $det\left(\frac{\partial}{\partial y} g\right) \neq 0$.

Here f is called the *differential equations* and g the *algebraic equations*.

Relationship between fully-implicit and semi-explicit DAE

A fully implicit DAE

$$F(\dot{x}(t),\, x(t),\, t) = 0$$

can always be rewritten as a semi-explicit DAE by introducing a new algebraic variable $x^{'}$

$$\begin{aligned} \dot{x}(t) &= x^{'}(t) \\ 0 &= F(x^{'}(t),\, x(t),\, t) \end{aligned}$$

Semi-Explicit DAE

The *semi-explicit DAE* representation

$$\begin{aligned} \dot{x}(t) &= f(x(t),\, y(t),\, t) \\ 0 &= g^1(x(t),\, y(t),\, t) \\ 0 &= g^2(x(t),\, y(t),\, t) \\ 0 &= g^3(x(t),\, y(t),\, t) \\ &\vdots \end{aligned}$$

with g^1 solvable for y, $det\left(\frac{\partial}{\partial y} g^1\right) \neq 0$ and $\left\{\frac{d}{dt} g_i^k\right\} \subseteq \{g_i^{k-1}\},\ k>1$.

Mass matrix DAE

$$M\dot{x}(t) = f(x(t),\, t)$$

where *mass matrix* M is singular.

An equivalent explicit formulation:

$$\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{f}(x(t),\, y(t),\, t) \\ g(x(t),\, y(t),\, t) \end{bmatrix}$$

See [cui2020mass].

Linear time-invariant DAE

$$E\dot{x}(t) = Ax(t)$$

where matrix E is singular.

Indices

There are multiple, sometimes closely related, definitions of indices for DAE systems. Usually these indices are a measure of how far from an ODE the DAE is.

Differential Index

The minimum number of times a DAE

$$F(\dot{x}(t), x(t), t) = 0$$

has to be differentiated with respect to time t to be able to determine $\dot{x}(t)$ as a function of t and x is called the *differential index*.

See [hairer2006numerical].

Perturbation Index

The *perturbation index* is a measure of the constraints among equations. An index-0 DAE contains neither algebraic loops nor structural singularities. An index-1 DAE contains algebraic loops, but no structural singularities. A DAE with a perturbation index greater than 1, a so-called *higher-index DAE*, contains structural singularities [2].

See [hairer2006numerical].

Example of Structural Singularity

[cellier2006continuous, chapter 7.7 Structural Singularity Elimination] provides a simple electric circuit model that is an higher-index DAE.

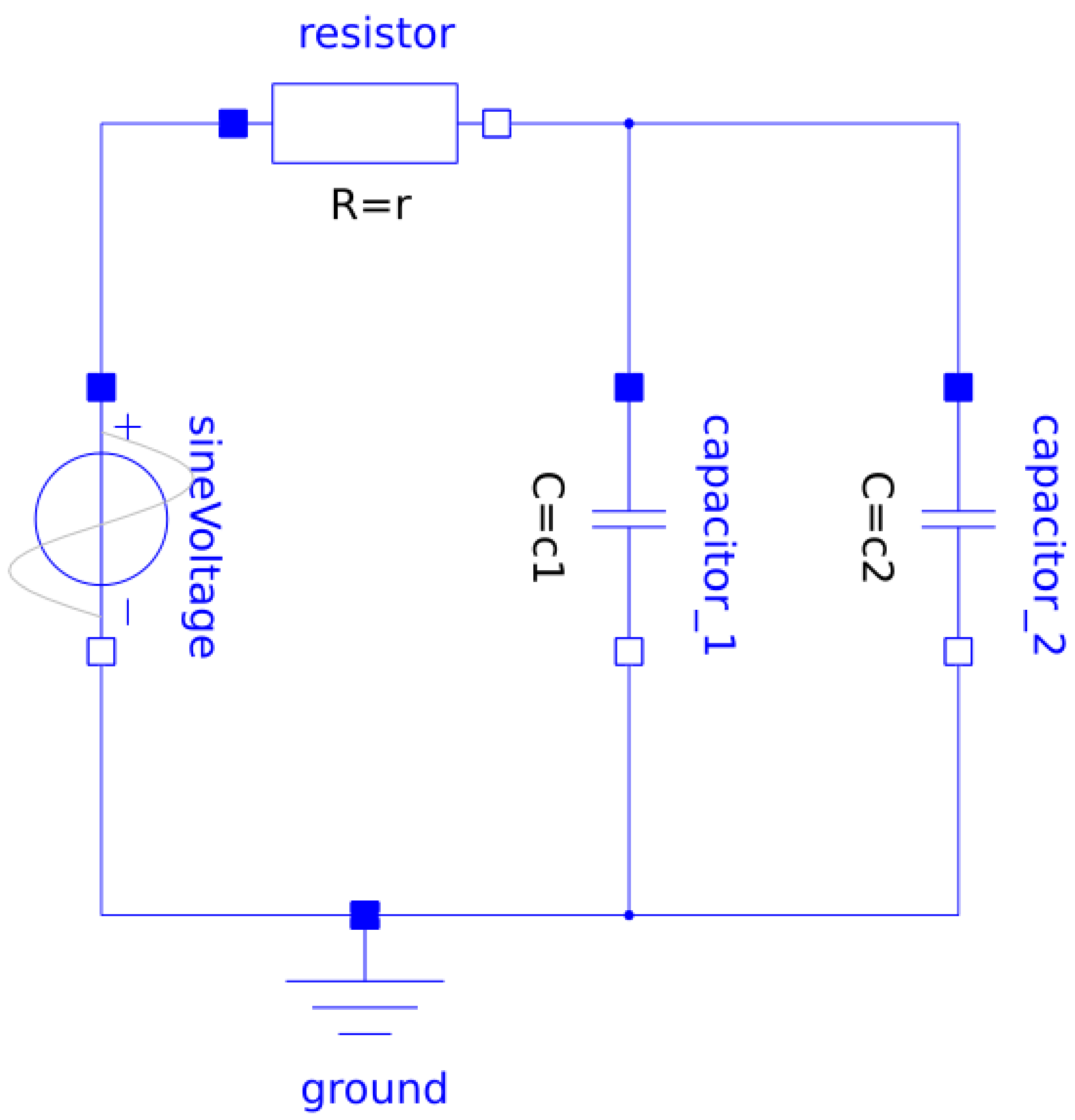

Figure 1. Modelica model of an electric circuit with two capacitors in parallel.

The circuit has two capacitors in parallel and can be represented with equations:

$$\begin{aligned} v_0 &= f(t) \\ v_R &= R \cdot i_0 \\ i_1 &= C_1 \cdot \dot{v}_1 \\ i_2 &= C_2 \cdot \dot{v}_2 \\ v_0 &= v_R + v_1 \\ v_2 &= v_1 \\ i_0 &= i_1 + i_2 \end{aligned}$$

When both capacitive voltages v_1 and v_2 are chosen as state variables equation eqref $constraintEquationExampleCellier$ has no unknowns left, so it is a *constraint equation*.

There are different ways to solve this. The modeler could change the causality or exporting tools can try to reduce the perturbation index symbolically [3] to end up with a consistent initialization for the system.

References

- [cui2020mass] Cui, Hantao, Fangxing Li, and Joe H. Chow. "Mass-matrix differential-algebraic equation formulation for transient stability simulation." arXiv preprint arXiv:2008.03883, 2020.
- [cellier2006continuous] Cellier, François E., and Ernesto Kofman. "Continuous system simulation". Boston, MA: Springer US, 2006.
- [hairer2006numerical] Hairer, Ernst, Christian Lubich, and Michel Roche. "The numerical solution of differential-algebraic systems by Runge-Kutta methods". Vol. 1409. Springer, 2006.
- [cellier1993automated] Cellier, Francois E., and Hilding Elmqvist. "Automated formula manipulation supports object-oriented continuous-system modeling". IEEE Control Systems Magazine 13.2 (1993): 28-38.

[1] Source: https://en.wikipedia.org/wiki/Differential-algebraic_system_of_equations

[2] From [cellier2006continuous], p. 285.

[3] e.g. by using Pantelides algorithm, see [cellier1993automated]