HW 2

due Wed Feb 4

Modeling Democracy
Duchin, Winter 2026

General instructions: To be graded, your problem set must indicate who you worked with. If you do use AI tools to complete this assignment, please include a page that lists all the prompts you used. (This is helpful for me so I can understand how you are using it.)

Problem 1. Suppose three districts (districts B, P, and R) have populations $M_B = 54, M_P = 243, M_R = 703$. Apportion the seats by Hamilton's method with $m = 10$ and $m = 11$ seats, with no constitutional requirement that everybody gets a seat on the council. Also consider $M'_B = 56, M'_P = 255, M'_R = 789$ at the same council sizes of 10 and 11. Which combinations of these conditions produce apportionment paradoxes?

Problem 2. The Balinski–Young impossibility result applies to deterministic apportionment methods. If we are allowed to use randomness, Grimmett showed in 2004 that we can circumvent Balinski–Young. In this problem we construct a random apportionment method that always satisfies the quota rule and achieves perfect proportionality in expectation.

Give each state its lower quota, and then (randomly) award some states one more seat, as follows: Let the fractional parts of the quota for the states be $\delta_1, \ldots, \delta_n \in [0, 1)$, and let k denote their sum, which is the number of seats not yet awarded. We line up n intervals on the number line without gaps, where the ith interval has length δ_i. We then shift all intervals to the right by a random amount u, drawn uniformly from $[0, 1)$. This now produces subintervals that cover $[u, u + k]$. Award an extra seat to exactly those states whose interval contains an integer.

For a state with $q + \delta$ seats, what is the probability that this process will round them up or down? Conclude that the method works as advertised.

Problem 3. We will write $\mathbf{n} = (n_1, \ldots, n_r)$ for the populations of r towns; $\mathbf{w}$ is the vector of voting weights; T is a *strict* threshold of success for a vote (so a winning coalition must have weight $> T$); and $\boldsymbol{\beta}$ is the vector of Banzhaf powers. Assume all vectors are normalized to sum to 1. Write f for the Banzhaf objective $|\mathbf{n} - \boldsymbol{\beta}|_1$ to be minimized.

(a) Prove that if $n_i \geq n_j$, then there is a weight vector at optimum discrepancy with $w_i \geq w_j$.

(b) By contrast, show that if $\mathbf{n}$ has two equal entries $n_i = n_j$, there may be NO optimal solution with $w_i = w_j$. (For example, there exist four-player games with this symmetry-breaking property.)

(c) BONUS: can a four-player game have the symmetry-breaking property at $T = 1/2$? Can any three-player game have the symmetry-breaking property?

Note: for the Banzhaf problem, you should not need to write code to assist you, but you are welcome to do that for the purpose of testing examples. With or without the use of code, a complete solution must include a proof of correctness.

Problem 4. (a) Show that not every 4-candidate preference profile has a (Euclidean) planar embedding.

(b) Show that every finite preference profile admits a metric embedding.

Problem 5. This one is going to require either some prior knowledge about recent members of Congress or the willingness to do a bit of web sleuthing!

Interpret the y axis in the DW-NOMINATE plot from the 117th Congress shown here, or give a plausible explanation of what kind of historical voting behavior explains the proximity of certain Ds to certain Rs.

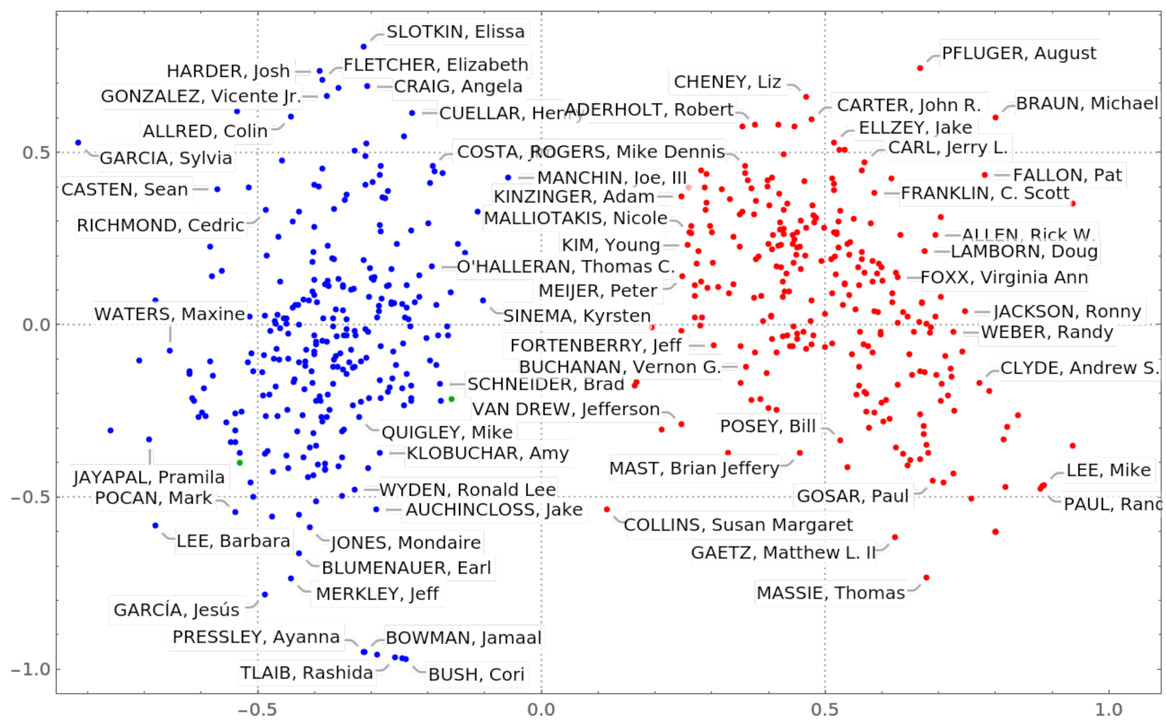

Figure 1: Plot found HERE; full data and notebook found HERE. (For Chris Wolfram's related work, see here, here, and here.)