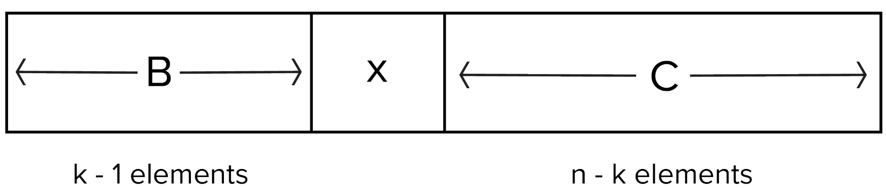
Median Finding Algorithms

There are a few approaches to finding the median of a list. One approach is to sort the elements of the list and then compute the position of the median. That approach takes $\Theta(n \log n)$ time.

Another approach inspired by quicksort is to find a pivot, and recurse on a subarray based on the rank of the pivot. The corresponding pseudocode is presented below.

```
SELECT(S, i):
  Pick x in S cleverly
  Compute k = rank(x)
  B = Set of elements in S, such that elements < x
  C = Set of elements in S, such that elements > x
  if k == i:
    return x
  else if k > i:
    return SELECT(B, i)
  else if k < i:
    return SELECT(C, i - k)
```

The intuition here is if we can eliminate either B or C cleverly, and prove that the subarray we recurse on is always a fraction of the previous array, the algorithm will run in $\Theta(n)$ time.



Without a clever strategy, if we're selecting the $n-1$ element and $k=1$ every time, then we're eliminating only one element on every iteration. There would be n iterations, each taking $\Theta(n)$ time to partition the subarray. Therefore, the algorithm would run in $\Theta(n^2)$.

Clever Partitioning

1. Arrange S into columns of size 5 ($\lceil n/5 \rceil$ columns).
2. Sort each column in linear time (top down)
3. Find the medians of medians as x.

Half of the $\lceil n/5 \rceil$ groups contribute at least 3 elements $> x$ except for 1 trailing group which could contain fewer than 5 elements and 1 group containing x. Therefore, there are at least $3(\lceil n/10 \rceil - 2)$ elements $> x$ and at least $3(\lceil n/10 \rceil - 2)$ elements $< x$.

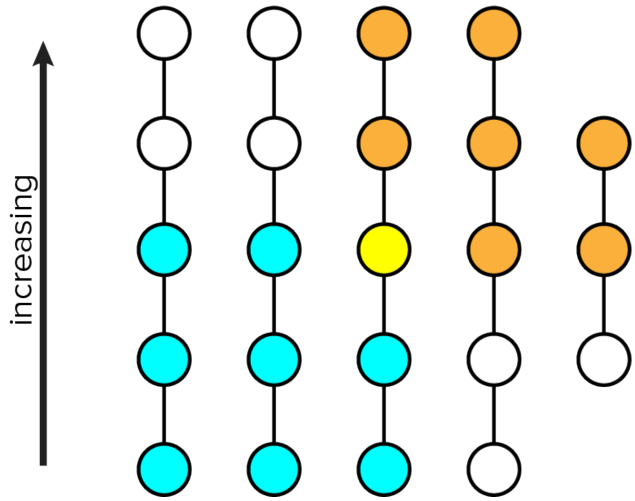


Figure 1: The cyan circles are the elements lower than x (denoted by the yellow circle) and the orange circles are the elements greater than x

The recurrence can now be written as

$$f(n) = \begin{cases} \Theta(1), & \text{for } n \leq 140 \\ T(\lceil n/5 \rceil) + T(\frac{7n}{10} + 6) + \Theta(n), & \text{for } n > 140 \end{cases}$$

Proof

Let's prove that $T(n) < cn$ via induction. The base case is when $n \leq 140$, in which case, $T(n) < cn$ for some arbitrarily large c.

Let's now consider the case when $n > 140$. First, we can substitute the base case into our equation. Then we can add an additional c term.

$$\begin{aligned} T(n) &\leq c\lceil n/5 \rceil + c(\frac{7n}{10} + 6) + an \\ &\leq \frac{cn}{5} + c + \frac{7nc}{10} + 6c + an \\ &= cn + (-\frac{cn}{10} + 7c + an) \end{aligned}$$

It's important to now check for when $(-\frac{cn}{10} + 7c + an) = 0$.

$$\begin{aligned}(-\frac{cn}{10} + 7c + an) &\leq 0\\ \frac{cn}{10} &\geq 7c + an\\ c &\geq \frac{70c}{n} + 10a\end{aligned}$$

The expression is always true for $n \geq 140$ and $c \geq 20a$.

Code

```
import math
import random

const = 20
sub_size = 5
def SELECT(S, i):
  if len(S) < const:
    return sorted(S)[i] # assuming rank is zero indexed
  num_medians = int(math.ceil(len(S) / sub_size))
  medians = []
  for curr_index in range(0, len(S), sub_size):
    sub = S[curr_index : min(curr_index + sub_size, len(S))]
    mid_index = int(math.floor(len(sub) / 2))
    medians.append(sorted(sub)[mid_index])

  median = SELECT(medians, int(math.floor(len(medians) / 2)))
  B = []
  C = []
  for element in S:
    if element < median:
      B.append(element)
    elif element > median:
      C.append(element)
  if len(B) == i:
    return median
  elif len(B) > i:
    return SELECT(B, i)
  else:
    return SELECT(C, i - len(B) - 1)

arr = range(40)
random.shuffle(arr)
```

```
print(arr)
print(SELECT(arr, 15))
```