MATHEMATICS FOR JUNIOR HIGH SCHOOLS

Christian Akrong Hesse
University of Professional Studies

MATHEMATICS

FOR

JUNIOR HIGH SCHOOLS

CHRISTIAN A. HESSE, BSc, MPhil
Lecturer in the Department Mathematics and Statistics
Methodist University College Ghana

ISBN: 978–9988–1–6260–3

Published and distributed by

AKRONG PUBLICATIONS LTD.
P. O. BOX M. 31
ACCRA, GHANA

0244 648757, 0264 648757, 0233 648757

Published, 2012

PREFACE

This book is intended to encourage an understanding and appreciation of Mathematics at the Junior High School level in West Africa. Providing appropriate solutions to examination problems is of particular importance in the study of mathematics. As a mathematics lecturer, the author has discovered the weaknesses and shortcomings of students in the handling of examination questions. Subsequently, to guide students in answering typical questions in mathematics as set out in recent examinations, the writer has paid particular attention to those areas of the syllabus, which many students find difficult.

A prominent feature of this book is the inclusion of many examples. Each example is carefully selected to illustrate the application of a particular mathematical technique and interpretation of results. Another feature is that each chapter has an extensive collection of exercises. It is important that students have several exercises to practice.

This book is therefore designed to help students to:

1. acquire the basic skills and understanding which is vital to examination success.
2. appreciate the use of mathematics as a tool for analysis and effective thinking.
3. discover order, patterns and relations.
4. communicate their thoughts through symbolic expressions and graphs.
5. develop mathematical abilities useful in commerce, industry and public service.

I have gone to great lengths to make this text both pedagogically sound and error-free. If you have any suggestions, or find potential errors, please contact the writer at **akrongh@yahoo.com**.

C. A. Hesse

November, 2012

ACKNOWLEDGEMENT

First and foremost, I express my sincere thanks to the Almighty God both for the gift of writing he has endowed me with and for the endurance to complete this particular book.

I am also grateful to Prof. A. Y. Jackson (Professor of Statistics) of Methodist University College Ghana and Mr. A. Acquaye (Former Head of Mathematics Department, Accra Academy SHS) for reading a draft of the book and offering helpful comments and suggestions.

I am also indebted to Dr. Nana Owusu Mensah Essel and Paapa Kwabena Aseda Essel who assisted me greatly by spending much time in editing every single chapter of this book.

Finally, my sincere gratitude goes to Mr. Fred Armah Hesse (Hesse Armah Chartered Accountants) and Mr. Ludwig Hesse (Department of Urban Roads, Accra) for their moral support, encouragement and for providing professional guidelines.

This book was typeset by
Akrong Publications Ltd.
0244648757

This book is dedicated to my wife, Mrs. Caroline Hesse, whose unceasing prayers kept me through to the successful completion of this publication. May the Lord grant her long life and good health to enjoy the fruits of her labour.

CONTENTS

CHAPTER ONE

Numbers and Numerals

1.1 Counting and writing numbers

In primary school you were introduced to the set of ***counting numbers*.** The members of this set are the numbers beginning with 1, with each successive number greater than its predecessor by 1. The set of counting numbers is denoted by N, that is $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\}$. These numbers serve as mathematical symbols used in counting and measuring. This set is also called the set of ***natural numbers***, or the set of ***positive integers***. Natural numbers is the most basic and familiar set of numbers. If we add the number zero to this set, we obtain the set $W = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\}$. This is referred to as the set of ***whole numbers***. A notational symbol which represents a number is called a ***numeral***. For instance, the numeral "5" is the notational symbol which represents the number "five" whilst "8" is used to represent the number "eight". What numeral is used to represent the number "nine"?

The number system, in universal use today for arithmetic operations is ***base ten number system***. This means we use ***ten*** numbers or ***digits***. The set of digits for base ten is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. This numeration system is also known as the ***denary*** or ***decimal*** system. In decimal numerals, we count in tens. For example; ***56 in base ten*** (or in decimal numerals) means ***5 tens and 6 ones***.

Place value

Place value is an important concept that is often misunderstood and sometimes misplaced. The decimal system depends on ***place value***. This means, the value of a digit varies depending on its position. For instance in 3**4**5, the number "**4**" stands for ***forty*** or **4** ***tens*** whilst in **4**53 the value of the number "**4**" is ***four hundred***. In 53**4**, the value of "**4**" is simply four since it is in the unit position. In **4** 350, the value of **4** is four thousand since it is in thousands position. The numeral 82 537 has 5 digits. This is read as "eighty two thousand, five hundred and thirty seven. These five digits represent number of ones, a number tens, the number of hundreds, a number thousands and a number ten thousands. Table 1.1 shows the place value and the value of the digits in the numeral 82 537.

Table 1.1: Place value and value of 82 537

	Digits				
	8	**2**	**5**	**3**	**7**
Place value of digits	ten thousands 10 000	Thousands 1 000	Hundreds 100	Tens 10	Ones 1
Value of digits	80 000	2 000	500	30	7

It can be seen that the place value of each digit in the decimal system of numeration is 10 times the place value of the next digit to right of it.

Therefore, 82 537 is a short way of writing:

80 000 + 2 000 + 500 + 30 + 7.

Examples 1.1

Write down: (i) the place value, (ii) the value, of each of the digits in the following numbers:

(a) 84 , (b) 784, (c) 495,786.

Solution

(a) 84 has two digits. Therefore, there are 8 sets of ten, plus 4 ones in the number 84.

		Digits	
		8	**4**
(i)	***Place value of digits***	tens	ones
(ii)	***Value of digits***	80	4

(b) 784 has three digits. Therefore, there are 7 sets of hundred, plus 8 sets of ten, plus 4 ones in the number 784.

		Digits		
		7	**8**	**4**
(i)	***Place value of digits***	hundreds	tens	ones
(ii)	***Value of digits***	700	80	4

(c) 495,786 has six digits. There are 4 sets of one hundred thousand, 9 sets of ten thousand, 5 sets of one thousand, 7 sets of one hundred, 8 sets of ten, and 6 ones in the number 495,786.

		Digits					
		4	**9**	**5**	**7**	**8**	**6**
(i)	***Place value of digits***	hundred thousands	ten thousands	thousands	hundreds	tens	ones
(ii)	***Value of digits***	400 000	90 000	5 000	700	80	6

Examples 1.2

Write down: (i) the place value, (ii) the value, of the digit "7" in each of the following decimal numbers:

(a) 57, (b) 278, (c) 7 234, (d) 495 786, (e) 274 581 936.

Solution

(a) (i) ones, (ii) 7, (b) (i) tens, (ii) 70, (c) (i) thousands, (ii) 7 000,
(d) (i) hundreds, (ii) 700, (e) (i) ten millions, (ii) 70 000 000.

Example 1.3

Write the following numerals in words:

(a) 13 457, (b) 258 674, (c) 3 918 243, (d) 75 243 968, (e) 5 613, (f) 43 286 714.

Solution

(a) 1 is in the ten thousands position.
3 is in the thousands position. } 13 thousands
4 is in the hundreds position. } 4 hundreds
5 is in the tens position.
7 is in the ones (or units) position } 57 units
The number is ***thirteen thousand, four hundred and fifty seven***.

(b) 2 is in the hundred thousands position.
5 is in the ten thousands position.
8 is in the thousands position. } 258 thousands
6 is in the hundreds position. } 6 hundreds
7 is in the tens position.
4 is in the ones (or units) position. } 74 units
The number is ***two hundred and fifty eight thousand, six hundred and seventy four***.

(c) 3 is in the millions position } 3 millions
9 is in the hundred thousands position.
1 is in the ten thousands position.
8 is in the thousands position. } 918 thousands
2 is in the hundreds position. } 2 hundreds
4 is in the tens position.
3 is in the ones (or units) position. } 43 units
The number is ***three million, nine hundred and eighteen thousand, two hundred and forty three.***

(d) Seventy five million, two hundred and forty three thousand, nine hundred and sixty eight.
(e) Five thousand, six hundred and thirteen.
(f) Forty three million, two hundred and eighty six thousand, seven hundred and fourteen.

Example 1.4

Write the following in figures.

(a) Fifty six million, seven hundred and sixty thousand, three hundred and thirty three.
(b) Six hundred and eighty seven thousand, nine hundred and twenty four.
(c) Seven hundred and twelve million, five hundred and thirty two thousand, two hundred and ninety six.

Solution

(a) 56 760 333, (b) 687 924, (c) 712 532 296.

Exercise 1(a)

1. Write down: (i) the place value, (ii) the value, of each of the digits in the following numbers: (a) 93, (b) 274, (c) 5 821, (d) 923 567.

2. Write down: (i) the place value, (ii) the value, of the digit "5" in each of the following decimal numbers:
   (a) 53, (b) 375, (c) 25 243, (d) 2 569, (e) 1 526 234 987.

3. Write down: (i) the place value, (ii) the value, of each of the underlined digits in the following numbers:
   (a) 245 **7**45 821, (b) 567 3**2**1, (c) 4**6** 912 193, (d) **3**8 357 234,
   (e) 2 **1**54, (f) **4**9 782, (g) 2 456 29**5** 764, (h) 34 234 93**8**.

4. Write the following numerals in words:
   (a) 56 734, (b) 569 345, (c) 5 654 295, (d) 63 465 237,
   (e) 7 132, (f) 38 142 576, (g) 23 952, (h) 192 387.

5. Write the following in figures.
   (a) Two million, seven hundred and thirty two thousand, five hundred and sixty two.
   (b) Thirty nine thousand, six hundred and eighty seven.
   (c) Five hundred and two million, six hundred and seventy three thousand, eight hundred and forty three.
   (d) Seventy two million, three hundred and ninety five thousand, seven hundred and ninety three.

6. (a) What are the values of the fives in the number 235 985 223?
   (b) What is the difference of these values?

7. What is the value of the digit 3 in the number 531 847 when it is
   (a) divided by 10, (b) multiplied by 10?

8. What is the sum of the values of the odd digits in the number 3 526?

1.2 Ordering numbers

The number line

Fig. 1.1 is a number line. As you may have noticed, the numbers are ordered and the marks for the numbers are equally spaced along the line.

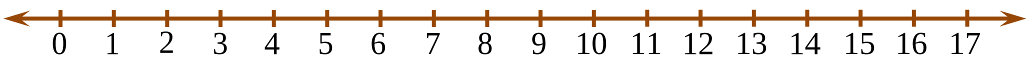


Fig. 1.1

A number on the number line is always ***greater*** than any number on its ***left*** and ***smaller*** than any number on its ***right***. The symbol "<" is used to represent "is less than", and ">" to represent "is greater than". Therefore, for any two numbers on the number line, the number to the left is ***less than*** the number to the right. It means that if $x < y$, then the mark for x is always positioned to left of the mark for y. For instance $6 < 9$, since the mark for 6 is to the left of the mark for 9.

Example 1.5

Fig. 1.2 shows a number line with missing numbers marked with letters x, y and z. Find the values of x, y and z.

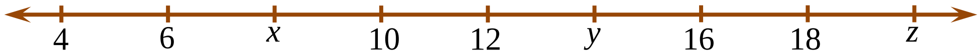


Fig. 1.2

Solution

Since the marks are equally spaced along the number line, it follows that $x = 8$, $y = 14$, $z = 20$.

Example 1.6

Find the missing numerals marked with letters p, q and r on the number line in Fig. 1.3.

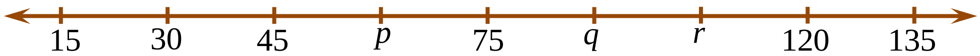


Fig. 1.3

Solution

$p = 60$, $q = 90$, $r = 105$.

Example 1.7

Copy and complete the number line in Fig. 1.4.

Fig. 1.4

Solution

Example 1.8

Use < or > to show which number is greater in each of the following pairs.

(a) 34 630, 41 735 (b) 265 368, 265 374 (c) 27 982, 2 798.

Solution

(a) 34 630 and 41 735 have the same number of digits, so we compare the first digits from the left, i.e. 3 and 4. Since $4 > 3$, it follows that $41\,735 > 34\,630$.

(b) 265 368 and 265 374 have the same number of digits. The first four digits, 2 6 5 3, are common to both numbers, so we compare 6 and 7, the fifth digits from the left. Since $7 > 6$, it follows that $265\,374 > 265\,368$.

(c) 27 982 has five (5) digits while 2 798 has four (4) digits. Hence, $27\,982 > 2\,798$.

Example 1.9

Use the Inequality signs < or > to indicate the relationship among the following numbers.

(a) 2 345, 2 317, 2 361, 2 352 (b) 65 234, 6 923, 64 234, 65 324.

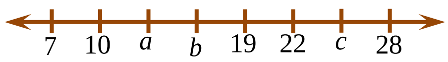
Solution

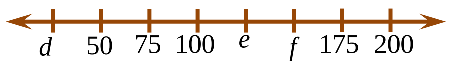
(a) Ascending order: 2 317 < 2 345 < 2 352 < 2 361 or
  Descending order: 2 361 > 2 352 > 2 345 > 2 317.

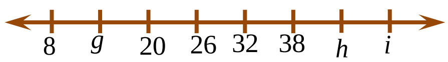
(b) Ascending order: 6 923 < 64 234 < 65 234 < 65 324 or
  Descending order: 65 324 > 65 234 > 64 234 > 6 923.

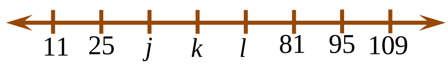
Exercise 1(b)

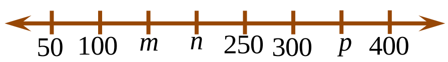
1. Find the missing numerals marked with letter in the following number lines.

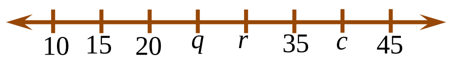
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

2. Use < or > to show which number is greater in each of the following pairs.
   (a) 543 234, 543 245 (b) 28 760, 2 876 (c) 3 767 596, 3 676 596

3. Use the Inequality signs < or > to indicate the relationship among the following numbers.
   (a) 6 578, 6 549, 6 518, 6 537 (b) 54 396 512, 54 396 534, 54 396 506,
   (c) 65 345, 6 934, 65 317, 65 305 (d) 5 672, 6 101, 4 321, 5 671.

4. Use the sign < or > to make the following number statements correct.
   (a) 23 456 … 23 546, (b) 4 194 … 4 094, (c) 467 983 965 … 98 899 796
   (d) 512 … 518, (e) 76 194 … 76 195, (f) 765 164 163 … 765 164 063,
   (g) 9 989 … 12 131, (h) 395 … 384, (i) 300 000 … 3 000 000.

5. Arrange each of the following sets of numbers in an ascending order of magnitude. (i.e. from lowest to highest).
   (a) 34, 45, 42, 31, 15, 39 (b) 354, 479, 321, 429, 510, 493
   (c) 1 243, 1 024, 1 103, 1 209, 1 078 (d) 23 934, 23 541, 23 168, 23 099, 23 473.

6. Arrange each of the following sets of numbers in descending order of magnitude. (i.e. from highest to lowest).
   (a) 45, 23, 54, 35, 34, 20 (b) 6 832, 6 457, 6 749, 6 946, 6 591
   (c) 354, 294, 345, 263, 412, 300 (d) 65 182, 65 253, 65 231, 65 923, 65 359.

7. Which of the following statements are true?
   (a) 5 is less than 12 and so it is to left of 12 on the number line.
   (b) 9 is to the right of 2 on the number line and it is greater than 2.
   (c) 4 is to the left of 11 on the number line and so 11 is greater than 4.

1.3 Rounding numbers

Numbers can be rounded to the nearest ten, hundred, thousand and million. For example, the numbers 10, 11, 12, 13 and 14 are nearer to 10 than 20, while the numbers 16, 17, 18 and 19 are nearer to 20 than 10. So if we want to write numbers as multiples of 10 then we say to the nearest 10. The sign ≈ means approximately equal to.

$12 \approx 10$, $13 \approx 10$, $14 \approx 10$, $15 \approx 20$, $16 \approx 20$, $17 \approx 20$

$21 \approx 20$, $24 \approx 20$, $27 \approx 30$, $35 \approx 40$, $59 \approx 60$, $94 \approx 90$

1.3.1 Rounding a number to the nearest ten

To round a number to the nearest ten, go to the digit in the ones position. If the digit is:

- less than 5, then just change it to 0 and write the new number,
- greater than or equal to 5, then add 1 to the digit in the tens position and change the digit in the ones position to 0 and write the new number.

Example 1.10

Round the following numbers to the nearest ten.

(a) 7, (b) 4, (c) 23, (d) 157, (e) 1 362, (f) 12 288, (g) 43 398.

Solution

(a) 10, (b) 0, (c) 20, (d) 160, (e) 1 360, (f) 12 290, (g) 43 400.

1.3.2 Round a number to the nearest hundred

To round a number to the nearest hundred, go to the digit in the tens position. If the digit is:

- less than 5, then just change it to 0 and the digit after it.
- greater than or equal to 5, then add 1 to the digit in the hundreds position and change the digit in the tens position to 0 and the digit after it.

Example 1.11

Round the following numbers to the nearest hundred.

(a) 45, (b) 51, (c) 93, (d) 432, (e) 456, (f) 23 763, (g) 65 951.

Solution

(a) 0, (b) 100, (c) 100, (d) 400, (e) 500, (f) 23 800, (g) 66 000.

1.3.3 Rounding a number to the nearest thousand

To round a number to the nearest thousand, go to the digit in the hundreds position: If the digit is:

- less than 5, then just change it to 0 and all the digits after it.
- greater than or equal to 5, then add 1 to the digit in the thousands position and change the digit in the hundreds position to 0 and all the digits after it.

Example 1.12

Round the following numbers to the nearest thousand.

(a) 516, (b) 978, (c) 486, (d) 3 478, (e) 56 578, (f) 629 512.

Solution

(a) 1 000, (b) 1 000, (c) 0, (d) 3 000, (e) 57 000, (f) 630 000.

1.3.4 Rounding a number to the nearest million

To round a number to the nearest ten, go to the digit in the hundred-thousands position: If the digit is:

- less than 5, then just change it to 0 and all the digits after it.
- greater than or equal to 5, then add 1 to the digit in the millions position and change the digit in the hundred-thousands position to 0 and all the digits after it.

Example 1.13

Round the following numbers to the nearest million.

(a) 506 876, (b) 924 654, (c) 465 193, (d) 7 572 495, (e) 58 783 483.

Solution

(a) 1 000 000, (b) 1 000 000, (c) 0, (d) 8 000 000, (e) 59 000 000

Exercise 1(c)

1. Round 3 827 194 to the nearest
   (a) 10, (b) 100, (c) 1 000, (d) 10 000, (e) 100 000, (f) 1 000 000.
2. Round the following numbers to the nearest ten.
   (a) 12, (b) 57, (c) 315, (d) 671, (e) 2 392, (f) 3, (g) 23 296.
   (h) 32, (i) 59, (j) 64, (k) 5 845, (l) 6 544, (m) 4, (n) 2 267.
3. Round the following numbers to the nearest hundred.
   (a) 50, (b) 123, (c) 89, (d) 573, (e) 827, (f) 1 645,
   (g) 42, (h) 4 567, (i) 23 746, (j) 9 374, (k) 6 796, (l) 11 555.
4. Round the following numbers to the nearest thousand.
   (a) 450, (b) 9 999, (c) 1 394, (d) 500, (e) 897, (f) 76 937,
   (g) 99, (h) 678 987, (i) 7 123 827, (j) 11 112, (k) 67 987, (l) 3 712.
5. Round the following numbers to the nearest million.
   (a) 13 162 736, (b) 123 746 346 (c) 434 234, (d) 1 293 534, (e) 503 654,
   (f) 12 524 635, (g) 72 273 273, (h) 99 723 354, (i) 9 998 028, (j) 863 457.
6. Round the following numbers to the nearest 1000:
   (a) 124, (b) 1 420, (c) 1 650, (d) 134 765, (e) 179 981, (f) 892 678.
7. Round the following to the nearest 10,000
   (a) 4 500, (b) 6 320, (c) 14 102, (d) 17 893, (e) 5 767 988, (f) 45 898 654.

1.4 Prime and composite numbers

1.4.1 Factors

Factors are the numbers you multiply to get another number. For instance, since $2 \times 3 = 6$, it follows that 2 and 3 are factors of 6. The product 2×3 is therefore referred to as a ***factorization*** of 6. Likewise, because $3 \times 5 = 15$, it means 3 and 5 are factors of 15. Some numbers have more than one factorization (more than one way of being factorized).

For example, 24 can be factorized as

1×24, 2×12, 3×8, or 4×6.

Therefore the factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

Since $1 \times n = n$, for every n, the number ***1 (one) is a factor of every number and every number is a factor of itself***. For example 1 and 9 are factors of 9. Likewise 1 and 5 are factors of 5.

Example 1.14

Find the factors of the following numbers.

(a) 12, (b) 63, (c) 42, (d) 45. (e) 5.

Solution

(a) 12 can be factored as 1×12, 2×6 or 3×4. Therefore, the factors of 12 are 1, 2, 3, 4, 6 and 12.

(b) 63 can be factored as 1×63, 3×21 or 7×9. Therefore, the factors of 63 are 1, 3, 7, 9, 21 and 63.

(c) 42 can be factorized as 1×42, 2×21, 3×14 or 6×7, Therefore, the factors of 42 are 1, 2, 3, 6, 7, 14, 21, and 42.

(d) 45 can be factorized as 1×45, 3×15 or 5×9. Therefore, the factors 45 are 1, 3, 5, 9, 15 and 45.

(e) 5 can factorized as 1×5. Therefore, the factors of 5 are 1 and 5.

1.4.2 Prime numbers

Some numbers have exactly two factors, the number itself and the number 1 (one), as in Example 1.14(e). For example, 13 can only be divided by itself and the number 1 (one) and therefore has exactly two factors. Such numbers are called ***prime numbers***.

> A *prime number* is a natural number greater than 1 that can be divided without remainder only by itself and by 1.

It is useful to know the first ten prime numbers. These are 2, 3, 7, 11, 13, 17, 19, 23 and 29. It is important to note that the number 1 (one) is not a prime number because it does not have exactly two factors. The only factor of 1 (one) is itself.

Numbers such as 9 that have more than two factors are called ***composite numbers***. Composite means "something made by combining things". Composite numbers are made up of prime numbers multiplied together.

A *composite number* can be divided evenly by numbers other than 1 or itself.

For example, 2 and 3 (other than 1 and 6) can divide 6 without a remainder, therefore 6 (six) is a composite number. Likewise 15 is a composite number since 3 and 5 (other than 1 and 15) can divide it evenly. Let's study carefully the Table 1.2 for more examples:

Table 1.2

Number	Can be evenly divided by	Prime or Composite?
1	(1 is not considered prime or composite)	
2	1, 2	Prime
3	1, 3	Prime
4	1, 2, 4	Composite
5	1, 5	Prime
6	1, 2, 3, 6	Composite
7	1, 7	Prime
8	1, 2, 4, 8	Composite
9	1, 3, 9	Composite
10	1, 2, 5, 10	Composite
11	1, 11	Prime
12	1, 2, 3, 4, 6, 12	Composite
...	...	...

Example 1.15

Which of the following numbers are composite numbers?

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37 and 39.

Solution

9, 15, 21, 25, 27, 33, 35 and 39 are not prime numbers. Hence, they are composite numbers.

The sieve of Eratosthenes

A method created by an ancient Greek mathematician, called ***Eratosthenes***, can be used to identify all prime numbers up to a given number. In mathematics, the ***Sieve of Eratosthenes*** is a simple method for finding all prime numbers up to a specified integer. The method is demonstrated in the following activity to determine the prime numbers less than 50.

Activity 1.1

1. Make a list of all natural numbers from 2 to 50.
2. Find the square root of 50, i.e. $\sqrt{50} = 7.07$.
3. Find all prime numbers less than or equal to 7.07. These are 2, 3, 5 and 7.
4. Circle 2 and strike out all multiples of 2 from list of natural numbers in (1).
5. Circle 3 and strike out all multiples of 3 from list of natural numbers in (1).
6. Circle 5 and strike out all multiples of 5 from list of natural numbers in (1).
7. Circle 7 and strike out all multiples of 7 from list of natural numbers in (1).

Notice that the next number left, 11, is larger than the square root of 50, so there are no multiples of 11 to cross off that haven't already been crossed off (22 by 2 and 33 by 3), and therefore the sieve is complete.

8. Circle the remaining natural numbers that do not get crossed out.
9. Make a list of the numbers that that are circled.

The result of Activity 1.1 is given in Table 1.3. Therefore all of the numbers circled are primes: {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}. Notice we just found these primes without dividing.

Table 1.3: The sieve of Eratosthenes

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~

1.4.3 Prime factorization

You most often want to find the list of all the prime-number factors of a given number. For instance, the composite number 12 can be written as the product of prime numbers, i.e. $12 = 2 \times 2 \times 3$. If a number is expressed as a product of prime numbers, then each of the prime numbers is called ***prime factor*** of the given number. The process of splitting into a product of prime factors is called ***prime factorization*** of that number.

> *Prime factorization* is finding which prime numbers is needed to multiply together to get the original number.

Example 1.16

Find the prime factorization of: (a) 8, (b) 84, (c) 189.

Solution

(a) $8 = 2 \times 2 \times 2$.

(b) $84 = 2 \times 42 = 2 \times 2 \times 21 = 2 \times 2 \times 3 \times 7$.

(c) $189 = 3 \times 63 = 3 \times 3 \times 21 = 3 \times 3 \times 3 \times 7$.

The index notation

Index or power indicates the number of times a number is repeated in the product. For example, we write $5 \times 5 \times 5 \times 5$ as 5^4. Thus 5^4 is a short way of writing $5 \times 5 \times 5 \times 5$. Similarly, $2 \times 2 \times 2$ can be written as 2^3. In the expression such as 2^3, the number 2 is the ***base*** and the number 3 is called the ***index, power*** *or* ***exponent***.

Example 1.17

Find the prime factorization of the following, expressing your answer in index form.

(a) 72, (b) 675, (c) 392.

Solution

(a) $72 = 2 \times 36 = 2 \times 2 \times 18 = 2 \times 2 \times 2 \times 9 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$.

(b) $675 = 3 \times 225 = 3 \times 3 \times 75 = 3 \times 3 \times 3 \times 25 = 3 \times 3 \times 3 \times 5 \times 5 = 3^3 \times 5^2$.

(c) $392 = 2 \times 196 = 2 \times 2 \times 98 = 2 \times 2 \times 2 \times 49 = 2 \times 2 \times 2 \times 7 \times 7 = 2^3 \times 7^2$.

Example 1.18

Find the prime factors of (a) 90 (b) 330.

Solution

(a) We first find the prime factorization of 90.

$$90 = 2 \times 45 = 2 \times 3 \times 15 = 2 \times 3 \times 3 \times 5 = 2 \times 3^2 \times 5.$$

Hence, the prime factors of 90 are 2, 3 and 5.

(b) The prime factorization of 330 is given by

$$330 = 2 \times 165 = 2 \times 3 \times 55 = 2 \times 3 \times 5 \times 11.$$

Hence, the prime factors of 330 are 2, 3, 5 and 11.

Factor tree

The factor tree is a structure used to find the prime factorization of natural numbers. Let's use 36 to demonstrate how the factor tree is used in determining its factorization.

- ♣ Choose any 2 factors: let's use 4 and 9.
- ♣ We then factor 4 and 9 as shown in Fig. 1.5.
- ♣ Next, arrange the prime factorization of 36 from least to greatest as

  $$36 = 2 \times 2 \times 3 \times 3.$$

  We now write the prime factorization of 36, in index form, as

  $$36 = 2^2 \times 3^2.$$

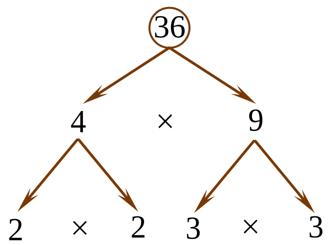


Fig. 1.5

Let us use 36 again as shown in Fig. 1.6.

- Alternatively, we can use 3 and 12 as our initial factors of 36.
- 12 is not a prime number, so we choose any two factors of 12: let's use 2 and 6.
- 6 is not prime, so let's factor again. We can use 2 and 3.
- We now write the prime factorization of 36, arranging from least to greatest, as $36 = 2 \times 2 \times 3 \times 3$.

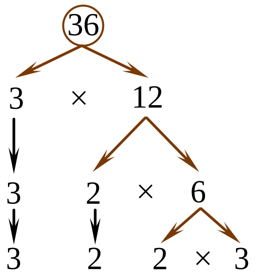


Fig. 1.6

> In prime factor tree, the goal is to keep reducing each factor to its lowest possible prime factor.

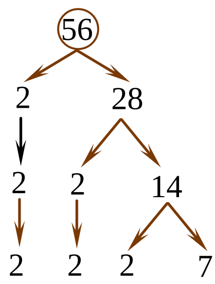
Example 1.19

Find the prime factors of the following using the factor tree.

(a) 56, (b) 135, (c) 540.

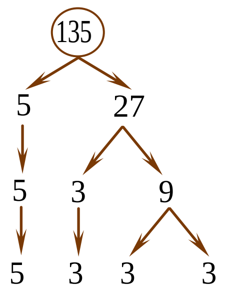
Solution

(a)



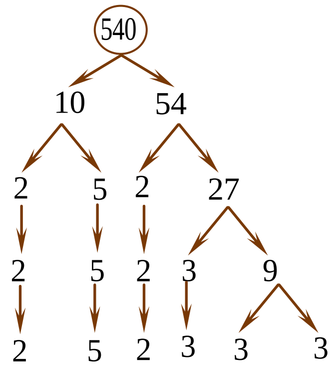
The prime factors of 56 are 2 and 7

(b)



The prime factors of 135 are 3 and 5.

(c)



The prime factors of 540 are 2, 3 and 5

1.4.4 The divisibility rules

Divisibility rules help us test if one number can be evenly divided by another, without having to do too much calculation! These can be useful tricks, especially for large numbers. Table 1.4 gives useful tests of determining if a given number is divisible by another number.

Table 1.4: Test of divisibility

A number is divisible by	If	For Example
2	The last digit is even: 0, 2, 4, 6, 8…	(a) 128, 364, 766 are divisible by 2. (b) 123, 167, 641 are not divisible by 2.

A number is divisible by	If	For Example
3	The sum of the digits is divisible by 3	The number 258 is divisible by 3, since the sum of its digits (2 + 5 + 8 = 15) is divisible by 3.
4	The number formed by its last 2 digits is divisible by 4	The number 1312 is divisible by 4, because 12, the number formed by its last two digits, is divisible by 4.
5	The last digit is 0 or 5	(a) 345 and 680 are divisible by 5. (b) 59 and 861 are **not** divisible by 5.
6	The number is divisible by both 2 and 3	756 is divisible by 6, since it is divisible by both 2 and 3.
7	If you double the last digit and subtract it from the number formed by the rest of the digits and the answer is: 0, or divisible by 7	The number 672 is divisible by 7, since: ♣ double of the last digit (2), is 4, and ♣ 67 – 4 = 63, is divisible by 7, i.e. $63/7 = 9$.
8	The number formed by its last three digits are divisible by 8	The number 389 112 is divisible by 8 because 112, the number formed by its last three digits, is divisible by 8.
9	The sum of the digits is divisible by 9	The number 1 629 is divisible by 9 since the sum of its digits (1 + 6 + 2 + 9 = 18) is divisible by 9.
10	The number ends in 0	(a) 230 and 970 are divisible by 10. (b) 93 and 784 are **not** divisible by 10.
11	If the difference between the sum of its digits in the odd places and the sum of its digits in the even places is: 0 or divisible by 11.	The number 71 962 is divisible by 11 because (7 + 9 + 2 = 18) and (1 + 6 = 7) and 18 and 7 differ by 11, which is divisible by 11.
12	The number is divisible by both 3 and 4.	The number 672 is divisible by 12, since it divisible by both 3 and 4.

Note the following:

- If a number is not divisible by 3, then the remainder when it is divided by 3 is the same as the remainder when the sum of its digits is divided by 3.
- If a number is not divisible by 4, then the remainder when the number is divided by 4 is the same as the remainder when the last two digits are divided by 4.
- If a number is not divisible by 5, then the remainder when it is divided by 5 is the same as the remainder when the last digit is divided by 5.

- If a number is not divisible by 8, then the remainder when the number is divided by 8 is the same as the remainder when the last three digits are divided by 8.
- If a number is not divisible by 9, then the remainder when it is divided by 9 is the same as the remainder when the sum of its digits is divided by 9.
- If a number is not divisible by 10, then the remainder when it is divided by 10 is the same as the units digit.

Example 1.20

Determine whether these numbers are divisible by 4 or 5.

(a) 3 124, (b) 95, (c) 345, (d) 5 632, (e) 1 253 980.

Solution

(a) 3 124 is not divisible by 5, but is divisible by 4 since the number formed by its last two digits, 24, is divisible by 4, i.e. $24/4 = 6$.

(b) 95 is divisible by 5, since the last digit is 5, but not divisible by 4.

(c) 345 is divisible by 5, but is not divisible by 4, since the number formed by its last two digits, 45, is not divisible by 4.

(d) 5 632 is not divisible by 5, but is divisible by 4, since the number formed by its last two digits, 32, is divisible by 4, i.e. $32/4 = 8$.

(e) The number 1 253 980 is divisible by both 4 and 5, since the number formed by its last two digits, 80, is divisible by 4 and the last digit is 0.

Example 1.21

Determine whether the following numbers are divisible by 8, 9 and 10.

(a) 3 389 032, (b) 58 618 431, (c) 1 229 570, (d) 2 239 920.

Solution

(a) The number 3 389 032 is **not** divisible by 9 and 10, but is divisible by 8 because 032, the number formed by its last three digits, is divisible by 8.

(b) The number 58 618 431 is **not** divisible by 8 and 10, but is divisible by 9 is divisible by 9, since the sum of its digits (5 + 8 + 6 + 1 + 8 + 4 + 3 + 1 = 36) is divisible by 9.

(c) The number 1229570 is divisible by 10 since the last digit is 0, but not divisible by 8 and 9.

(d) The number 2 239 920 is divisible by 8, 9 and 10.

1.4.5 Square numbers and square root

Square

The ***square*** of any number is the number multiplied by itself. Thus, if n is a natural number such that $k = n^2$, the k is referred to as ***a square number*** or a ***perfect square***. For example, 25 is a square number, since $25 = 5^2$. Like 4 and 9 are square numbers, since $4 = 2^2$ and $9 = 3^2$. The first 12 square numbers are

$1 = 1^2$,	$4 = 2^2$,	$9 = 3^2$,	$16 = 4^2$,	$25 = 5^2$,	$36 = 6^2$,
$49 = 7^2$,	$64 = 8^2$,	$81 = 9^2$,	$100 = 10^2$,	$121 = 11^2$,	$144 = 12^2$.

Square numbers can, always, be expressed in terms of prime factors with even indices. For example, 36 can be expressed as a product of two perfect squares 4 and 9. Therefore, $36 = 4 \times 9 = 2^2 \times 3^2$. Likewise, $144 = 9 \times 16 = 3^2 \times 2^4$.

Example 1.22

Which of the following products are perfect squares.

(a) $2^2 \times 5^2$, (b) $5^2 \times 7^2 \times 11$, (c) $2^4 \times 3^2 \times 5^6$, (d) $11^3 \times 13^2$, (e) $5^4 \times 17^6 \times 23^2$.

Solution

(a) $2^2 \times 5^2$ is a perfect square, since the indices of both 2 and 5 are both even.
(b) $5^2 \times 7^2 \times 11$ is not a perfect square, since the index of 11 is odd.
(c) $2^4 \times 3^2 \times 5^6$ is a perfect square, since the indices of 2, 3 and 5 are all even.
(d) $11^3 \times 13^2$ is not a perfect square, since the index of 11 is odd.
(e) $5^4 \times 17^6 \times 23^2$ is a perfect square, since all the indices are even numbers.

Example 1.23

Given that $180 = 2^2 \times 3^2 \times 5$, find the least number that should be multiplied by 180 to make the product a square number.

Solution

The number 180 is not a perfect square, since the index of the number 5, in the product $2^2 \times 3^2 \times 5$, is odd. If the number 5 is squared (i.e. 5^2), then the product $2^2 \times 3^2 \times 5^2$ becomes a perfect square. Thus, the least number that should be multiplied by 180 to make the index of 5 even, is 5. Hence, the least number required to make the product a perfect square is 5.

Example 1.24

Given that $4\ 536 = 2^3 \times 3^4 \times 7$, find the least number that should be multiplied by 4 536 to make the product a square number.

Solution

The number 4 536 is not a perfect square, since the indices of the prime numbers 2 and 7, in the product $2^3 \times 3^4 \times 7$, are odd. If 2 is raised to the power 4 (i.e. 2^4) and 7 is squared (i.e. 7^2), then the product $2^4 \times 3^4 \times 7^2$ becomes a perfect square. Thus, the smallest number by which 4 536 must be multiplied in order to make the indices of 2 and 7 even, is 2×7. Hence, the least number required that should be multiplied by 4 536 to make the product a perfect square is 2×7 or 14

Example 1.25

Find the smallest number by which 252 must be multiplied so that the product is a perfect square.

Solution

The prime factorization of 252 is

$$252 = 2 \times 126 = 2 \times 2 \times 63 = 2 \times 2 \times 9 \times 7 = 2^2 \times 3^2 \times 7.$$

The indices of 2 and 3 in 252 are even while the index of 7 is odd. The smallest number by which 252 must be multiplied so that the product becomes perfect square is 7.

Square root

If $x = n^2$, then we say that n is the square root of x. We therefore write $n = \sqrt{x}$. $\sqrt{\ }$ is the symbol for square root. For example, if $k^2 = 25$, then $k = \sqrt{25} = 5$. Alternatively, if $k^2 = 25$, then $k^2 = 5^2$ and hence $k = 5$. Also if $k^2 = 2^2 \times 3^2$, then $k = \sqrt{2^2 \times 3^2} = 2 \times 3$.

Example 1.26

Without using calculator, evaluate the following.

(a) $\sqrt{2^2 \times 5^2}$, (b) $\sqrt{3^4 \times 11^2}$, (c) $\sqrt{2^6 \times 3^4}$, (d) $\sqrt{2^4 \times 3^2 \times 5^2}$.

Solution

(a) $\sqrt{2^2 \times 5^2} = 2 \times 5 = 10.$

(b) $\sqrt{3^4 \times 11^2} = 3^2 \times 11 = 9 \times 11 = 99.$

(c) $\sqrt{2^6 \times 3^4} = 2^3 \times 3^2 = 8 \times 9 = 72.$

(d) $\sqrt{2^4 \times 3^2 \times 5^2} = 2^2 \times 3 \times 5 = 4 \times 3 \times 5 = 60.$

Example 1.27

Find the square root of the following numbers which are perfect squares.

(a) 225, (b) 400, (c) 648, (d) 441.

Solution

(a) We first find the prime factorization of 225.

$$225 = 5 \times 45 = 5 \times 5 \times 9 = 5^2 \times 3^2.$$

Thus, $\sqrt{225} = 5 \times 3 = 15.$

(b) The prime factorization of 400 is given by

$$400 = 4 \times 100 = 2^2 \times 4 \times 25 = 2^2 \times 2^2 \times 5^2 = 2^4 \times 5^2.$$

Thus, $\sqrt{400} = 2^2 \times 5 = 4 \times 5 = 20.$

(c) The factorization of 648 is given by

$$324 = 4 \times 81 = 2^2 \times 9^2.$$

Thus, $\sqrt{324} = 2 \times 9 = 18$.

(d) The prime factorization of 441 is given by

$$441 = 9 \times 49 = 3^2 \times 7^2.$$

Thus, $\sqrt{441} = 3 \times 7 = 21$.

To find the square root of a fraction, find the square root of the numerator and denominator separately. For example, $\sqrt{\frac{4}{9}} = \sqrt{\frac{2^2}{3^2}} = \frac{2}{3}$.

Example 1.28

Find the square root of:

(a) $\frac{9}{16}$, (b) $2\frac{14}{25}$, (c) $\frac{27}{75}$, (d) $\frac{81}{144}$, (e) $2\frac{7}{9}$.

Solution

(a) $\frac{9}{16} = \frac{3^2}{4^2} = \frac{3}{4}$.

(b) We first express $2\frac{14}{25}$ as an improper fraction. That is

$$2\frac{14}{25} = \frac{2 \times 25 + 14}{25} = \frac{64}{25} = \frac{8^2}{5^2} \Rightarrow \sqrt{2\frac{14}{25}} = \sqrt{\frac{8^2}{5^2}} = \frac{8}{5}.$$

(c) $\frac{27}{75} = \frac{3 \times 9}{3 \times 25} = \frac{9}{25} = \frac{3^2}{5^2} \Rightarrow \sqrt{\frac{27}{75}} = \sqrt{\frac{3^2}{5^2}} = \frac{3}{5}$.

(d) $\frac{81}{144} = \frac{9^2}{12^2} \Rightarrow \sqrt{\frac{81}{144}} = \frac{9}{12} = \frac{3 \times 3}{3 \times 4} = \frac{3}{4}$.

(e) We first express $2\frac{7}{9}$ as an improper fraction.

$$2\frac{7}{9} = \frac{2 \times 9 + 7}{9} = \frac{25}{9} = \frac{5^2}{3^2} \Rightarrow \sqrt{2\frac{7}{9}} = \sqrt{\frac{5^2}{3^2}} = \frac{5}{3}.$$

Exercise 1(d)

1. Find the factors of the following numbers.
   (a) 24, (b) 42, (c) 75, (d) 105, (e) 126, (f) 182.
2. Which of the following numbers are prime numbers?
   2, 12, 13, 21, 27, 29, 31, 33, 35, 37, 39, 43, 45, 47, 49, 51, 53, 55, 61, 63, 65 and 69.
3. There are 12 prime numbers between 1 and 40. What are they.
4. List the composite numbers between
   (a) 31 and 37, (b) 5 and 19, (c) 41 and 51.
5. Use the sieve of Eratosthenes to find the all prime numbers less than 80.
6. Find the prime factorization of following.

(a) 48, (b) 63, (c) 539, (d) 16, (e) 30, (f) 110, (g) 62, (h) 40, (i) 135.

7. Find the prime factors of the following.
   (a) 24, (b) 45, (c) 128, (d) 120, (e) 270, (f) 539, (g) 88, (h) 104.

8. Find the prime factors of the following using the factor tree.
   (a) 48, (b) 136, (c) 162, (d) 150, (e) 70, (f) 76, (g) 42, (h) 92.

9. Determine which of the numbers: 182, 759, 3 978 and 360 are divisible by
   (a) 2, (b) 3, (c) 6.

10. Which of the following numbers are divisible by 4 or 5.
    (a) 924, (b) 7 564, (c) 98 135, (d) 1 820, (e) 198 220.

11. Determine whether the following numbers are divisible by 8, 9 or 10.
    (a) 249 256 (b) 3 884 870, (c) 5 699 979, (d) 15 273 576.

12. Which of the following products are perfect squares.
    (a) $7^4 \times 11^2$, (b) $3^2 \times 5^4 \times 7^6$, (c) $11^8 \times 13^2 \times 17^6$, (d) $5^5 \times 23^2$, (e) $2 \times 3^8 \times 5^2$.

13. Given that $1\ 008 = 2^4 \times 3^2 \times 7$, find the least number that should be multiplied by 1 008 to make the product a square number.

14. Given that $1\ 440 = 2^5 \times 3^2 \times 5$, find the least number that should be multiplied by 1 440 to make the product a square number.

15. Find the smallest number by which 720 must be multiplied so that the product is a perfect square.

16. Without using calculator, evaluate the following.
    (a) $\sqrt{7^2 \times 3^2}$, (b) $\sqrt{2^4 \times 5^2}$, (c) $\sqrt{2^6 \times 11^2}$, (d) $\sqrt{2^2 \times 3^4 \times 5^2}$.

17. Find the square root of the following numbers which are perfect squares.
    (a) 729, (b) 1 024, (c) 324, (d) 576.

18. Find the square root of:
    (a) $\frac{36}{81}$, (b) $2\frac{1}{4}$, (c) $\frac{50}{18}$, (d) $\frac{121}{100}$, (e) $3\frac{6}{25}$.

19. Find the prime factorization of the following numbers
    (a) 147, (b) 90, (c) 330, (d) 60, (e) 102.

20. A number is expressed as a product of its prime factors as: $2^3 \times 3^2 \times 5$. Find the number.

21. Write each of the numbers below as a product of its prime factors using the factor tree:
    (a) 62, (b) 38, (c) 82, (d) 320, (e) 54, (f) 1 000.

22. A number is expressed as a product of its prime factors as: $2^3 \times 3 \times 5^2$. What is the number?

23. A particular number has prime factors 2, 3 and 7. What are the 3 smallest values the number could be?

24 What is the smallest number that has:
(a) four different prime factors? (b) five prime factors?.

25. Write down two numbers, neither of which must end in 0, and which:
(a) multiply together to give 1 000, (b) multiply together to give 1 000 000.

26. Determine whether the following numbers are divisible by 2 or 3.
(a) 3 248, (b) 84, (c) 33 336, (d) 177, (e) 12 970.

1.5 Highest common factor (HCF) or Greatest common factor (GCF)

Common factor

When two (or more) numbers have ***the same factor***, that factor is called a ***common factor***. For example, to find the common factors of 12 and 18, we first list the factors of the two numbers.

Factors of 12 are **1**, **2**, **3**, 4, **6**, 12.
Factors of 18 are **1**, **2**, **3**, **6**, 18.

The common factors of 12 and 18 are 1, 2, 3 and 6.

Example 1.29

Find the common factors of 36 and 48.

Solution

Factors of 36: **1**, **2**, **3**, **4**, **6**, 9, **12**, 18, 36.
Factors of 30: **1**, **2**, **3**, **4**, **6**, 8, **12**, 16, 48.
The common factors of 24 and 30 are 1, 2, 3, 4, 6 and 12.

Highest common factor

It can be seen from Example 1.28 that the common factors of 36 and 48 are 1, 2, 3, 4, 6 and 12. The largest factor that is common to both numbers is 12. We therefore call the number 12 the ***highest common factor*** (HCF) of 36 and 48.

> The *highest common factor* (HCF) of two or more natural numbers is the biggest number that will divide into (is a factor of) both numbers.

In other words, it is the largest natural number which divides each of the numbers without a remainder. The highest common factor of two or more natural numbers is also called the ***greatest common factor*** (GCF) of the numbers.

To find the highest common factor (HCF) of two or more numbers:

1. we first find the prime factorization of each number,
2. identify the prime numbers common to all the numbers,
3. take the least number of times each common prime factor occur in each of the given numbers.

The HCF of the numbers is the product of these prime factors in (3), if there are any; otherwise the HCF of the numbers is 1.

Example 1.30
Find the highest common factor (HCF) of 3, 6, and 8.

Solution
First we factorize the numbers and list their prime factorizations:

$$3 = 3.$$
$$6 = 2 \times 3.$$
$$8 = 2 \times 2 \times 2 = 2^3.$$

Since there is no common prime factor, the highest common factor of 3, 6 and 8 is 1 (one).

Note that 3, 6, and 8 share no common factors. While 3 and 6 share a factor, and 6 and 8 share a factor, there is no prime factor that ***all three of them*** share. Since 1 divides into everything, then the greatest common factor in this case is just 1. When 1 is the GCF, the numbers are said to be "***relatively***" prime; that is, they are prime, relative to each other.

Example 1.31
Find the highest common factor (HCF) of 36, 60, and 84.

Solution
We first find the prime factorizations of 36, 60 and 84.

$$36 = 2 \times 18 = 2 \times 2 \times 9 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$
$$60 = 2 \times 30 = 2 \times 2 \times 15 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5.$$
$$84 = 2 \times 42 = 2 \times 2 \times 21 = 2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7.$$

It can be seen that the prime numbers 2 and 3 are common factors of all three numbers. 2 occurs twice (i.e. 2^2) and 3 occurs at least once in all three cases.

Thus, the highest common factor of 36, 60 and 84 is $2^2 \times 3 = 4 \times 3 = 12$.

Example 1.32
Find the highest common factor of 40, 100 and 200.

Solution
We first find the prime factorizations of 120, 100 and 200.

$$120 = 2 \times 60 = 2 \times 2 \times 30 = 2 \times 2 \times 2 \times 15 = 2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3 \times 5.$$
$$100 = 2 \times 50 = 2 \times 2 \times 25 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2.$$
$$200 = 2 \times 100 = 2 \times 2 \times 50 = 2 \times 2 \times 2 \times 25 = 2 \times 2 \times 2 \times 5 \times 5 = 2^3 \times 5^2.$$

It can be seen that 2 occurs at least twice (i.e. 2^2) and 5 occurs at least once in all three cases.

Thus, the highest common factor of 120, 100 and 200 is $2^2 \times 5 = 4 \times 5 = 20$.

Example 1.33
Find the highest common factor of 288 and 168.

Solution

We first find the prime factorizations of 288 and 168.

$$288 = 2\times144 = 2\times2\times72 = 2^2\times2\times36 = 2^3\times2\times18 = 2^4\times2\times9 = 2^5\times3^2.$$

$$168 = 2\times84 = 2\times2\times42 = 2^2\times2\times21 = 2^3\times3\times7.$$

It can be seen that 2 occurs at least thrice (i.e. 2^3) and 3 occurs at least once in both cases.

Thus, the highest common factor of 288 and 168 is $2^3\times3=8\times3=24.$

Practical applications of HCF

Example 1.34

There are 72 oranges and 60 mangoes in two different boxes. Find the greatest number of fruits that can be taken from each box an exact number of times.

Solution

Any fruit that can be taken an exact number of times from each box of 72 oranges and 60 mangoes must be a factor of 72 and 60. The greatest numbest number of fruits that can be taken from each box an exact number of times is therefore the HCF of 72 and 60.

$$72 = 2\times36 = 2\times2\times18 = 2\times2\times2\times9 = 2\times2\times2\times3\times3 = 2^3\times3^2.$$

$$60 = 2\times30 = 2\times2\times15 = 2\times2\times3\times5 = 2^2\times3\times5.$$

The HCF of 72 and 60 is $2^2\times3=4\times3=12.$ Hence, the greatest number of fruits that can be taken from each box an exact number of times is 12.

Example 1.35

A gardener wishes to make a rectangular hen-run of dimension 81 m by 45 m. Wire nets are arranged end to end to fence the hen-run. If the wire nets are all cut to the same length, and do not overlap, find their maximum length.

Solution

The common length of the wire nets must be a common divisor of 81 m and 45 m. The maximum length of wire nets is therefore the HCF of 81 m and 45 m.

$$81 = 3\times27 = 3\times3\times9 = 3\times3\times3\times3 = 3^4.$$

$$45 = 3\times15 = 3\times3\times5 = 3^2\times5.$$

The HCF of 81 and 45 is $3^2=9.$ Hence, the maximum length of wire net is 9 m.

Exercise 1(e)

1. Find highest common factor (HCF) of each of the following:
   (a) $2^4\times3^2$ and $2^2\times3^3$, (b) $3^2\times5^5$ and $3\times5^2\times7$,
   (c) $2^3\times11$ and $2^2\times5^6\times11^2$, (d) $3^4\times13\times23^2$ and $3\times13^3\times11$.

2. (a) Write down all the factors of 60,
   (b) Write down all the factors of 48,

(c) Which factors are common to 60 and 48?
(d) What is the greatest common factor of 60 and 48?

3. Find highest common factor (HCF) of each of the following:
(a) 16 and 36, (b) 30 and 45, (c) 144 and 60, (d) 720 and 324,
(e) 165 and 75, (f) 126 and 84, (g) 72, 54 and 90, (h) 60, 72 and 220,
(i) 78, 104 and 130, (j) 504, 540 and 198, (k) 120, 84 and 66, (l) 75, 105 and 165.

4. Find the highest common factor (HCF) of each of the following:
(a) 324 and 360, (b) 350 and 105, (c) 138 and 190,
(d) 300 and 80, (e) 63 and 45, (f) 275 and 325,
(g) 245, 343 and 196, (h) 242, 363 and 484, (i) 180, 396 and 252.

5. Mr. Jackson has a garden measuring 12 m by 16 m. He wants to divide it into square plots of equal size. What is the largest size of square plots that he can use?

6. The volume of three reservoirs of a school are 360 litres, 252 litres and 132 litres. What is the volume of the largest container that can be used to fill each of them an exact number of times?

7. Square boards are placed end to end to fence a rectangular farm measuring 144 m by 540 m. If the boards are all cut to the same length, and do not overlap, find their maximum length.

8. Find the greatest number which divides 242 and 507, leaving remainder of 2 and 3 respectively.

9. Three empty bottles with capacity of 1 440 mm^3, 1 008 mm^3 and 1 080 mm^3 are to be filled with water. What is the volume of the largest container that can be used to fill each of them an exact number of times?

10. Five tank of different sizes contain 108 litres, 120 litres, 168 litres, 216 litres and 324 litres of water. Find the capacity of the largest container that can be used to bale out the water in each tank an exact number of times.

11. Find the greatest mass that can be taken an exact number of times from 315 g and 675 g.

1.6 Least common multiple (LCM)

Multiples

Multiple of a number x refers to any number formed when x is multiplied by any integer. For example, since $7 \times 9 = 63$, it follows that 63 is a multiple of both 7 and 9. As you can see, 7 and 9 are also factors of 63. If n is a positive integer, then the multiples of n are n, $2n$, $3n$, $4n$, $5n$, $6n$, … . The first multiple of a number is always the number itself. For example,
the multiples of 3: 3, 6, 9, 12, 15, 18, 21, 24, …
the multiples of 4: 4, 8, 12, 16, 20, 24, 28, 32, …

Example 1.36

(a) Write down the multiples of 6, (b) Write down the multiples of 8,

(c) Which multiples are common to 6 and 8?

Solution

(a) The multiples of 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, …
(b) The multiples of 8: 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, …
(c) The common multiples of 6 and 8: 24, 48, 72, …

Least common multiple

It can be seen from Example 1.35 (c) that the smallest of the common multiples of 6 and 8 is 24. That is, 24 is the smallest number that contains both 6 and 8 as factors. We therefore call 24 the ***least common multiple*** (LCM) of 6 and 8.

> The *least common multiple* (LCM) of two or more numbers is the smallest natural number that is a multiple of these numbers.

To find the least common multiples (LCM) of two or more numbers:

1. we first find the prime factorization of each number,
2. take the greatest number of times each prime factor occur in any one of the numbers.

The LCM of the numbers is the product of these prime factors in (2).

Example 1.37

Find the least common multiple (LCM) of 12 and 18.

Solution

We first find the prime factorization of 24 and 18.

$$24 = 2\times2\times2\times3$$
$$18 = 2\times3\times3$$

The greatest number of times prime number, 2, occurred in any of the two numbers is 3, that is $2\times2\times2$.

The greatest number of times prime number, 3, occurred in any of the two numbers is 2, that is 3×3.

Multiplying these prime factors, we obtain $2\times2\times2\times3\times3=72$. Hence, the least common multiple (LCM) of 24 and 18 is 72.

Example 1.38

Find the LCM of the following:

(a) 20 and 18, (b) 36 and 24, (c) 5, 15 and 18.

Solution

(a) We first find the prime factorization of 20 and 18.

$$20 = 2\times10 = 2\times2\times5 = 2^2\times5.$$
$$18 = 2\times9 = 2\times3\times3 = 2\times3^2.$$

Hence, the LCM of 20 and 18 $= 2^2\times3^2\times5 = 4\times9\times5 = 180$.

(b) $36 = 2\times18 = 2\times2\times9 = 2\times2\times3\times3 = 2^2\times3^2$.
$24 = 2\times12 = 2\times2\times6 = 2\times2\times2\times3 = 2^3\times3$.
Hence, the LCM of 36 and 24 $= 2^3\times3^2 = 8\times9 = 72$.

(c) $5 = 1\times5$, $15 = 3\times5$, $18 = 2\times9 = 2\times3\times3 = 2\times3^2$.
Hence, the LCM of 5, 15 and 18 $= 2\times3^2\times5 = 2\times9\times5 = 90$.

Example 1.39

Find the LCM of the following:
(a) 27, 84 and 90, (b) 120 and 180.

Solution

(a) We first find the prime factorizations of 27, 84 and 90
$27 = 3\times3\times3 = 3^3$.
$84 = 2\times2\times3\times7 = 2^2\times3\times7$.
$90 = 2\times3\times3\times5 = 2\times3^2\times5$.
Hence, the LCM of 27, 84 and 90 = $2^2\times3^3\times5\times7 = 3\,780$.

(b) Prime factorizations of 120 and 180:
$120 = 2\times2\times2\times3\times5 = 2^3\times3\times5$
$180 = 2\times2\times3\times3\times5 = 2^2\times3^2\times5$.
Hence the LCM of 120 and 180 is $2^3\times3^2\times5 = 360$.

Practical applications of LCM

We now consider some practical applications of LCM.

Example 1.40

Mr. Peterson gave to each of his three children three different denominations of Ghana cedi notes. He gave 10 Ghana cedi notes to the first child and 20 Ghana cedi notes to the second child while the third child received 50 Ghana cedi notes.
(a) Find the smallest sum of Ghana cedi notes that can be given out to each child in equal sum.
(b) Hence, determine the least number of Ghana cedi notes that can be given out to each child in equal sum.

Solution

(a) The required sum of money is the smallest amount that can be divided by GH¢ 10, GH¢ 20 and GH¢ 50, without a remainder. This is the LCM of GH¢ 10, GH¢ 20 and GH¢ 50. Now,
$10 = 2\times5$, $20 = 2\times2\times5 = 2^2\times5$, $50 = 2\times5\times5 = 2\times5^2$.
Hence the LCM of 10, 20 and 50 is $(2^2\times5^2) = 100$. The required amount of money is therefore GH¢ 100.

(b) The least number of 10 Ghana cedi notes received by the first child $= \frac{100}{10} = 10$.

The least number of 20 Ghana cedi notes received by the second child $= \frac{100}{20} = 5$.

The least number of 50 Ghana cedi notes received by the third child $= \frac{100}{50} = 2$.

This means that in order for the children to have the least equal sum of GH¢ 100, the first child receives ten (10) of the 10 Ghana cedi note and the second child receives five (5) of the 20 Ghana cedi note while the third child receives two (2) of the 50 Ghana cedi notes.

Example 1.41

Two kinds of cement blocks have heights 12 cm and 15 cm.

(a) Find the least height of wall that can be erected of each kind that will have equal heights,

(b) Hence, determine the least number of rows of each kind that will have equal heights.

Solution

(a) The required height is the least height that can be divided by 12 cm and 15 cm, without a remainder. This is the LCM of 12 cm and 15 cm.

$12 = 2 \times 2 \times 3 = 2^2 \times 3, \qquad 15 = 3 \times 5.$

Hence, the LCM of 12 and 15 $(2^2 \times 3 \times 5) = 60$. The required height is therefore 60 cm.

(b) The number of rows of the 12 cm blocks $= \frac{60}{12} = 5$.

The number of rows of the 15 cm blocks $= \frac{60}{15} = 4$.

Exercise 1(f)

1. (a) Write down the multiples of 9.
   (b) Write down the multiples of 12.
   (c) Which multiples in (a) and (b) are common to 9 and 12.
   (d) What is the least common multiples of 9 and 12.

2. Find the LCM of each of the following:

   (a) $2^2 \times 5$ and $2^3 \times 3$, (b) $2^2 \times 3^2 \times 5$ and $2^3 \times 3$, (c) $2^2 \times 5$ and $2^2 \times 7$,

   (d) 2^4 and $2^2 \times 3^2$, (e) $2^2 \times 5$, $2^3 \times 3$ and $2 \times 3 \times 5$, (f) 5×7^2 and $2 \times 3 \times 7$,

   (g) 5, 3^2 and $2^2 \times 3$, (h) $2^2 \times 3 \times 5$, $2^2 \times 5^2$ and 5^3, (i) 7, 2×7, 2×3^2.

3. Find the least common factor (LCM) of the following:

   (a) 12 and 9, (b) 6 and 8, (c) 16 and 36, (d) 24 and 40,
   (e) 28 and 44, (f) 39 and 65, (g) 46 and 69, (h) 58 and 87,
   (i) 10, 15 and 35, (j) 14, 21 and 35, (k) 12, 20 and 28, (l) 62 and 93,
   (m) 60, 180 and 300, (n) 58, 87 and 145, (o) 30, 90 and 60, (p) 20 and 80.

4. Find the smallest mass that can be measured out in equal amount of 10 g, 14 g and 22 g.
5. Three lighthouse flash their lights at intervals of 10, 15 and 35 seconds. If they flash together. If they flash together at 12 noon, after how many seconds will they flash together?
6. Two kinds of square boxes are arranged separately on top each other. If the boxes have heights 20 cm 28 cm,
   (a) what is the shortest height of boxes, of each kind, that can be arranged in equal height,
   (b) what is the least number of boxes of each kind that will have equal heights?
7. If one car gets 20 km per gallon and 12 km per gallon,
   (a) find the shortest distance each car will need to travel exactly the same distance as the other car,
   (b) what is the smallest whole number of gallons of fuel each car will need to travel exactly the same distance as the other car?
8. Find the length of the shortest piece of string that can be cut into equal lengths, each 26 cm or 39 cm or 65 cm long.
9. Find the smallest number of mobile phones that can be put into bags which all contain 24 or 40 or 56 mobile phones with none left.
10. Find the smallest natural number which is divisible by 80 and 112, leaving a remainder of 3 in each case.
11. Three bells ring at intervals 14 seconds, 21 seconds and 28 seconds. If they start ringing together, after how many seconds will they ring together again?

1.7 Operations on whole numbers

1.7.1 Additions and subtraction of whole numbers

For addition and subtraction, you must always remember to line up the number from the units digits.

Example 1.42

Simplify the following:

(a) 76 386 734 + 84 735 434 + 65 487, (b) 37 987 653 + 72 653 + 736 592

(c) 938 475 674 + 7 492 554 + 48 765 927.

Solution

(a)
$$\begin{array}{r} 76\,386\,734 \\ 84\,735\,434 \\ +\quad 65\,487 \\ \hline 161\,187\,655 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 37\,987\,653 \\ 72\,653 \\ +\quad 736\,592 \\ \hline 38\,796\,898 \\ \hline \end{array}$$

(c)
$$\begin{array}{r} 938\,475\,674 \\ 7\,492\,554 \\ +\ 48\,765\,927 \\ \hline 994\,734\,155 \\ \hline \end{array}$$

Example 1.43

Simplify the following:

(a) 76 345 213 – 4 321 768, (b) 8 989 342 – 931 689, (c) 293 848 572 – 9 282 736.

Solution

(a)
$$\begin{array}{r} 76\,345\,213 \\ \underline{-4\,321\,768} \\ \underline{72\,023\,445} \end{array}$$

(b)
$$\begin{array}{r} 8\,989\,342 \\ \underline{-\;\;931\,689} \\ \underline{8\,057\,653} \end{array}$$

(c)
$$\begin{array}{r} 293\,848\,572 \\ \underline{-\;\;9\,282\,736} \\ \underline{284\,565\,836} \end{array}$$

1.7.2 Multiplication and division of whole numbers

Example 1.44

Simplify the following

(a) 653 × 47, (b) 2 734 × 316, (c) 374 × 623.

Solution

(a)
$$\begin{array}{r} 653 \\ \underline{\times \quad 47} \\ 4571 \\ \underline{2612\;} \\ \underline{30691} \end{array}$$

(b)
$$\begin{array}{r} 2734 \\ \underline{\times \quad 316} \\ 16404 \\ 2734\; \\ \underline{8202\;\;\;} \\ \underline{863944} \end{array}$$

(c)
$$\begin{array}{r} 374 \\ \underline{\times \quad 623} \\ 1122 \\ 748\; \\ \underline{2244\;\;} \\ \underline{233002} \end{array}$$

Example 1.45

Simplify the following:

(a) 62 744 ÷ 124, (b) 13 575 ÷ 25, (c) 15 939 ÷ 69.

Solution

(a)
$$\begin{array}{r} 506 \\ 124\overline{)62744} \\ \underline{620\;\;} \\ 744 \\ \underline{744} \\ 0 \end{array}$$

(b)
$$\begin{array}{r} 543 \\ 25\overline{)13575} \\ \underline{125\;\;} \\ 107\; \\ \underline{100\;} \\ 75 \\ \underline{75} \\ 0 \end{array}$$

(c)
$$\begin{array}{r} 231 \\ 69\overline{)15939} \\ \underline{138\;\;} \\ 213\; \\ \underline{207\;} \\ 69 \\ \underline{69} \\ 0 \end{array}$$

Exercise 1(g)

1. Simplify the following:
   (a) 58 654 985 + 254 658, (b) 45 987 534 + 3 876 234 + 293 487,
   (c) 812 542 256 + 233 765 + 752 894, (d) 54 736 843 + 23 547 + 764 893.

2. Simplify the following:

(a) 237 879 375 – 89 354 854, (b) 345 654 543 – 8 765 876,
(c) 45 768 132 – 854 876, (d) 47 645 736 – 9 765 987.

3. Simplify the following:
   (a) $24 \times 5\,765$, (b) 345×765, (c) $543 \times 12\,876$, (d) $65 \times 5\,832$.

4. Simplify the following:
   (a) $34\,662 \div 53$, (b) $39\,916 \div 68$, (c) $71\,149 \div 13$, (d) $2\,400 \div 25$

1.8 Properties of operations

1.8.1 Order of Operations – BODMAS

"Operations" refer to things like add (+), subtract (–), multiply (×), divide (÷), etc.
What do you think the answer to **3 + 4 × 5** is?
Is it $(3 + 4) \times 5 = 7 \times 5 = 35$? or $3 + (4 \times 5) = 3 + 20 = 23$?
"BODMAS" is an acronym, which give rules to follow so that we always get the right answer for such a problem. The following are the rules:

(**B**)rackets:	Number inside brackets must be worked out first
(**O**)f:	"Of" is worked out before (÷), (×), (+) and (–).
(**D**)ivision:	Division (÷) should be done before (×), (+) and (–).
(**M**)ultiplication:	Multiplication (×) should be done before (+) and (–).
(**A**)ddition:	Addition (+) should be done before (–).
(**S**)ubtraction:	Subtraction (–) is always the last operation to perform.

The following example shows us how to use all the features of BODMAS.

Example 1.46

Perform the operations (a) $1 + 20 \times (6 + 2) \div 2$, (b) $21 \div 7 \times \frac{1}{2}$ of $4 - 3$.

Solution

(a) $1 + 20 \times (6 + 2) \div 2 = 1 + 20 \times 8 \div 2 = 1 + 20 \times 4 = 1 + 80 = 81$.

(b) $21 \div 7 \times \frac{1}{2}$ of $4 - 3 = 21 \div 7 \times \left(\frac{1}{2} \times 4\right) - 3$

$= 21 \div 7 \times 2 - 3 = 3 \times 2 - 3 = 6 - 3 = 3$.

When horizontal line is used to represent fraction, its operation is similar to that of bracket. The numbers above and below the line must therefore be worked first out first before performing the division. This is illustrated in the following example.

Example 1.47

Perform the operation $\dfrac{22 - 6 \div 3}{4 + 3 \times 2}$.

Solution

$$\frac{22 - 6 \div 3}{4 + 3 \times 2} = \frac{22 - (6 \div 3)}{4 + (3 \times 2)} = \frac{22 - 2}{4 + 6} = \frac{20}{10} = 2.$$

Example 1.48

Calculate the following operations:

(a) 16 – (3 × 2), (b) 5 + (7 × 9), (c) (4 + 8) × 2,
(d) 28 ÷ (18 – 11), (e) 2 + (8 × 3) – (5 + 6), (f) 10 + [30 – (2 × 9)],
(g) (17 + 8) + (9 + 20), (h) (55 – 6) + (13×2).

Solution

(a) 16 – (3 × 2) = 16 – 6 = 10.
(b) 5 + (7 × 9) = 5 + 63 = 68.
(c) (4 + 8) × 2 = 12 × 2 = 24.
(d) 28 ÷ (18 – 11) = 28 ÷ 7 = 4.
(e) 2 + (8 × 3) – (5 + 6) = 2 + 24 –11 = 26 – 11 = 15.
(f) 10 + [30 – (2 × 9)] = 10 + [30 – 18] = 10 + 12 = 22.
(g) (17 + 8) + (9 + 20) = 25 + 29 = 54.
(h) (55 – 6) + (13 × 2) = 49 + 26 = 75.

Example 1.49

Determine the value of each of the following.

(a) 23 + (12 ÷ 4) + (11 × 2), (b) 86 + [14 + (10 – 8)],
(c) 31 + {9 + [1 + (35 – 2)]}, (d) {6 – [24 ÷ (4 × 2)]} × 3,
(e) 3 + 15 ÷ 3 + 5 × 22 + 3, (f) (15 – 8) + 5(6 + 4),
(g) 63 – (4 + 6 × 3) + 76 – 4, (h) 6 × (32 + 22) + 42.

Solution

(a) 23 + (12 ÷ 4) + (11 × 2) = 23 + 3 + 22 = 48.
(b) 86 + [14 + (10 – 8)] = 86 + [14 + 2] = 86 + 16 = 102.
(c) 31 + {9 + [1 + (35 – 2)]} = 31 + {9 + [1 + 33]} = 31 + {9 + 34} = 31 + 41 = 72.
(d) {6 – [24 ÷ (4 × 2)]} × 3 = {6 – [24 ÷ 8]} × 3 = {6 – 3} × 3 = 3 × 3 = 9.
(e) 3 + 15 ÷ 3 + 5 × 22 + 3 = 3 + 5 + 110 + 3 = 128.
(f) (15 – 8) + 5(6 + 4) = 17 + 5(10) = 17 + 50 = 67.
(g) 63 – (4 + 6 × 3) + 76 – 4 = 63 – (4 + 18) + 76 – 4 = 63– 22 + 76 – 4
= 41 + 76 – 4 = 117 – 4 = 113.
(h) 6 × (32 + 22) + 42 = 6 × 54 + 42 = 324 + 42 = 366.

1.8.2 Commutative property of addition and multiplication

Addition and Multiplication are commutative: switching the order of two numbers being added or multiplied does not change the result. That is, for the numbers *a* and *b*;

(i) $a + b = b + a$ and (ii) $a \times b = b \times a$.

For example:
$100 + 8 = 8 + 100 = 108$, $100 \times 8 = 8 \times 100 = 800$.

1.8.3 Associative property of addition and multiplication

Addition and multiplication are associative: the order that numbers are grouped in addition and multiplication does not affect the result. That is, for the numbers a, b and c;
(i) $(a + b) + c = a + (b + c)$ and (ii) $(a \times b) \times c = a \times (b \times c)$.

For example: $(2 + 10) + 6 = 2 + (10 + 6) = 18$,
$(2 \times 10) \times 6 = 2 \times (10 \times 6) = 120$.

1.8.4 Distributive Property

The distributive property of multiplication over addition or subtraction: multiplication may be distributed over addition or subtraction. That is, for the numbers a, b and c;
(i) $a \times (b + c) = (a \times b) + (a \times c)$ and (ii) $a \times (b - c) = (a \times b) - (a \times c)$

For example:
$10 \times (50 + 3) = (10 \times 50) + (10 \times 3)$,
$3 \times (12 - 99) = (3 \times 12) - (3 \times 99)$.

1.8.5 The zero property of addition

Adding zero (0) to a number leaves it unchanged. We call 0 the additive identity. That is, for the number a;
$a + 0 = 0 + a = a$.

For example:
$88 + 0 = 0 + 88 = 88$ and $0 + 53 = 53 + 0 = 53$.

1.8.6 The zero property of multiplication

Multiplying any number by zero (0) gives zero (0). That is, for the number a;
$a \times 0 = 0 \times a = 0$

For example:
$88 \times 0 = 0 \times 88 = 0$ and $0 \times 1003 = 1003 \times 0 = 0$.

1.8.7 The multiplicative identity

We call 1 the multiplicative identity. Multiplying any number by 1 leaves the number unchanged. That is, for the number a;
$a \times 1 = 1 \times a = a$

For example:
$88 \times 1 = 1 \times 88 = 88$ and $1 \times 35 = 35 \times 1 = 35$.

Exercise 1(h)

Calculate the following:

(a) $144 \div 12 - 10$,

(b) $20 \div (12 - 7)$,

(c) $(24 \div 2 - 10) \times 7$,

(d) $4 + 2 \times 10 - 12 \div 2 + (6 + 3) \div 3$,

(e) $12 \times 2 \div 3 + 5 \times 6 \div 10 - 12 \times 2 \div 6$,

(f) $\{10 \div (10 \div 2 + 5) \times 4\} \div 2$,

(g) $\{10 \div (10 \div 2 + 5) \times (4 + 3 \times 4)\} \div 2$,

(h) $3 \times 6 - 12 \div 2 + 15 \div 3 + 7$,

(i) $(4 + 5 - 7) \times (13 - 3) \times 10 - 98$,

(j) $(5 \times 8 + 20) \div 10 + 20 - 10 \div 2$,

(k) $(5 \times 8 + 20) \div (10 + 20) - 4 \div 2$,

(l) $\{(10 \times 8 + 20) \div (5 + 20) - 2\} \div 2$,

(m) $(20 \div 2 - 12 \div 2) \times (12 \div 3) \div 2$,

(n) $2 + (23 - 20) \times (2 + 8) \div 2 - 3$,

(o) $30 \times 3 - 6 \times 2 + 9 + 4 \times 8 \div 2 - 2 + 7 \times 4$,

(p) $8 \times 8 - 50 \div 5 + 6 - 7 + 8 - 3 + 7 \times 7$,

(q) $12 - 4 \times 3 + (12 + 2 \times 3) \div 9 + 18 - 2 \times 7$,

(r) $(4 \times 4 - 16 \div 4) + 10 - 3 + 5 + (8 - 2 \times 3)$,

(s) $6 + 3 - 8 + 7 \times 5 - 14 \times 4 \div 8 + 6 - 3 + 40 \div 4$,

(t) $(4 \times 4 - 16 \div 4) + 10 - 3 + 5 + (8 - 2 \times 3) \div 2$,

(u) $\{(4 \times 4 - 16 \div 4) + 10 - 3 + 5 + (8 - 2 \times 3)\} \div 2$,

(v) $\{8 \times 5 - 4 + 5 - 20 \div 5 + 3 \times 4 + (6 + 3 - 8 \div 2)\} \div 2$,

(w) $(2 \times 4 + 6 \times 4 + 4 \times 4) \div 8 + (3 \times 4 - 2 \times 9 + 3 \times 5) \div 3$,

(x) $8 \times 8 \div 2 + 50 - 20 \div 2 \times \{10 - 24 \div (2 + 2 \times 5) \div 2\} \div 2$,

(y) $(2 + 5 - 9 \div 3 + 2) + \{(6 \div 2 + 2 \times 15) \div 11\} \times \{(7 + 2 - 6) + (18 - 2 \times 6)\}$,

(z) $\{(2 + 5 - 9 \div 3 + 2) + (6 \div 2 + 3 \times 10) \div 11\} \times \{(7 + 2 - 6) + (21 - 2 \times 6) \div 9\}$.

1.9 Estimation of natural numbers

Estimation has become very necessary in mathematics programs in recent years. Often in everyday applications we need to make a quick calculation that does not have to be exact to serve the purpose at hand. For example, when shopping, you may want to estimate the total cost of the items selected in order to avoid an unpleasant surprise at the checkout counter.

Estimation is important for developing “number sense” and predicting the reasonableness of answers. An estimation mindset helps to determine when it is appropriate to estimate and how close an estimate is required in a given situation. Indeed, estimation is a legitimate part of mathematics.

There are many techniques for estimating. After obtaining an estimation, we sometimes need to know if it is less than or greater than the actual answer. This can often be determined from the method of estimation used. We’ll use the method of ***rounding***.

Rounding

If an approximate sum or difference is all that is needed, we can round the numbers before computing. The type of problem will often determine to what place value the numbers will be rounded.

1.9.1 Sums and Differences

Example 1.50

Write estimation by rounding each number to the place value of the leading digit.

(a) 4 723 + 419 +1 040, (b) 624 – 289 – 132, (c) 812 – 245.

Solution

By rounding to the place value of the first digits, we have

(a) 47 23 + 419 +1 040 = 5 000 + 400 + 1000 = 6 400,

(b) 624 – 289 – 132 = 600 – 300 – 100 = 200,

(c) 812 – 245 = 800 – 200 = 600.

Example 1.51

Obtain an estimate for the following operations by rounding.

(a) 88 + 37 + 66 + 24, (b) 142 – 119, (c) 127 + 416 – 288.

Solution

Here are some estimations by rounding. Others may occur to you.

(a) 88 + 37 + 66 + 24 = 90 + 40 + 70 + 20 = 220,
which is greater than the actual answer.

(b) 142 – 119 = 140 – 120 = 20,
which is less than the actual answer.

(c) 127 + 416 – 288 = 130 + 400 – 300 = 230,
which is less than the actual answer.

Example 1.52

Use the rounding method of estimation to estimate each sum or difference.

(a) 1 306 + 7 247 + 3 418, (b) 4 718 – 1 335,

(c) 527 + 4215 + 718, (d) 7316 – 547.

Solution

(a) 1 306 + 7 247 + 3 418 = 1 000 + 7 000 + 3 000 = 11 000.

(b) 4 718 – 1 335 = 4 000 – 1 000 = 3 000.

(c) 527 + 4 215 + 718 = 500 + 4 000 + 700 = 5 200.

(d) 7 316 – 547 = 7 000 – 500 = 6 500.

1.9.2 Product and Quotient

Example 1.53
Use rounding to estimate each product and quotient.
(a) 27 × 63, (b) 81 × 57, (c) 194 × 26, (d) 1 225 ÷ 35.

Solution
Here are some estimations by rounding.
(a) 27 × 63 = 30 × 60 = 1 800.
(b) 81 × 57 = 80 × 60 = 4 800.
(c) 194 × 26 = 190 × 30 = 5 700.

Example 1.54
Obtain an estimate for these products and quotients.
(a) 2 × 117 × 49, (b) 5609 ÷ 79, (c) 63 ÷ 23, (d) 43 × 72.

Solution
(a) 2 × 117 × 49 = 2 × 120 × 50 = 12 000, which is greater than the actual answer.
(b) 5 609 ÷ 79 = 6 000 ÷ 80 = 75, which is greater than the actual answer.
(c) 63 ÷ 23 = 60 ÷ 20 = 3, which is less than the actual answer.
(d) 43 × 72 = 40 × 70 = 2 800, which is less than the actual answer.

Example 1.55
Use the rounding method of estimation to estimate each product and quotient.
(a) 625 ÷ 25, (b) 237 × 76, (c) 30,328 × 419 (d) 432 ÷ 38.

Solution
(a) 625 ÷ 25 = 630 ÷ 30 = 21.
(b) 237 × 76 = 240 × 80 = 19 200.
(c) 30 328 × 419 = 30 330 × 420 = 12 738 600.
(d) 432 ÷ 38 = 400 ÷ 40 = 10.

CHAPTER TWO

Sets

2.1 Sets of objects and numbers

2.1.1 Definition

If you want to prepare a cake, you need flour, eggs, margarine, baking powder and sugar. When you buy these ingredients in a shop, you probably do not buy them one at a time. It is easier and cheaper to buy them in a set. We often use the word '***set***' to describe a collection of objects, quantities or numbers. We have a set of living room furniture: what does it contain? We speak of cutlery set, a math set, the set of students under 14 in your class and so on. Can you think of any more?

> A set is a well-defined collection of objects, quantities or numbers that have a given property or properties in common.

2.1.2 Notation

A set is usually denoted by capital letters such as A, B, P, Q, X and Y. The objects that make up a set are called ***members*** or ***elements*** of the set. The elements of a set may be named in a list or may be given by a description enclosed in braces { }. For instance, the set of numbers between 1 and 6 may be given as {2, 3, 4, 5} or as {the numbers between 1 and 6}.

For example, in the set $Q = \{2, 4, 6, 8, 10, 12\}$, 4 is a member or element of the set Q. In set operations, the symbol $\in$ is used to denote the clause '***is a member of***' or '***is an element of***' or '***belongs to***'. So the statement '4 is a member of Q' can be written as $4 \in Q$. Can you name the other elements of the set Q? Similarly the statement '***5 is not a member of Q***' may abbreviated to $5 \notin Q$, $\notin$ standing for '***is not an element of***' or '***does not belong to***'. The number of elements in set Q is 6 and this is denoted as $n(Q) = 6$.

Example 2.1

If $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 5, 7, 9\}$, complete the following statements by inserting $\in$, $\notin$, A, B or elements of the sets A and B

(a) $4 \ldots A$ (b) $6 \in \ldots$ (c) $2 \notin \ldots$ (d) $8 \ldots B$ (e) $\ldots \notin A$
(f) $10 \ldots B$ (g) $5 \in \ldots$ (h) $7 \notin \ldots$ (i) $7 \ldots B$ (j) $\ldots \notin B$.

Solution

(a) $4 \in A$, (b) $6 \in A$, (c) $2 \notin B$, (d) $8 \notin B$, (e) $9 \notin A$,
(f) $10 \notin B$, (g) $5 \in B$, (h) $7 \notin A$, (i) $7 \in B$, (j) $2 \notin B$.

Example 2.2

If $A = \{1, 3, 5, 7, 9, 11, 13\}$ indicate whether the following statements are true or false:

(a) $5 \in A$, (b) $9 \notin A$, (c) $15 \in A$, (d) $3.5 \in A$, (e) $0 \in A$, (f) $6\frac{1}{2} \notin A$.

Solution

(a) True, (b) False, (c) False, (d) False, (e) False, (f) True.

Example 2.3

Write out the following statements in full.

(a) $36 \in$ {multiple of 4}, (b) Togo $\notin$ {state where English is the official language},
(c) A snake $\notin$ {bird}, (d) Canada $\notin$ {African countries},
(e) $6 \in$ {factor of 48}, (f) A quadrilateral $\in$ {polygons},
(g) Lizard $\in$ {reptiles}, (h) $7 \in$ {prime numbers}.

Solution

(a) 36 is a member of the set multiples of 4,
(b) Togo is not a member of the set of states where English is the official language,
(c) A snake is not member of the set of birds,
(d) Canada is not a member of the set of African countries,
(e) 6 is an element of the set of factors of 48,
(f) A quadrilateral is not member of the set of polygons,
(g) Lizard is a member of the set of reptiles,
(h) 7 is an element of the set of prime numbers.

Exercise 2(a)

1. If $P = \{1, 2, 5, 7, 8, 9\}$ and $Q = \{3, 4, 6, 10\}$, complete the following statements by inserting $\in$, $\notin$, P, Q or elements of the sets P and Q
   (a) $2 \ldots P$, (b) $3 \in \ldots$, (c) $6 \notin \ldots$, (d) $4 \ldots Q$, (e) $\ldots \notin P$,
   (f) $1 \ldots Q$, (g) $7 \in \ldots$, (h) $8 \notin \ldots$, (i) $6 \ldots Q$, (j) $\ldots \notin Q$.
2. If $P = \{2, 4, 6, 8, 10, 14, 18\}$ indicate whether the following statements are true or false:
   (a) $12 \in P$, (b) $14 \notin P$, (c) $6 \in P$, (d) $3 \notin P$, (e) $1 \in P$, (f) $8 \in P$.
3. Rewrite the following, first in set notation.
   (a) My cat is not a bird, (b) Ghana is a country in West Africa,
   (c) 4 is an even number, (d) Mensah is a student at Methodist High School,
   (e) My dog is an animal, (f) June is a month in the year.
4. Write the following in ordinary English, without using the word "set" or "belong".
   (a) Mensah $\in$ {students of Peace JHS}, (b) Essien $\in$ {players in the national team},
   (c) Lion $\notin$ {reptiles}, (d) Lagos $\notin$ {cities in Ghana},
   (e) Ghana $\in$ {countries in West Africa}, (f) An eagle $\notin$ {fishes}.

5. Write the following statements using set notation.
   (a) Kofi is my friend,
   (b) Oxygen is a gas,
   (c) Gambia is a country in West Africa,
   (d) Mansah is a student of Peace JHS,
   (e) A tiger is not a bird,
   (f) Osu is a town in Greater Accra}.

2.2 Describing and writing Sets

Not all sets have precise descriptions of any sort; they may simply be arbitrary collections, with no expressible "rule" saying what elements are in or out.

1. Some sets may be ***described in words***, for example:
   A is the set whose members are the first five positive whole numbers.
   B is the set whose members are the colours of the Ghanaian Flag.
2. A set can also be defined by explicitly ***listing its elements***. For example,
   $P = \{1, 2, 3, 4, 5\}$ or $Q = \{\text{red, yellow, green, black}\}$.

   For sets with many elements, an abbreviated list is sometimes used. For example, the first one thousand positive whole numbers can be described using the symbolic shorthand: $R = \{1, 2, 3, ..., 1\,000\}$, where the ellipsis (...) indicates that the list continues in the obvious way. Similarly the set of even numbers can be described by the notation: $T = \{2, 4, 6, 8, ...\}$.
3. Sometimes, we can describe a set using a ***set-builder notation***. For example, the set P can also be described using a notation or symbol such as 'x' to represent any member of the set of even numbers between 1 and 20. Thus we can write P as follows:
   $P = \{x: x \text{ is an even number, and } 1 < x < 20\}$.

   In the above expression for P, the colon ':' means 'such that', and is followed by the property that x is an even number between 1 and 20. The set Q of integers greater than 100 can be written as
   $Q = \{x : x \text{ is an integer, and } x > 100\}$

   The set R of regions in Ghana can also be written as
   $R = \{x : x \text{ is a region in Ghana}\}$

   Other letters such as y and z can also be used as notations for representing sets.

Some useful definitions

The following are some few definitions that will enable us to define the elements of sets in problems.

1. An ***integer*** is a positive or negative whole number. Examples are … –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, …
2. An ***odd number*** is a number which cannot be divided exactly by two (2). Odd numbers can be expressed as $2n + 1$, where n is an integer.
   Example: … –5, –3, –1, 1, 3, 5, …
3. An ***even number*** is a number leaving no remainder when divided by two.
   Example: … –6, –4, –2, 0, 2, 4, 6 …

4. An integer x is said to be ***a factor*** of another integer y if x can divide y without leaving any remainder. Example: The set factors of 48 = {1, 2, 4, 6, 8, 12, 24, 48}
5. A ***prime number*** is any positive number that is exactly divisible only by itself and one. Example: 2, 3, 5, 7, 11, 13, 17, etc.
6. The ***prime factors*** of a number n refer to the factors of n that are prime numbers. Example: The set of prime factors of 36 = {2, 3}
7. ***Multiple of a number*** x refers to any number formed when x is multiplied by any integer. Example: Multiples of 3 are 3, 6, 9, 12, 16, etc.

Example 2.4

List the elements of the following sets:

(a) P is the set of months of the year with names beginning with the letter J,
(b) Q is the set of vowels of the English alphabets,
(c) R is the set of first six positive integers.
(d) S is the set of consecutive even numbers greater than 4 but less than 12.

Solution

(a) P = {January, June, July}, (b) $Q = \{a, e, i, o, u\}$,
(c) $R = \{1, 2, 3, 4, 5, 6\}$, (d) $S = \{6, 8, 10\}$.

Example 2.5

List all the elements of the following sets:

(a) P = {natural numbers less than 10},
(b) Q = {prime numbers greater than 2 but less than 24}.

Solution

(a) $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, (b) $Q = \{3, 5, 7, 11, 13, 17, 19, 23\}$.

Example 2.6

Write each of the following sets in standard notation, by listing its elements which describes its elements.

(a) M is the set of natural numbers that are multiples of 5.
(b) N is the set of fractions between (but not including) –1 and 1 that have 3 as a denominator.
(c) P is the set of integers between –5.1 and 2.1.
(d) Q is the set of natural numbers less than 20 and divisible by 3.

Solution

(a) $M = \{5, 10, 15, ...\}$, (b) $N = \left\{-\frac{2}{3}, -\frac{1}{3}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}\right\}$,

(c) $P = \{-5, -4, -3, -2, -1, 0, 1, 2\}$, (d) $Q = \{3, 6, 9, 12, 15, 18\}$.

Example 2.7

List the elements of the following sets:

(a) A = {factors of 12}, (b) B = {multiples of 4 less than 20}.

Solution

(a) $A = \{1, 2, 3, 4, 6, 12\}$, (b) $B = \{4, 8, 12, 16\}$.

Example 2.8

Find the next three elements in each of the following sets:

(a) $\{50, 46, 42, 38, \ldots\}$, (b) $\{3, 6, 9, 12, \ldots\}$

(c) $\{22, 33, 44, 55, \ldots\}$, (d) $\{12, 112, 1112, \ldots\}$

Solution

(a) 34, 30, 26, (b) 15, 18, 21,

(c) 66, 77, 88, (d) 11112, 111112, 1111112.

Example 2.9

Use set builder notation to describe the following:

(a) The set of odd numbers greater than 30,
(b) The set of prime numbers greater than 2 but less than 24,
(c) The set of triangles,
(d) The set of positive integers less than 100,
(e) The set of rivers in Ghana.

Solution

(a) $\{x : x$ is an odd number, and $x > 30\}$,
(b) $\{x : x$ is a prime number, and $2 < x < 24\}$,
(c) $\{x : x$ is a triangle$\}$
(d) $\{x : x$ is a positive integer, $x < 100\}$,
(e) $\{x : x$ is a river in Ghana$\}$.

Exercise 2(b)

1. List the elements of the following sets and determine the number of elements.
   (a) $A = \{x$: x is a factor of 36$\}$,
   (b) $B = \{x$: x is a multiple of 4 less 25$\}$,
   (c) $C = \{x$: x is an odd number between 3 and 19$\}$,
   (d) $D = \{x$: $4 < x \leq 12$, where x is an even number$\}$,
   (e) $P = \{x$: $-3 \leq x < 2$, where x is an integer$\}$,
   (f) $Q = \{x$: x is a prime number less 12$\}$,
   (g) $R = \{x$: x is a month of the year with name beginning with the letter A$\}$

2. Match the set ***P*** with the set ***Q***.

P	***Q***
(a) {March, May},	(a) The set of whole numbers greater than 6.
(b) {2, 3, 5, 7, 11},	(b) The set of months in the year with name beginning with *M*.
(c) {7, 8, 9, 10, …},	(c) The set of multiples of 3 between 8 and 20.
(d) {9, 12, 15, 18},	(d) The set of factors of 12.
(e) {1, 2, 3, 4, 6, 12}.	(e) The set of West African countries with name beginning with the letter G.
(f) {Ghana, Gambia, Guinea}	(f) The set of prime numbers less than 13.

3. Write down the next three elements of each of the following sets.
   (a) {2, 4, 6, 8, …}, (b) {4, 7, 10, 13, …}, (c) {1, 4, 9, 16, …},
   (d) {4, 8, 12, 16, …}, (e) {42, 39, 36, 33, …}, (f) {15, 13, 11, 9, …}.

4. List the elements of the following sets:
   (a) The set whose elements are the letters of the word "arrangement",
   (b) The set of countries in West Africa,
   (c) The set of months of the year with 30 days,
   (d) The set of days of the week with names beginning with letter *T*,
   (e) {integers x: $6 \leq x < 13$},
   (f) {x: x is a prime factor of 48}

5. How many elements are in the following sets?
   (a) {days of the week}, (b) {rectangle with three sides},
   (c) {toes of your left foot}, (d) {x: x is a prime factor of 36},
   (e) {triangles with four sides}, (f) {players in a football team},
   (g) {present regional capitals of Ghana}, (h) {months beginning with J},
   (i) {7, 8, 9, …, 15}, (j) {squares with two sides}.

6. Describe the following sets in words:
   (a) {March, May}, (b) {1, 3, 5, 7},
   (c) {*a*, *e*, *i*, *o*, *u*}, (d) {Sunday, Saturday},
   (e) { October, November, December}, (f) {5, 7, 11, 13, 17}.

7. Describe the following sets in words:
   (a) {Togo, Mali, Benin, Senegal, Guinea, Ivory Coast},
   (b) {12, 16, 20, 24, 28, 32, 36}, (c) {–5, –4, –3, …, 5},
   (d) {January, June, July}.

2.3 Types of sets

1. Finite and infinite set

When the elements of a set are arranged in increasing order of magnitude, the first element (the least member) is called the ***lower limit*** whilst the last element (the greatest member) is the ***upper limit***.

Example: If $A = \{2, 4, 6, 8\}$, then the Lower Limit = 2 and the Upper Limit = 8

(a) A set is said to be ***finite*** if it has both lower and upper limits. In other words a set is finite if the first and the last members can be found. A finite set is also called a ***bounded set***. For example, the following sets are finite: (i) $A = \{2, 4, 6, 8\}$, (ii) $B = \{a, b, c, d, \ldots, x, y, z\}$, (iii) C = {red, yellow, green, blue, black}.

(b) A set without a lower or upper limit or both is called an ***infinite set***. An infinite set is also called an ***unbounded set***. For example, the following sets are infinite: (i) $N = \{1, 2, 3, 4,\ldots\}$, (ii) $Z = \{\ldots,–3, –2, –1, 0, 1, 2, 3,\ldots\}$, (iii) $P = \{\ldots,7, 9, 13, 15\}$.

Example 2.10

Which of the following are finite and which are infinite?

(a) {1, 3, 5, …, 101}, (b) {3, 6, 9, 12, 15, …},

(c) {…, –2, –1, 0, 1, 2, 3}, (d) {Sunday, Monday, Tuesday, …, Friday}.

Solution

(a) and (d) are finite, while (b) and (c) are infinite.

2. Empty set

A set, which contains no elements, is called an ***empty (or null) set***. It is usually denoted by { } or ∅. The following sets are empty:

(a) {days of the week with names beginning with the letter *Y*},

(b) {prime numbers greater than 8 but less than 11},

(c) {three sided squares}.

3. Unit set

A set with only one member is called a ***unit set***. A unit set is sometimes called a ***singleton***. For example, $A = \{3\}$ or $B = \{7\}$ is a unit set.

Example 2.11

Which of the following are unit sets?

(a) {days of the week with names beginning with the letter *M*},

(b) {the football team of your school},

(c) {present capitals of Ghana},

(d) {Francophone countries in Africa}.

Solution

(a) and (c) are unit sets.

Exercise 2(c)

1. Which of the following are empty sets?

   (a) {months whose name begins with T}, (b) {squares with six sides},

(c) {days whose name begins with M},
(d) {prime numbers less than 2},
(e) {present regional capital in Ghana},
(f) {prime numbers between 7 and 11}.

2. Which of the following sets are unit sets?
   (a) {prime numbers between 19 and 24},
   (b) {months which begin with N},
   (c) {positive integers less than 2},
   (d) {triangles with four sides},
   (e) {days whose name begins with T},
   (f) {months with 24 days}.

3. Which of the following sets are finite and which are infinite?
   (a) The set of positive integers,
   (b) $\{5, 10, 15, 20, \ldots\}$,
   (c) The set of cities in Ghana,
   (d) $\{1, 4, 9, \ldots, 100\}$,
   (e) $\{\ldots, -5, -4, -3, -2, -1, 0, 1\}$,
   (f) The set of countries in West Africa.

2.4 Equal set and equivalent sets

1. Equal sets

The sets $A = \{6, 2, 1, 5, 3\}$ and $B = \{3, 1, 2, 5, 6\}$ contain the same elements. The difference between A and B exist only in the order in which the elements are arranged. In listing the elements of a set, the elements may be in any order, and each element must be written once. Thus A and B refer to the same set. Therefore, the sets A and B can be written as $A = B$.

Two sets are equal if they contain ***identical elements***.

Example 2.12

If $A = \{5, 7, 9, 12, 15\}$ and $B = \{9, 15, 5, x, 12\}$ are equal, find the value of x.

Solution

$A = \{5, 7, 9, 12, 15\} = \{9, 15, 5, 7, 12\}$. Since A and B are equal, it follows that $A = B$. Thus, $\{9, 15, 5, 7, 12\} = \{9, 15, 5, x, 12\}$, which gives $x = 7$.

Example 2.13

Which of the following pairs of sets are equal?

(a) $\{u, v, w, x, y, z\}$, $\{x, w, u, z, v, y\}$,
(b) $\{a, b, c, d, e\}$, $\{d, c, f, a, e, d\}$,
(c) {odd number between 2 and 8}, {prime numbers between 2 and 11},
(d) {prime numbers less than 2}, {triangles with four sides}.

Solution

(a) $\{u, v, w, x, y, z\} = \{x, w, u, z, v, y\}$,
(b) $\{a, b, c, d, e\} \neq \{d, c, f, a, e, d\}$,
(c) {odd number between 2 and 8} = {prime numbers between 2 and 11} = $\{3, 5, 7\}$,
(d) {prime numbers less than 2} = {triangles with four sides} = $\varnothing$.

2. Equivalent sets

The two sets $C = \{a, b, c, d, e\}$ and $D = \{1, 2, 3, 4, 5\}$ are not equal. It can be seen, however, that C and D has the same number of elements, that is $n(C) = n(D) = 5$.

Two sets are equivalent if they have the *same number of elements.*

Example 2.14

Which of the following sets are equal and which ones are equivalent?
$A = \{a, f, j, q\}$, $B = \{1, 2, 3, 5, 8\}$, $C = \{x, y, z, w\}$ and $D = \{8, 1, 3, 5, 2\}$.

Solution

B and D have the same element. B and D are therefore equal ($B = D$).
Each of the sets A and C has 4 elements, that is $n(A) = n(C)$. A and C are therefore equivalent.

Example 2.15

Determine which of the following pairs of sets are equivalent.

(a) $A = \{1, 3, 5, \ldots, 15, 17\}$, $B = \{12, 14, 16, \ldots, 26, 28\}$,
(b) $P = \{p, q, r, s, t\}$, $Q = \{3, 1, 2, 5\}$,
(c) $S = \{a, b, c, d, e\}$, T = {toes of your left foot}.

Solution

(a) Since each of the sets A and B has 9 elements, it implies that they are equivalent.
(b) The set P has 5 elements while the set Q has 4 elements. Since P and Q do not have the same number of elements, it implies that they are not equivalent.
(c) Each of the sets S and T has 5 elements. The two sets are therefore equivalent.

Exercise 2(d)

1. If $A = \{1, 3, 6, 7, 9\}$ and $B = \{9, 6, 1, x, 7\}$ are equal, find the value of x.
2. If $\{4, 2, 7, 10, 12, 14\} = \{12, 10, x, 4, 14, 7\}$, find the value of x.
3. Which of the following sets are equivalent sets?
   $A = \{a, f, j, q\}$, $B = \{1, 2, 3, 5, 8\}$, $C = \{x, y, z, w\}$ and $D = \{8, 1, 3, 5, 2\}$.
4. Which, if any, of the following sets are equal?
   $A = \{1, 2, 3\}$, $B = \{\ \}$ and $C = \{-1, 0, 1, 2, 3\}$
5. Which of the following pairs of sets are equivalent?
   (a) $\{a, s, d, f, g\}$, $\{3, 5, 8, 9, 2\}$,
   (b) $\{z, x, c, v, b, n, m\}$, $\{l, k, j, h, g, f, d, s\}$,
   (c) $\{1, 4, 6, 2, 3\}$, $\{1, 2, 3, 4, 5, 6\}$
   (d) {months whose name begins with J}, $\{q, w, e\}$,
   (e) {even number between 6 and 8}, {prime number between 7 and 11}
   (f) {days whose name begins with T}, {months whose name begins with M}

6. Which of the following pairs of sets are equal?
   (a) $\{t, y, u, o, p\}$, $\{u, t, y, p, o\}$,
   (b) {3, 2, 1, 4}, {1, 2, 3},
   (c) {factors of 6}, {prime numbers less than 5},
   (d) {6, 2, 5, 7, 3}, {2, 3, 5, 6, 7},
   (e) $\{x$: x is a prime number between 23 and 29$\}$, {squares with six sides},
   (f) {odd numbers less than 5}, {1, 3, 5},
7. Which of the following statements are true?
   (a) $\{1, 3, 5, 7\} = \{1, 3, 5, 7, \ldots\}$,
   (b) $\{4, 8, 12, 16, \ldots\} = \{4, 8, 12, 16, 20, \ldots\}$,
   (c) {rectangles with 5 sides} = {months with 20 days},
   (d) {2, 9, 5, 6} is equivalent to $\{a, b, c, d\}$,
   (e) $\{t, h, s, f\} = \{h, t, f, s\}$,
   (f) $\{10, 12, 14\ldots\} = \{14, 12, 10, \ldots\}$,
8. Consider the following sets:
   $P = \{3, 5, 7, 8\}$, $Q = \{5, 7, 23, 31\}$, $R = \{8, 3, 5, 7\}$ and $S = \{1, 2, 3, 4\}$.
   (a) Are all the four sets equivalent? Why. (b) Which of the sets are equal?

2.5 Subsets

Consider the following sets $P = \{2, 5, 8\}$ and $Q = \{1, 2, 3, 5, 7, 8\}$. It can be seen that every member of the set P is also a member of the set Q. If all Ghanaians are Africans, then {Ghanaians} is a subset of {Africans}. All prime numbers are whole numbers; therefore {prime numbers} is a subset of {whole numbers}.

> A set P is said to be the subset of the set Q if every elements of P belong to the set Q.

The symbol $\subset$ is used to denote the phrase **'subset of'**. P is a subset of Q is therefore written as $P \subset Q$. For example, If $P = \{2, 5, 8\}$ and $Q = \{1, 2, 3, 5, 7, 8\}$, then $P \subset Q$. If the set A is not a subset if the set B, we write $A \not\subset$ B.

It is important not to confuse the symbols $\subset$ and $\in$. The symbol $\subset$ connects two sets while $\in$ connect a member and its set.

Example 2.16

$A = \{x$: x is an animal$\}$, $B = \{y$: y is an animal with tail$\}$, C = {horses, kangaroos} and D = {cows, guinea pigs, horses, kangaroos}. Which of the following statements are true?

(a) $B \subset A$, (b) $D \subset B$, (c) $C \subset B$, (d) $C \subset D$, (e) $A \subset B$, (f) $A \supset D$.

Solution

(a) True, (b) False: guinea pigs do not have tails, (c) True,
(d) True, (e) False: guinea pigs $\in A$ but $\notin B$, (f) True.

Example 2.17

If $A=\{q, r, s\}$, $B=\{p, q, r, s, u, v\}$ and $D=\{u, v\}$, are the following statements true or false? (a) $A \subset B$, (b) $B \subset A$, (c) $D \subset A$, (d) $D \subset B$, (e) $A \supset D$, (f) $A = B$.

Solution

(a) True, (b) False, (c) False, (d) True, (e) False, (f) False.

Example 2.18

Indicate whether the following statements are true or false if: $A=\{1, 2, 3, 4\}$, $B=\{1, 2\}$, $D=\{3, 4, 5\}$, and $E=\{4, 5, 3\}$.

(a) $B \subset A$, (b) $D \subseteq E$, (c) $D = E$, (d) $4 \in E$, (e) $E \subset B$, (f) $4 \notin B$.

Solution

(a) True, (b) True, (c) True, (d) True, (e) False, (f) True.

Example 2.19

Rewrite the following in ordinary English.

(a) {Ghanaians} $\subset$ {Africans},
(b) {sparrow, eagle} $\subset$ {birds},
(c) {bullies} $\subset$ {strong people},
(c) {goats} $\subset$ {animals that eat grass}

Solution

(a) All Ghanaians are Africans.
(b) Sparrow and eagle are birds.
(c) All bullies are strong people.
(c) Goats are animal that eat grass.

Example 2.20

Rewrite the following in 'set language' using the notation "$\subset$" or "$\not\subset$".

(a) All Akans are Ghanaians,
(b) All rectangles are parallelograms,
(c) All my friends are intelligent,
(d) Not all prefects play football.

Solution

(a) {Akans} $\subset$ {Ghanaians},
(b) {rectangles} $\subset$ {parallelograms},
(c) {my friends} $\subset$ {intelligent people},
(d) {prefects} $\not\subset$ {footballers}.

Universal set

Consider the set $\mathcal{U}$ = {whole numbers}. Can you suggest some of the subsets of $\mathcal{U}$. For instance, {odd numbers}, {even number}, {prime numbers} are subsets of $\mathcal{U}$. All the sets you suggested are subsets of one set, that is, the set of all whole numbers.

> The set of all objects under discussion is called the *universe* or *universal set*.

We use the symbol $\mathcal{U}$ to denote the universal set.

Example 2.21

Suggest a universal set for each of the following subsets.

(a) {Francophone countries in Africa}, (b) {isosceles triangles},
(c) {students in your class}, (d) {horses}.

Example 2.22

The sets P = {factors of 36} and Q = {prime numbers} are subsets of $\mathcal{U}$ = {x: x is a whole number less than 10}. List the elements of P and Q.

Solution

$\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$P = \{1, 2, 3, 4, 6, 9\}$,
$Q = \{2, 3, 5, 7\}$.

2.5.1 Using Venn diagrams

So far, we have discovered that a set can be described by using words/set builder notation or listing the members of the set. We also discussed the connection between two sets. The ideas that we have met so far can be represented very simply by means of diagram.

Consider: $\mathcal{U}$ = {all people},
P = {people who wear uniform},
J = {JHS pupils}.

If all JHS pupils wear uniform, then {JHS pupils} $\subset$ {people who wear uniform} or we write $J \subset P$. We know that J and P are both subsets of $\mathcal{U}$, that is $J, P \subset \mathcal{U}$. This information can be represented diagrammatically.

Fig. 2.1 shows the relationship between the sets J, P and $\mathcal{U}$. Since $P \subset \mathcal{U}$, the circle representing P is drawn inside the rectangle which represents $\mathcal{U}$. Furthermore, since $J \subset P$, the circle representing J is inside that of P. A diagram like this is called a ***Venn diagram***, named after the English mathematician John Venn (1834 – 1923).

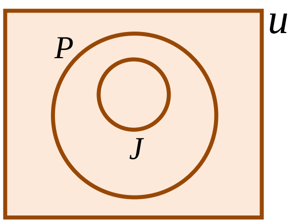


Fig. 2.1

Example 2.23

Draw a Venn diagram of

$\mathcal{U}$ = {plane geometrical figures},
P = {polygons},
T = {triangles},

where $T \subset P \subset \mathcal{U}$.

Solution

Fig. 2.2 shows the required Venn diagram. Notice that since $T \subset P$, the circle representing T lies entirely inside that of P.

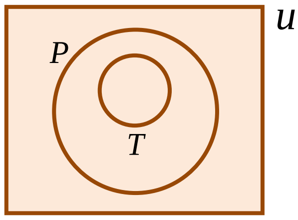


Fig. 2.2

Example 2.24

Consider the following sets:

$\mathcal{U}$ = {all students},
M = {students with measles},
B = {students in the sick bay},

If all students with measles stay in the sick bay, illustrate the information on a Venn diagram.

Solution

If all students with measles stay in the sick bay, then it implies that $M \subset B \subset \mathcal{U}$. Therefore, Fig. 2.3 shows the required Venn diagram. Notice that since $M \subset B$, the circle representing M lies entirely inside that of B.

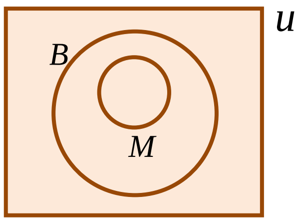
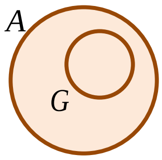


Fig. 2.3

Example 2.25

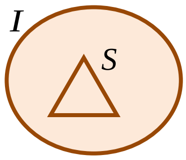
What information do the Venn diagrams below give?

(a)



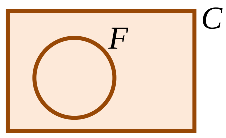
G = {Ghanaians}
A = {Africans}

(b)



S = {scientists}
I = {introverts}

(c)



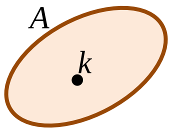
F = {my friends}
C = {those who attend church on Sunday}

Solution

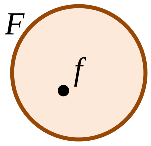
(a)	Symbols:	$G \subset A$ or {Ghanaians} $\subset$ {Africans},
	Set language:	{Ghanaians} is a subset of {Africans},
	Ordinary English:	All Ghanaians are Africans.
(b)	Symbols:	$S \subset I$ or {scientists} $\subset$ {introverts},
	Set language:	{scientists} is subset of {introverts},
	Ordinary English:	All scientists are introverts.
(c)	Symbols:	$F \subset C$ or {my friends} $\subset$ {those who attend church on Sunday},
	Set language:	{my friends} subset of {those who attend church on Sunday},
	Ordinary English:	All my friends attend church on Sundays.

Example 2.26

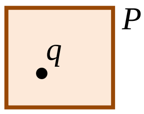
What information do the Venn diagrams below give?

(a) 

k = Kofi
A = {Akans}

(b) 

f = Frank
F = {my friends}

(c) 

q = 2
P = {prime numbers}

Solution

(a) Symbols: $k \in A$ or Kofi $\in$ {Akans},
Set language: Kofi belongs to the set of all Akans.
Ordinary English: Kofi is an Akan.

(b) Symbol: $f \in F$ or Frank $\in$ {my friends},
Set language: Frank belongs to the set of all friends of mine.
Ordinary English: Frank is my friend.

(c) Symbols: $q \in P$ or $2 \in$ {prime numbers},
Set language: 2 belongs to the set of prime numbers,
Ordinary English: 2 is a prime number.

2.5.2 Subsets of a set with *n* elements

Note that ***the null set* $\varnothing$ *is a subset of every set***. Also since every element of any set S belongs to S, it follows from our definition of the term "subset" that ***every set S is a subset of itself***. Thus, $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$.

If $S = \{\ \}$, then the possible subset of S is $\{\ \}$ or $\varnothing$.

If $S = \{1\}$ then the possible subsets of S are $\varnothing$ and $\{1\}$.

If $S = \{1, 2\}$ then the possible subsets of S are $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$.

If $S = \{1, 2, 3\}$ then the possible subsets of S are $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$.

If $S = \{1, 2, 3, 4\}$ then the possible subsets of S are $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$.

These results can be summarized in a tabular form as shown in Table 2.1 on the next page.

We can therefore conclude that the number of ***subsets of a set with n elements*** is given by 2^n where n is the number of elements in the set. The set of all subsets of a set is called the ***power set*.** Therefore, the power set of a set with n elements can be determined by the expression 2^n.

Table 2.1

Set	Subsets	Number of subsets
{ }	{ } or $\varnothing$	$1 = 2^0$
{1}	$\varnothing$, {1}	$2 = 2^1$
{1, 2}	$\varnothing$, {1} , {2}, {1, 2}	$4 = 2^2$
{1, 2, 3}	$\varnothing$, {1} , {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}.	$8 = 2^3$
{1, 2, 3, 4}	$\varnothing$, {1}, { 2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, { 2, 3, 4}, {1 , 2, 3, 4}.	$16 = 2^4$

Example 2.27

If S is the set $\{x, y, z\}$, then the complete list of subsets of S is as follows:
(i) { } (also denoted $\varnothing$, the empty set), (ii) $\{x\}$, (iii) $\{y\}$, (iv) $\{z\}$, (v) $\{x, y\}$, (vi) $\{x, z\}$, (vii) $\{y, z\}$, (viii) $\{x, y, z\}$.

Hence the power set of S is $P(S) = \big\{\{\ \}, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\big\}$.

The number of elements in $P(S)$, $n\big(P(S)\big)$, is $2^3 = 8$.

Exercise 2(e)

1. If $P = \{2, 3, 5\}$, $Q = \{1, 2, 3, 4, 5, 6\}$ and $R = \{3, 5, 2\}$, are the following statements true or false?
   (a) $P \subset R$, (b) $Q \subset P$, (c) $R \subset P$, (d) $R \subset Q$, (e) $R \supset Q$, (f) $P = R$.

2. If $P = \{2, 3, 5, 7, 11, 13, 17\}$, which of the following are subsets of set P?
   (a) {3, 9, 11, 15}, (b) {3, 5, 7, 11}, (c) {3, 7, 13, 17}.

3. In the following pairs of sets, state whether the second set is a subset of the first set.
   (a) {2, 4, 6, 8, 10}, {4, 8}, (b) {Africans}, {Nigerians},
   (c) {prime numbers}, {9, 15, 21}, (d) {animals}, {reptiles},
   (e) {months of the year}, {June, July}, (f) {3, 6, 9}, {3, 4, 9}.

4. Indicate whether the following statements are true or false if: $A = \{a, b, c, d\}$, $B = \{b, d\}$, $D = \{a, c, e\}$, and $E = \{e, a, c\}$.
   (a) $B \subset A$, (b) $D \subseteq E$, (c) $D = E$, (d) $e \in E$, (e) $E \subset B$, (f) $e \notin B$.

5. If $P = \{p, q, r\}$ and $Q = \{q, r\}$ and $R = \varnothing$, which of the following are true?
   (a) $Q \subset P$, (b) $R \subset Q$, (c) $R \subset P$, (d) $P \subset Q$.

6. Rewrite the following in 'set language' using the notation "$\subset$" or "$\not\subset$".
   (a) All goats eat grass, (b) All students are hardworking,

(c) Not all prime numbers are odd,
(d) Not all Senior High School pupils are well behaved,
(e) All Senior High School pupils wear uniform,
(f) Not all bullies are strong people.

7. Rewrite the following in ordinary English.
(a) {scientists} $\subset$ {introverts},
(b) {final year students} $\not\subset$ {students in the SRC},
(c) {my friends} $\subset$ {those who like Coca-cola},
(d) {Form 3 students} $\not\subset$ {studious}.

8. Find all the subsets of:
(a) $\varnothing$, (b) $\{r\}$, (c) $\{x, y\}$, (d) {cat, dog}.

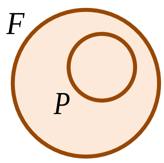
9. (a) Which of $\varnothing$, $\{p\}$, $\{p, q\}$ and $\{p, q, r\}$ are subsets of $\{p, q\}$?
(b) If A = {Tuesday, Thursday}, and B = {days whose names begin with the letter T}, which of the following are true of A and B?
(i) $A \subset B$, (ii) $B \subset A$, (iii) $A = B$.

10. Consider the following sets:
$\mathcal{U}$ = {students}, J = {Junior High School pupils},
W = {well behaved students}.
If all Junior High School pupils are well behaved, illustrate the above information on a Venn diagram.

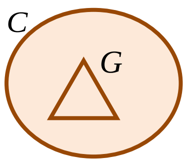
11. Consider the following statement: "all students of Peace JHS are hardworking". Let
$\mathcal{U}$ = {students}, P = {students of Peace JHS},
H = {hardworking students}.
Illustrate the above information on a Venn diagram.

12. Consider the following statement: "all the good students of Mathematics are in the football team". Let,
$\mathcal{U}$ = {students}, M = {good Mathematics students},
F = {students in the football team}.
Illustrate the above information on a Venn diagram.

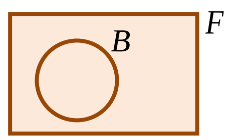
13. What information do the Venn diagrams below give? Write down your answer in ordinary English.

(a)


P = {prefects}
F = {Form 3 students}

(b)


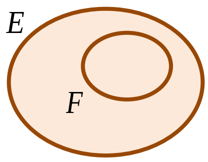
G = {girls in my class}
C = {people who like chocolate}

(c)


B = {boys in my form}
F = {people who like playing football}

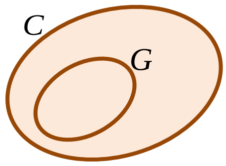
14. What information do the Venn diagrams below give? Write down your answer in ordinary English.

(a)



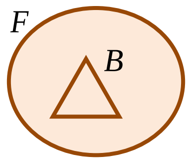
F = {my friends}
E = {people who do well in examinations}

(b)



G = {equilateral triangles}
C = {symmetric figures}

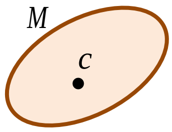
(c)



B = {goats}
F = {animals that eat grass}.

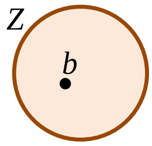
15. What information do the Venn diagrams below give? Write down your answer in ordinary English.

(a)



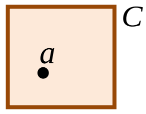
k = Caleb
M = {people who three metres tall}

(b)



b = 7
Z = {integers}

(c)



a = Akos
C = {people who like chocolate}

2.6 Intersection and union of sets

In the preceding section, we learnt about subsets and how the idea of subsets can represented by means of Venn diagram. In this section, we will study about two basic operations on sets, namely intersection and union of sets.

2.6.1 Intersection

The headmistress of Peace Junior High School invited the school athletic team for a dinner at his residence. The team is made up of 8 sprinters and 5 hurdlers. She realised that, there were 10 athletes in his residence. She checked and found that all the athletes were present. But 8 + 5 > 10. Can you explain it?

The solution is much easier using a Venn diagram. We shall use S and H to denote the sets of student who are sprinters and hurdler respectively. The information is illustrated in Fig. 2.4. If every member of the athletic team were present at the dinner, then it follows that 3 members of the team who are sprinters must also be hurdlers. The set of elements which are common to both S and N is called the ***intersection*** of S and N.

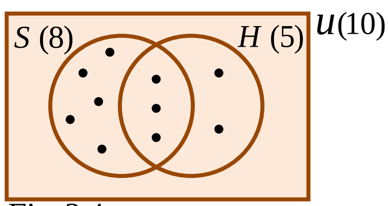


Fig. 2.4

The intersection of two sets A and B is defined as the set of all elements that belong to both A and B.

The operation '$\cap$' is used to define the intersection between two sets. Intersection of A and B is written as $A \cap B$. For example, if $A = \{1, 2, 3, 4, 5, 8\}$ and $B = \{2, 4, 6, 8, 10\}$, then $A \cap B = \{2, 4, 8\}$.

The shaded regions Fig. 2.5 show the intersection between the sets A and B.

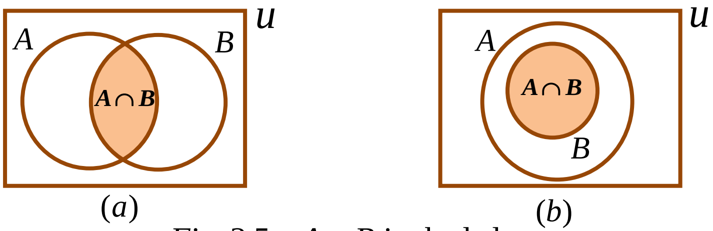


Fig. 2.5: $A \cap B$ is shaded

As illustrated in Fig. 2.5(b), if $B \subset A$, then $A \cap B = B$.

Example 2.28

If $A = \{2, 3, 5, 8\}$ and $B = \{1, 2, 5, 7, 9\}$, find $A \cap B$.

Solution

$A \cap B = \{2, 3, 5, 8\} \cap \{1, 2, 5, 7, 9\} = \{2, 5\}$.

Example 2.29

If $A = \{p, q, r\}$, $B = \{q, r, t\}$ and $C = \{p, q, t\}$, find: (a) $A \cap B$, (b) $A \cap C$.

Solution

(a) $A \cap B = \{p, q, r\} \cap \{q, r, t\} = \{q, r\}$. (b) $A \cap C = \{p, q, r\} \cap \{p, q, t\} = \{p, q\}$.

Disjoint set

If two sets A and B have no elements in common, then the sets A and B are ***disjoint sets***. Which implies A and B do not intersect. Thus, $A \cap B = \varnothing$. For example, if $P = \{a, b, c\}$ and $Q = \{p, q, r\}$, then P and Q are disjoint set since P and Q have no elements in common, that is $P \cap Q = \varnothing$.

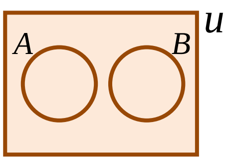


Fig. 2.6

Example 2.30

If $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8, 10\}$, find $A \cap B$.

Solution

$A \cap B = \{1, 3, 5, 7\} \cap \{2, 4, 6, 8, 10\} = \varnothing$.

Example 2.31

Let $A = \{a, b, c, d, e\}$ and $B = \{b, c, f, g\}$.

(a) Draw a Venn diagram of the two sets A and B. Show all the members of each set.

(b) Using this diagram, find the intersection of A and B.

Solution

(a) Fig. 2.7 shows the required Venn diagram.

(b) From the diagram $A \cap B = \{b, c\}$

Can you see another way of finding this intersection, without drawing the diagram?

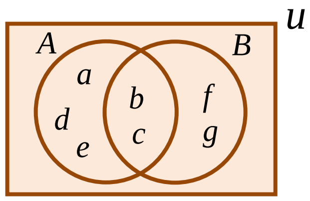


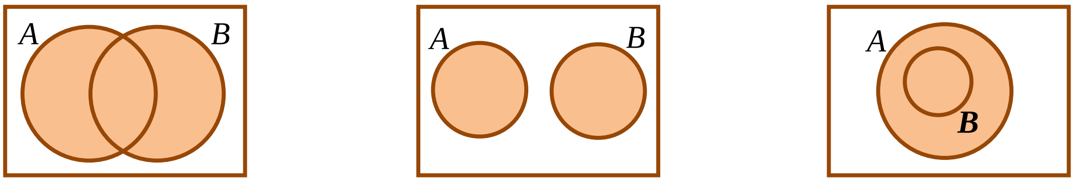
Fig. 2.7

2.6.2 Union of sets

The union of two sets A and B is defined as the set of all elements that belong to either A or B or both. The operation $\cup$ is used to define the union between two sets. The union of A and B is written as $A \cup B$.

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then $A \cup B = \{1, 2, 3, 4, 6, 8\}$.

The union of A and B is shaded in Fig. 2.8.



(a) (b) (c)

Fig. 2.8: $A \cup B$ is shaded

It can be seen that Fig. 2.8(c) that if $B \subset A$, then $A \cup B = A$.

Example 2.32

If $P = \{1, 3, 5\}$ and $Q = \{2, 3, 6\}$, find $P \cup Q$.

Solution

$P \cup Q = \{1, 3, 5\} \cup \{2, 3, 6\} = \{1, 2, 3, 5, 6\}$.

Example 2.33

If $A = \{p, q, r\}$, $B = \{q, r, t\}$ and $C = \{p, q, t\}$, find: (a) $B \cup C$, (b) $A \cap (B \cup C)$.

Solution

(a) $B \cup C = \{p, q, r, t\}$. (b) $A \cap (B \cup C) = \{p, q, r\} \cap \{p, q, r, t\} = \{p, q, r\}$.

Example 2.34

The sets P = {prime factors of 30} and Q = {factors of 36} are subsets of U = {integers}. List the elements of: (a) $P \cap Q$, (b) $P \cup Q$.

Solution

P = {prime factors of 30} = $\{2, 3, 5\}$,

Q = {factors of 36} = $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.

(a) $P \cap Q = \{2, 3\}$, (b) $P \cup Q = \{1, 2, 3, 4, 5, 6, 9, 12, 18, 36\}$.

Example 2.35

The sets A = {even numbers}, B = {factors of 42} and C = {prime numbers} are the subsets of $\mathcal{U}$ = {natural numbers less than 10}. Find:

(a) $A \cap B$, (b) $A \cap B \cap C$, (c) $B \cup C$.

Solution

u = {natural numbers less than 10} = {1, 2, 3, 4, 5, 6, 7, 8, 9}

A = {even numbers} = {2, 4, 6, 8}

B = {factors 42} = {1, 2, 3, 6, 7}

C = {prime numbers} = {2, 3, 5, 7}

(a) $A \cap B = \{2, 6\}$, (b) $A \cap B \cap C = \{2\}$, (c) $B \cup C = \{1, 2, 3, 5, 6, 7\}$.

Properties of set operations

1. Commutative properties:

The union ($\cup$) and the intersection ($\cap$) are both commutative. It follows that, for any two sets A and B,

(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$.

Example 2.36

Let $A = \{a, b, c, d\}$ and $B = \{a, d, e\}$.

(a) Find $A \cup B$ and $B \cup A$. Is $A \cup B = B \cup A$.

(b) Find $A \cap B$ and $B \cap A$. Is $A \cap B = B \cap A$.

Solution

(a) $A \cup B = \{a, b, c, d, e\}$. (b) $B \cup A = \{a, b, c, d, e\}$.

It can be seen that $A \cup B = B \cup A$.

(b) $A \cap B = \{a, d\}$. (b) $B \cap A = \{a, d\}$.

It can be seen that $A \cap B = B \cap A$.

2. Associative properties

The union ($\cup$) and the intersection ($\cap$) are also associative. For any three sets A, B and C,

(a) $(A \cup B) \cup C = A \cup (B \cup C)$ (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

Example 2.37

Let $A = \{a, b, c, d\}$, $B = \{a, b, f\}$ and $C = \{b, e, f\}$.

(a) Find $(A \cup B) \cup C$ and $A \cup (B \cup C)$. Is $(A \cup B) \cup C = A \cup (B \cup C)$.

(b) Find $(A \cap B) \cap C$ and $A \cap (B \cap C)$. Is $(A \cap B) \cap C = A \cap (B \cap C)$.

Solution

(a) $A \cup B = \{a, b, c, d, f\}$.

$(A \cup B) \cup C = \{a, b, c, d, f\} \cup \{b, e, f\} = \{a, b, c, d, e, f\}$.

$B \cup C = \{a, b, e, f\}$.

$A \cup (B \cup C) = \{a, b, c, d\} \cup \{a, b, e, f\} = \{a, b, c, d, e, f\}$.

It can be seen that $(A \cup B) \cup C = A \cup (B \cup C)$.

(b) $A \cap B = \{a, b\}$.

$(A \cap B) \cap C = \{a, b\} \cap \{b, e, f\} = \{b\}$.

$B \cap C = \{b, f\}$.

$A \cap (B \cap C) = \{a, b, c, d\} \cap \{b, f\} = \{b\}$.

It can be seen that $(A \cap B) \cap C = A \cap (B \cap C)$.

3. Distributive Properties

The intersection ($\cap$) is distributive over the union ($\cup$). For three sets A, B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Also the union ($\cup$) is distributive over the intersection ($\cap$). That is

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example 2.38

Let $A = \{a, b, c, d\}$, $B = \{a, e, f\}$ and $C = \{b, f, g\}$. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

$B \cup C = \{a, b, e, f, g\}$, $A \cap B = \{a\}$, $A \cap C = \{b\}$

$A \cap (B \cup C) = \{a, b, c, d\} \cap \{a, b, e, f, g\} = \{a, b\}$.

$(A \cap B) \cup (A \cap C) = \{a\} \cup \{b\} = \{a, b\}$.

It can be seen that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Complement of a set

Let A be a subset of the universal set $\mathcal{U}$. The ***complement of A*** is the set of members that belong to the universal set $\mathcal{U}$ but do not belong to A. The complement of A is written as A'.

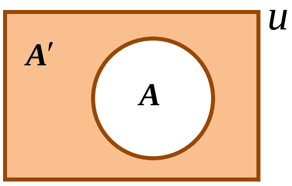


Fig. 2.8

For example, if $A = \{1, 3, 5\}$ and $B = \{2, 4, 5\}$ are subsets of the universal set $\mathcal{U} = \{1, 2, 3, 4, 5, 7\}$ then the complements of A and B are $A' = \{2, 4, 7\}$ and $B' = \{1, 3, 7\}$ respectively. The complement of the universal set is the empty set.

Example 2.39

The set A is a subset of the universal set $\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$. If $A = \{2, 4, 6\}$, then $A' = \{1, 3, 5\}$.

Note: $A \cap A' = \varnothing$ and $A \cup A' = \mathcal{U}$.

Example 2.40

The sets P = {factors of 36} and Q = {prime numbers} are subsets of $\mathcal{U}$ = {x: x is a whole number less than 10}. List the elements of

(a) P', (b) Q', (c) $P' \cap Q$, (d) $P \cap Q'$, (e) $(P \cup Q)'$.

Solution

$\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $P = \{1, 2, 3, 4, 6, 9\}$, $Q = \{2, 3, 5, 7\}$.

(a) If $P = \{1, 2, 3, 4, 6, 9\}$, then $P' = \{5, 7, 8, 10\}$.

(b) If $Q = \{2, 3, 5, 7\}$, then $Q' = \{1, 4, 6, 8, 9, 10\}$.

(c) If $P = \{1, 2, 3, 4, 6, 9\}$ and $Q' = \{1, 4, 6, 8, 9, 10\}$, then $P \cap Q' = \{1, 4, 6, 9\}$.

(d) $P \cup Q = \{1, 2, 3, 4, 5, 6, 7, 9\} \quad \Rightarrow \quad (P \cup Q)' = \{8, 10\}$.

Example 2.41

P = {multiples of 3} and Q = {factors of 12} are subsets of the universal set $U = \{x: 1 \leq x \leq 12\}$.

(a) Draw a Venn diagram to illustrate the above information.

(b) List the elements of (i) $P \cap Q$ (ii) $P \cup Q$ (iii) $P \cup Q'$ (iv) $P' \cap Q$ (v) $(P \cap Q)'$ (vi) $(P \cup Q)'$

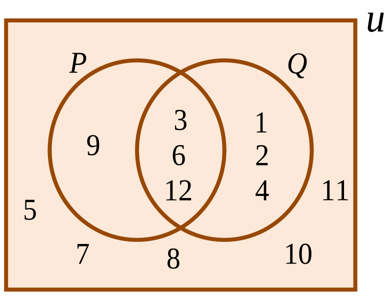
Solution

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

P = {multiples of 3} = {3, 6, 9, 12}

Q = {factors of 12} = {1, 2, 3, 4, 6, 12}

(a)



(b) (i) $P \cap Q$ = {3, 6, 12}

(ii) $P \cup Q$ = {1, 2, 3, 4, 6, 9, 12}

(iii) $P \cup Q'$ = {3, 6, 9, 12} $\cup$ {5, 7, 8, 9, 10, 11}
= {3, 5, 6, 7, 8, 9, 10, 11}

(iv) $P' \cap Q$ = {1, 2, 4, 5, 7, 8, 10, 11} $\cap$
= {1, 2, 3, 4, 6, 12} = {1, 2, 4}

(v) $(P \cap Q)'$ = {1, 2, 4, 5, 7, 8, 9, 10, 11}

(vi) $(P \cup Q)'$ = {5, 7, 8, 10, 11}

Exercise 2(f)

1. If P = {2, 6, 8, 9} and Q = {3, 5, 6, 7, 9}, find (a) $P \cap Q$, (b) $P \cup Q$.
2. If A = {2, 4, 6}, B = {2, 3, 6} and C = {2, 4, 5}, find: (a) $A \cap B$, (b) $A \cap C$.
3. Let A = {3, 5, 7, 9}, B = {2, 3, 7 } and C = {4, 7, 9}.

(a) Find $A \cup B$ and $B \cap C$.
(b) Find $(A \cup B) \cap C$ and $A \cup (B \cap C)$. Is $(A \cup B) \cup C = A \cup (B \cup C)$.

4. If $A = \{5, 7, 9\}$, $B = \{5, 7, 8\}$ and $C = \{3, 5, 7\}$, find:
(a) $B \cup C$, (b) $A \cap (B \cup C)$.

5. The sets P = {factors of 18} and Q = {prime factors 24} are subsets of $\mathcal{U}$ = {integers}. List the elements of: (a) $P \cap Q$, (b) $P \cup Q$.

6. The sets A = {odd numbers}, B = {factors of 36} and C = {prime numbers} are the subsets of $\mathcal{U}$ = {natural numbers less than 10}. Find:
(a) $A \cap B$, (b) $A \cap B \cap C$, (c) $B \cup C$.

7. Let $P = \{a, b, c, d\}$, $Q = \{a, c, d, e\}$ and $R = \{b, d, e, f\}$,
(a) Find $P \cup Q$ and $Q \cup R$, (b) Find $(P \cup Q) \cup R$ and $P \cup (Q \cup R)$,
(c) What can you say about $(P \cup Q) \cup R$ and $P \cup (Q \cup R)$,
(d) Find $P \cap Q$ and $Q \cap R$, (e) Find $(P \cap Q) \cap R$ and $P \cap (Q \cap R)$,
(f) What can you say about $(P \cap Q) \cap R$ and $P \cap (Q \cap R)$.

8. Let A = {odd numbers less than 25} and B = {factors of 25}. Find:
(a) $A \cup B$, (b) $A \cap B$.

9. Let P = {multiples of 4 less than 40} and Q = {multiples of 3 less than 32}. Find:
(a) $P \cup Q$, (b) $P \cap Q$.

10. Let A = {even numbers less than 16}, B = {multiples of 3 less than 20} and C = {factors of 24}.
(a) List the members of A, B and C.
(b) Find: (i) $A \cap B$, (ii) $A \cap C$, (iii) $A \cup B$, (iv) $B \cap C$, (v) $B \cup C$, (vi) $A \cup C$.

11. Let T = {prime numbers less than 16} and S = {odd numbers less than 11}.
(a) List the elements of T and S.
(b) List the elements of (i) $T \cap S$, (ii) $T \cup S$.

12. Let X = {composite numbers less than 14}, Y = {even numbers less than 14} and Z = {factors of 16}.
(a) List the elements of X, Y and Z.
(b) Find (i) $X \cap Y$, (ii) $X \cup Y$, (iii) $X \cap Z$, (iv) $Y \cap Z$.

13. The sets A and B are subsets of the universal set $\mathcal{U} = \{1, 2, 3, \ldots, 12\}$. If $A = \{1, 2, 5, 8\}$ and $B = \{2, 8, 9, 10, 11\}$, list the elements of the following.
(a) A', (b) B', (c) $A' \cap B$, (d) $A \cap B'$, (e) $A' \cup B$.

14. The sets P = {multiples of 3} and Q = {even numbers} are subsets of $\mathcal{U} = \{x$: x is an integer between 2 and 13$\}$. List the elements of
(a) P', (b) Q', (c) $P' \cap Q$, (d) $P \cap Q'$, (e) $(P \cup Q)'$.

15. A = {2, 3, 4, 5, 7} and B = {2, 3, 4, 7, 10, 19} are subsets of the universal set $\mathcal{U}$ = {2, 3, 4, 5, 7, 10, 13, 19, 37}. List the elements of: (a) $A' \cap B$ (b) $(A' \cap B)'$.

16. P, Q, and R are subsets of the universal set $\mathcal{U} = \{1 \leq x \leq 10$: x is an integer} and P = {x: x is a factor of 20}, Q = {x: x is a multiple of 5} and $R = \{x: x \geq 5\}$. Find:
(a) $P \cup Q$, (b) $P' \cap R'$, (c) $Q \cap R$, (d) $(P \cap Q')'$, (e) $P' \cup Q$.

17. The sets P = {x: x is a prime factor of 42} and Q = {x: x is a factor of 24} are subsets of the U = {x: x is an integer}. List the elements of (a) $P \cap Q$, (b) $P \cup Q$.

18. The sets A = {x: x is an odd number}, B = {x: x is a factor of 60} and C = {x: x is a prime number} are the subsets of $\mathcal{U}$ = {x: x is a natural number and $x < 9$}. Find :
(a) $A \cap B$, (b) $B' \cap C$, (c) $A \cap B \cap C$, (d) $B \cup C$.

19. A = {10, 11, 12, 13, 14} and B = {10, 12, 14, 16, 18} are subsets of the universal set $\mathcal{U}$ = {10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}.
List the elements of: (a) $A' \cap B$, (b) $(A' \cap B)'$.

20. P, Q, and R are subsets of the universal set $\mathcal{U} = \{1 \leq x \leq 12$: x is an integer} and P = {x: x is a factor of 12}, Q = {x: x is a multiple of 3} and $R = \{x: x \geq 4\}$. Find:
(a) $P \cup Q$ (b) $P' \cap R'$ (c) $Q \cap R$ (d) $(P \cap Q')'$ (e) $P' \cup Q$.

2.7 Using intersection of sets to find the LCM and HCF of numbers

In Chapter 1, we learnt how we can use prime factorization of numbers to find the LCM and HCF of numbers. In this section, apply the idea of intersection to determine the LCM and HCF of numbers.

2.7.1 The Least Common Multiple

Let A = {multiples of 4} and B = {multiples of 6}. Thus,
A = {4, 8, 12, 16, 20, 24, 28, 32, 36 …}, B = {6, 12, 18, 24, 30, 36, 42, 48, 54, …} and
$A \cap B$ = {12, 24, …}.
$A \cap B$ is the set of all common multiples to 4 and 6. The smallest number of the set is the Least Common Multiple (LCM) of 4 and 6. Thus, the LCM of 4 and 6 is 12.

Example 2.42
Find the LCM of 3 and 4.

Solution
Let P = {multiples of 3} = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, …}
Q = {multiples of 4} = {4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, …}
$P \cap Q$ = {12, 24, 36, …}.
The smallest multiple of 3 and 4 is 12. Hence, the LCM of 3 and 4 is 12.

Example 2.43

Find the LCM of 4, 6 and 8.

Solution

We consider two methods.

First method

Let P = {multiples of 4} = {4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, …},
Q = {multiples of 6} = {6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78 …},
R = {multiples of 8} = {8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, …}.
$P \cap Q \cap R$ = {24, 48, …}

The smallest multiple of 4, 6 and 8 is 24. Hence, the LCM of 4, 6 and 8 is 24.

Second method **(see Chapter 1)**

First we factor the numbers and list their prime factorizations:

$$4 = 1 \times 2 \times 2 = 1 \times 2^2, \qquad 6 = 1 \times 2 \times 3, \qquad 8 = 1 \times 2 \times 2 \times 2 = 1 \times 2^3.$$

The LCM of 4, 6 and 8 is the highest powers of the prime factors in 4, 6 and 8. The highest power of 2 is 2^3 while the highest power of 3 is 3. Hence, the LCM of 4, 6 and 8 is $2^3 \times 3 = 24$.

2.7.2 Highest Common Factor

Let A = {factors of 36} and B = {factors of 48}. Thus, A = {1, 2, 3, 4, 6, 9, 12, 18, 36} and B = {1, 2, 3, 4, 6, 8, 12, 16, 24, 48} and $A \cap B$ = {1, 2, 3, 4, 6, 12}. $A \cap B$ is the set of all common factors to 36 and 48. The largest number of the set is the Highest Common Factor (HCF) of 36 and 48. Therefore, the HCF of 36 and 48 is 12.

Example 2.44

Find the HCF of 24 and 18.

Solution

First Method

Let A = {factors of 24} = {1, 2, 3, 4, 6, 8, 12, 24} and B = {factors of 18} = {1, 2, 3, 6, 9, 18}, therefore, $A \cap B$ = {1, 2, 6}. The largest number in $A \cap B$ is 6. Hence, the HCF of 24 and 18 is 6.

Second Method **(see Chapter 1)**

Express 24 and 18 as product of the prime factors.

$$24 = 2 \times 12 = 2 \times 2 \times 6 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3.$$
$$18 = 2 \times 9 = 2 \times 3 \times 3 = 2 \times 3^2.$$

The prime factors common to 24 and 18 are 2 and 3. The HCF of 24 and 18 is the product of the lowest power of these common prime factors. Hence, the HCF of 24 and 18 is $2 \times 3 = 6$.

Exercise 2(g)

1. Let A = {factors of 42} and B = {factors of 36}. Find:
   (a) $A \cap B$, (b) the HCF of 42 and 72.
2. Let P = {multiples of 6} and Q = {multiples of 8}.
   (a) Find $P \cap Q$, (b) Use (a) to find the LCM of 6 and 8.
3. Let A = {factors of 18} and B = {factors of 20}. Find:
   (a) $A \cap B$, (b) the HCF of 18 and 20.
4. Let P = {multiples of 4} and Q = {multiples of 10}. Find:
   (a) Find $P \cap Q$, (b) Use (a) to find the LCM of 4 and 10.
5. Let A = {factors of 45} and B = {factors of 36}. Find:
   (a) $A \cap B$, (b) the HCF of 45 and 36.
6. Let P = {multiples of 7} and Q = {multiples of 14}.
   (a) Find $P \cap Q$, (b) Use (a) to find the LCM of 7 and 14.
7. Let A = {factors of 36}, B = {factors of 90}, C = {factors of 60} and D = {factors of 54}. Find: (a) $A \cap B \cap C \cap D$, (b) the HCF of 36, 90, 60 and 54.
8. Let P = {multiples of 4}, Q = {multiples of 8} and R = {multiples of 12}.
   (a) List the elements of P, Q and R. (b) Find the LCM of 4, 8 and 12.
9. Let A = {factors of 56}, B = {factors of 48} and C = {factors of 40}.
   (a) List the elements of A, B and C. (b) Find the HCF of 56, 48 and 40.
10. Let P = {multiples of 4}, Q = {multiples of 6} and R = {multiples of 8}.
    (a) List the elements of P, Q and R. (b) Find the LCM of 4, 6 and 8.

2.8 Practical applications of sets

Two-set problems

We learnt in Section 2.6 that it is often useful to represent sets in a Venn diagram. In this section, we consider some application of sets to real life situations. Fig. 2.9 shows two sets A and B represented on a Venn diagram. The circles representing A and B divide the rectangle for the universal set $\mathcal{U}$ into four regions R_1, R_2, R_3, and R_4.

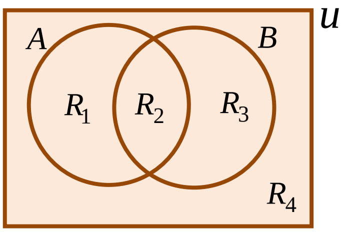


Fig. 2.9

The sets

R_1 consists of elements which belong to only A,

R_2 consists of elements which belong to both A and B,

R_3 consists of elements which belong to only B,

R_4 consists of elements which belong to neither A nor B.

Notice that:

$$n(\mathcal{U}) = n(R_1) + n(R_2) + n(R_3) + n(R_4).$$

Example 2.45

The Fig. 2.10 shows the results of an interview of students of Peace Junior High School. M = {students who like Mathematics} and E = {Students who like English}.

(a) How many students were interviewed?
(b) How many students like Mathematics?
(c) How many students like only one subject?
(d) How many students like English only?
(e) How many students like English and Mathematics?
(f) How many students like English or Mathematics?
(g) How many students like none of the two subjects?

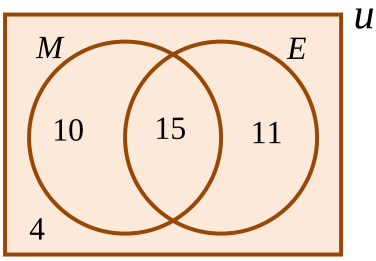


Fig. 2.10

Solution

(a) The number of students interviewed = 10 + 15 + 11 + 4 = 40.
(b) The number of students who like Mathematics = 10 + 15 = 25.
(c) The number of students who like only one subject = 10 + 11 = 21.
(d) The number of students who like English only = 11.
(e) In set operations the word ‘and’ implies intersection ($\cap$)

$\therefore$ The number of students who like English and Mathematics = $n(P \cap C)$ = 15.

(f) In set operations the word ‘or’ implies union ($\cup$)

$\therefore$ The number of students who like English or Mathematics $= n(P \cup C)$
$= 10 + 15 + 11 = 36.$

(g) The number of students who like none of the two subjects = 4.

Example 2.46

In a school, 27 students were asked their preference for two brands of soft drinks Fanta and Coca-Cola. 15 liked Fanta and 17 like Coca-Cola. Each student liked at least one of the two soft drinks.

(a) Draw a Venn diagram to illustrate this information.
(b) How many students liked both Fanta and Coca-Cola?

Solution

(a) Let $\mathcal{U}$ = {students},

F = {students who liked Fanta}, and

C = {students who liked Coca-Cola}.

Then $n(\mathcal{U}) = 27$, $n(F) = 15$ and $n(C) = 17$. Let x denote the number of students who liked both Fanta and Coca-Cola, that is $n(F \cap C) = x$. The Venn diagram is as shown in Fig. 2.11. Notice that $(15 - x)$ students like Fanta only and $(17 - x)$ liked Coca-Cola only.

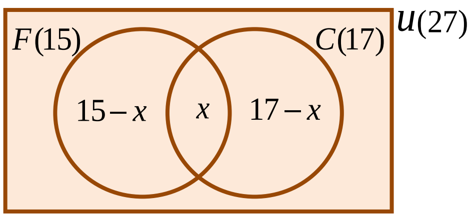


Fig. 2.11

(b) $n(\mathcal{U}) = (15-x) + x + (17-x) = 15 + 17 - x = 32 - x.$

But $n(\mathcal{U}) = 27$. Hence,

$32 - x = 27$, which gives $x = 32 - 27 = 5$.

Thus, 5 students liked both Fanta and Coca-Cola.

Example 2.47

In a group of 30 traders, 23 sell rice, 20 sell maize and 4 sell neither rice nor maize.

(a) Draw a Venn diagram to illustrate this information.

(b) How many traders sell

(i) both items, (ii) only rice, (iii) exactly one of the two items?

Solution

(a) Let $\mathcal{U}$ = {traders},

R = {traders who sell rice},

M = {traders who sell maize}.

Then, $n(\mathcal{U}) = 30$, $n(R) = 23$ and $n(M) = 20$.

Fig. 2.12 shows the required Venn diagram, where x denote the number of traders who sell both items. Notice that, since x traders sell both items and 23 traders sell rice, $(23 - x)$ traders sell only rice. Similarly, $(20 - x)$ traders sell only maize.

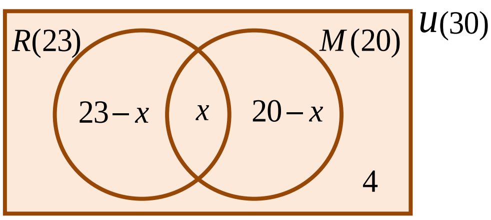


Fig. 2.12

(b) $n(\mathcal{U}) = 4 + (23-x) + x + (20-x) = 4 + 23 + 20 - x = 47 - x.$

But $n(\mathcal{U}) = 30$. Thus,

$47 - x = 30$ which gives $x = 47 - 30 = 17$.

(i) 17 traders sell both items.

(ii) The number of traders who sell only rice is

$$23 - x = 23 - 17 = 6.$$

(iii) The number of traders who sell exactly one of the two items is

$$(23 - x) + (20 - x) = (23 - 17) + (20 - 17) = 6 + 3 = 9.$$

Example 2.48

A survey of the reading habits of 130 students showed that 30 read both Comics and Novels, 10 read neither Comics nor Novels and 20 read Novels only.

(a) How many students read Novels? (b) How many read Comics?

(c) How many read only Comics?

Solution

Let $\mathcal{U}$ = {students},

C = {students who like Comics},

N = {students who like Novels}.

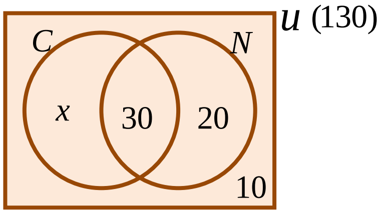


Fig. 2.13

Then, $n(\mathcal{U}) = 130$ and $n(C \cap N) = 30$. Fig. 2.13 shows a Venn diagram for the above information, where x denotes the number of students who read Comics only. Thus,

$$n(\mathcal{U}) = x + 30 + 20 + 10 = x + 60.$$

But $n(\mathcal{U}) = 130$. Thus,

$$x + 60 = 130 \text{ which gives } x = 130 - 60 = 70.$$

(a) The number of students who read Novels $= 20 + 30 = 50$.

(b) The number of students who read Comics $= x + 30 = 70 + 30 = 100$.

(c) The number of students who read Comics only $= x = 70$.

Example 2.49

(a) List the members of each of the sets

B = {whole numbers from 20 to 30} and D = {factors of 63}.

List the members of (i) $B \cap D$, (ii) $B \cup D$.

(b) In a class of 60 students, 46 passed Mathematics and 42 passed English Language. Every student passed at least one of the two subjects.

(i) Illustrate this information on a Venn diagram.

(ii) How many students passed in both subjects?

Let n represent the number of students who passed in both subjects. **1990**.

Ans: (a) $B = \{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$, $D = \{1, 3, 7, 9, 21, 63\}$

(i) $B \cap D = \{21\}$, (ii) $B \cup D = \{1, 3, 7, 9, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 63\}$

(b) (ii) 28 students.

Example 2.50

(a) If X = {prime numbers less than 13} and Y = {odd numbers less than 13}.
(i) List the members of X and Y. (ii) List the members of $X \cap Y$ and $X \cup Y$. **1991**.

(b) There are 20 students in a hostel; 16 of them are fluent in French and 10 of them are fluent in English. Each student is fluent in at least one of the two languages.
(i) Draw a Venn diagram to illustrate this information.
(ii) How many students are fluent in both English and French? **1995**.

***Ans*:** (a) (i) $X = \{2, 3, 5, 7, 11\}$, $Y = \{1, 3, 5, 7, 9, 11\}$,
(ii) $X \cap Y = \{3, 5, 7, 11\}$, $X \cup Y = \{1, 2, 3, 5, 7, 9, 11\}$. (b) (ii) 6 students.

Example 2.51

(a) M is a set consist of all positive integers between 1 and 10. P and Q are subsets of M such that P = {factors of 6} and Q = {multiple of 2}.
(i) List the elements of M, P and Q.
(ii) Represent M, P and Q on a Venn diagram.
(iii) Find $P \cap Q$. **2001**.

(b) There are 30 boys in a sporting club, 20 of them play hockey and 15 play volleyball. Each boy plays at least one of the two games.
(i) Illustrate the information on a Venn diagram.
(ii) How many boys play volleyball only? **2005**.

***Ans*:** (a) (i) $M = \{2, 3, 4, 5, 6, 7, 8, 9\}$, $P = \{2, 3, 6\}$, $Q = \{2, 4, 6, 8\}$, (ii) $P \cap Q = \{2, 6\}$.
(b) (ii) 10 boys.

Example 2.52

$\xi = \{1, 2, 3, 4, ..., 18\}$; A = {prime numbers} and B = {odd numbers greater than 3}.

(a) If A and B are subsets of the universal set ξ, list the members of A and B.

(b) Find the set (i) $A \cap B$, (ii) $A \cup B$.

(c) (i) Illustrate ξ, A and B on a Venn diagram.
(ii) Shade the region for prime factors of 18 on the Venn diagram. **2003**.

***Ans*:** (a) $A = \{2, 3, 5, 7, 11, 13, 17\}$, $B = \{5, 7, 9, 11, 13, 15, 17\}$.
(b) $A \cap B = \{5, 7, 11, 13, 17\}$, $A \cup B = \{2, 3, 5, 7, 9, 11, 13, 15, 17\}$.

Exercise 2(h)

1. There are 40 pupils in a class. 33 of them study French and 37 study English. Each student in the class studies at least one of the two subjects.
(a) Represent this information on a Venn diagram.
(b) How many pupils study all two subjects?

2. There are 22 players in a football team. 9 play defence, 12 play attack and 7 play neither defence nor attack.

(a) Represent this information on a Venn diagram
(b) How many play:
(i) both defence and attack, (ii) defence only?

3. In an athletic team of 36, there are 20 sprinters, 12 hurdlers and 8 are neither sprinters nor hurdlers.
(a) Represent this information on a Venn diagram.
(b) How many are
(i) both sprinters and hurdlers, (ii) hurlers only.

4. 12 teachers in Methodist Junior High School were asked their preference for two FM stations in Accra, Joy and Peace. 6 like Joy, 4 like Peace only and 2 like neither Joy nor Peace FM.
(a) Represent this information on a Venn diagram.
(b) How many teachers like
(i) exactly one of the two stations, (ii) Peace FM?

5. A recent survey revealed that the number of students studying one or more of the two subjects Mathematics and Science is as follows:

Mathematics	25
Science	30
Only Two Subjects	15.

(a) Represent this information on a Venn diagram.
(b) (i) Determine the number of students who study both subjects.
(ii) How many students were interviewed?

6. 36 teachers were asked their preference for two newspapers, Graphic and Times. 18 liked Graphic and 10 like both Graphic and Times. If 6 of the teachers liked neither Graphic nor times, illustrate this information on a Venn diagram. Find the number of teachers who like (a) Times only, (b) Times.

7. There are 40 students in a hostel; 25 of them are fluent in French and 20 of them are fluent in English. Each student is fluent in at least one of the two languages.
(a) Draw a Venn diagram to illustrate this information.
(b) How many students are fluent in both English and French?

8. In a class of 42 students, 26 offer Mathematics and 28 offer Chemistry. If each student offers at least one of the two subjects, find the number of students who offer both subjects.

9. In a class of 42 students each student studies either Economics or Accounting or both. If 12 students study both subjects and the number of students who study Accounting only is twice that of those who study Economics only, find how many students study (a) Economics, (b) Accounting.

10. In a school of 320 students, 85 students are in the band, 200 students are on sports teams, and 60 students participate in both activities. How many students are involved in either band or sports?

11. Of 100 farmers in a village, 65 grow tomatoes and 75 grow onions. Each farmer is known to grow at least one of the two crops. How many farmers grow both crops?

12. In a sport contingent there were 20 players in the football team and 18 players in the hockey team. 4 players play both football and hockey.
(a) Find the number of players in the contingent.
(b) Find the number who play only football or only hockey.

13. In a class, the number of students studying French or History is 40. 20 study both subjects and those who study French is 10 more than the number who study history.
(a) How many study French? (b) How many study History?

14. In an examination each student each of 100 students sat for Integrated Science and Technical Skills. All passed at least one of the two subjects. 55 passed Integrated Science and 75 passed Technical Skills. How many students passed exactly:
(a) two subjects, (b) one subject?

15. A survey of the reading habits of 65 people revealed that 33 read Graphic only, 10 read both Graphic and Times whilst 5 read none of the two news papers.
(a) Illustrate the information on a Venn diagram.
(b) Find the number of people who read Times.

Revision Exercise 2

1. (a) List the members of each of the sets
B = {whole numbers from 20 to 30} and D = {factors of 63}
List the members of (i) $B \cap D$, (ii) $B \cup D$.
(b) In a class of 60 students, 46 passed Mathematics and 42 passed English language. Every student passed at least one of the two subjects.
(i) Illustrate this information on a Venn diagram.
(ii) How many students passed in both subjects?

2. (a) If X = {prime numbers less than 13} and Y = {odd numbers less than 13},
(i) List the members of X and Y, (ii) List the members of $X \cap Y$ and $X \cup Y$.
(b) There are 20 students in a hostel, 16 of them are fluent in French and 10 of them are fluent in English. Each student is fluent in at least one of the two languages.
(i) Illustrate this information on a Venn diagram.
(ii) How many students are fluent in both English and French?

3. 25 students in a class took an examination in Mathematics and Science. 17 of them passed in Science and 8 passed in both mathematics and Science. 3 students did not pass in any of the subjects.
(a) How many passed in Mathematics?
(b) Find the probability of meeting a student who passed in one subject only.

4. (a) $\mathcal{U}$ is the set of odd numbers between 1 and 12. P and Q are subsets of $\mathcal{U}$ such that P = {factors of 63} and Q = {prime number}.
   (i) List the elements of $\mathcal{U}$, P and Q.
   (ii) Represent $\mathcal{U}$, P and Q on a Venn diagram. (iii) Find $P \cap Q$.
   (a) $\mathcal{U} = \{1, 3, 5, 7, 9\}$, $P = \{1, 3, 7, 9\}$, $Q = \{3, 5, 7\}$
   (b) There are 50 pupil in a class. Out of this number, 5 speak French only and 45 speak English. If each student speaks at least one of the langauges, find the number of students who speak English only.

5. $\mathcal{U} = \{1, 2, 3, \ldots, 18\}$; A = {prime numbers} and B = {odd numbers greater than 3}.
   (a) If A and B are subsets of the universal set, $\mathcal{U}$, list the members of A and B.
   (b) Find the set: (i) $A \cap B$, (ii) $A \cup B$.
   (c) (i) Illustrate $\mathcal{U}$, A and B on a Venn diagram.
   (ii) Shade the region for prime factors of 18 on the Venn diagram.

6. (a) There are 30 boys in a sporting club. 20 of them play hockey and 15 play volleyball. Each boy play at least one of the two games.
   (i) Illustrate the information on a Venn diagram.
   (ii) How many boys play volleyball only?
   (b) The sets P = {factors of 42} and Q = {prime numbers} are subsets of $\mathcal{U}$ = {x: x is a positive integer less than 15}. List the elements of
   (i) P', (ii) Q', (iii) $P' \cap Q$.

7. (a) In class of 39 students, 19 offer French and 25 Offer Ga. Five students do not offer any of the two languages. How many students offer only French?
   (b) E and F are subsets of the universal set $\mathcal{U}$ such that
   $\mathcal{U}$ = {natural numbers less than 15},
   E = {even numbers between 1 and 15} and
   F = {multiples of 4 between 9 and 15}.
   (i) List the elements of $\mathcal{U}$, E and F.
   (ii) Draw a Venn diagram to show the sets. **2008**.

8. (a) The sets P = {x: x is a prime factor of 63} and Q = {multiples of 3} are subsets of the U = {x: x is a whole number less than 22}. List the elements of
   (i) P, (ii) Q, (iii) $P \cap Q$, (iv) $P \cup Q$.
   (b) A survey of 65 farmers revealed that 50 cultivate maize and 30 cultivate yam. If 5 of them cultivate none of the two crops,
   (i) illustrate this information on a Venn diagram,
   (ii) find the number of farmers who cultivate yam only.

CHAPTER THREE

Fractions

3.1 Introduction

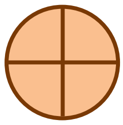


Fig. 3.1

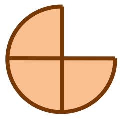


Fig. 3.2

Fig. 3.1 shows a circle which has been divided into four equal parts. One part is cut off and the remaining three parts is as shown in Fig. 3.2. We say that the remaining part is three out of four equal divisions which represents three quarters, $\frac{3}{4}$, of the whole circle and the part removed represents one quarter, $\frac{1}{4}$, of the whole circle. The resulting numeral, $\frac{3}{4}$, is an example of a ***fraction***.

A *fraction* is the result of dividing something into a number of parts.

Example 3.1

What fraction of the whole figure does the shaded part represent in each of the following?

(a)

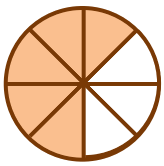


Fig. 3.3

(b)

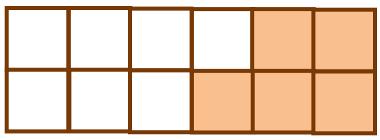


Fig. 3.4

(c)

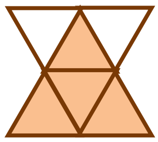


Fig. 3.5

Solution

(a) Fig. 3.3 is a circle divided into 8 equal parts and 5 of them are shaded. The shade region is therefore three eighth of the whole. The numeral for three eighths is $\frac{5}{8}$.

(b) Fig. 3.4 is a rectangle divided into 12 equal squares and 5 of them are shaded. The shaded part therefore represents five twelfths of the whole. The numeral for five twelfths is $\frac{5}{12}$.

(c) Fig. 3.5 is made up of six equal triangles of which four of them are shaded, which represent four sixths of the whole. The numeral for four sixths is $\frac{4}{6}$.

The numerals $\frac{5}{8}$, $\frac{5}{12}$ and $\frac{4}{6}$ in the solution to Example 3.1 are examples of fractions. Every fractional numeral can be expressed in the form $\frac{x}{y}$, where x and y are natural numbers. The quantity x is called the ***numerator*** while y is the ***denominator***. The denominator indicates the number equal parts the whole has been divided into while the numerator shows the number parts in the fraction.

$$\text{Fraction} = \frac{\text{numerator}}{\text{denominator}}.$$

For example, $\frac{5}{7}$ is a fraction that has 5 as its numerator and 7 as its denominator. Fractions are also called ***rational numbers***. It will sometimes be useful to remember that fractions can indicate division, since the fraction $\frac{a}{b}$ means $a \div b$, b cannot be zero. For instance, $\frac{1}{3}$ can mean "one divided by three", as well as "one part out of three parts".

Example 3.2

Fig. 3.6 shows a rectangle containing circles, rectangles and a triangle.

(a) Write the fraction that shows what part of the set are circles.

(b) Write the fraction that shows what part of the set are squares.

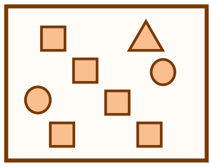


Fig. 3.6

Solution

(a) $\frac{2}{8}$ of the objects are circles. (b) $\frac{5}{8}$ of the objects are squares.

Example 3.3

A class has 20 girls and 30 boys. What part of the class are boys?

Solution

Numerator: boys = 30, Denominator: class = 20 + 30 = 50.

Part or fraction: $\frac{30}{50} = \frac{3}{5}$. So, $\frac{3}{5}$ of the class are boys.

Representing fractions on the number line

In Chapter 1 of this book, we learnt about how to represent natural numbers on the number line. We can also represent factions on the number line. Fig. 3.7 shows part of the number line with some fractions located on it.

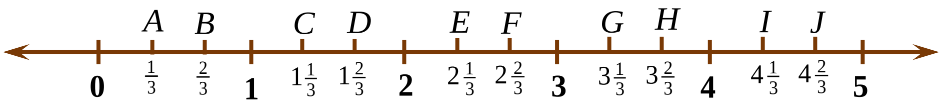


Fig. 3.7

When we divide the interval between 0 and 1 into three equal parts, we obtain the points A and B as shown in Fig. 3.7. A and B therefore corresponds to the fractions $\frac{1}{3}$ and $\frac{2}{3}$ respectively on the number line. Similarly, when the second unit interval, between 1 and 2, is divided into three equal parts and one part of them is taken, starting from 1 and moving towards 2, we locate the points C and D which corresponds to the fractions $1\frac{1}{3}$ and $2\frac{2}{3}$ respectively.

Types of fraction

***Proper Fraction*:** In a proper fraction the numerator is less than the denominator, as in $\frac{2}{3}, \frac{1}{2}, \frac{4}{9}$ etc.

***Improper Fraction*:** In an improper fraction the numerator is greater than the denominator, as in $\frac{5}{3}, \frac{11}{8}, \frac{12}{5}$ etc.

***Mixed Fraction*:** A mixed fraction contains both a whole number and a fractional part, as in $2\frac{2}{3}, 5\frac{1}{2}, 7\frac{5}{8}$ etc.

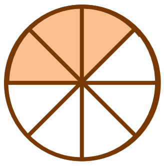
***Like Fraction*:** Like fractions are fractions with a common denominator. For example $\frac{2}{7}, \frac{3}{7}$ and $\frac{4}{7}$ are like fractions.

***Unlike Fraction*:** Unlike fractions have different denominators. For example, $\frac{1}{2}, \frac{5}{6}$ and $\frac{2}{7}$ are unlike fractions.

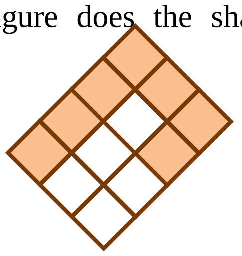
Exercise 3(a)

1. What fraction of the whole figure does the shaded part represent in each of the following?

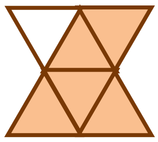
(a)



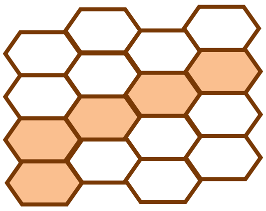
(b)



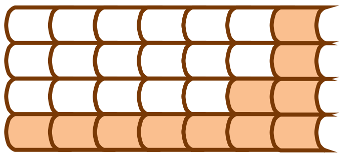
(c)



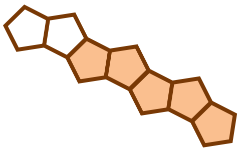
(d)



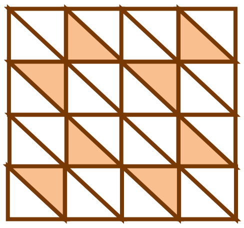
(e)



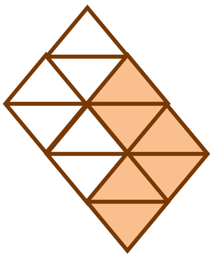
(f)



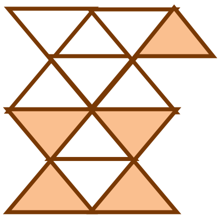
(g)



(h)



(i)



2. Kofi and Atta shared 10 oranges in such a way that Kofi receives 3 oranges. What fraction of the oranges did Atta receive?

3. In a class of 32 students, 5 of them were born on Sunday. Find the fraction of students who were born on Sunday.

4. A field is divided into 30 equal plots. If Mr. Jackson own 13 of these plots, what fraction of the field does he own?
5. A factory found that 50 of the goods which they produced were below standard. If the factory produced 200 articles, what fraction of the goods were below standard?
6. The profit which a bookshop makes in a month is GH¢ 600. If the total sales for that month was GH¢ 1 200, find what fraction of the total sales was made as profit?
7. Isaac achieves four fifths of the total marks in a mathematics test marked out of 30. What was his actual mark out of 30?
8. A housewife spent the following sums of money in buying ingredients for a family Christmas cake in 2006. She spent GH¢ 20 of the money on flour, GH¢ 15 on margarine and GH¢ 13 on sugar. Express amount spent on flour as a fraction of the total sum of money spent.
9. Mr. Ansah gave GH¢ 20 to his daughter and GH¢ 30 to his son. Find the fraction of the total sum given to his son.
10. Yaw Mensah withdrew a sum of money from the bank. He gave GH¢ 32 of it to his son and GH¢ 28 to his daughter. If he had GH¢ 40 left, what fraction of the sum was given to the daughter?
11. Alex scores 21 out of 70 in a Mathematics examination. What fraction of the total marks did he score?

3.2 Equivalent fractions

Some fractions may look different, but are really the same. For instance, when we multiplies both the numerator and the denominator of the fraction $\frac{1}{2}$ by 2, we obtain

$$\frac{1 \times 2}{2 \times 2} = \frac{2}{4}.$$

If we multiply by 4, we get

$$\frac{1 \times 4}{2 \times 4} = \frac{4}{8}.$$

Therefore, $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.

Because the fractions $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ have the same value, we say they are ***equivalent***.

> *Equivalent* fractions are fractions that have the same value but look different.

If the numerator and denominator of the fraction $\frac{a}{b}$ are both multiplied by the same integer the resulting fraction will be equivalent to $\frac{a}{b}$. This is so because any number divided by itself, is just one (1), that is, if n is any natural number, then $\frac{n}{n} = 1$. Thus,

$$\frac{a}{b} = \frac{a}{b} \times 1 = \frac{a}{b} \times \frac{n}{n} = \frac{a \times n}{b \times n}.$$

We use this fact in finding equivalent fractions and in reducing fractions. For example,

$$\frac{7}{5} = \frac{7 \times 4}{5 \times 4} = \frac{28}{40}.$$

Notice that if $\frac{a}{b} = \frac{p}{q}$, then $a \times q = b \times p$ and the vice versa.

Example 3.4

Determine which of the following pairs of fractions are equivalent.

(a) $\frac{2}{7}$ and $\frac{18}{63}$, (b) $\frac{3}{5}$ and $\frac{24}{40}$, (c) $\frac{4}{7}$ and $\frac{32}{42}$, (d) $\frac{5}{8}$ and $\frac{30}{46}$.

Solution

(a) *First method*

$$\frac{2}{7} = \frac{2 \times 9}{7 \times 9} = \frac{18}{63}.$$

Hence, $\frac{2}{7}$ and $\frac{18}{63}$ are equivalent fractions.

Second method

$2 \times 63 = 126$ and $7 \times 18 = 126.$

Hence, $\frac{2}{7}$ and $\frac{18}{63}$ are equivalent fractions.

(b) *First method*

$$\frac{3}{5} = \frac{3 \times 8}{4 \times 8} = \frac{24}{40}.$$

Hence, $\frac{3}{5}$ and $\frac{24}{40}$ are equivalent fractions.

Second method

$3 \times 40 = 120$ and $5 \times 24 = 120.$

Hence, $\frac{3}{5}$ and $\frac{24}{40}$ are equivalent fractions.

(c) $4 \times 42 = 168$ and $7 \times 32 = 224.$

It follows that $\frac{4}{7}$ and $\frac{32}{42}$ are not equivalent fractions.

(d) $5 \times 46 = 230$ and $8 \times 30 = 240.$

It follows that $\frac{5}{8}$ and $\frac{30}{46}$ are not equivalent fractions.

Example 3.5

For each of the following fractions, find an equivalent fraction which has the denominator indicated.

(a) $\frac{8}{9}$, denominator 27

(b) $\frac{4}{5}$, denominator 35

(c) $\frac{5}{12}$, denominator 72

(d) $\frac{7}{13}$, denominator 52.

Solution

(a) $\frac{8}{9} = \frac{8 \times 3}{9 \times 3} = \frac{24}{27}$,

(b) $\frac{4}{5} = \frac{4 \times 7}{5 \times 7} = \frac{28}{35}$,

(c) $\frac{5}{12} = \frac{5 \times 6}{12 \times 6} = \frac{30}{72}$,

(d) $\frac{7}{13} = \frac{7 \times 4}{13 \times 4} = \frac{28}{52}$.

3.2.1 Lowest term

It is usually best to show an answer using the simplest fraction. That is called ***simplifying*** or ***reducing*** the fraction. When the denominator and numerator of a fraction are simplified until they have no common factor, we say that the fraction is in its **lowest term** or **simplest form**. The following two methods can be used in simplifying a fraction to its lowest term.

Method 1

Try dividing both the top and bottom of the fraction by any common factor until you can't go any further. This is illustrated in the following example.

Example 3.6

Simplify the fraction $\frac{36}{60}$.

Solution

$$\frac{36}{60} \overset{\div 2}{=} \frac{18}{30} \overset{\div 2}{=} \frac{9}{15} \overset{\div 3}{=} \frac{3}{5}$$

Therefore, $\frac{36}{60} = \frac{3}{5}$.

Method 2

Divide both the top and bottom of the fraction by the ***highest common factor (HCF)***, (you have to work it out first!).

Example 3.7

Simplify the fraction $\frac{8}{12}$.

Solution

1. The highest common factor (HCF) of 8 and 12 is 4.
2. Divide both top and bottom by 4:

   Therefore, $\frac{8}{12} = \frac{2}{3}$.

$$\frac{8}{12} \overset{\div 4}{=} \frac{2}{3}$$

Example 3.8

Write each fraction in the simplest form.

(a) $\frac{4}{8}$, (b) $\frac{35}{40}$, (c) $\frac{36}{42}$, (d) $\frac{8}{16}$.

Solution

(a) The GCF is 4, so

$$\frac{4}{8} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2}.$$

(b) The GCF is 5, so

$$\frac{35}{40} = \frac{35 \div 5}{40 \div 5} = \frac{7}{8}.$$

(c) The GCF is 6, so

$$\frac{36}{42} = \frac{36 \div 6}{42 \div 6} = \frac{6}{7}.$$

(d) The GCF is 8, so

$$\frac{8}{16} = \frac{8 \div 8}{16 \div 8} = \frac{1}{2}.$$

3.2.2 Mixed numbers

As stated earlier, improper fractions have the numerator part ***greater or equal*** to the denominator part, for example $\frac{5}{2}$ or $\frac{7}{3}$. A mixed fraction is a ***whole number plus a fraction***, for example $2\frac{1}{2}$ or $5\frac{2}{3}$. Mixed Fractions are sometimes called ***mixed numbers*** or ***mixed numerals***. It is not always easy to see how big an improper fraction is. Therefore, when you do a calculation, it is better to give the result as a mixed number.

To calculate the numerator when converting mixed to improper fractions:

- ♣ multiply the whole number of the mixed fraction by the denominator, then
- ♣ add on the numerator of the fraction part.

This is illustrated in the following example.

Example 3.9

Express the following mixed numbers as improper fraction.

(a) $4\frac{2}{3}$, (b) $6\frac{4}{5}$, (c) $12\frac{1}{3}$, (d) $11\frac{2}{5}$.

Solution

(a) $4\frac{2}{3} = \frac{3 \times 4 + 2}{3} = \frac{12 + 2}{3} = \frac{14}{3}.$

(b) $6\frac{4}{5} = \frac{5 \times 6 + 4}{5} = \frac{30 + 4}{5} = \frac{34}{5}.$

(c) $12\frac{1}{3} = \frac{3 \times 12 + 1}{3} = \frac{36 + 1}{3} = \frac{37}{3}.$

(d) $11\frac{2}{5} = \frac{5 \times 11 + 2}{5} = \frac{55 + 2}{5} = \frac{57}{5}.$

To change the improper fraction to a mixed fraction, divide the numerator by the denominator.

- ♣ The quotient (i.e. the result of division) is the whole number part of the mixed fraction
- ♣ The remainder is the numerator of the fraction part of the mixed fraction

Example 3.10

Express the following improper fractions as mixed numbers.

(a) $\frac{9}{2}$, (b) $\frac{38}{3}$, (c) $\frac{35}{9}$, (d) $\frac{23}{8}$.

Solution

(a) $\frac{9}{2} = 9 \div 2 = 4$ remainder 1. Hence $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

(b) $\frac{38}{3} = 38 \div 3 = 12$ remainder 2. Hence $\frac{38}{3} = 12 + \frac{2}{3} = 12\frac{2}{3}$.

(c) $\frac{35}{9} = 35 \div 9 = 3$ remainder 8. Hence $\frac{35}{9} = 3 + \frac{8}{9} = 3\frac{8}{9}$.

(d) $\frac{23}{8} = 23 \div 8 = 2$ reminder 7. Hence $\frac{23}{8} = 2 + \frac{7}{8} = 2\frac{7}{8}$.

Exercise 3(b)

1. Determine which of the following pairs of fractions are equivalent:
   (a) $\frac{7}{11}$ and $\frac{21}{33}$, (b) $\frac{13}{24}$ and $\frac{39}{72}$, (c) $\frac{5}{8}$ and $\frac{45}{72}$, (d) $\frac{5}{9}$ and $\frac{45}{54}$.
   (e) $\frac{3}{13}$ and $\frac{24}{104}$, (f) $\frac{3}{5}$ and $\frac{27}{40}$, (g) $\frac{8}{9}$ and $\frac{48}{54}$, (h) $\frac{5}{8}$ and $\frac{40}{48}$.

2. For each of the following fractions, find an equivalent fraction which has the denominator indicated.
   (a) $\frac{4}{5}$, denominator 65 (b) $\frac{7}{8}$, denominator 56
   (c) $\frac{11}{7}$, denominator 63 (d) $\frac{5}{17}$, denominator 68.

3. Express the following fractions in their lowest terms.
   (a) $\frac{24}{36}$, (b) $\frac{40}{72}$, (c) $\frac{65}{91}$, (d) $\frac{96}{108}$, (e) $\frac{45}{75}$, (f) $\frac{64}{88}$,
   (g) $\frac{64}{112}$, (h) $\frac{42}{70}$, (i) $\frac{68}{85}$, (j) $\frac{38}{57}$, (k) $\frac{84}{63}$, (l) $\frac{108}{81}$.

4. Complete the following equivalent fraction.
   (a) $\frac{2}{3} = \frac{}{12} = \frac{}{18}$, (b) $\frac{3}{5} = \frac{}{25} = \frac{21}{}$, (c) $\frac{7}{11} = \frac{21}{} = \frac{35}{}$,
   (d) $\frac{4}{7} = \frac{}{56} = \frac{48}{}$, (e) $\frac{8}{9} = \frac{}{18} = \frac{}{45}$, (e) $\frac{11}{12} = \frac{33}{} = \frac{}{84}$.

5. A housewife spent the GH¢ 270 of her monthly allowance of GH¢ 900 in buying ingredients for a family Christmas cake. What fraction of her allowance did she spend on the ingredients?

6. The development budget of a District Council includes expenditure on feeder roads, schools and water supply. The expenditure on roads, schools and water supply are 14 000, 30 000 and 4 000 respectively. What fraction of the total budget is used on schools?

7. In an election, the number of votes won by political parties *A*, *B* and *C* in a village are as follows.

Party	*A*	*B*	*C*
Number of votes	320	560	120

   What percentage of the total votes did the winner obtain?

8. Write the following mixed fractions as improper fraction:

   (a) $1\frac{2}{5}$, (b) $4\frac{2}{5}$, (c) $11\frac{2}{3}$, (d) $2\frac{11}{13}$, (e) $3\frac{11}{14}$, (f) $1\frac{15}{17}$, (g) $20\frac{1}{2}$, (h) $8\frac{1}{5}$.

9. Express the following improper fractions as mixed numbers.

   (a) $\frac{9}{5}$, (b) $\frac{19}{7}$, (c) $\frac{55}{3}$, (d) $\frac{43}{9}$, (e) $\frac{37}{13}$, (f) $\frac{65}{17}$, (g) $\frac{51}{4}$, (h) $\frac{103}{6}$.

10. Simplify each of the following:

   (a) $\frac{26 \times 4}{13 \times 20}$, (b) $\frac{18 \times 12}{3 \times 36}$, (c) $\frac{45 \times 35}{42 \times 60}$, (d) $\frac{27 \times 16}{4 \times 52}$, (e) $\frac{24 \times 12}{18 \times 21}$, (f) $\frac{9 \times 8}{4 \times 24}$.

3.3 Comparing and ordering fractions

What is the easiest way of telling that one fraction is bigger than another, for example, which of the fractions $\frac{5}{9}$ and $\frac{7}{11}$, is greater? How do you place fractions in descending or ascending order? The sign < is used in ordering fractions in ascending, increasing order or from the smallest to the largest. For example, $\frac{1}{7} < \frac{2}{7} < \frac{3}{7} < \frac{4}{7}$. The sign > is used in ordering fractions in descending, decreasing order or from the largest to the smallest. For example, $\frac{4}{7} > \frac{3}{7} > \frac{2}{7} > \frac{1}{7}$.

Fractions with different denominators can be compared and ordered using equivalent fractions with the same denominators. The following three examples illustrate the method.

Example 3.11

Determine an order (inequality), either < or >, for each of the following pairs of fractions.

(a) $\frac{3}{4}$, $\frac{5}{6}$ (b) $\frac{5}{8}$, $\frac{2}{3}$ (c) $\frac{2}{7}$, $\frac{3}{10}$ (d) $\frac{3}{4}$, $\frac{5}{7}$

Solution

(a) ♣ We first determine the LCM of the two denominators, 4 and 6. Now, the LCM of 4 and 6 is 12.

♣ We then express each of the fractions to an equivalent fraction with 12 as the denominator.

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12} \quad \text{and} \quad \frac{5}{6} = \frac{5 \times 2}{6 \times 2} = \frac{10}{12}.$$

Since $\frac{9}{12} < \frac{10}{12}$, it follows that $\frac{3}{4} < \frac{5}{6}$.

(b) Writing the fractions with same denominator – equivalent fractions;

$$\frac{5}{8} = \frac{5 \times 3}{8 \times 3} = \frac{15}{24} \quad \text{and} \quad \frac{2}{3} = \frac{2 \times 8}{3 \times 8} = \frac{16}{24}.$$

Since $\frac{16}{24} > \frac{15}{24}$, we conclude that $\frac{2}{3} > \frac{5}{8}$.

(c) Writing the fractions with same denominator – equivalent fractions;

$$\frac{2}{7} = \frac{2 \times 10}{7 \times 10} = \frac{20}{70} \quad \text{and} \quad \frac{3}{10} = \frac{3 \times 7}{10 \times 7} = \frac{21}{70}.$$

Since $\frac{20}{70} < \frac{21}{70}$, we conclude that $\frac{2}{7} < \frac{3}{10}$.

(d) Writing the fractions with same denominator – equivalent fractions;

$$\frac{3}{4} = \frac{3 \times 7}{4 \times 7} = \frac{21}{28} \quad \text{and} \quad \frac{5}{7} = \frac{5 \times 4}{7 \times 4} = \frac{20}{28}.$$

Since $\frac{21}{28} > \frac{20}{28}$, we conclude that $\frac{3}{4} > \frac{5}{7}$.

Example 3.12

In each of the following sets, put the fractions in a ascending order of magnitude (that is, from lowest to highest).

(a) $\frac{8}{9}, \frac{11}{12}, \frac{2}{3}, \frac{5}{6}$ (b) $\frac{3}{4}, \frac{7}{16}, \frac{5}{6}, \frac{1}{2}$

Solution

(a) ♣ We first determine the LCM of the four denominators, 9, 12, 6 and 3.

$$9 = 3^2, \quad 12 = 2^2 \times 3, \quad 6 = 2 \times 3, \quad 3 = 3.$$

The LCM of 9, 6, 3, and 12 $= 2^2 \times 3^2 = 4 \times 9 = 36$.

♣ We then express each of the fractions to an equivalent fraction with 36 as the denominator.

$$\frac{8}{9} = \frac{8 \times 4}{9 \times 4} = \frac{32}{36}, \quad \frac{5}{6} = \frac{5 \times 6}{6 \times 6} = \frac{30}{36}, \quad \frac{2}{3} = \frac{2 \times 12}{3 \times 12} = \frac{24}{36}, \quad \frac{11}{12} = \frac{11 \times 3}{12 \times 3} = \frac{33}{36}.$$

Since $\frac{24}{36} < \frac{30}{36} < \frac{32}{36} < \frac{33}{36}$, we can conclude that $\frac{2}{3} < \frac{5}{6} < \frac{8}{9} < \frac{11}{12}$.

(b) ♣ We first determine the LCM of the four denominators, 4, 16, 6 and 2.

$$4 = 2^2, \quad 16 = 2^4, \quad 6 = 2 \times 3, \quad 2 = 2.$$

The LCM of 9, 6, 3, and 12 $= 2^4 \times 3 = 16 \times 3 = 48$.

♣ We then express each of the fractions to an equivalent fraction with 48 as the denominator.

$$\frac{3}{4} = \frac{3 \times 12}{4 \times 12} = \frac{36}{48}, \quad \frac{7}{16} = \frac{7 \times 3}{16 \times 3} = \frac{21}{48}, \quad \frac{5}{6} = \frac{5 \times 8}{6 \times 8} = \frac{40}{48}, \quad \frac{1}{2} = \frac{1 \times 24}{2 \times 24} = \frac{24}{48}.$$

Since $\frac{21}{48} < \frac{24}{48} < \frac{36}{48} < \frac{40}{48}$, we can conclude that $\frac{7}{16} < \frac{1}{2} < \frac{3}{4} < \frac{5}{6}$.

Example 3.13

(a) A housewife spent the following sums of money in buying ingredients for a family Christmas cake in 2006. She spent $\frac{2}{15}$ of the money on margarine, $\frac{1}{6}$ of the money on flour and $\frac{5}{36}$ of the money on sugar. Which of these three ingredients did she spend the largest amount?

(b) Arrange the following fractions in descending order of magnitude (that is, from highest to lowest) $\frac{5}{12}, \frac{2}{5}, \frac{1}{2}$.

Solution

(a) We first arrange the three fractions, $\frac{2}{15}, \frac{1}{6}$ and $\frac{5}{36}$ in increasing order of magnitude. We now determine the LCM of the denominator 15, 6 and 36.

$$15 = 3 \times 5, \qquad 6 = 2 \times 3, \qquad 36 = 2^2 \times 3^2.$$

The LCM of 9, 6 and 36 = $2^2 \times 3^2 \times 5$ = 180.

$$\frac{2}{15} = \frac{2 \times 12}{15 \times 12} = \frac{24}{180}, \qquad \frac{1}{6} = \frac{1 \times 30}{6 \times 30} = \frac{30}{180}, \qquad \frac{5}{36} = \frac{5 \times 5}{36 \times 5} = \frac{25}{180}.$$

Since $\frac{30}{180} > \frac{25}{180} > \frac{2}{15}$, we can conclude that $\frac{1}{6} > \frac{5}{36} > \frac{2}{15}$.

Hence, she spent the largest amount on floor.

(b) We first find the LCM of the denominators 12, 5, and 2. The LCM is 60. We then express each of the fractions to an equivalent fraction with 60 as the denominator.

$$\frac{5}{12} = \frac{5 \times 5}{12 \times 5} = \frac{25}{60}, \qquad \frac{2}{5} = \frac{2 \times 12}{5 \times 12} = \frac{24}{60}, \qquad \frac{1}{2} = \frac{1 \times 30}{2 \times 30} = \frac{30}{60}.$$

Since $\frac{30}{60} > \frac{25}{60} > \frac{24}{60}$, we can conclude that $\frac{1}{2} > \frac{5}{12} > \frac{2}{5}$. Hence the required order is $\frac{1}{2}, \frac{5}{12}, \frac{2}{5}$.

Exercise 3(c)

1. Determine an order (inequality), either < or >, for each of the following pairs of fractions.

(a) $\frac{4}{9}, \frac{7}{15}$ (b) $\frac{3}{10}, \frac{1}{2}$ (c) $\frac{5}{12}, \frac{4}{9}$ (d) $\frac{1}{4}, \frac{5}{24}$ (e) $\frac{2}{13}, \frac{3}{26}$

(f) $\frac{3}{10}, \frac{7}{30}$ (g) $\frac{8}{25}, \frac{2}{5}$ (h) $\frac{1}{6}, \frac{5}{24}$ (i) $\frac{13}{16}, \frac{3}{4}$ (j) $\frac{2}{17}, \frac{5}{34}$

2. In each of the following sets, put the fractions in ascending order of magnitude (that is, from lowest to highest).

(a) $\frac{9}{13}, \frac{8}{13}, \frac{7}{26}, \frac{11}{26}$ (b) $\frac{3}{8}, \frac{7}{12}, \frac{11}{24}, \frac{5}{16}$ (c) $\frac{3}{16}, \frac{11}{48}, \frac{5}{32}, \frac{5}{24}$

(d) $\frac{1}{6}, \frac{5}{18}, \frac{11}{36}, \frac{5}{24}$ (e) $\frac{3}{20}, \frac{7}{50}, \frac{6}{25}, \frac{1}{5}$ (f) $\frac{5}{36}, \frac{1}{9}, \frac{11}{72}, \frac{7}{48}$.

3. In a football team, $\frac{2}{3}$ of the players are attackers, $\frac{5}{12}$ of the players are defenders and $\frac{1}{2}$ of the players are midfielders. Find which of these three positions has the least number of players.

4. In an athletic team, $\frac{1}{2}$ of the athletes are hurdlers, $\frac{5}{8}$ athletes are sprinters and $\frac{5}{12}$ of the athletes are pole-vaulters. Determine which of the three events has the maximum number of athletes.
5. In a certain school, the students were asked their preference for three brands of soft drinks Fanta, Coca-Cola, and Sprite. $\frac{5}{14}$ of the students liked Fanta, $\frac{10}{21}$ of the students liked Coca-Cola and $\frac{2}{7}$ of the students liked Sprite. Which of the three soft drinks is most patronized by the students?
6. In a group of traders, $\frac{5}{18}$ of them sell gari, $\frac{5}{12}$ sell rice and $\frac{1}{3}$ sell maize. Which of the three items is sold by most of the traders?

3.4 Addition and subtraction of Fractions

You can add or subtract fractions easily if the bottom number (that is the denominator) is the same. To add two fractions with the same denominator, you simply add the numerators and keep the denominator the same.

Example 3.14

(a) $\frac{5}{7} + \frac{4}{7} = \frac{5+4}{7} = \frac{9}{7} = 1\frac{2}{7}$, (b) $\frac{5}{8} - \frac{3}{8} = \frac{5-3}{8} = \frac{2}{8} = \frac{1}{4}$.

If the denominators are ***not*** the same, you may apply the technique mentioned above to make them the same before doing the addition or the subtraction. To achieve this, we can use the following steps:

1. Find the LCM of the denominators of the fractions.
2. Express each of the fractions to an equivalent fraction with the LCM as the denominator.
3. Add or subtract as illustrated in Example 3.14.

Example 3.15

Simplify the following:

(a) $\frac{1}{4} + \frac{2}{5}$, (b) $\frac{3}{8} + \frac{5}{6}$, (c) $\frac{5}{6} + \frac{4}{9}$, (d) $\frac{6}{15} + \frac{2}{9}$.

Solution

(a) The LCM of the denominators, 4 and 5, is 20.

$$\frac{1}{4} + \frac{2}{5} = \frac{1\times 5}{4\times 5} + \frac{2\times 4}{5\times 4} = \frac{5}{20} + \frac{8}{20} = \frac{13}{20}.$$

(b) The LCM of the denominators, 8 and 6, is 24.

$$\frac{3}{8} + \frac{5}{6} = \frac{3\times 3}{8\times 3} + \frac{5\times 4}{6\times 4} = \frac{9}{24} + \frac{20}{24} = \frac{9+20}{24} = \frac{29}{24} = 1\frac{5}{24}.$$

(c) The LCM of the denominators, 6 and 9, is 18.

$$\frac{5}{6} + \frac{4}{9} = \frac{5 \times 3}{6 \times 3} + \frac{4 \times 2}{9 \times 2} = \frac{15}{18} + \frac{8}{18} = \frac{15 + 8}{18} = \frac{23}{18} = 1\frac{5}{18}.$$

(d) The LCM of the denominators, 15 and 9, is 45.

$$\frac{6}{15} + \frac{2}{9} = \frac{6 \times 3}{15 \times 3} + \frac{2 \times 5}{9 \times 5} = \frac{18}{45} + \frac{10}{45} = \frac{18 + 10}{45} = \frac{28}{45}.$$

Example 3.16

Simplify the following:

(a) $\frac{3}{4} - \frac{1}{6}$, (b) $\frac{5}{6} - \frac{7}{12}$, (c) $\frac{5}{8} - \frac{5}{12}$, (d) $\frac{7}{9} - \frac{11}{15}$.

Solution

(a) The LCM of the denominators, 4 and 6, is 12.

$$\frac{3}{4} - \frac{1}{6} = \frac{3 \times 3}{4 \times 3} - \frac{1 \times 2}{6 \times 2} = \frac{9}{12} - \frac{2}{12} = \frac{9 - 2}{12} = \frac{7}{12}.$$

(b) The LCM of the denominators, 6 and 12, is 12.

$$\frac{5}{6} - \frac{7}{12} = \frac{5 \times 2}{6 \times 2} - \frac{7 \times 1}{12 \times 1} = \frac{10}{12} - \frac{7}{12} = \frac{10 - 7}{12} = \frac{3}{12} = \frac{1}{4}.$$

(c) The LCM of the denominators, 8 and 12, is 24.

$$\frac{5}{8} - \frac{5}{12} = \frac{5 \times 3}{8 \times 3} - \frac{5 \times 2}{12 \times 2} = \frac{15}{24} - \frac{10}{24} = \frac{15 - 10}{24} = \frac{5}{24}.$$

(d) The LCM of the denominators, 9 and 15, is 45.

$$\frac{7}{9} - \frac{11}{15} = \frac{7 \times 5}{9 \times 5} - \frac{11 \times 3}{15 \times 3} = \frac{35}{45} - \frac{33}{45} = \frac{35 - 33}{45} = \frac{2}{45}.$$

Example 3.17

Simplify the following:

(a) $6\frac{2}{5} + 3\frac{3}{10}$, (b) $5\frac{2}{3} - 3\frac{1}{2}$.

Solution

Note carefully how these sums are done;

(a) *First method*

We first express each mixed fraction as improper fraction and then simplify.

$$6\frac{2}{5} + 3\frac{3}{10} = \frac{32}{5} + \frac{33}{10} = \frac{32 \times 2}{5 \times 2} + \frac{33 \times 1}{10 \times 1} = \frac{64}{10} + \frac{33}{10} = \frac{97}{10} = 9\frac{7}{10}.$$

Second method

$$6\frac{2}{5} + 3\frac{3}{10} = (6 + 3) + \left(\frac{2}{5} + \frac{3}{10}\right) = 9 + \left(\frac{2 \times 2}{5 \times 2} + \frac{3}{10}\right) = 9 + \left(\frac{4}{10} + \frac{3}{10}\right) = 9 + \frac{7}{10} = 9\frac{7}{10}.$$

(b) *First method*

We first express each mixed fraction as improper fraction and then simplify.

$$5\frac{2}{3} - 3\frac{1}{2} = \frac{17}{3} - \frac{7}{2} = \frac{17 \times 2}{3 \times 2} - \frac{7 \times 3}{2 \times 3} = \frac{34}{6} - \frac{21}{6} = \frac{13}{6} = 2\frac{1}{6}.$$

Second method

$$5\frac{2}{3}-3\frac{1}{2} = (5-3)+\frac{2}{3}-\frac{1}{2} = 2+\frac{2\times 2}{3\times 2}-\frac{1\times 3}{2\times 3} = 2+\frac{4}{6}-\frac{3}{6} = 2+\frac{1}{6} = 2\frac{1}{6}.$$

Word problems

Example 3.18

Mensah walked $\frac{1}{3}$ of a mile yesterday and $\frac{3}{4}$ of miles today. How many miles has Mensah walked?

Solution

This word problem requires addition of fraction.

Total distance covered $= \frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12} = 1\frac{1}{12}.$

So, Mensah walked a total of $1\frac{1}{12}$ miles.

Example 3.19

Kojo bought $2\frac{1}{3}$ gallons of paint but he only used $\frac{1}{2}$ gallon of the paint. How much paint was left?

Solution

This word problem requires subtraction of fraction.

Gallons of paint left $= 2\frac{1}{3} - \frac{1}{2} = \frac{7}{3} - \frac{1}{2} = \frac{14}{6} - \frac{3}{6} = \frac{11}{6} = 1\frac{5}{6}.$

Example 3.20

Mr. Lartey gave $\frac{1}{5}$ of his salary to his son and $\frac{8}{15}$ of his salary to the daughter. What fraction of his salary is left?

Solution

Total fraction given $= \frac{1}{5} + \frac{8}{15} = \frac{3}{15} + \frac{8}{15} = \frac{3+8}{15} = \frac{11}{15}.$

Fraction left $= 1 - \frac{11}{15} = \frac{15}{15} - \frac{11}{15} = \frac{15-11}{15} = \frac{4}{15}.$

Example 3.21

Four friends, Yeboah, Mills, Cudjoe and Sekyi, entered into a business partnership. Yeboah invested $\frac{1}{4}$ of the total capital, Mills invested $\frac{1}{8}$ of the total capital and Cudjoe invested $\frac{3}{12}$ of the total capital, while the remaining amount is invested by Sekyi. What fraction of the total capital is invested by Sekyi?

Solution

Total fraction of the capital invested by Yeboah, Mills and Cudjoe

$$\frac{1}{4}+\frac{1}{8}+\frac{3}{12} = \frac{1\times 6}{4\times 6}+\frac{1\times 3}{8\times 3}+\frac{3\times 2}{12\times 2} = \frac{6}{24}+\frac{3}{24}+\frac{6}{24} = \frac{15}{24} = \frac{5}{8}.$$

Hence the fraction of the total capital invested by Sekyi is

$$1-\frac{5}{8} = \frac{8}{8}-\frac{5}{8} = \frac{8-5}{8} = \frac{3}{8}.$$

Example 3.22

The monthly expenditure of Mr. Kwao is on transportation, food and utilities. He spent $\frac{1}{6}$ of his monthly salary on transportation, $\frac{1}{2}$ of his monthly salary on food and $\frac{5}{18}$ of his monthly salary on utilities. What fraction of his monthly salary is left?

Solution

Total fraction of his monthly salary spent on transportation, food and utilities is

$$\frac{1}{6}+\frac{1}{2}+\frac{5}{18} = \frac{3}{18}+\frac{9}{18}+\frac{5}{18} = \frac{17}{18}.$$

Fraction of Mr. Kwao's monthly salary left is

$$1-\frac{17}{18} = \frac{18}{18}-\frac{17}{18} = \frac{1}{18}.$$

Exercise 3(d)

1. Simplify the following:

(a) $\frac{1}{8}+\frac{1}{6}$, (b) $\frac{4}{9}+\frac{1}{15}$, (c) $\frac{3}{5}+\frac{2}{7}$, (d) $\frac{5}{12}+\frac{7}{15}$,

(e) $\frac{2}{3}-\frac{5}{15}$, (f) $\frac{5}{16}-\frac{5}{24}$, (g) $\frac{4}{5}-\frac{4}{15}$, (h) $\frac{5}{12}-\frac{7}{36}$,

(i) $\frac{7}{18}+\frac{5}{24}$, (j) $\frac{2}{14}+\frac{1}{28}$, (k) $\frac{3}{15}+\frac{1}{25}$, (l) $\frac{6}{18}+\frac{1}{21}$,

(m) $\frac{6}{7}-\frac{7}{14}$, (n) $\frac{5}{9}-\frac{7}{21}$, (o) $\frac{8}{15}-\frac{5}{18}$, (p) $\frac{7}{11}-\frac{5}{22}$.

2. Simplify the following:

(a) $3\frac{1}{2}+4\frac{1}{3}$, (b) $1\frac{1}{9}+2\frac{1}{3}$, (c) $4\frac{3}{4}+2\frac{1}{3}$, (d) $5\frac{1}{5}+2\frac{2}{10}$,

(e) $3\frac{2}{3}-2\frac{1}{6}$, (f) $5\frac{4}{5}-3\frac{3}{4}$, (g) $5\frac{5}{6}-4\frac{3}{8}$, (h) $8\frac{1}{12}-6\frac{5}{24}$.

3. Simplify the following:

(a) $\frac{2}{3}+\frac{1}{2}+\frac{5}{6}$, (b) $2\frac{1}{2}-3\frac{1}{4}+4\frac{3}{8}$, (c) $\frac{13}{25}+\frac{4}{5}-\frac{3}{20}$,

(d) $6\frac{3}{4}-1\frac{1}{3}-2\frac{1}{12}$, (e) $4\frac{2}{5}+3\frac{3}{4}+2\frac{1}{8}$, (f) $6\frac{2}{3}-7\frac{4}{5}+2\frac{1}{15}$.

4. Appiah spent $\frac{1}{6}$ of his money on Monday and $\frac{7}{12}$ on Tuesday. What fraction of his money is left?
5. On Sunday, Kwame spent $\frac{1}{12}$ of his time at church, $\frac{5}{16}$ of his time in watching football and $\frac{1}{8}$ of his time doing homework. What fraction of his time on that Sunday is left for doing other things?
6. Apreko and Boateng entered into a business partnership. Apreko contributed $\frac{2}{5}$ of the total capital and Boateng contributed $\frac{1}{3}$ of the total capital while the remaining amount was borrowed from a bank. What fraction of the total capital was borrowed?
7. Mr. Yeboah gave his wife an amount of money to buy some food items from the market. She spends $\frac{4}{9}$ of the money on rice, $\frac{1}{3}$ of the money on tubers of yam and $\frac{2}{15}$ of the money on some miscellaneous items. What fraction of the money is left?
8. Tawiah, Gruma and Osei together invest an amount of money in a business and agree to share the profit as follows: Tawiah receives $\frac{1}{8}$ of the total profit and Gruma receives $\frac{1}{6}$ of the total profit while Osei receives $\frac{7}{12}$ of the total profit. The remaining amount was paid to the government as income tax. What fraction of the profit is paid as tax?

3.5 Multiplication of Fractions

To multiply two fractions, multiply the two numerators and multiply the two denominators (the denominators need not be the same). When possible, we simplify. For example; $\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$.

Example 3.23

Simplify the following:

(a) $\frac{2}{3} \times \frac{5}{7}$, (b) $\frac{4}{21} \times \frac{7}{12}$, (c) $\frac{8}{15} \times \frac{5}{12}$, (d) $\frac{11}{16} \times \frac{8}{22}$.

Solution

(a) $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$.

(b) $\frac{4}{21} \times \frac{7}{12} = \frac{4 \times 7}{21 \times 12} = \frac{\cancel{4} \times \cancel{7}}{3 \times \cancel{7} \times 3 \times \cancel{4}} = \frac{1}{9}$.

(c) $\frac{8}{15} \times \frac{5}{12} = \frac{8 \times 5}{15 \times 12} = \frac{2 \times \cancel{4} \times \cancel{5}}{3 \times \cancel{5} \times \cancel{4} \times 3} = \frac{2}{9}$.

(d) $\frac{11}{16} \times \frac{8}{22} = \frac{\cancel{11} \times \cancel{8}}{2 \times \cancel{8} \times 2 \times \cancel{11}} = \frac{1}{4}$.

Example 3.24

Simplify the following:

(a) $3\frac{3}{4} \times \frac{8}{25}$, (b) $1\frac{1}{5} \times 2\frac{1}{12}$, (c) $\frac{9}{24} \times 4\frac{4}{5}$, (d) $2\frac{2}{9} \times 2\frac{5}{8}$.

Solution

In each case, we first convert the mixed numbers to an improper fraction before simplifying the expression.

(a) $3\frac{3}{4} \times \frac{8}{25} = \frac{15}{4} \times \frac{8}{25} = \frac{15 \times 8}{4 \times 25} = \frac{3 \times \cancel{5} \times 2 \times \cancel{4}}{\cancel{4} \times \cancel{5} \times 5} = \frac{6}{5} = 1\frac{1}{5}$.

(b) $1\frac{1}{5} \times 2\frac{1}{12} = \frac{6}{5} \times \frac{25}{12} = \frac{6 \times 25}{5 \times 12} = \frac{\cancel{6} \times \cancel{5} \times 5}{\cancel{5} \times 2 \times \cancel{6}} = \frac{5}{2} = 2\frac{1}{2}$.

(c) $\frac{9}{24} \times 4\frac{4}{5} = \frac{9}{24} \times \frac{24}{5} = \frac{9 \times \cancel{24}}{\cancel{24} \times 5} = \frac{9}{5} = 1\frac{4}{5}$.

(d) $2\frac{2}{9} \times 2\frac{5}{8} = \frac{20}{9} \times \frac{21}{8} = \frac{20 \times 21}{9 \times 8} = \frac{\cancel{4} \times 5 \times \cancel{3} \times 7}{3 \times \cancel{3} \times 2 \times \cancel{4}} = \frac{35}{6} = 5\frac{5}{6}$.

Word problems

Example 3.25

Boakye's car gets him $12\frac{1}{3}$ kilometers per gallon. Suppose Boakye's tank is empty and he puts $4\frac{1}{2}$ gallons, how far can Boakye go with the car?

Solution

If for 1 gallon Boakye travel $12\frac{1}{3}$ kilometer, then for $4\frac{1}{2}$ gallons he can travel

$$12\frac{1}{3} \times 4\frac{1}{2} = \frac{37}{3} \times \frac{9}{2} = \frac{37 \times 9}{3 \times 2} = \frac{37 \times \cancel{3} \times 3}{\cancel{3} \times 2} = \frac{111}{2} = 55\frac{1}{2} \text{ km.}$$

Example 3.26

Tuffour weighs $20\frac{1}{4}$ kg. What is the weight of Tuffour's junior sister who weighs four fifths of his weight?

Solution

The weight of Tuffour's sister $= \frac{4}{5}$ of (the weight of Tuffour) $= \frac{4}{5}$ of $20\frac{1}{4}$ kg

$$= \frac{4}{5} \times \frac{81}{4} = \frac{\cancel{4} \times 81}{5 \times \cancel{4}} = \frac{81}{5} = 16\frac{1}{5} \text{ kg.}$$

Example 3.27

Yaw, Kofi and Atta share some money. If Yaw gets $\frac{2}{5}$ of the money and Kofi gets $\frac{5}{9}$ of the remainder, what fraction of the money does Atta get?

Solution

If Yaw received $\frac{2}{5}$ of the money, then the fraction of the money left is

$$1 - \frac{2}{5} = \frac{5}{5} - \frac{2}{5} = \frac{3}{5}.$$

If Kofi received $\frac{5}{9}$ of the remainder, then the fraction Kofi gets is

$$\frac{5}{9} \text{ of } \frac{3}{5} = \frac{5}{9} \times \frac{3}{5} = \frac{1}{3}.$$

Total fraction Yaw and Kofi received is

$$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}.$$

Therefore, the fraction Atta gets is

$$1 - \frac{11}{15} = \frac{15}{15} - \frac{11}{15} = \frac{4}{15}.$$

Example 3.28

Bernard and Richard entered into a business partnership. They agreed to share the yearly profit in the following manner: Richard as Managing Director is paid $\frac{1}{24}$ of the total profit and gets in addition $\frac{4}{7}$ of the remainder of the profit as the majority share holder. Bernard then gets the rest of the total profit. What fraction of the total profit does Bernard get?

Solution

Fraction of Richard's share of the total profit as the Managing Director $= \frac{1}{24}$.

The fraction of the total profit left $= 1 - \frac{1}{24} = \frac{24}{24} - \frac{1}{24} = \frac{24-1}{24} = \frac{23}{24}$.

Fraction of Richard's share of the total profit as Majority share holder

$$= \frac{4}{7} \text{ of } (\text{the fraction of the total profit left})$$

$$= \frac{4}{7} \times \frac{23}{24} = \frac{4 \times 23}{7 \times 24} = \frac{\cancel{4} \times 23}{7 \times \cancel{4} \times 6} = \frac{23}{42}.$$

Total fraction of the profit received by Richard $= \frac{1}{24} + \frac{23}{42} = \frac{33}{56}$.

Fraction of Bernard's share of the total profit

$$= 1 - \frac{33}{56} = \frac{56}{56} - \frac{33}{56} = \frac{56-33}{56} = \frac{23}{56}.$$

Example 3.29

Machine A can mow $2\frac{1}{2}$ plots of grassland in 1 hour while Machine B can mow the $1\frac{1}{3}$ plots of the same grassland in 1 hour.

(a) How many plots of grass land can both machines mow in 1 hour if they are used simultaneously?

(b) How many plots of grassland can Machine A mow in $3\frac{1}{3}$ hours?

(c) How many plots of grassland can Machine B mow in $3\frac{1}{3}$ hours?

(d) How many plots of grass land can both machines mow in $3\frac{1}{3}$ hours if they are used simultaneously?

Solution

(a) If the two machines are used simultaneously, then the number plots both machine mow in 1 hour is

$$2\frac{1}{2} + 1\frac{1}{3} = \frac{5}{2} + \frac{4}{3} = \frac{15}{6} + \frac{8}{6} = \frac{23}{6} = 3\frac{5}{6}.$$

(b) If in 1 hour Machine A can mow $2\frac{1}{2}$ plots, then in $3\frac{1}{3}$ hours, Machine A is expected to mow

$$2\frac{1}{2} \times 3\frac{1}{3} = \frac{5}{2} \times \frac{10}{3} = \frac{50}{6} = \frac{25}{3} = 8\frac{1}{3} \text{ plots.}$$

(c) If in 1 hour Machine B can mow $1\frac{1}{3}$ plots, then in $3\frac{1}{3}$ hours, Machine B is expected to mow

$$1\frac{1}{3} \times 3\frac{1}{3} = \frac{4}{3} \times \frac{10}{3} = \frac{40}{9} = 4\frac{4}{9} \text{ plots.}$$

(d) If Machine A can mow $8\frac{1}{3}$ plots in $3\frac{1}{3}$ hours and Machine B can mow $4\frac{4}{9}$ plots $3\frac{1}{3}$ hours, then the number of plots both machines can mow in $3\frac{1}{3}$ hours is

$$8\frac{1}{3} \times 4\frac{4}{9} = \frac{25}{3} \times \frac{40}{9} = \frac{1000}{27} = 37\frac{1}{27}.$$

Exercise 3(e)

1. Simplify the following:

(a) $\frac{3}{4} \times \frac{2}{9}$, (b) $\frac{5}{48} \times \frac{12}{15}$, (c) $\frac{7}{16} \times \frac{8}{21}$, (d) $\frac{13}{36} \times \frac{16}{26}$,

(e) $4 \times \frac{7}{36}$, (f) $\frac{5}{8} \times 1\frac{1}{15}$, (g) $2\frac{4}{5} \times \frac{10}{36}$, (h) $\frac{25}{81} \times 5\frac{2}{5}$,

(i) $1\frac{1}{5} \times 3\frac{1}{2}$, (j) $3\frac{3}{7} \times 1\frac{1}{2}$, (k) $1\frac{3}{17} \times \frac{34}{45}$, (l) $\frac{39}{48} \times 1\frac{19}{65}$.

2. Simplify the following:

(a) $\frac{2}{3} \times \frac{12}{14} \times 1\frac{2}{5}$, (b) $\frac{1}{5} \times 1\frac{2}{3} \times \frac{7}{21}$, (c) $\frac{5}{6} \times \frac{18}{55} \times \frac{11}{27}$,

(d) $\frac{2}{3} \times \frac{12}{14} \times 1\frac{1}{5}$, (e) $\frac{3}{4} \times 1\frac{1}{5} \times \frac{8}{9}$, (f) $2\frac{1}{3} \times \frac{6}{21} \times 1\frac{13}{14}$.

3. Water is flowing into a tank at the rate of $20\frac{1}{2}$ litres per minute. Suppose the tank is empty at the start and the water flows for $15\frac{1}{3}$ minutes, find the volume of water in the tank.
4. John's truck car gets him $2\frac{1}{3}$ miles per litre. Suppose John's tank is empty and he puts $18\frac{2}{7}$ litres, how far can John go with the truck?
5. A cup of rice weighs $2\frac{1}{4}$ kg. What is the weight of three quarters of the cup of rice?
6. In a class, $\frac{8}{9}$ of the students like Integrated Science. If $\frac{3}{4}$ of those who like Integrated Science like Mathematics, what fraction of students in the class like Mathematics.
7. The monthly expenditure of Mr. Kwao is on transportation, food and utilities. He spent $\frac{5}{11}$ of his monthly salary on transportation and $\frac{7}{12}$ of the remainder on food. If the remaining portion goes into utilities, what fraction of his monthly salary is spent on utilities?
8. A bag of rice is three quarters full at the beginning of January. At the end of January, two thirds of this rice is used. How full is the bag at the end of January?
9. A ball always bounces $\frac{3}{5}$ of the height from which it falls. What fraction of its initial height does it rise after the second bounce?
10. Tawiah and Gruma together invest an amount of money in a business. Tawiah receives $\frac{7}{16}$ of the total profit and Gruma receives $\frac{1}{6}$ of the remaining profit. What fraction of the total profit did Gruma get?

3.6 Division of Fractions

Dividing fractions is just about as easy as in multiplication; there's just one extra step. To divide one fraction by another, first ***turn upside down*** ("find the reciprocal") the fraction you are dividing by, and then proceed as in multiplication.

For example, $\frac{8}{9} \div \frac{4}{3} = \frac{8}{9} \times \frac{3}{4} = \frac{2}{3}$.

Example 3.30

Simplify the following:

(a) $2\frac{2}{5} \div 4\frac{1}{5}$, (b) $\frac{8}{21} \div 5\frac{1}{3}$, (c) $3\frac{3}{16} \div 5\frac{1}{4}$, (d) $\frac{9}{14} \div 2\frac{25}{28}$.

Solution

(a) $\frac{12}{5} \div \frac{21}{5} = \frac{12}{5} \times \frac{5}{21} = \frac{12 \times 5}{5 \times 21} = \frac{\cancel{3} \times 4 \times \cancel{5}}{\cancel{5} \times \cancel{3} \times 7} = \frac{4}{7}.$

(b) $\frac{8}{21} \div 5\frac{1}{3} = \frac{8}{21} \div \frac{16}{3} = \frac{8}{21} \times \frac{3}{16} = \frac{8 \times 3}{21 \times 16} = \frac{\cancel{8} \times \cancel{3}}{\cancel{3} \times 7 \times 2 \times \cancel{8}} = \frac{1}{14}.$

(c) $3\frac{3}{16} \div 5\frac{1}{4} = \frac{51}{16} \div \frac{21}{4} = \frac{51}{16} \times \frac{4}{21} = \frac{51 \times 4}{16 \times 21} = \frac{\cancel{3} \times 17 \times \cancel{4}}{\cancel{4} \times 4 \times \cancel{3} \times 7} = \frac{17}{28}.$

(d) $\frac{9}{14} \div 2\frac{25}{28} = \frac{9}{14} \div \frac{81}{28} = \frac{9}{14} \times \frac{28}{81} = \frac{9 \times 28}{14 \times 81} = \frac{\cancel{9} \times 2 \times \cancel{14}}{\cancel{14} \times \cancel{9} \times 9} = \frac{2}{9}.$

Word problem

Example 3.31

A truck is traveling at $53\frac{1}{2}$ kilometers per hour. How long does it take to travel $285\frac{1}{3}$ kilometers.

Solution

The time taken for the truck to travel $285\frac{1}{3}$ km is

$$285\frac{1}{3} \div 53\frac{1}{2} = \frac{856}{3} \div \frac{107}{2} = \frac{856}{3} \times \frac{2}{107}$$
$$= \frac{856 \times 2}{3 \times 107} = \frac{8 \times 107 \times 2}{3 \times 107} = \frac{16}{3} = 5\frac{1}{3} \text{ hours.}$$

Example 3.32

A reservoir has a capacity of $503\frac{1}{4}$ litres. How many buckets holding $16\frac{1}{2}$ litres can be filled from this reservoir?

Solution

The number of buckets which can be filled from a reservoir of capacity $500\frac{1}{4}$ litres is

$$503\frac{1}{4} \div 16\frac{1}{2} = \frac{2\,013}{4} \div \frac{33}{2} = \frac{2\,013}{4} \times \frac{2}{33} = \frac{2\,013 \times 2}{4 \times 33} = \frac{61}{2} = 30\frac{1}{2}.$$

Example 3.33

A board, $4\frac{1}{4}$ m long, weighs $11\frac{1}{3}$ kg. What is the weight of one meter of the board?

Solution

The weight of 1 m of the board is

$$4\frac{1}{4} \div 11\frac{1}{3} = \frac{17}{4} \div \frac{34}{3} = \frac{17}{4} \times \frac{3}{34} = \frac{17 \times 3}{4 \times 34} = \frac{3}{4 \times 2} = \frac{3}{8}.$$

Exercise 3(f)

1. Simplify the following:

   (a) $1\frac{5}{7} \div 1\frac{1}{14}$, (b) $\frac{8}{9} \div \frac{12}{15}$, (c) $2\frac{2}{7} \div 3\frac{3}{7}$, (d) $2\frac{1}{8} \div \frac{21}{48}$,

   (e) $\frac{7}{12} \div \frac{5}{12}$, (f) $1\frac{1}{4} \div 3\frac{1}{2}$, (g) $1\frac{9}{16} \div 2\frac{11}{32}$, (h) $\frac{13}{14} \div \frac{39}{56}$,

   (i) $\frac{6}{11} \div \frac{18}{22}$, (j) $2\frac{2}{3} \div 1\frac{19}{21}$, (k) $2\frac{1}{7} \div \frac{24}{35}$, (l) $1\frac{7}{9} \div 1\frac{5}{27}$.

2. If a tortoise is timed traveling an average of $1\frac{3}{4}$ kilometres per hour, how long would it take the tortoise to travel $6\frac{1}{2}$ kilometres?

3. The ingredients you use to make a family Christmas cake uses $1\frac{2}{3}$ cups of flour for each cake. How many cakes can you make with the $9\frac{1}{6}$ cups of flour?

4. How many $2\frac{3}{4}$ metres long strips of ribbon can be cut from a ribbon that is $27\frac{1}{2}$ metres long?

5. An area of $\frac{1}{6}$ cm^2 required 1 tile. How many tiles would be needed for a room with floor area of $630\frac{2}{3}$ cm^2?

6. The fare for Metro buses for $\frac{3}{20}$ Ghana cedis for each kilometre. How far does a bus cover in order to charge $4\frac{1}{5}$ Ghana cedis?

7 A textbook weighs $\frac{2}{5}$ kg. How many of the same books would be needed to weigh $10\frac{2}{5}$ kg?

8. It cost a school $1\frac{1}{5}$ Ghana cedis to feed a student. How many students can be fed with $45\frac{3}{5}$ Ghana cedis?

3.7 Order of operations

As we learn in Chapter 1, when working problems involving fractions which have more than one of the operations 'addition (+), subtraction (–), multiplication (×) and division (÷)', we follow the rules of BODMAS (see page 28 of Chapter 1).

Example 3.34
Evaluate the following

(a) $8\frac{1}{3} \div \left(3\frac{1}{2} - 2\frac{1}{4}\right)$, (b) $2\frac{1}{3} + \frac{1}{2}$ of $1\frac{1}{3} - \frac{5}{6}$, (c) $\frac{1}{4} + \frac{1}{2}$ of $2\frac{3}{4} \div \frac{33}{16}$.

Solution

(a) $8\frac{1}{3} \div \left(3\frac{1}{2} - 2\frac{1}{4}\right) = \frac{25}{3} \div \left(\frac{7}{2} - \frac{9}{4}\right) = \frac{25}{3} \div \left(\frac{14-9}{4}\right) = \frac{25}{3} \div \frac{5}{4}$

$= \frac{25}{3} \times \frac{4}{5} = \frac{25 \times 4}{3 \times 5} = \frac{\cancel{5} \times 5 \times 4}{3 \times \cancel{5}} = \frac{20}{3} = 6\frac{2}{3}$.

(b) $2\frac{1}{3} + \frac{1}{2}$ of $1\frac{1}{3} - \frac{5}{6} = 2\frac{1}{3} + \left(\frac{1}{2} \text{ of } 1\frac{1}{3}\right) - \frac{5}{6} = \frac{7}{3} + \left(\frac{1}{2} \times \frac{4}{3}\right) - \frac{5}{6}$

$= \left(\frac{7}{3} + \frac{2}{3}\right) - \frac{5}{6} = \frac{9}{3} - \frac{5}{6} = \frac{18-5}{6} = \frac{13}{6} = 2\frac{1}{6}$.

(c) $\frac{1}{4} + \frac{1}{2}$ of $2\frac{3}{4} \div \frac{33}{16} = \frac{1}{4} + \left(\frac{1}{2} \text{ of } 2\frac{3}{4}\right) \div \frac{33}{16} = \frac{1}{4} + \left(\frac{1}{2} \times \frac{11}{4}\right) \div \frac{33}{16}$

$= \frac{1}{4} + \frac{11}{8} \div \frac{33}{16} = \frac{1}{4} + \left(\frac{11}{8} \times \frac{16}{33}\right)$

$= \frac{1}{4} + \frac{2}{3} = \frac{3+8}{12} = \frac{11}{12}$.

Exercise 3(g)

Evaluate the following:

(a) $2\frac{2}{5} - \left(2\frac{1}{3} \div 1\frac{2}{3}\right)$, (b) $3\frac{1}{3} + \frac{1}{3}$ of $2\frac{1}{2}$, (c) $\left(3\frac{1}{2} + 2\frac{1}{3}\right) \div 4\frac{1}{6}$,

(d) $2\frac{2}{5} \times \left(2\frac{2}{3} + 1\frac{5}{6}\right)$, (e) $\left(2\frac{2}{3} \times 1\frac{1}{4}\right) + \left(1\frac{2}{3} \div \frac{5}{6}\right)$, (f) $\dfrac{2\frac{1}{3} - 1\frac{2}{3}}{1\frac{1}{6}}$,

(g) $\frac{3}{4} \times \left(2\frac{1}{2} + 5\frac{1}{4} \div \frac{7}{8}\right)$, (h) $\left(3\frac{1}{4} + 1\frac{1}{2}\right) \div \left(2\frac{1}{2} - 1\frac{3}{4}\right)$, (i) $\left(\frac{3}{4} + \frac{1}{2}\right) \div \frac{5}{6}$ of $2\frac{2}{5}$,

(j) $\dfrac{\frac{3}{4} - \frac{1}{6} + \frac{1}{8}}{\frac{5}{12} + \frac{7}{24}}$, (k) $\dfrac{\frac{2}{3} - \frac{1}{4} \text{ of } \frac{8}{9} + \frac{1}{9}}{1\frac{5}{9} - \frac{8}{9}}$, (l) $\dfrac{\frac{4}{5} + \frac{7}{15} - \frac{2}{3}}{\frac{8}{15} - \frac{1}{30}}$,

(m) $\dfrac{\frac{5}{9} \times 2\frac{1}{4} + \frac{3}{4} \times \frac{2}{3}}{\frac{1}{2} \times \frac{3}{2} - \frac{3}{4} \times \frac{1}{8}}$, (n) $\left(\frac{1}{2}+\frac{1}{4}-\frac{3}{8}\right)\Big/\left(1\frac{3}{8} \text{ of } \frac{4}{11}\right)$, (o) $\frac{5}{8} - \frac{1}{6} \div \frac{1}{3} + \frac{5}{6}$.

3.8 Finding the fraction of a given quantity

A fraction of a quantity implies multiplying the given fraction by that quantity. For example, to find $\frac{1}{5}$ of 500, we multiply $\frac{1}{5}$ by 500, that is $\frac{1}{5} \times 500 = 100$.

Example 3.35

A survey of the reading habits of 150 students showed $\frac{3}{5}$ of the students read Comics and $\frac{1}{5}$ of the students read Novels.

(a) How many students read Comics? (b) How many read Novels?

Solution

(a) The number of students who read Comics $= \frac{3}{5}$ of $150 = \frac{3}{5} \times 150 = 3 \times 30 = 90$.

(b) The number of students who read Novels $= \frac{1}{5}$ of $150 = \frac{1}{5} \times 150 = 30$.

Example 3.36

There are 50 pupils in a class. Out of this number $\frac{1}{10}$ speak French only and $\frac{4}{5}$ of the remainder speak both French and English. If the rest speak English only,

(a) Find the number of students who speak

(i) both French and English, (ii) only English.

(b) Draw a Venn diagram to illustrate the above information. **2002**.

Solution

(a) (i) Pupils who speak French only $= \frac{1}{10}$ of $50 = \frac{1}{10} \times 50 = 5$.

The remainder $= 50 - 5 = 45$.

The number pupil who speak French and English $= \frac{4}{5}$ of the remainder

$= \frac{4}{5} \times 45 = 36$.

(ii) If $\frac{4}{5}$ of the remainder speak both French and English, then $\frac{1}{5}$ $\left(\text{i.e. } 1 - \frac{4}{5}\right)$ of the remainder speak English only.

$\therefore$ The number of pupils who speak only English $= \frac{1}{5}$ of the remainder

$= \frac{1}{5} \times 45 = 9$.

Alternative approach

If 5 speak French only and 36 speak both French and English, then the number of pupils that speak only English $= 50 - 5 - 36 = 9$.

(b) Let $\mathcal{U}$ = {pupils in the class},

F = {pupils who speak French},

E = {pupils who speak English}.

Then, $n(\mathcal{U}) = 50$ and $n(F \cap E) = 36$. Fig. 3.8 shows a Venn diagram for the above information.

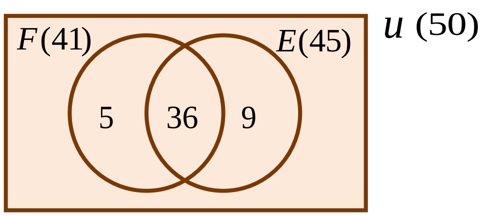


Fig. 3.8

Example 3.37

Kofi and Yaw entered into a business partnership with a total capital of GH¢ 2 400. Kofi contribute $\frac{5}{12}$ the total capital and Yaw contributed $\frac{3}{5}$ of the remainder. If the rest of the of the capital is borrowed from the bank, find

(a) Yaw's contribution to the total capital, (b) the amount borrowed from the bank.

Solution

(a) Kofi's contribution to the total capital $= \frac{5}{12}$ of the total capital

$= \frac{5}{12} \times 2\,400 =$ GH¢ 1 000.

The remainder of the contribution = GH¢ 2 400 – GH¢ 1 000 = GH¢ 1 400.

Yaw's contribution to the total capital $= \frac{3}{5}$ of the remainder

$= \frac{3}{5} \times$ GH¢ 1 400 = GH¢ 840.

(b) If Kofi and Yaw contributed GH¢ 1 000 and GH¢ 840 respectively to the total capital, then the amount borrowed from the bank = GH¢ 2 400 – GH¢ 1 000 – GH¢ 840

= GH¢ 560.

Alternative approach

If Yaw contribute $\frac{3}{5}$ of the remaining amount, then the amount borrowed from the bank is $\frac{2}{5}$ $\left(\text{i.e. } 1-\frac{3}{5}\right)$ of the remaining amount. Thus,

the amount borrowed from the bank $= \frac{2}{5} \times$ GH¢ 1 400 = GH¢ 560.

Exercise 3(h)

1. In a school, 42 students were asked their preference for two brands of soft drinks, Fanta and Coca-Cola. $\frac{3}{7}$ of the students like Coca-Cola only and $\frac{3}{4}$ of the remaining students like Coca-Cola and Fanta. Each student liked at least one of the two soft drinks.
   (a) Find the number of students who like (i) both Coca-Cola and Fanta, (ii) only Fanta.
   (b) Draw a Venn diagram to illustrate this information.

2. In a group of 30 traders, $\frac{5}{6}$ sell rice, $\frac{2}{3}$ sell maize and $\frac{2}{15}$ sell neither rice nor maize.
   (a) Draw a Venn diagram to illustrate this information.
   (b) How many traders sell (i) both items, (ii) only rice, (iii) exactly one of the two items?

3. A survey of the reading habits of 100 students showed that $\frac{6}{25}$ read both Comics and Novels, $\frac{1}{10}$ read neither Comics nor Novels and $\frac{3}{20}$ read Novels only.

(a) How many students read Novels? (b) How many read Comics?
(c) How many read only Comics?

4. There are 40 students in a class. $\frac{3}{4}$ of them study French and $\frac{9}{10}$ study English. Each student in the class studies at least one of the subjects.
   (a) Represent this information on a Venn diagram.
   (b) How many students study all two subjects?

5. There are 21 players in a football team. $\frac{2}{7}$ play defence, $\frac{2}{3}$ play midfield and $\frac{1}{7}$ play neither defence nor midfield.
   (a) Illustrate this information on a Venn diagram.
   (b) How many play: (i) both defence and midfield, (ii) defence only?

6. Amartey and Nortey entered into a business partnership with a total capital of GH¢ 3 750. Nortey contribute $\frac{2}{5}$ the total capital and Amartey contributed the rest of the capital, find the amount each contributed to the total capital.

7. The monthly expenditure of Mr. Donkor is on transportation, food and utilities. The expenditure on transportation, food and utilities are respectively $\frac{1}{5}$, $\frac{3}{5}$ and $\frac{1}{5}$. If expenditure on food is GH¢ 210, find the
   (a) expenditure on (i) utilities, (ii) transportation., (b) total expenditure.

8. The following information gives the fraction in which Mr. Tekorang spend his annual salary.

Item	Fraction	Item	Fraction
Food	$\frac{3}{10}$	Income Tax	$\frac{1}{5}$
Rent	$\frac{3}{20}$	Savings	$\frac{1}{20}$
Transport	$\frac{3}{40}$	Miscellaneous	Remainder

   If Mr. Tekorang's annual salary is GH¢ 4 320, find the
   (a) amount he spends on each of the items,
   (b) part of Mr. Tekorang's annual salary spent on Miscellaneous.

9. The following table shows the fraction of students who offer certain subjects in a school.

Subject	Fraction
Biology	$\frac{7}{90}$
Chemistry	$\frac{7}{45}$

Economics	$\frac{1}{5}$
History	$\frac{1}{10}$
Mathematics	$\frac{1}{4}$
Physics	$\frac{13}{60}$

If there are 180 students in the school, find the number of students offering each of the subjects.

10. The following table shows the expenditure distribution of essential public services by a district council in fractions.

Services	Expenditure
Health	$\frac{1}{5}$
Basic Education	$\frac{9}{40}$
Secondary Education	$\frac{1}{8}$
Transport	$\frac{3}{16}$
Police	$\frac{1}{8}$
Housing	$\frac{1}{5}$

If GH¢ 400 million is allocated to the district council, find the expenditure distribution by the district council in millions of Ghanaian cedis.

Revision Exercise 3

1. Robert has 816 marbles. He puts $\frac{3}{8}$ of them in his pocket for a game. How many does he put in his pocket?
2. Rachael has 5 packets of 20 biscuits. She puts three tenths of them on a plate. How many biscuits are on the plate?
3. Sarah has 348 gel pens. She gives $\frac{3}{4}$ of them away. How many does she give away?
4. James buys 65 sweets. He eats $\frac{4}{5}$ of them. How many does he have left?
5. A small bag of flour is 3kg and a large bag is 9kg. What fraction of the large bag is the small bag?

6. I have GH¢ 10. I give GH¢ 6 to my friend Lucy. What fraction of GH¢ 10 have I given her?

7. What fraction of 1 meter is 25cm?

8. What fraction of GH¢ 2 is GH 0.40p?

9. I have 608 sweets I eat three quarters of them. How many have I eaten?

10. If Michael Essien earns \$ x in a week and spends \$ y of it, what part of his weekly salary did he save.

11. Determine which of the following pairs of fractions are equivalent:

    (a) $\frac{2}{5}$ and $\frac{8}{20}$, (b) $\frac{10}{12}$ and $\frac{30}{36}$, (c) $\frac{3}{7}$ and $\frac{21}{49}$, (d) $\frac{1}{8}$ and $\frac{8}{64}$.

    (e) $\frac{5}{15}$ and $\frac{30}{105}$, (f) $\frac{4}{9}$ and $\frac{36}{72}$, (g) $\frac{6}{7}$ and $\frac{54}{70}$, (h) $\frac{1}{6}$ and $\frac{40}{48}$.

12. For each of the following fractions, find an equivalent fraction which has the denominator indicated.

    (a) $\frac{3}{2}$, denominator 26 (b) $\frac{5}{3}$, denominator 21

    (c) $\frac{7}{5}$, denominator 45 (d) $\frac{11}{7}$, denominator 28.

13. Express the following fractions in their lowest terms.

    (a) $\frac{26}{39}$, (b) $\frac{45}{81}$, (c) $\frac{70}{98}$, (d) $\frac{120}{135}$, (e) $\frac{30}{50}$, (f) $\frac{88}{121}$,

    (g) $\frac{32}{56}$, (i) $\frac{105}{175}$, (j) $\frac{17}{60}$, (k) $\frac{37}{56}$, (l) $\frac{56}{42}$, (m) $\frac{100}{75}$.

14. Simplify the following

    (a) $\left(\frac{2}{3} \div \frac{5}{6} - \frac{3}{10}\right) \times \frac{5}{3}$, (b) $\left(5\frac{1}{4} - 2\frac{1}{2} + \frac{7}{8}\right) \div \frac{3}{8}$, (c) $2\frac{2}{5} \times \left(1\frac{5}{6} - 2\frac{2}{3}\right)$,

    (d) $2\frac{2}{5} + \left(2\frac{1}{3} \div \frac{3}{7}\right)$, (e) $\left(3\frac{1}{2} - 2\frac{1}{3}\right) \times 1\frac{5}{7}$, (f) $\left(3\frac{1}{4} - 1\frac{1}{4}\right) \div \left(2\frac{1}{2} + 1\frac{1}{4}\right)$.

15. Write the following fractions $\frac{4}{15}$, $\frac{6}{17}$ and $\frac{3}{16}$ from least to greatest.

16. Order the following fractions from the smallest to the largest: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$.

17. Evaluate the following:

    (a) $\left[\left(\frac{3}{4} - \frac{1}{2}\right) \div \frac{5}{4}\right] \times 1\frac{1}{5}$, (b) $\left(3\frac{1}{3} - \frac{1}{3}\right) \div 3\frac{1}{2}$, (c) $\left(3\frac{1}{5} + \frac{1}{5}\right) \times 2\frac{1}{3}$,

    (d) $\dfrac{\frac{3}{4} - \frac{1}{6} + \frac{1}{8}}{\frac{5}{12} + \frac{7}{24}}$, (e) $\dfrac{\frac{2}{3} - \frac{1}{4} \text{ of } \frac{8}{9} + \frac{1}{9}}{1\frac{5}{9} - \frac{8}{9}}$, (f) $\dfrac{\frac{1}{5} + \frac{7}{15} \text{ of } \frac{2}{3}}{\frac{4}{3} - \frac{2}{9}}$.

18. McDonalds sell milkshakes in two sizes. A small milkshake contains 300ml and a large milkshake contains $\frac{2}{3}$ more.
    (a) How much does a large milkshake contain?
    (b) If Mr Murrin drinks $\frac{2}{3}$ of a small milkshake and Miss Hoyne $\frac{1}{2}$ of a large milkshake who drinks the most?
19. Skateboards cost £36 each in my local store. The shopkeeper says if I buy one I can buy another for only $\frac{7}{9}$ of the normal price. How much would a second skateboard cost?
20. Steve's gas tank holds 12 gallons of gas. The gas gauge shows the gas tank is $\frac{1}{8}$ full. If Steve is attending a pool tournament that takes $3\frac{1}{2}$ gallons of gas to go to and from the tournament site, how much gas will he need to buy to go to the tournament and return?
21. During September, Riley traveled a total distance of 180 miles to go back and forth to work. In August, he traveled $\frac{5}{6}$ of that distance going back and forth to work. In July, he traveled $\frac{3}{4}$ of August's distance to go back and forth to work. How many miles did Riley travel in all during those three months?
22. If it rained $1\frac{1}{4}$ inch on Monday, $2\frac{3}{8}$ inches on Tuesday, and $1\frac{5}{16}$ inches on Wednesday, what is the total rainfall amount for those three days?
23. A Magic Square has rows and columns that add to be a constant number. In the case below, each row and column should add to be 1. Fill in the remaining fractions.

$\frac{2}{15}$		
	$\frac{1}{3}$	
$\frac{2}{5}$		

24. Tara is using boards and cement blocks to make shelves for her dorm room. Each shelf uses $\frac{1}{2}$ of a board and 2 cement blocks. If Tara has 7 boards and 16 cement blocks, how many shelves can she make?
25. Debbie is making a Halloween costume for her daughter and needs $4\frac{3}{8}$ yards of gold trim and $3\frac{7}{8}$ yards of pink trim. If the trim costs \$2 a yard, how much will both trims cost?

CHAPTER FOUR

Integers

4.1 The idea of integers (negative and positive integers)

In Chapter 1, we discussed the concept of whole numbers and how they are represented on the number line. When whole numbers became inadequate for the purpose of society, this number system was enlarged.

Consider a point which moves along the line segment $\overline{AB}$. This point can only move in one of two directions. Fig. 4.1 shows the position of this point relative to the fixed point marked zero (0) on $\overline{AB}$. If we start from the point 7 units to the right side of 0 and move 5 units to the left, we end up at the point marked 2, which shows that 7 – 5 = 2. Suppose from the point marked 2, we move another 5 units to the left we will end up at the point marked -3 on $\overline{AB}$. This final position is 3 units to the left of 0, which indicates that $2-5=-3$. The symbol "$-$" in front of 3 is pronounce "***negative***". So -3 is read as "negative three". Likewise -2 is read "negative two".

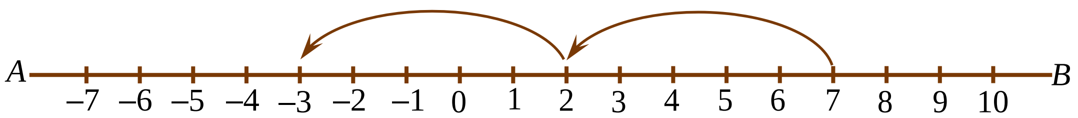


Fig. 4.1

The numbers to the left of 0 are called negative integers. The set of negative integers is {..., –7, –6, –5, –4, –3, –2, –1}. While the set of numbers, {1, 2, 3, 4, 5, 6, 7, …}, to the right of 0 is called the set of positive integers. The numbers …, –5, –4, –3, –2, –1, 0, 1, 2, 4, 5, …, are called integers. Thus,

$$\{\text{integers}\} = \{\text{negative integers}\} \cup \{0\} \cup \{\text{positive integers}\}.$$

Practical applications of integers

The concept of integers is useful whenever we wish to count on both sides of a fixed point of reference. The positive numbers indicate one direction, and the negative numbers indicate the opposite direction. The following examples are some uses of integers.

Example 4.1

In a football tournament, the national team scored 4 goals, while it conceded 7 goals. What aggregate number goals did the team score?

Solution

The number of goals scored is considered to be positive while the number of goals conceded is taken to be negative. If the team conceded 7 goals, then it implies the team scored -7 goals. The 4 goals the team scored will reduce the aggregate number goal it conceded to 3. Hence, the aggregate number of goals scored by the team is –3, indicating that $4-7=-3$.

Example 4.2

A storey building has seven floors above the ground floor and two floors below the ground floor. The building might look like Fig. 4.2.

Instead of talking about "floor above the ground floor" and "floor below the ground floor", we could use integers. Floors above the ground floor are considered to be positive while floor below are considered negative. If for instance a man started from the fifth floor, which is marked 5 and move 3 floors downstairs, he will end up on the second floor, marked 2, showing that 5 – 3 = 2.

Suppose, from the 2nd floor, the man descended further 3 floors down. He will end up at one floor below the ground floor, which is marked –1, indicating that 2 – 3 = –1.

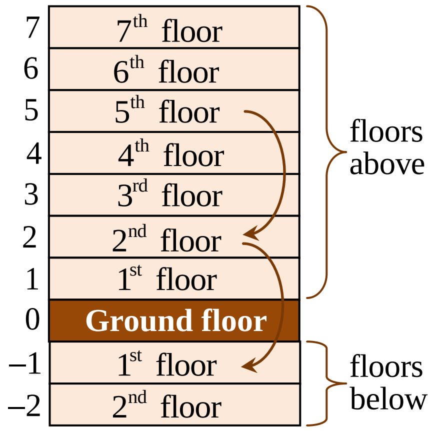


Fig. 4.2

Example 4.3

We can consider the year Ghana attained independence, 1957, to be year zero (0). We can therefore take years after this as positive, and years before this as negative.

Instead saying "50 years after independence" we can say "year 50". Instead of "5 years before independence", we say "year –5". If Kofi was born in 1947, that is 10 years before independence, then we can say "Kofi was born in year –10".

Example 4.4

Kojo is walking along a straight road running from east to west. A drinking spot, O, is on the road. We can measure his distance from O at any time. If he is East of O, we can say his distance is positive. If he is West of O, his distance will be negative. So, Instead of saying "3 km East of O", we can say "distance 3 km". Instead of "2 km West of O", we can say "distance –2 km".

Suppose Kojo is at distance 4 km, and walked 9 km westward, what will be his position with respect to O? Fig. 4.3 shows the destination of Kojo, which is at distance –5 km, indicating that 4 km – 9 km = –5 km.

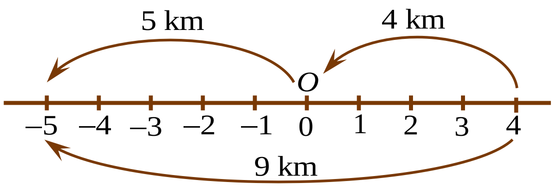


Fig. 4.3

In all these four examples, we see why negative numbers are useful. Can you think of any other examples?

Exercise 4(a)

1. The temperature of a liquid is 10 °C. The temperature drops 12 °C. What is the new temperature?
2. At a meat factory in Ghana, there is a cold room where meat is stored. On a certain day the temperature outside was 35 °C. The temperature of the cold room was 40 °C colder than the outside temperature. What was the temperature of the cold room?
3. The highest elevation in Ghana is Kwahu, which is 20 320 feet above sea level. The lowest elevation is Valley, which is 282 feet below sea level. What is the distance from the top of Kwahu to the bottom of Valley?
4. A road runs from East to West through a Cathedral. The East direction is taken as positive. At 6 am, a man is on the road -2 km from the Cathedral and walked a distance of 5 km eastward. Find his destination from the Cathedral.
5. A teacher gave a quiz to a group of students. The teacher gave each student 10 marks to start with and subtracted 1 mark for each wrong answer. At the end of the quiz, John gave 15 wrong answers. How many marks should John have?
6. A man stands on a hill which is 100 m above sea level. He climbs down 120 m. What is his height now?
7. Year zero in Ghana's History is 1957. What year was 1970? What year was 1950?
8. Zero hour on a certain day was 12 noon. What was 5 pm on that day? What time was 5 am?
9. Atta is 4 km East of a village. He walks 6 km West. How far East of the town is he.
10. In top four football competition, Accra United Football Club conceded 9 goals while it scored 5 goals. If an aggregate number of goals of zero implies goals conceded is equal to goals scored, find the aggregate number of goals scored by Accra United.
11. When the temperature of a body of 70 °C drops 5 °C. What is the new temperature?
12. If the temperature of a body is 5° below zero and goes up 17°. What is the new temperature?
13. Joe's account was GH¢ 40 debt and he deposited GH¢ 35 into it. What is his balance?

4.2 Comparing and ordering integers

We can compare two different integers by looking at their positions on the number line. For any two different places on the number line, the integer on the right is greater than the integer on the left. Note that every positive integer is greater than any negative integer. These numbers are shown in Fig. 4.4. For instance $-7 < -5$, since the mark for -7 is to the left of the mark for -5.

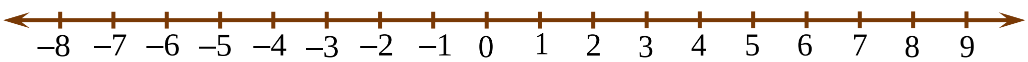


Fig. 4.4

It can be seen, from Fig. 4.2, that …, $9 > 8 > 7 > 6 > 5 > 4 > 3 > 2 > 1 > 0 > -1 > -2 > -3 > -4 > -5 > -6 > -7 > -8 > -9 >$ … An integer is positive if it is greater than zero and negative if it is less than zero. Zero is defined as neither negative nor positive.

Example 4.5

Use the sign < or > to make the following number statements correct.

(a) 9 … 4, (b) 6 … –9, (c) –2 … –8, (d) 0 … –5,
(e) –2 … 1, (f) 8 … 11, (g) –7 … –5, (h) –10 … 0.

Solution

(a) $9 > 4$, (b) $6 > -9$, (c) $-2 > -8$, (d) $0 > -5$,
(e) $-2 < 1$, (f) $8 < 11$, (g) $-7 < -5$, (h) $-10 < 0$.

Example 4.6

Arrange the following numbers in increasing order of magnitude (from the least to greatest)

(a) –9, –6, 5, 0, 4, 3 (b) 0, 7, 2, –13, –17 (c) –4, 3, 0, 8, –10, 1.

Solution

(a) –9, –6, 0, 3, 4, 5 (b) –17, –13, 0, 2, 7 (c) –10, – 4, 0, 1, 3, 8.

Examples 4.7

Use the Inequality signs < or > to indicate the relationship among the following numbers:

(a) – 9, – 6, 5, 0, 4, 3 (b) 0, 7, 2, –13, –17 (c) – 4, 3, 0, 8, –10, 1

Solution

(a) Ascending order: $-9 < -6 < 0 < 3 < 4 < 5$
Descending order: $5 > 4 > 3 > 0 > -6 > -9$

(b) Ascending order: $-17 < -13 < 0 < 2 < 7$
Descending order: $7 > 2 > 0 > -13 > -17$

(c) Ascending order: $-10 < -4 < 0 < 1 < 3 < 8$
Descending order: $8 > 3 > 1 > 0 > -4 > -10$

Exercise 4(b)

1. Arrange the following numbers in increasing order of magnitude (from the least to greatest)
   (a) –3, –7, 6, –2, 0, 8, –9 (b) 2, –8, –6, 3, –10, 5, –5 (c) –7, 7, –15, 2, 4, –10
   (d) –7, 0, –12, 3, 5, –8, (e) 6, –8, 3, –7, –3, –24, 1 (f) 4, –6, –9, 7, –11, 2, –12.
2. Use the sign < or > to make the following number statements correct.
   (a) –4 … –2, (b) –5 … 0, (c) –1 … –5, (d) –7 … –20, (e) –23 … –4,

(e) –4 … –1, (f) 5 … –5, (g) –24 … –26, (h) –3 … –1, (i) –6 … –7.

3. Place either < or > between each of the following
(a) –12 –14, (b) –8 1, (c) –9 –2, (d) –12 –13, (e) –24 –34,
(e) –65 –76, (f) 4 –4, (g) –56 –1, (h) –24 –54, (i) –76 –67.

4.3 Addition and subtraction of integers

4.3.1 Addition of integers

1. When adding integers of the same sign, we add their absolute values, and give the result the same sign.

Examples 4.8

Compute the following sums.

(a) 2 + 5, (b) (–7) + (–2), (c) (–10) + (–4), (d) (–21) + (–3).

Solution

(a) 2 + 5 = 7, (b) (–7) + (–2) = – (7 + 2) = –9,
(c) (–10) + (–4) = – (10 + 4) = –14, (d) (–21) + (–3) = –(21 + 3) = –24.

We can use the number line to verify these answers.

(a) If we start from the point marked 2 and move 5 units to the right, we end up at the point marked 7 as shown in Fig. 4.5, indicating that 2 + 5 = 7.
Note: To add a positive integer to any number, we move to the right.

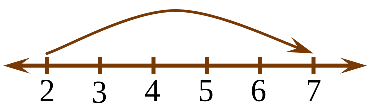


Fig. 4.5

(b) If we start from the point marked (–7) and move 2 units to the left we end up at (–9) as shown in Fig. 4.6, indicating that (–7) + (–2) = –9.
Note: To add negative integer to any number, we move the left.

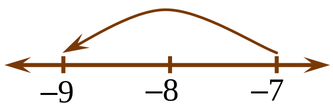


Fig. 4.6

(c) Fig. 4.7 shows the addition of (–10) + (–4). On the number line, we start from (–10) and move 4 units to the left, which takes us to the point marked (–14), indicating that (–10) + (–4) = –14

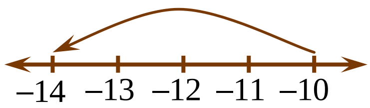


Fig. 4.7

(d) Fig. 4.8 shows the addition of (–21) + (–3). On the number line, we start from (–21) and move 3 units to the left, which takes us to the point marked (–24), indicating that (–21) + (–3) = –24

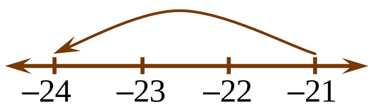


Fig. 4.8

2. When adding integers of the opposite signs, we take their absolute values (positive value), subtract the smaller from the larger, and give the result the sign of the integer with the larger absolute value.

Example 4.9

Find the following sums.

(a) 8 + (–3), (b) 2 + (–5), (c) –9 + 4, (d) 53 + (–53).

Solution

(a) The absolute values of (8 and –3) are (8 and 3). Subtracting the smaller from the larger gives 8 – 3 = 5, and since the larger absolute value was 8, we give the result the same sign as 8. Thus, 8 + (–3) = 5.

Fig. 4.9 shows the addition of 8 + (–3).

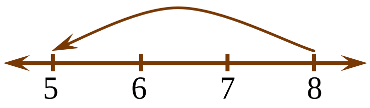


Fig. 4.9

(b) The absolute values of (2 and –5) are (2 and 5). Subtracting the smaller from the larger gives 5 – 2 = 3, and since the larger absolute value was 5, we give the result the same sign as –5, so 2 + (–5) = –3.

Fig. 4.10 shows the addition of 2 + (–5).

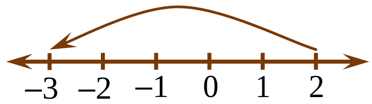


Fig. 4.10

(c) The absolute values of (–9 and 4) are (9 and 4). Subtracting the smaller from the larger gives 9 – 4 = 5, and since the larger absolute value was 9, we give the result the same sign as –9. Thus, –9 + 4 = –5.

Fig. 4.11 shows the addition of –9 + 4.

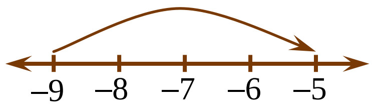


Fig. 4.11

(d) The absolute values of 53 and –53 are 53 and 53. Subtracting the smaller from the larger gives 53 – 53 = 0. The sign in this case does not matter, since 0 and –0 are the same. Note that 53 and –53 are opposite integers. All opposite integers have this property that their sum is equal to zero. Two integers that add up to zero are also called additive inverses.

Example 4.10

(a) $3+5=8$, (b) $-3+(-5)=-8$, (c) $-3+5=2$. (d) $3+(-5)=-2$,

(e) $-9+2=-7$, (f) $9+2=11$, (g) $9+(-2)=7$, (h) $-9+(-2)=-11$,

(i) $8+(-12)=-4$, (j) $-8+12=4$, (k) $-8+(-12)=-20$, (l) $8+12=20$,

(m) $-5+(-9)=-14$, (n) $5+(-9)=-4$, (o) $5+9=14$, (p) $-5+9=4$.

Example 4.11

Compute the following.

(a) 50 + (–23) + (–60), (b) 15 + (–26) + 10 + (–5), (c) –5 + 18 + (–2).

Solution

(a) 50 + (–23) + (–60) = 50 + (–83) = –33.

(b) 15 + (–26) + 10 + (–5) = 15 + 10 + (–26) + (–5) = 25 + (–31) = –6.

(c) –5 + 18 + (–2) = (–5) + (–2) + 18 = –7 + 18 = 11.

4.3.2 Subtraction of integers

The pairs of integers, 2 and –2, 5 and –5, 7 and –7, etc are called ***opposite*** or ***negatives*** of each other. Subtracting an integer is the same as adding it's opposite.

Examples 4.12

In the following examples, we convert the subtracted integer to it's opposite, and add the two integers.

(a) $7 - 4 = 7 + (-4) = 3$, (b) $12 - (-5) = 12 + 5 = 17$,
(c) $-8 - 7 = -8 + (-7) = -15$, (d) $-22 - (-40) = -22 + 40 = 18$.

Note that the result of subtracting two integers could be positive or negative.

Example 4.13

Compute the following differences:

(a) $15 - 7$, (b) $-14 - 3$, (c) $22 - (-5)$, (d) $-15 - (-10)$.

Solution

(a) $15 - 7 = 15 + (-7) = 8$, (b) $-14 - 3 = -14 + (-3) = -17$,
(c) $22 - (-5) = 22 + 5 = 27$, (d) $-15 - (-10) = -15 + 10 = -5$.

Example 4.14

(a) $-3-(-2) = -3+2 = -1$, (b) $7-(-3) = 7+3 = 10$, (c) $-4-5 = -4+(-5) = -9$,
(d) $-5-(-9) = -5+9 = 4$, (e) $3-8 = 3+(-8) = -5$, (f) $-6-(-2) = -6+2 = -4$.

Example 4.15

Compute the following.

(a) $50 - (-23) - (-60)$, (b) $15 - (-26) - 10 - (-5)$, (c) $-5 - 18 - (-2)$.

Solution

(a) $50 - (-23) - (-60) = 50 + 23 + 60 = 133$
(b) $15 - (-26) - 10 - (-5) = 15 + 26 - 10 + 5 = 41 - 10 + 5 = 31 + 5 = 36.$
(c) $-5 - 18 - (-2) = -5 - 18 + 2 = -23 + 2 = -21.$

Exercise 4(c)

1. Compute the following sums.
   (a) $-3+(-5)$, (b) $3+5$, (c) $-3+5$, (d) $3+(-5)$,
   (e) $5+6$, (f) $-5+(-6)$, (g) $-5+6$, (h) $5+(-6)$,
   (i) $-7+3$, (j) $-7+(-3)$, (k) $7+3$, (l) $7+(-3)$,
   (m) $8+(-15)$, (n) $-8+(-15)$, (o) $-8+15$, (p) $8+15$.

2. Compute the following sums.
   (a) $2+(-3)$, (b) $-4+(-2)$, (c) $-9+(-3)$, (d) $-4+7$,

(e) $-6+1$, (f) $-5+(-2)$, (g) $5+(-8)$, (h) $-8+9$,
(i) $-8+13$, (j) $18+(-8)$, (k) $-5+(-8)$, (l) $5+17$,
(m) $7+(-20)$, (n) $-16+9$, (o) $-7+(-6)$, (p) $-11+9$.

3. Compute the following differences.
(a) $-3-(-5)$, (b) $3-5$, (c) $-3-5$, (d) $3-(-5)$,
(e) $5-6$, (f) $-5-(-6)$, (g) $-5-6$, (h) $5-(-6)$,
(i) $-7-3$, (j) $-7-(-3)$, (k) $7-3$, (l) $7-(-3)$,
(m) $8-(-15)$, (n) $-8-(-15)$, (o) $-8-15$, (p) $8-15$.

4. Compute the following differences.
(a) $2-(-3)$, (b) $-4-(-2)$, (c) $-9-(-3)$, (d) $-4-7$,
(e) $-6-1$, (f) $-5-(-2)$, (g) $5-(-8)$, (h) $-8-9$,
(i) $-8-13$, (j) $18-(-8)$, (k) $-5-(-8)$, (l) $5-17$,
(m) $7-(-20)$, (n) $-16-9$, (o) $-7-(-6)$, (p) $-11-9$.

5. Compute the following.
(a) $-23+20$, (b) $-12-10$, (c) $50+(-53)$, (d) $20-(-4)$,
(e) $24-36$, (f) $23+27$, (g) $-53+(-20)$, (h) $-15-(-7)$,
(i) $-43-7$, (j) $-54+30$, (k) $-23-(-13)$, (l) $-14+12$,
(m) $23-(-27)$, (n) $-13-(-7)$, (o) $29-50$, (p) $-12+(-13)$,
(q) $45-17$, (r) $23-57$, (s) $-36-24$, (t) $76-89$,
(u) $-23+(-32)$, (v) $12+(-29)$, (w) $-83+53$, (x) $27+(-20)$.

6. Compute the answer for each of the following.
(a) $3+5+(-4)-(-2)$, (b) $3-7+(-4)+9$, (c) $-7+8-(-6)-4$,
(d) $-12+10-(-5)-4$, (e) $-7-(-8)+(-5)+9$ (f) $12+(-13)-(-3)-7$.

7. Compute the answer for each of the following.
(a) $125 - 17 - (-25) - 13$, (b) $700 - (-298) - 135$, (c) $70 - 43 - (-60)$,
(d) $260 - (-49)$, (e) $-487 - (-653)$, (f) $360 - (-241)$.

4.4 Multiplying and dividing of integers

In the previous section, we have considered how to add and subtract integers. In this section we shall look at how to multiply and divide integers.

4.4.1 Multiplication of integers

The rules for multiplication of integers are as follows.

- The product of two negative numbers is a positive number. It means that, if both numbers have the ***same*** sign, their product is the product of their absolute values.

$(\text{positive})\times(\text{positive})=(\text{positive})$, $(\text{negative})\times(\text{negative})=(\text{positive})$.

Example: $-3\times(-4)=3\times4=12$, $-5\times(-2)=5\times2=10$, $-6\times(-4)=6\times4=24$.

- The product of a positive number and a negative number is a negative number. That is, if the numbers have ***opposite*** signs, their product is the ***opposite*** of the product of their absolute values.

  $(\text{negative})\times(\text{positive})=(\text{negative})$, $(\text{positive})\times(\text{negative})=(\text{negative})$.

  Example: $-3\times4=-(3\times4)=-12$, $5\times(-2)=-(5\times2)=-10$, $6\times(-4)=-(6\times4)=-24$.

- If one or both numbers is zero (0), the product is zero (0).

  Example: $3\times0=0$, $0\times(-5)=0$, $-6\times0=0$,

Example 4.16

(a) $4\times3=12$, (b) $-4\times(-3)=12$, (c) $-4\times3=-12$, (d) $4\times(-3)=-12$,
(e) $6\times(-5)=-30$, (f) $-6\times5=-30$, (g) $6\times5=30$, (h) $-6\times(-5)=30$,
(i) $-12\times2=-24$, (j) $12\times2=24$, (k) $-12\times(-2)=24$, (l) $12\times(-2)=-24$.

Example 4.17

(a) $-10\times3=-30$, (b) $7\times(-5)=-35$, (c) $-6\times(-8)=48$, (d) $-2\times(-25)=50$,
(e) $9\times(-3)=-27$, (f) $20\times15=300$, (g) $-30\times3=-90$, (h) $-11\times(-4)=44$,
(i) $-12\times-4=48$, (j) $-13\times4=-52$, (l) $16\times(-2)=-32$, (m) $25\times3=75$.

To multiply any number of integers:

1. Count the number of negative numbers in the product.
2. Find the size of the product by multiplying the numbers without the '–' sign.
3. If the number of negative integers counted in step 1 is even, the product is just the product from step 2,
4. If the number of negative integers is odd, the product is the opposite of the product in step 2 (give the product in step 2 a negative sign).
5. If any of the integers in the product is 0, the product is 0.

Example 4.18

Determine the product $5 \times (-2) \times 3 \times (-10) \times (-2)$.

Solution

$5 \times (-2) \times 3 \times (-10) \times (-2) = ?$

There are 3 negative integers:

-2, -10, and -2.

Next, find the size of the product:

$5 \times 2 \times 3 \times 10 \times 2 = 600$.

Since there were an odd number of integers, the product is the negation of 600. Thus

$5 \times (-2) \times 3 \times (-10) \times (-2) = -600$.

Example 4.19

Find the following products.

(a) $-4\times6\times(-2)$, (b) $12\times5\times(-3)$, (c) $-5\times7\times2$,

(d) $-2\times4\times(-3)\times(-1)$, (e) $-10\times(-5)\times(-2)\times(-3)$, (f) $15\times(-2)\times3\times(-5)$.

Solution

(a) $-4\times6\times(-2)=48$, (b) $12\times5\times(-3)=-180$,

(c) $-5\times7\times2=-70$, (d) $-2\times4\times(-3)\times(-1)=-24$,

(e) $-10\times(-5)\times(-2)\times(-3)=300$, (f) $15\times(-2)\times3\times(-5)=450$.

4.4.2 Dividing Integers

To divide a pair of integers:

- In a division of two integers, if one is positive and the other is negative, the answer is negative. We find the size of answer by dividing the two integers without the "–' sign and then give the result a negative sign.

$$\frac{\text{negative}}{\text{positive}}=\text{negative} \quad \text{and} \quad \frac{\text{positive}}{\text{negative}}=\text{negative}.$$

Example: $-15\div5=-\frac{15}{5}=-3$, $\quad 14\div(-2)=-\frac{14}{2}=-7$, $\quad -8\div6=-\frac{8}{6}=-\frac{4}{3}$.

- If both integers are negative, the answer is positive. We find the size of the answer by dividing the two integers without the '–' sign and then maintain the positive sign.

$$\frac{\text{negative}}{\text{negative}}=\text{positive} \quad \text{and} \quad \frac{\text{positive}}{\text{positive}}=\text{positive}.$$

Example: $-18\div(-6)=\frac{-18}{-6}=\frac{18}{6}=3$, $\quad -28\div(-7)=\frac{-28}{-7}=\frac{28}{7}=4$.

Example 4.20

Compute the following quotients.

(a) $-4\div2=-\frac{4}{2}=-2$, (b) $12\div(-3)=-\frac{12}{3}=-4$, (c) $-18\div(-6)=\frac{18}{6}=3$,

(d) $-81\div(-9)=\frac{81}{9}=9$, (e) $36\div(-4)=-\frac{36}{4}=-9$, (f) $-66\div11=-\frac{66}{11}=-6$.

Example 4.21

Simplify the following.

(a) $\frac{5\times(-4)}{-6}$, (b) $\frac{6\times5}{-30}$, (c) $-3\times\left(\frac{4\times(-6)}{-11+2}\right)$, (d) $\left(\frac{16+(-9)}{-6+13}\right)\times(-4)$.

Solution

(a) $\frac{5\times(-4)}{-6}=\frac{-20}{-6}=\frac{20}{6}=\frac{10}{3}=3\frac{1}{3}$.

(b) $\frac{6 \times 5}{-30} = \frac{30}{-30} = -\frac{30}{30} = -1.$

(c) $-3\times\left(\frac{4 \times (-6)}{-11 + 2}\right) = -3\times\left(\frac{-24}{-9}\right) = -\cancel{3}\times\frac{8}{\cancel{3}} = -8.$

(d) $\left(\frac{16 + (-9)}{-6 + 13}\right)\times(-4) = \frac{7}{7}\times(-4) = -4.$

Example 4.22

(a) $\left(\frac{-14 + 20}{7 - (-5)}\right) \times \left(\frac{20 + (-24)}{-4 - 2}\right)$, (b) $\left(\frac{-4 - (-6) + (-5)}{9 - 3 + (-2)}\right) \div \left(\frac{3 - (-4) + 1}{-16 - (-8) - 4}\right)$.

Solution

(a) $\left(\frac{-14 + 20}{7 - (-5)}\right) \times \left(\frac{20 + (-24)}{-4 - 2}\right) = \left(\frac{-14 + 20}{7 + 5}\right) \times \left(\frac{20 + (-24)}{-4 + (-2)}\right)$

$= \left(\frac{6}{12}\right) \times \left(\frac{-4}{-6}\right) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}.$

(b) $\left(\frac{-4 - (-6) + (-5)}{9 - 3 + (-2)}\right) \div \left(\frac{-16 - (-8) - 4}{3 - (-4) + 1}\right) = \left(\frac{-4 + 6 + (-5)}{9 - 3 + (-2)}\right) \div \left(\frac{-16 + 8 - 4}{3 + 4 + 1}\right)$

$= \left(\frac{-3}{4}\right) \div \left(\frac{-12}{8}\right) = \frac{-3}{4} \div \frac{-3}{2}$

$= \frac{-3}{4} \times \frac{2}{-3} = \frac{2}{4} = \frac{1}{2}.$

Example 4.23

Evaluate the following

(a) $\frac{-12 \div (-4 - 2)}{(-5 + 4) \div 4}$, (b) $\frac{2 + \frac{1}{4} \text{ of } (-12) - 3}{(-4 + 6) \div (-1 - 2)}$, (c) $\frac{-2 + 4 \text{ of } \left(-\frac{1}{8}\right) \div \frac{1}{2}}{-5 + (-2)\times 3 + 7}$.

Solution

(a) $\frac{-12 \div (-4 - 2)}{(-5 + 4) \div 4} = \frac{-12 \div (-6)}{-1 \div 4} = \frac{-12}{-6}\times\frac{4}{-1} = 2\times(-4) = -8.$

(b) $\frac{2 + \frac{1}{4} \text{ of } (-12) - 3}{(-4 + 6) \div (-1 - 2)} = \frac{2 + \frac{1}{4} \times (-12) - 3}{2 \div (-3)}$

$= \frac{2 - 3 - 3}{-\frac{2}{3}} = \frac{-4}{-\frac{2}{3}} = -4\times\left(-\frac{3}{2}\right) = 2\times 3 = 6.$

(c) $\frac{-2 + 4 \text{ of } \left(-\frac{1}{8}\right) \div \frac{1}{2}}{-5 + (-2) \times 3 + 7} = \frac{-2 + 4 \times \left(-\frac{1}{8}\right) \times \frac{2}{1}}{-5 + (-2) \times 3 + 7}$

$= \frac{-2 + \frac{-4 \times 2}{8}}{-5 + (-6) + 7} = \frac{-2 + (-1)}{-11 + 7} = \frac{-3}{-4} = \frac{3}{4}.$

Exercise 4(d)

1. Compute the following products.
   (a) 6×4, (b) $-6\times(-4)$, (c) -6×4, (d) $6\times(-4)$,
   (e) $2\times(-8)$, (f) -2×8, (g) 2×8, (h) $-2\times(-8)$,
   (i) -13×3, (j) 13×3, (k) $-13\times(-3)$, (l) $13\times(-3)$.

2. Compute the following products.
   (a) -15×4, (b) $8\times(-6)$, (c) $-9\times(-3)$, (d) $-3\times(-15)$,
   (e) $18\times(-2)$, (f) 12×10, (g) -14×4, (h) $-12\times(-5)$,
   (i) -16×-3, (j) -19×4, (k) $17\times(-3)$, (l) 23×7.

3. Find the following products.
   (a) $-3\times 2\times(-5)$, (b) $10\times 2\times(-6)$, (c) $-7\times 8\times 2$,
   (d) $-10\times 5\times(-2)\times(-3)$, (e) $-12\times(-7)\times(-2)\times(-5)$, (f) $15\times(-4)\times 2\times(-5)$,
   (g) $-12\times 5\times 10\times(-2)$, (h) $25\times(-4)\times(-20)\times(-5)$, (i) $-2\times 50\times(-20)\times(-4)$.

4. Compute the following quotients.
   (a) $81\div(-27)$, (b) $-24\div 6$, (c) $-35\div(-5)$, (d) $135\div(-15)$,
   (e) $-72\div(-9)$, (f) $64\div(-16)$, (g) $75\div 15$, (h) $96\div(-12)$,
   (i) $-42\div 3$, (j) $-52\div(-4)$, (k) $32\div 2$, (l) $150\times(-30)$.

5. Simplify the following.
   (a) $\dfrac{(-6)\times 3}{2}$, (b) $\dfrac{-7\times 5}{15}$, (c) $\dfrac{(-8)\times 2}{-4}$, (d) $\dfrac{4\times(-9)}{6}$, (e) $\dfrac{-3\times 9}{30}$,
   (f) $5\times\left(\dfrac{3\times(-6)}{-11+6}\right)$, (g) $\left(\dfrac{12+(-9)}{-6+16}\right)\times(-5)$. (h) $-3\times\left(\dfrac{4\times(-6)}{-11+2}\right)$,

6. Simplify the following.
   (a) $\left(\dfrac{-14+20}{7-(-5)}\right)\times\left(\dfrac{20+(-24)}{-4-2}\right)$, (b) $\left(\dfrac{-4-(-6)+(-5)}{9-3+(-2)}\right)\div\left(\dfrac{3-(-4)+1}{-16-(-8)-4}\right)$.
   (c) $\left(\dfrac{-14+20}{7-(-5)}\right)\times\left(\dfrac{20+(-24)}{-4-2}\right)$, (d) $\left(\dfrac{-3+(-3)-(-5)}{7-9-(-4)}\right)\div\left(\dfrac{2-(-2)-3}{-14-(-8)+4}\right)$.

7. Evaluate the following
   (a) $\dfrac{-12\div(-4-2)}{(-5+4)\div 4}$, (b) $\dfrac{4+\frac{1}{3}\text{ of }(-18)-2}{(5+(-3))\div(7-10)}$, (c) $\dfrac{-2+4\text{ of }\left(-\frac{1}{8}\right)\div\frac{1}{2}}{-5+(-2)\times 3+7}$.
   (d) $\dfrac{-12\div(-4-2)}{(-5+4)\div 4}$, (e) $\dfrac{2+\frac{1}{4}\text{ of }(-12)-3}{(-4+6)\div(-1-2)}$, (f) $\dfrac{4+5\text{ of }\left(-\frac{1}{30}\right)\div\frac{1}{12}}{8-(-3)\times 12-43}$.