# Extrema on a closed interval

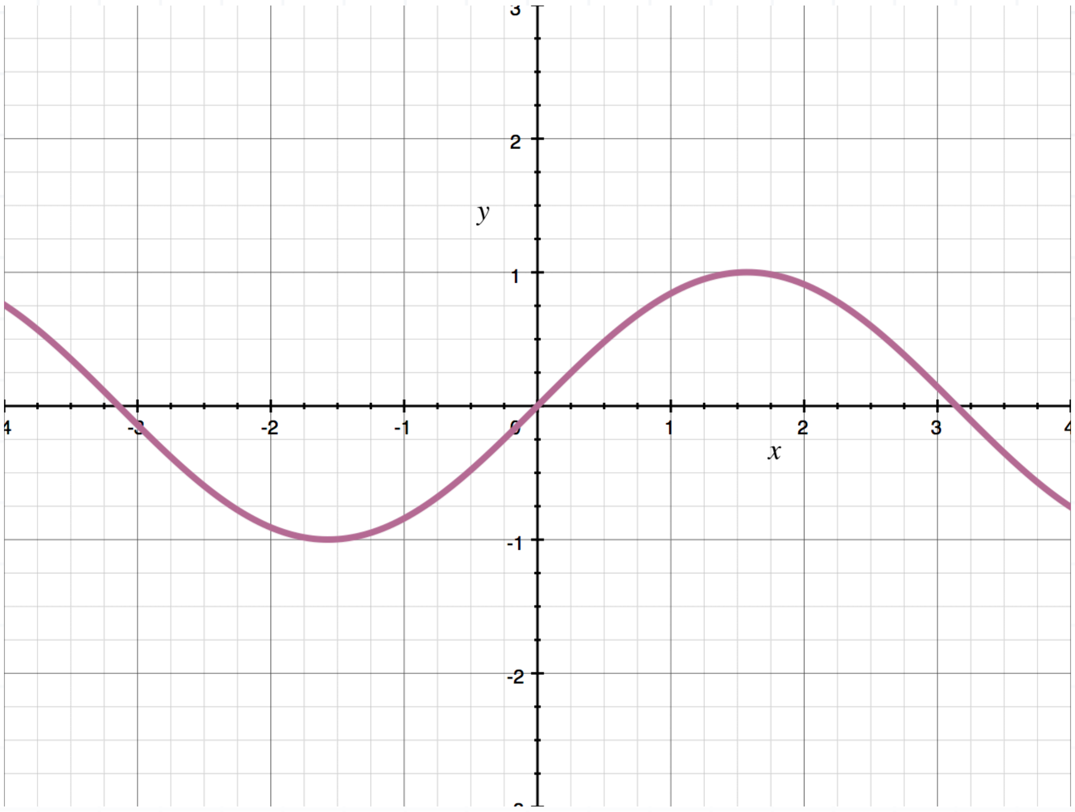
We already know how to find the extrema of a function by finding critical points, and then using either the first derivative test or second derivative test in order to classify them.

And we know how to classify those critical points as local maxima or minima, or global maxima or minima.

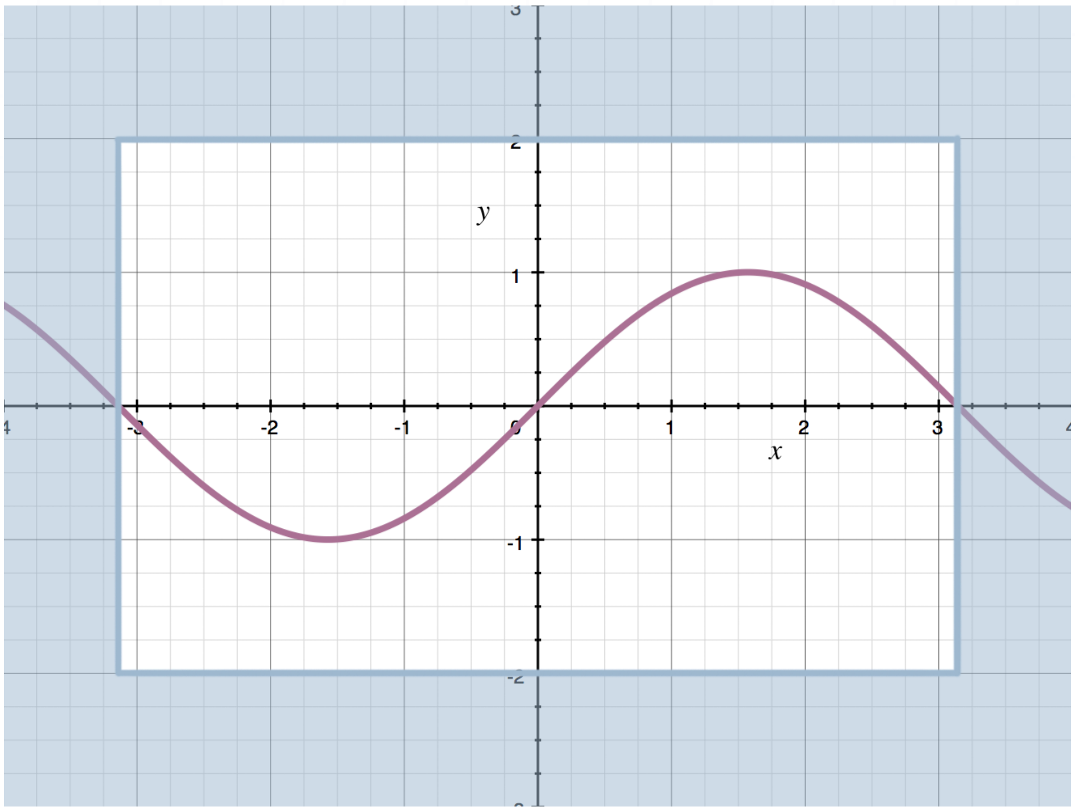
But the process of classifying extrema has always been based on the function's entire domain. In other words, we've never limited the interval in which we were looking for critical points, but that's what we want to do now.

## Extrema in closed intervals

Sometimes we'll only be interested in extrema within a particular interval. For instance, instead of looking at the entire domain of $y = \sin x$,




maybe instead we just want to focus on the interval $[-\pi, \pi]$, and look for extrema there.




When we want to find and classify extrema on a closed interval like this, all of our steps we'll be the same as what we've done in the past, except for one key difference: we'll need to consider the values of the function at the endpoints of the interval.

So, in the case of this $y = \sin x$ function on $[-\pi, \pi]$, we'd need to look for critical points like normal, but then we'd also need to look at the values $y = \sin(-\pi)$ and $y = \sin \pi$.

The reason we consider those values is because the function may have its least or greatest value, within the interval, at the edge of the interval. So when we're determining which maxima will be local maxima and global maximum, and which minima will be the local minima and global minimum, we'll consider the values of the function at every critical point in the interval, and at both endpoints of the interval.

Let's work through an example so that we can see how to classify the extrema of a function when we've restricted its domain with a closed interval.

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**Example**

Calculate the extrema of the function on $[-2,2]$. Distinguish between absolute and relative extrema.

$$y = x^3 - 2x + 1$$

First, take the derivative.

$$y' = 3x^2 - 2$$

Find critical points.

$$3x^2 - 2 = 0$$

$$3x^2 = 2$$

$$x^2 = \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}}$$

We'll use the first derivative test to classify these critical points, using test values of $x = -1$, $x = 0$, and $x = 1$.

$$y' = 3(-1)^2 - 2$$

$$y' = 3 - 2$$

$$y' = 1 > 0$$

and

$$y' = 3(0)^2 - 2$$

$$y' = 0 - 2$$

$$y' = -2 < 0$$

and

$$y' = 3(1)^2 - 2$$

$$y' = 3 - 2$$

$$y' = 1 > 0$$

If we summarize these findings, we can say that the function is

- increasing on $-\infty < x < -\sqrt{2/3}$
- decreasing on $-\sqrt{2/3} < x < \sqrt{2/3}$
- increasing on $\sqrt{2/3} < x < \infty$

Based on the increasing/decreasing behavior of the function, it has a local maximum at $x = -\sqrt{2/3}$ and a local minimum at $x = \sqrt{2/3}$.

Now that we've tested the critical points, we need to plug each of them, and both endpoints of the interval, into the original function, $y = x^3 - 2x + 1$.

For $x = -\sqrt{2/3} \approx -0.82$,

$$y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 1$$

$$y = -\frac{2\sqrt{2}}{3\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{3}} + 1$$

$$y \approx 2.09$$

For $x = \sqrt{2/3} \approx 0.82$,

$$y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right) + 1$$

$$y = \frac{2\sqrt{2}}{3\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}} + 1$$

$$y \approx -0.09$$

For $x = -2$,

$$y = (-2)^3 - 2(-2) + 1$$

$$y = -8 + 4 + 1$$

$$y = -3$$

For $x = 2$,

$$y = 2^3 - 2(2) + 1$$

$$y = 8 - 4 + 1$$

$$y = 5$$

We have four points. If we arrange them in order of $y$-values from least to greatest, we get

$$(-2, -3)$$

Local minimum $\left(\sqrt{\frac{2}{3}}, -0.09\right)$

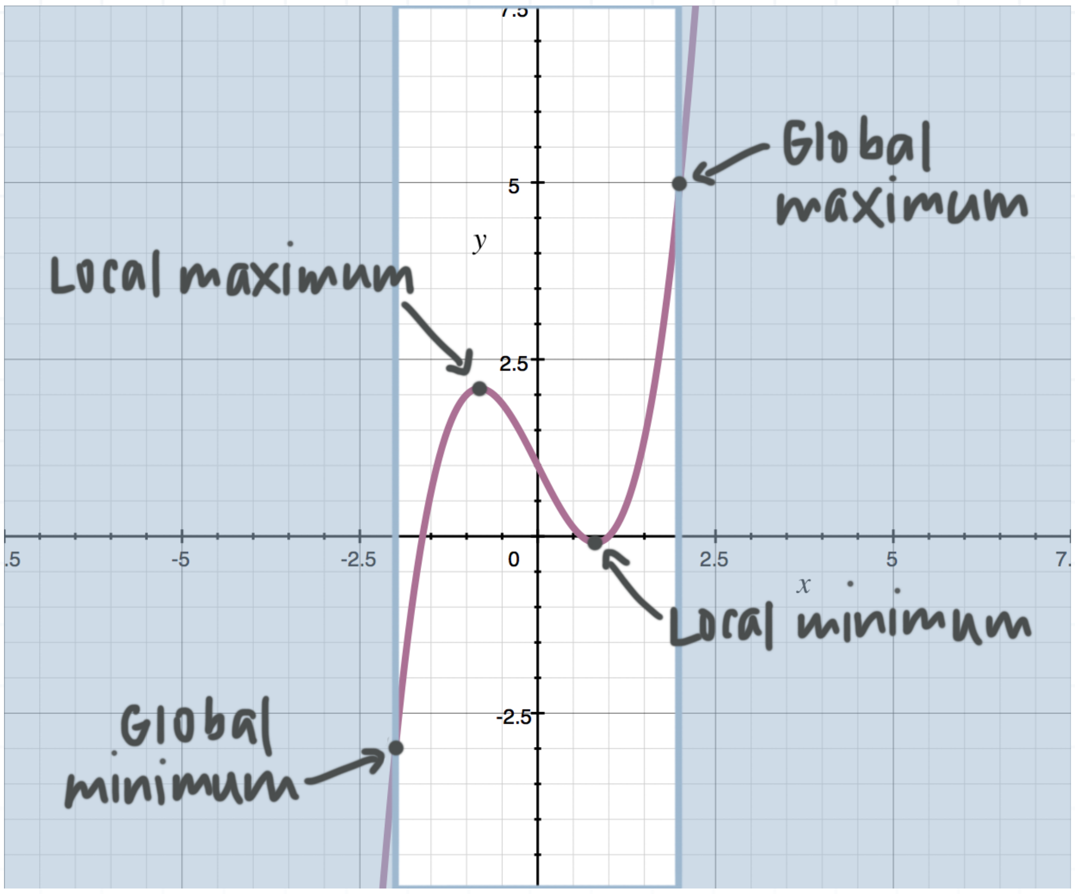
Local maximum $\left(-\sqrt{\frac{2}{3}}, 2.09\right)$

$$(2,5)$$

The second point is a local minimum, but the function has a lower value at the endpoint $(-2,-3)$, so the function's global minimum in the interval is at $(-2,-3)$. The third point is a local maximum, but the function has a higher value at the endpoint $(2,5)$, so the function's global maximum in the interval is at $(2,5)$.

| | |
|---|---|
| Global minimum | $(-2,-3)$ |
| Local minimum | $\left(\sqrt{\frac{2}{3}},-0.09\right)$ |
| Local maximum | $\left(-\sqrt{\frac{2}{3}},2.09\right)$ |
| Global maximum | $(2,5)$ |

If we sketch the function, and the interval $[-2,2]$ that we're interested in, we see these results.



Local maximum
Global maximum
Local minimum
Global minimum
y
x
7.5
5
2.5
-2.5
0
.5
-5
-2.5
2.5
5
7.