Water level in the tank

In the same way that we used the formula for the volume of a sphere to solve the inflating/deflating balloon problems, we can use the formulas for the volume of other geometric figures to solve "water level in the tank" problems.

Formulas and variables

The shape of the tank in the problem will dictate which volume formula we need to use. As a reminder, here are the volume formulas for different geometric figures:

Tank shape	Volume formula
Cube	$V = s^3$
Rectangular prism	$V = lwh$
Triangular prism	$V = (1/2)whl$
Pyramid	$V = (1/3)s^2h$
Cone	$V = (1/3)\pi r^2h$
Sphere	$V = (4/3)\pi r^3$
Cylinder	$V = \pi r^2h$

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In these kinds of related rates problems, a tank is either being filled up with water, or some other substance, or the tank is being emptied of that water or substance.

We may be given information about the amount of substance in the tank or how quickly the tank is being filled or emptied, about the height of the substance in the tank or how quickly that height is changing, or (for some tank shapes), the length of the radius of the substance, or how fast the length of the radius is changing.

As with other related rates problems, we'll start with the volume equation, then use implicit differentiation to take the derivative, treating each variable as a function of time t.

Once we've found the derivative, we'll substitute the values we know, and then solve for the value we're trying to find.

Tank vs. substance

There's one thing we need to be really careful of whenever we're doing "water level in the tank" problems, which is that we need to make sure we're distinguishing between the tank itself, and the substance in the tank.

If the tank is completely full, then the substance and the tank will have the same dimensions and the same volume. But if the tank isn't full, then the substance will have a smaller volume and smaller dimensions than the tank.

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Usually, we're focused on the substance itself, so when we apply the formula for volume, we're using it as a representation of the volume of the substance, not the volume of the tank.

Let's work through an example in which we're filling up a cone-shaped tank.

Example

An inverted cone-shaped tank with radius 8 meters and height 15 meters is being filled at a rate of 2 cubic meters of water per minute. How fast is the water level rising when the height of the cone of water is 12 meters?

Because the tank is cone-shaped, we'll be using the formula for the volume of a cone.

$$V = \frac{1}{3}\pi r^2 h$$

Eventually, we'll need to solve for dh/dt, which we'll introduce into the equation by differentiating. But first, we need to eliminate r from the volume equation by expressing it in terms of h.

Knowing that the height and radius of the tank are 15 and 8, respectively, we can say

$$\frac{8}{15} = \frac{r}{h}$$

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$$r = \frac{8}{15}h$$

This is the relationship between the radius and height of the tank, but the cone of water follows the same relationship, so we can substitute for r into the volume equation,

$$V = \frac{1}{3}\pi\left(\frac{8}{15}h\right)^2 h$$

$$V = \frac{64}{675}\pi h^3$$

then differentiate,

$$\frac{dV}{dt} = \frac{64}{225}\pi h^2\left(\frac{dh}{dt}\right)$$

and finally plug in what we know.

$$2 = \frac{64}{225}\pi(12)^2\left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{25}{512\pi}$$

$$\frac{dh}{dt} \approx 0.0155$$

This answer tells us that the water level is increasing at rate of $h' \approx 0.0155$ meters per minute.

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Let's do one more example, but this time we'll be emptying a cylindrical tank.

Example

Oil is being pumped out of a cylindrical tank with radius 3 feet at a rate of 10 gallons (0.13 cubic feet) per minute. How fast is the oil level changing?

Because the tank is cylindrical, we'll use the formula for the volume of a cylinder.

$$V = \pi r^2 h$$

The radius of the oil in the cylindrical tank never changes, so substitute $r = 3$.

$$V = \pi(3)^2 h$$

$$V = 9\pi h$$

Differentiate the equation.

$$1(V') = 9\pi(1)h'$$

$$V' = 9\pi h'$$

The volume of the oil is decreasing as oil is pumped out at 10 gallons (0.13 cubic feet) per minute, so we'll substitute $V' = -0.13$ to get h' in feet per minute.

$$-0.13 = 9\pi h'$$

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We want to know how fast the oil level is falling, so we'll solve for h'.

$$h' = -\frac{0.13}{9\pi}$$

$$h' \approx -0.0046$$

This result tells us that the height of the oil is changing at a rate of $h' \approx -0.0046$ feet per minute (falling by 0.0046 feet per minute).

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