Intercepts and vertical asymptotes

At this point, we understand the optimization process:

- We know how to find the critical points of a function.
- We know how to determine where the function is increasing and where it's decreasing.
- We know how to classify critical points as local and/or global maxima and minima.
- We know how to determine where the function is concave up and where it's concave down.

Interestingly, all of these optimization steps have actually produced a lot of information about the function's graph. Without having done anything else, we should already be starting to get a pretty clear picture of what the function's graph might look like.

For the rest of this section, we'll look at other pieces of information we can gather about the function, and then we'll bring everything together to sketch its graph.

In this lesson specifically, we'll look at the function's intercepts and vertical asymptotes.

Intercepts

One quick thing we can do to get a clearer picture of the graph of a function is to find its intercepts, which are the points at which the function crosses the x- and y-axes.

To find the points where the graph intersects the x-axis, we'll substitute $y = f(x) = 0$. And to find the points where the graph intersects the y-axis, we'll substitute $x = 0$.

Throughout the notes in this section, we've been working through the same example, and we'll continue with that example here.

Example

Find the function's intercepts.

$$f(x) = x + \frac{4}{x}$$

First, let's find the y-intercepts by substituting $x = 0$.

$$y = 0 + \frac{4}{0}$$

$$y = \frac{4}{0}$$

Because the fraction is undefined, this tells us that the function has no y-intercepts, which means it never crosses the y-axis. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x + \frac{4}{x}$$

$$0 = x^2 + 4$$

$$x^2 = -4$$

Since there are no real solutions to this equation, which tells us that the function also has no x-intercepts.

Let's do another example where we find the function's intercepts.

Example

Find the function's intercepts.

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

First, let's find the y-intercept by substituting $x = 0$.

$$y = 0^3 + \frac{1}{2}(0)^2 - 10(0) - 5$$

$$y = -5$$

So the function has a y-intercept at $(0, -5)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x^3 + \frac{1}{2}x^2 - 10x - 5$$

$$0 = \frac{1}{2}x^2(2x+1) - 5(2x+1)$$

$$0 = (2x+1)\left(\frac{1}{2}x^2 - 5\right)$$

$$x = -\frac{1}{2}, \ -\sqrt{10}, \sqrt{10}$$

So the function has x-intercepts at $(-1/2,0)$, $(-\sqrt{10},0)$ and $(\sqrt{10},0)$.

Vertical asymptotes

An **asymptote** is a line which a function's graph approaches, but never crosses. The graph will get closer and closer to the asymptote, but no matter how far you go out, the graph will never touch or cross the line of the asymptote.

Asymptotes can be perfectly vertical, perfectly horizontal, or neither, in which case we say that the asymptote is slanted.

For now, we'll look just at vertical asymptotes, and how to find them. They're actually the easiest kinds of asymptotes to test for, because **vertical asymptotes** only exist where the function is undefined.

Remember, a function is undefined whenever we have a value of 0 as the denominator of a fraction, or whenever we have a negative value inside a square root sign, or whenever the argument of the logarithmic function is equal to 0. Also, four of the standard trig functions, $\tan x$, $\cot x$, $\sec x$, and $\csc x$, have vertical asymptotes.

Let's continue with the same example to find the function's vertical asymptotes, if it has any.

Example

Find any vertical asymptotes of the function.

$$f(x) = x + \frac{4}{x}$$

We can see right away that this function includes a fraction. We know that a fraction will be undefined whenever its denominator is 0, which means this fraction will be undefined at $x = 0$.

Therefore, the entire function $f(x)$ will be undefined when $x = 0$, and we can say that the function has a vertical asymptote there.

That means the function's graph will come alongside the vertical line $x = 0$, skimming it very closely, but will never touch or cross it.

Realize that our conclusions about the vertical asymptote match the answer we got earlier when we looked at the function's intercepts.

In the intercepts example, we found that the graph never crossed the y-axis, because we got an undefined value when we substituted $x = 0$. Because we got a 0 in the denominator of the fraction in that example, we could have concluded right away that there was also a vertical asymptote at that point.

Let's do one more example with vertical asymptotes.

Example

Find any vertical asymptotes of the function.

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

We can see right away that this function doesn't include a fraction, square root, log function, or trig function. We know that the cubic polynomials are defined for all real numbers, so there's no vertical asymptote.