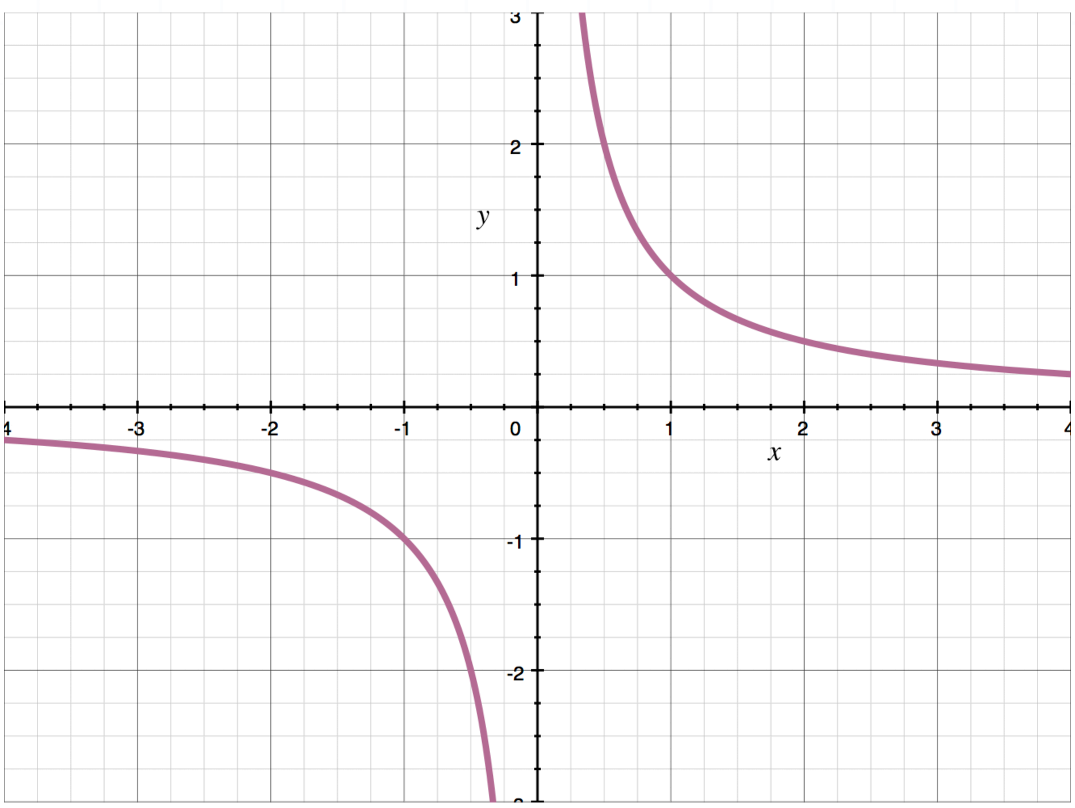
Infinite limits and vertical asymptotes

There's a difference between "limits at infinity" and "infinite limits." When we see *limits at infinity*, it means we're talking about the limit of the function as we approach ∞ or $-\infty$. Contrast that with *infinite limits*, which means that the value of the limit is ∞ or $-\infty$ as we approach a particular point.

Limits at infinity, infinite limits

In the graph of $f(x) = 1/x$,



the function has infinite, one-sided limits at $x = 0$. There's a vertical asymptote there, and we can see that the function approaches $-\infty$ from the left, and ∞ from the right.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Talking about limits at infinity for this function, we can see that the function approaches 0 as we approach either ∞ or $-\infty$.

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

How to find infinite limits

Infinite limits exist around vertical asymptotes in the function. Of course, we get a vertical asymptote whenever the denominator of a rational function in lowest terms is equal to 0.

Here's an example of a rational function in lowest terms, meaning that we can't factor and cancel anything from the fraction.

$$\lim_{x \to 1} \frac{1}{(x-1)^2}$$

We can see that setting $x = 1$ gives 0 in the denominator, which means that we have a vertical asymptote at $x = 1$. Therefore, we know we'll have infinite limits on either side of $x = 1$.

Once we've established that this is a rational function in lowest terms and that a vertical asymptote exists, all that's left to determine is whether the one-sided limits around $x = 1$ approach ∞ or $-\infty$.

In order to do that, we can substitute values very close to $x = 1$. If the result is positive, the limit will be ∞; if the result is negative, the limit will be $-\infty$.

$$f(0.99) = \frac{1}{(0.99 - 1)^2} = \frac{1}{(-0.01)^2} = \frac{1}{0.0001} = 10{,}000 = \infty$$

$$f(1.01) = \frac{1}{(1.01 - 1)^2} = \frac{1}{(0.01)^2} = \frac{1}{0.0001} = 10{,}000 = \infty$$

Because the value of the function tends toward ∞ on both sides of the vertical asymptote, we can say that the general limit of the function as $x \to 1$ is ∞.

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$