Compounding interest

We've been looking at exponential decay problems, like half-life, Newton's Law of Cooling, and sales decline, but now we want to turn from exponential decay to exponential growth, starting with compound interest problems.

Compound interest

In general, the idea of compound interest is that the interest our money earns, itself earns interest.

For instance, if we deposit $\$100$ into an account that earns 5% interest annually, then we'll earn $\$100 \cdot 5\% = \$5$ in interest the first year.

Then at the beginning of the second year, we'll be starting with $\$105$, instead of just $\$100$. Which means that, at 5%, we'll be earning interest not only on the initial $\$100$ deposit, but also on the $\$5$ of interest we earned in the first year. So during the second year, we'll earn $\$105 \cdot 5\% = \$5.25$.

And this pattern will continue year after year. We'll have only deposited the initial $\$100$, but the interest we earn each year will get compounded.

When interest is **compounded**, it means that the interest gets added to the principal that we initially deposited, and therefore will itself also earn interest. Because of this compounding, we'll earn more and more interest each year, which is what we see in the table.

Year	Interest earned	Running balance
1	\$5.00	\$105.00
2	\$5.25	\$110.25
3	\$5.51	\$115.76
4	\$5.79	\$121.55
5	\$6.08	\$127.63
6	\$6.38	\$134.01
7	\$6.70	\$140.71
8	\$7.04	\$147.75
9	\$7.39	\$155.14
10	\$7.76	\$162.90

Exponential growth

In the table above, we see the exponential growth of the compounding interest. The account is earning more and more interest each year, and the running balance in the account is therefore growing by a larger and larger amount each year.

It's this kind of exponential growth that we model with

$$A = Pe^{rt}$$

where P is the initial investment (the principal), r is annual interest rate, and A is the amount in the account after time t. Sometimes we'll write this formula as

$$FV = PVe^{rt}$$

where $A = FV$ is the "future value" in the account, and $P = PV$ is the "present value" in the account. We only use this formula when interest is compounded continuously. If instead interest is compounded a specific n times per year (monthly, quarterly, semi-annually, annually, etc.), we'd use

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the initial investment, r is the annual interest rate, A is the amount in the account after time t, and n is the number of times the interest is compounded per year.

Let's work through an example where we use the formula for continuous compounding to calculate the amount in the account after a certain period of time.

Example

We invest $\$2{,}000$ at a rate of $3\,\%$ compounded continuously. How much will the investment be worth after 3 years?

From the question, we know that the principal is $P = \$2{,}000$, the interest rate is $r = 0.03$, and the time is $t = 3$. Substitute these values into the compound interest formula

$$A = Pe^{rt}$$

$$A = 2{,}000e^{0.03(3)}$$

$$A = 2{,}000e^{0.09}$$

$$A \approx 2{,}188.35$$

So after 3 years, at $3\,\%$ interest, the investment is worth about $\$2{,}188.35$.

Let's try another example. In this one, we'll earn different interest rates over different periods of time, so we'll need to compute each of those time periods separately.

Example

We invest $\$6{,}000$ at $2\,\%$ for 2 years. After 2 years, the interest rate increases to $3.5\,\%$ for the next 2 years. Then, after those first 4 years, the interest rate increases to $5\,\%$. Find the value of the investment after 3 years, and the value after 6 years.

We'll need to handle each interest rate separately. Let's call the first 2 years the "first term," the second 2 years the "second term," and everything after that the "third term."

We'll use subscripts to denote whether the rate belongs to the first term, second term, or third term.

$$r_1 = 0.02$$

$$r_2 = 0.035$$

$$r_3 = 0.05$$

We know $P_1 = \$6{,}000$ but we don't know P_2 or P_3. Let's find those now since we'll need them to answer both parts of this question. P_2 will equal A_1, the amount of money we have at the end of the first term (after 2 years). But we can find $A_1 = P_1 e^{r_1 t}$ where $t = 2$.

$$A_1 = P_1 e^{r_1 t}$$

$$A_1 = 6{,}000 e^{0.02(2)}$$

$$A_1 = 6{,}244.86$$

$$P_2 = 6{,}244.86$$

Next we remember that P_3 will equal A_2, the amount of money we have at the end of the second term (2 years after the end of the first term, 4 years since the beginning of the first term). So $A_2 = P_2 e^{r_2 t}$ where $t = 2$.

$$A_2 = P_2 e^{r_2 t}$$

$$A_2 = 6{,}244.86 e^{0.035(2)}$$

$$A_2 = 6{,}697.66$$

$$P_3 = 6{,}697.66$$

Armed with r_1, r_2, and r_3, plus P_1, $P_2 = A_1$, and $P_3 = A_2$, we can tackle both parts of this question.

In order to solve for the value of the investment after 3 years, we'll use the data for the second term. But we have to remember that 3 years into the investment is 1 year into the second term, so $t = 1$, $P_2 = 6{,}244.86$ and $r_2 = 0.035$.

$$A_{3\text{ years}} = 6{,}244.86e^{0.035(1)}$$

$$A_{3\text{ years}} = 6{,}467.30$$

The investment would be worth $\$6{,}467.30$ after 3 years.

In order to solve for the value of the investment after 6 years, we'll use the data for the third term. But we have to remember that 6 years into the investment is 2 years into the third term so $t = 2$, $P_3 = 6{,}697.66$ and $r_3 = 0.05$.

$$A_{6\text{ years}} = 6{,}697.66e^{0.05(2)}$$

$$A_{6\text{ years}} = 7{,}402.06$$

The investment would be worth $\$7{,}402.06$ after 6 years.

Now let's do one where interest is compounded quarterly, instead of continuously, so that we can see how to use the other formula.

Example

We deposit $\$6{,}000$ into an account that pays $5.5\,\%$ interest, compounded quarterly. How long before the balance in the account reaches $\$10{,}500$?

Because interest is compounded quarterly, we set $n = 4$, and plug everything we've been given into the formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$10{,}500 = 6{,}000\left(1 + \frac{0.055}{4}\right)^{4t}$$

$$10{,}500 = 6{,}000(1.01375)^{4t}$$

$$1.75 = 1.01375^{4t}$$

Take the natural logarithm of both sides.

$$\ln(1.75) = \ln(1.01375^{4t})$$

$$\ln(1.75) = 4t\ln(1.01375)$$

$$4t = \frac{\ln(1.75)}{\ln(1.01375)}$$

$$t = \frac{\ln(1.75)}{4\ln(1.01375)}$$

$$t \approx 10.2446$$

It'll take a little over 10 years for the balance in the account to grow from $\$6,000$ to $\$10,500$.