Calculus 1 Workbook Solutions

Tangent and normal lines

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TANGENT LINES

■ 1. Find the equation of the tangent line to the graph of the equation at $(1/2,\pi)$.

$$f(x)=4\arctan 2x$$

Solution:

The derivative of $\arctan x$ is given by

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

So the derivative is

$$f'(x)=\frac{4}{1+(2x)^2}\cdot 2$$

$$f'(x)=\frac{8}{1+4x^2}$$

Evaluating the derivative at $(1/2,\pi)$, we get

$$f'\left(\frac{1}{2}\right)=\frac{8}{1+4\left(\frac{1}{2}\right)^2}=\frac{8}{1+1}=\frac{8}{2}=4$$

Now we can find the equation of the tangent line by plugging the slope $f'(1/2) = 4$ and the point $(1/2,\pi)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f\left(\frac{1}{2}\right) + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4x - 2$$

$$y = 4x + \pi - 2$$

2. Find the equation of the tangent line to the graph of the equation at $(-1, -9)$.

$$g(x) = x^3 - 2x^2 + x - 5$$

Solution:

The derivative is

$$g'(x) = 3x^2 - 4x + 1$$

Evaluating the derivative at $(-1, -9)$, we get

$$g'(-1) = 3(-1)^2 - 4(-1) + 1$$

$$g'(-1) = 3(1) + 4(1) + 1$$

$$g'(-1) = 3 + 4 + 1$$

$$g'(-1) = 8$$

Now we can find the equation of the tangent line by plugging the slope $g'(-1) = 8$ and the point $(-1, -9)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = g(-1) + 8(x - (-1))$$

$$y = -9 + 8(x + 1)$$

$$y = -9 + 8x + 8$$

$$y = 8x - 1$$

■ 3. Find the equation of the tangent line to the graph of the equation at $(0, -4)$.

$$h(x) = -4e^{-x} + 3x$$

Solution:

The derivative is

$$h'(x) = -4(-1)e^{-x} + 3$$

$$h'(x) = 4e^{-x} + 3$$

Evaluating the derivative at $(0, -4)$, we get

$$h'(0) = 4e^{-0} + 3$$

$$h'(0) = 4(1) + 3$$

$$h'(0) = 7$$

Now we can find the equation of the tangent line by plugging the slope $h'(0) = 7$ and the point $(0, -4)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = h(0) + 7(x - 0)$$

$$y = -4 + 7(x - 0)$$

$$y = -4 + 7x$$

$$y = 7x - 4$$

■ 4. Find the equation of the tangent line to the graph of the equation at $(1,1)$.

$$f(x) = -6x^4 + 4x^3 - 3x^2 + 5x + 1$$

Solution:

The derivative is

$$f'(x) = -24x^3 + 12x^2 - 6x + 5$$

Evaluating the derivative at $(1,1)$, we get

$$f'(1) = -24(1)^3 + 12(1)^2 - 6(1) + 5$$

$$f'(1) = -24 + 12 - 6 + 5$$

$$f'(1) = -13$$

Now we can find the equation of the tangent line by plugging the slope $f'(1) = -13$ and the point $(1,1)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f(1) - 13(x - 1)$$

$$y = 1 - 13x + 13$$

$$y = -13x + 14$$

■ 5. At what point(s) is the tangent line of $f(x) = 2x(3 - x)^2$ horizontal?

Solution:

A line is horizontal when it's slope is 0, so we need to find the point(s) at which the slope of the tangent line is 0.

Take the derivative of the function using the product rule.

$$f'(x) = 2(3 - x)^2 + (2x)(-2)(3 - x)$$

$$f'(x) = 2(9 - 6x + x^2) - 12x + 4x^2$$

$$f'(x) = 18 - 12x + 2x^2 - 12x + 4x^2$$

$$f'(x) = 6x^2 - 24x + 18$$

$$f'(x) = 6(x^2 - 4x + 3)$$

$$f'(x) = 6(x - 3)(x - 1)$$

Set the derivative equal to 0 and solve for x.

$$6(x - 3)(x - 1) = 0$$

$x = 3$ and $x = 1$

Plug these x-values into $f(x)$ to find the y-values where tangent line is horizontal.

$$f(1) = 2(1)(3 - 1)^2 = 2(2^2) = 8$$

$$f(3) = 2(3)(3 - 3)^2 = 6(0) = 0$$

So the function has horizontal tangent lines at $(1,8)$ and $(3,0)$.

■ 6. Find the constants a, b, and c such that the function $f(x) = ax^2 + bx + c$ intersects the point $(-2,5)$ and has a horizontal tangent line at $(0, -3)$.

Solution:

We know the function must intersect $(-2,5)$ and $(0,-3)$. We'll start by plugging $(0,-3)$ into the function,

$$-3 = a(0)^2 + b(0) + c$$

$$-3 = c$$

then we'll plug in $(-2,5)$.

$$5 = a(-2)^2 + b(-2) - 3$$

$$5 = 4a - 2b - 3$$

$$8 = 4a - 2b$$

$$4 = 2a - b$$

We know that the tangent line is horizontal at $(0,-3)$, so this means the derivative is zero when $x = 0$. So we'll take the derivative

$$f'(x) = 2ax + b$$

and then substitute $x = 0$ and $f'(x) = 0$ and solve for b.

$$0 = 2a(0) + b$$

$$b = 0$$

Substitute $b = 0$ into $4 = 2a - b$ to find the value of a.

$$4 = 2a - 0$$

$$a = 2$$

So the function $f(x) = ax^2 + bx + c$ will intersect $(-2,5)$ and $(0, -3)$ and have a horizontal tangent line at $(0, -3)$ when $a = 2$, $b = 0$, and $c = -3$. Therefore the function is

$$f(x) = 2x^2 + 0x - 3$$

$$f(x) = 2x^2 - 3$$

VALUE THAT MAKES TWO TANGENT LINES PARALLEL

■ 1. Find the value of a such that the tangent lines to $f(x) = 2x^3 + 2$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $f(x)$.

$$f'(x) = 6x^2$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 6a^2$$

$$f'(a + 1) = 6a^2 + 12a + 6$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$6a^2 = 6a^2 + 12a + 6$$

$$0 = 12a + 6$$

$$-12a = 6$$

$$a = -\frac{1}{2}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = -\frac{1}{2} + 1$$

$$a + 1 = \frac{1}{2}$$

Therefore, the function has parallel tangent lines one unit apart at $x = -1/2$ and $x = 1/2$.

■ 2. Find the value of a such that the tangent lines to $g(x) = x^3 + x^2 + 7$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $g(x)$.

$$g'(x) = 3x^2 + 2x$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$g'(a) = 3a^2 + 2a$$

$$g'(a + 1) = 3a^2 + 8a + 5$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 2a = 3a^2 + 8a + 5$$

$$-6a = 5$$

$$a = -\frac{5}{6}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = -\frac{5}{6} + 1$$

$$a + 1 = \frac{1}{6}$$

Therefore, the function has parallel tangent lines one unit apart at $x = -5/6$ and $x = 1/6$.

■ 3. Find the value of a such that the tangent lines to $h(x) = \tan^{-1} x$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $h(x)$.

$$h'(x) = \frac{1}{1 + x^2}$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$h'(a) = \frac{1}{1 + a^2}$$

$$h'(a + 1) = \frac{1}{a^2 + 2a + 2}$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$\frac{1}{1+a^2}=\frac{1}{a^2+2a+2}$$

$$1+a^2=a^2+2a+2$$

$$-1=2a$$

$$a=-\frac{1}{2}$$

If this is the value of a, then $a+1$ is

$$a+1=-\frac{1}{2}+1$$

$$a+1=\frac{1}{2}$$

Therefore, the function has parallel tangent lines one unit apart at $x=-1/2$ and $x=1/2$.

■ 4. Find parallel tangent lines to $f(x)=4x^3-6x+7$ at $x=a$ and $x=a+1$.

Solution:

We want to find the equation of the tangent lines at $x=a$ and $x=a+1$. We'll start by working on the line at $x=a$. We'll need $f(a)=4a^3-6a+7$, and the value of the derivative at $x=a$.

$$f'(x)=12x^2-6$$

$$f'(a)=12a^2-6$$

So the equation of the tangent line at $x=a$ is

$$y=f(a)+f'(a)(x-a)$$

$$y=4a^3-6a+7+(12a^2-6)(x-a)$$

Now we'll do the same thing at $x=a+1$. We know that

$$f(a+1)=4(a+1)^3-6(a+1)+7$$

$$f(a+1)=4a^3+12a^2+6a+5$$

and the derivative at $x=a+1$ will be

$$f'(x)=12x^2-6$$

$$f'(a+1)=12a^2+24a+6$$

So the equation of the tangent line at $x=a+1$ is

$$y=f(a+1)+f'(a+1)(x-(a+1))$$

$$y=4a^3+12a^2+6a+5+(12a^2+24a+6)(x-(a+1))$$

For the two tangent lines to be parallel, we'll set their slopes equal to each other and solve for a.

$$12a^2 - 6 = 12a^2 + 24a + 6$$

$$-12 = 24a$$

$$a = -\frac{1}{2}$$

The slope of $f(x)$ at $x = -1/2$ is -3, and the slope of $f(x)$ at $x = 1/2$ is -3.

So the equation of the tangent line at $x = a$ is

$$y = 4a^3 - 6a + 7 + (12a^2 - 6)(x - a)$$

$$y = 4\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right) + 7 + \left(12\left(-\frac{1}{2}\right)^2 - 6\right)\left(x - \left(-\frac{1}{2}\right)\right)$$

$$y = 4\left(-\frac{1}{8}\right) + 3 + 7 + \left(12\left(\frac{1}{4}\right) - 6\right)\left(x + \frac{1}{2}\right)$$

$$y = -3x + 8$$

and the equation of the tangent line at $x = a + 1$ is

$$y = 4a^3 + 12a^2 + 6a + 5 + (12a^2 + 24a + 6)(x - (a + 1))$$

$$y = 4\left(-\frac{1}{2}\right)^3 + 12\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 5$$

$$+\left(12\left(-\frac{1}{2}\right)^2 + 24\left(-\frac{1}{2}\right) + 6\right)\left(x - \left(-\frac{1}{2}\right) + 1)\right)$$

$$y = -3x + 6$$

5. Find the value of a such that the tangent lines to $g(x) = (x-2)^3 + x^2 + 3$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $g(x)$.

$$g'(x) = 3(x-2)^2 + 2x$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$g'(a) = 3(a-2)^2 + 2a$$

$$g'(a+1) = 3(a-1)^2 + 2(a+1)$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3(a-2)^2 + 2a = 3(a-1)^2 + 2(a+1)$$

$$3a^2 - 10a + 12 = 3a^2 - 4a + 5$$

$$7 = 6a$$

$$a = \frac{7}{6}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = \frac{7}{6} + 1$$

$$a+1=\frac{13}{6}$$

Therefore, the function has parallel tangent lines one unit apart at $x=7/6$ and $x=13/6$.

■ 6. Find the approximate value of a, rounded to the nearest hundredth, such that the tangent lines to $h(x)=e^x-3x^2$ at $x=a$ and $x=a+1$ are parallel.

Solution:

Start by finding the derivative of $h(x)$.

$$h'(x)=e^x-6x$$

Now we'll plug both $x=a$ and $x=a+1$ into the derivative.

$$h'(a)=e^a-6a$$

$$h'(a+1)=e^{a+1}-6(a+1)$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$e^a-6a=e^{a+1}-6(a+1)$$

$$e^a-6a=e^{a+1}-6a-6$$

$$e^a=e^{a+1}-6$$

$$e^{a+1} - e^a - 6 = 0$$

$$e^a e^1 + (-e^a) - 6 = 0$$

$$(-e^a)e^1 - (-e^a) + 6 = 0$$

Substitute $x = -e^a$, then solve for x.

$$xe - x + 6 = 0$$

$$x(e-1) + 6 = 0$$

$$x(e-1) = -6$$

$$x = -\frac{6}{e-1}$$

Back-substitute.

$$-e^a = -\frac{6}{e-1}$$

$$e^a = \frac{6}{e-1}$$

Take the natural log of both sides.

$$\ln(e^a) = \ln\left(\frac{6}{e-1}\right)$$

$$a = \ln\left(\frac{6}{e-1}\right)$$

$$a \approx 1.25$$

Therefore, the function has parallel tangent lines one unit apart at about $x \approx 1.25$ and $x \approx 2.25$.

VALUES THAT MAKE THE FUNCTION DIFFERENTIABLE

■ 1. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} x^2 & x \leq 3 \\ ax - b & x > 3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 3$,

$$\lim_{x \to 3^-} (x^2) = \lim_{x \to 3^+} (ax - b)$$

$$3^2 = a(3) - b$$

$$9 = 3a - b$$

$$b = 3a - 9$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 3$ must be equal to each other. So $2x = a$, and when $x = 3$, and $a = 6$. Therefore, $a = 6$ and

$$b = 3(6) - 9$$

$$b = 9$$

2. What value of a and b will make the function differentiable?

$$g(x)=\begin{cases} ax+b & x\le -1 \\ bx^2-1 & x>-1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x=-1$,

$$\lim_{x\to -1^-}(ax+b)=\lim_{x\to -1^+}(bx^2-1)$$

$$a(-1)+b=b(-1)^2-1$$

$$-a+b=b-1$$

$$a=1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x=-1$ must be equal to each other. So $a=2bx$ when $x=-1$, and $a=1$.

$$1=2b(-1)$$

$$b=-\frac{1}{2}$$

Therefore, $a=1$ and $b=-1/2$.

3. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^3 & x \le 2 \\ x^2 - b & x > 2 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 2$,

$$\lim_{x \to 2^-} (ax^3) = \lim_{x \to 2^+} (x^2 - b)$$

$$a(2)^3 = (2)^2 - b$$

$$8a = 4 - b$$

$$8a - 4 = -b$$

$$b = 4 - 8a$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 2$ must be equal to each other. So $3ax^2 = 2x$ when $x = 2$, and

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

To get b, we'll plug in $a = 1/3$.

$$b = 4 - 8\left(\frac{1}{3}\right) = \frac{4}{3}$$

Therefore, $a = 1/3$ and $b = 4/3$.

■ 4. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} 3 - x & x \le 1 \\ ax^2 - bx & x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 1$,

$$\lim_{x \to 1^-} (3 - x) = \lim_{x \to 1^+} (ax^2 - bx)$$

$$3 - (1) = a(1)^2 - b(1)$$

$$2 = a - b$$

$$a = 2 + b$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 1$ must be equal to each other. So

$$-1 = 2ax - b$$

$$-1 = 2a(1) - b$$

$$-1 = 2a - b$$

$$-1 - 2a = -b$$

$$b = 2a + 1$$

Now, since $a = 2 + b$ and $b = 2a + 1$,

$$a = 2 + 2a + 1$$

$$-a = 3$$

$$a = -3$$

Then

$$b = 2a + 1$$

$$b = 2(-3) + 1$$

$$b = -5$$

The answer is $a = -3$ and $b = -5$.

■ 5. What value of a and b will make the function differentiable?

$$g(x) = \begin{cases} x^3 & x \le 1 \\ a(x-2)^2 - b & x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = 1$,

$$\lim_{x \to 1^-} (x^3) = \lim_{x \to 1^+} (a(x-2)^2 - b)$$

$$(1)^3 = a(1-2)^2 - b$$

$$1 = a - b$$

$$-b = 1 - a$$

$$b = a - 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = 1$ must be equal to each other. So

$$3x^2 = 2a(x-2)$$

$$3(1) = 2a(1-2)$$

$$3 = 2a(-1)$$

$$a = -\frac{3}{2}$$

So $a = -3/2$ and $b = -3/2 - 1 = -5/2$.

■ 6. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^2 + b & x \le 3 \\ bx + 4 & x > 3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 3$,

$$\lim_{x \to 3^-} (ax^2 + b) = \lim_{x \to 3^+} (bx + 4)$$

$$a(3)^2 + b = b(3) + 4$$

$$9a + b = 3b + 4$$

$$9a = 2b + 4$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 3$ must be equal to each other. So

$$2ax = b$$

$$2a(3) = b$$

$$b = 6a$$

Plugging $b = 6a$ into the equation for a gives

$$9a = 2(6a) + 4$$

$$9a = 12a + 4$$

$$-3a = 4$$

$$a = -\frac{4}{3}$$

Then, $b = 6(-4/3) = -8$. The answer is $a = -4/3$ and $b = -8$.

NORMAL LINES

■ 1. Find the equation of the normal line to the graph of $f(x) = 5x^4 + 3e^x$ at $(0,3)$.

Solution:

Begin by finding the slope of the tangent line at $(0,3)$, starting with taking the derivative. Then evaluate the derivative at $(0,3)$.

$$f'(x) = 20x^3 + 3e^x$$

$$f'(0) = 20(0)^3 + 3e^0$$

$$f'(0) = 0 + 3(1)$$

$$f'(0) = 3$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/f'(a) = -1/3$, so the equation of the normal line is

$$y = f(a) - \frac{1}{f'(a)}(x - a)$$

$$y = 3 - \frac{1}{3}(x - 0)$$

$$y = -\frac{1}{3}x + 3$$

■ 2. Find the equation of the normal line to the graph of $g(x) = \ln e^{4x} + 2x^3$ at $(2,24)$.

Solution:

Begin by finding the slope of the tangent line at $(2,24)$, starting with taking the derivative. Then evaluate the derivative at $(2,24)$.

$$g'(x) = 4 + 6x^2$$

$$g'(2) = 4 + 6(2)^2$$

$$g'(2) = 4 + 24$$

$$g'(2) = 28$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/g'(a) = -1/28$, so the equation of the normal line is

$$y = g(a) - \frac{1}{g'(a)}(x - a)$$

$$y = 24 - \frac{1}{28}(x - 2)$$

$$y = -\frac{1}{28}x + \frac{337}{14}$$

■ 3. Find the equation of the normal line to the graph of $h(x) = 5\cos x + 5\sin x$ at $(\pi/2,5)$.

Solution:

Begin by finding the slope of the tangent line at $(\pi/2,5)$, starting with taking the derivative. Then evaluate the derivative at $(\pi/2,5)$.

$$h'(x) = -5\sin x + 5\cos x$$

$$h'\left(\frac{\pi}{2}\right) = -5\sin\left(\frac{\pi}{2}\right) + 5\cos\left(\frac{\pi}{2}\right)$$

$$h'\left(\frac{\pi}{2}\right) = -5(1) + 5(0)$$

$$h'\left(\frac{\pi}{2}\right) = -5$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/h'(a) = 1/5$, so the equation of the normal line is

$$y = h(a) - \frac{1}{h'(a)}(x - a)$$

$$y = 5 + \frac{1}{5}\left(x - \frac{\pi}{2}\right)$$

■ 4. Find the equation of the normal line to the graph of $f(x) = 7x^3 + 2x^2 - 5x + 9$ at $(2,63)$.

Solution:

Begin by finding the slope of the tangent line at $(2,63)$, starting with taking the derivative. Then evaluate the derivative at $(2,63)$.

$$f'(x) = 21x^2 + 4x - 5$$

$$f'(2) = 21(2)^2 + 4(2) - 5$$

$$f'(2) = 84 + 8 - 5$$

$$f'(2) = 87$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/f'(a) = -1/87$, so the equation of the normal line is

$$y = f(a) - \frac{1}{f'(a)}(x - a)$$

$$y = 63 - \frac{1}{87}(x - 2)$$

$$y = -\frac{1}{87}x + \frac{5{,}483}{87}$$

■ 5. Find the equation of the normal line to the graph of $g(x) = 5\sqrt{x^2 - 14x + 49}$ at $(2,25)$.

Solution:

Begin by finding the slope of the tangent line at $(2,25)$, starting with taking the derivative. Then evaluate the derivative at $(2,25)$.

$$g'(x) = \frac{5}{2\sqrt{x^2 - 14x + 49}} \cdot (2x - 14)$$

$$g'(x) = \frac{5x - 35}{\sqrt{x^2 - 14x + 49}}$$

$$g'(x) = \frac{5(x - 7)}{|x - 7|}$$

$$g'(2) = \frac{5(2 - 7)}{|2 - 7|}$$

$$g'(2) = \frac{-25}{5}$$

$$g'(2) = -5$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/g'(a) = 1/5$, so the equation of the normal line is

$$y = g(a) - \frac{1}{g'(a)}(x - a)$$

$$y = 25 + \frac{1}{5}(x - 2)$$

$$y = \frac{1}{5}x + \frac{123}{5}$$

■ 6. Find the equations of the tangent and normal lines of $g(x) = (2x^2 - 5x + 3)^2$ at $(0,9)$.

Solution:

Differentiate the function,

$$g'(x) = 2(2x^2 - 5x + 3)(4x - 5)$$

$$g'(x) = (2x^2 - 5x + 3)(8x - 10)$$

then evaluate at $(0,9)$.

$$g'(0) = (2(0)^2 - 5(0) + 3)(8(0) - 10)$$

$$g'(0) = 3(-10)$$

$$g'(0) = -30$$

With $g'(0) = -30$ and $(a, g(a)) = (0,9)$, the equation of the tangent line is

$$y = g(a) + g'(a)(x - a)$$

$$y = 9 - 30(x - 0)$$

$$y = 9 - 30x$$

$$y = -30x + 9$$

Since the normal line is perpendicular to the tangent line at the same point, the slope of the normal line is $-1/g'(a) = 1/30$, so the equation of the normal line is

$$y = g(a) - \frac{1}{g'(a)}(x - a)$$

$$y = 9 + \frac{1}{30}(x - 0)$$

$$y = \frac{x}{30} + 9$$

AVERAGE RATE OF CHANGE

■ 1. Find the average rate of change of the function over the interval $[4,9]$.

$$f(x)=\frac{5\sqrt{x}-2}{3}$$

Solution:

The values of $f(9)$ and $f(4)$ are

$$f(9)=\frac{5\sqrt{9}-2}{3}=\frac{5(3)-2}{3}=\frac{13}{3}$$

$$f(4)=\frac{5\sqrt{4}-2}{3}=\frac{5(2)-2}{3}=\frac{8}{3}$$

Therefore, average rate of change on $[a,b]=[4,9]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{\frac{13}{3}-\frac{8}{3}}{9-4}=\frac{\frac{5}{3}}{\frac{5}{1}}=\frac{5}{3}\cdot\frac{1}{5}=\frac{1}{3}$$

■ 2. Find the average rate of change of the function over the interval $[16,25]$.

$$g(x) = \frac{2x - 8}{\sqrt{x} - 2}$$

Solution:

The values of $g(25)$ and $g(16)$ are

$$g(25) = \frac{2(25) - 8}{\sqrt{25} - 2} = \frac{42}{3} = 14$$

$$g(16) = \frac{2(16) - 8}{\sqrt{16} - 2} = \frac{24}{2} = 12$$

Therefore, average rate of change on $[a, b] = [16,25]$ is given by

$$\frac{g(b) - g(a)}{b - a}$$

$$\frac{14 - 12}{25 - 16} = \frac{2}{9}$$

■ 3. Find the average rate of change of the function over the interval $[0,4]$.

$$h(x) = \frac{x^3 - 8}{x^2 - 4x - 5}$$

Solution:

In this question, $g(4)$ and $g(0)$ are

$$h(4) = \frac{4^3 - 8}{4^2 - 4(4) - 5} = \frac{64 - 8}{16 - 16 - 5} = \frac{56}{-5} = -\frac{56}{5}$$

$$h(0) = \frac{0^3 - 8}{0^2 - 4(0) - 5} = \frac{0 - 8}{0 - 0 - 5} = \frac{-8}{-5} = \frac{8}{5}$$

Therefore, average rate of change on $[a, b] = [0,4]$ is given by

$$\frac{h(b) - h(a)}{b - a}$$

$$\frac{\frac{-56}{5} - \frac{8}{5}}{4 - 0} = \frac{-\frac{64}{5}}{\frac{4}{1}} = -\frac{64}{5} \cdot \frac{1}{4} = -\frac{16}{5}$$

■ 4. Find the average rate of change of the function over the interval $[-2, -3/2]$.

$$f(x) = -\frac{1}{4 - x}$$

Solution:

The values of $f(-2)$ and $f(-3/2)$ are

$$f(-2) = -\frac{1}{4 - (-2)} = -\frac{1}{4 + 2} = -\frac{1}{6}$$

$$f\left(-\frac{3}{2}\right)=-\frac{1}{4-\left(-\frac{3}{2}\right)}=-\frac{1}{4+\frac{3}{2}}=-\frac{1}{\frac{8}{2}+\frac{3}{2}}=-\frac{1}{\frac{11}{2}}=-\frac{2}{11}$$

Therefore, average rate of change on $[a,b]=[-2,-3/2]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{-\frac{2}{11}-\left(-\frac{1}{6}\right)}{-\frac{3}{2}-(-2)}=\frac{-\frac{2}{11}+\frac{1}{6}}{-\frac{3}{2}+2}=\frac{-\frac{12}{66}+\frac{11}{66}}{-\frac{3}{2}+\frac{4}{2}}=\frac{-\frac{1}{66}}{\frac{1}{2}}=-\frac{1}{66}\cdot\frac{2}{1}=-\frac{1}{33}$$

■ 5. On Thursday, the price of a gallon of gas was $\$3.24$. What was the price of a gallon of gas on Sunday, if the average rate of change of the price of a gallon of gas from Thursday to Sunday is $\$0.09$ per day?

Solution:

From the problem above, we know x_1 is Thursday and x_2 is Sunday, so x_2-x_1 is 3 days.

Therefore, average rate of change on $[x_1,x_2]$ is given by

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$\frac{f(x_2)-3.24}{3}=0.09$$

$$f(x_2) - 3.24 = 0.09(3)$$

$$f(x_2) - 3.24 = 0.27$$

$$f(x_2) = 0.27 + 3.24$$

$$f(x_2) = 3.51$$

The price of a gallon of gas on Sunday was $\$3.51$.

■ 6. Find an expression in terms of a that models the average rate of change of the function $f(x) = 2x^2 + 5x - 4$ over the interval $[0,2a]$.

Solution:

The values are $f(2a)$ and $f(0)$ are

$$f(2a) = 2(2a)^2 + 5(2a) - 4$$

$$f(2a) = 2(4a^2) + 10a - 4$$

$$f(2a) = 8a^2 + 10a - 4$$

and

$$f(0) = 2(0)^2 + 5(0) - 4$$

$$f(0) = 0 + 0 - 4$$

$$f(0) = -4$$

Therefore, the average rate of change on $[a,b]=[0,2a]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{8a^2+10a-4-(-4)}{2a-0}=\frac{8a^2+10a}{2a}=4a+5$$

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