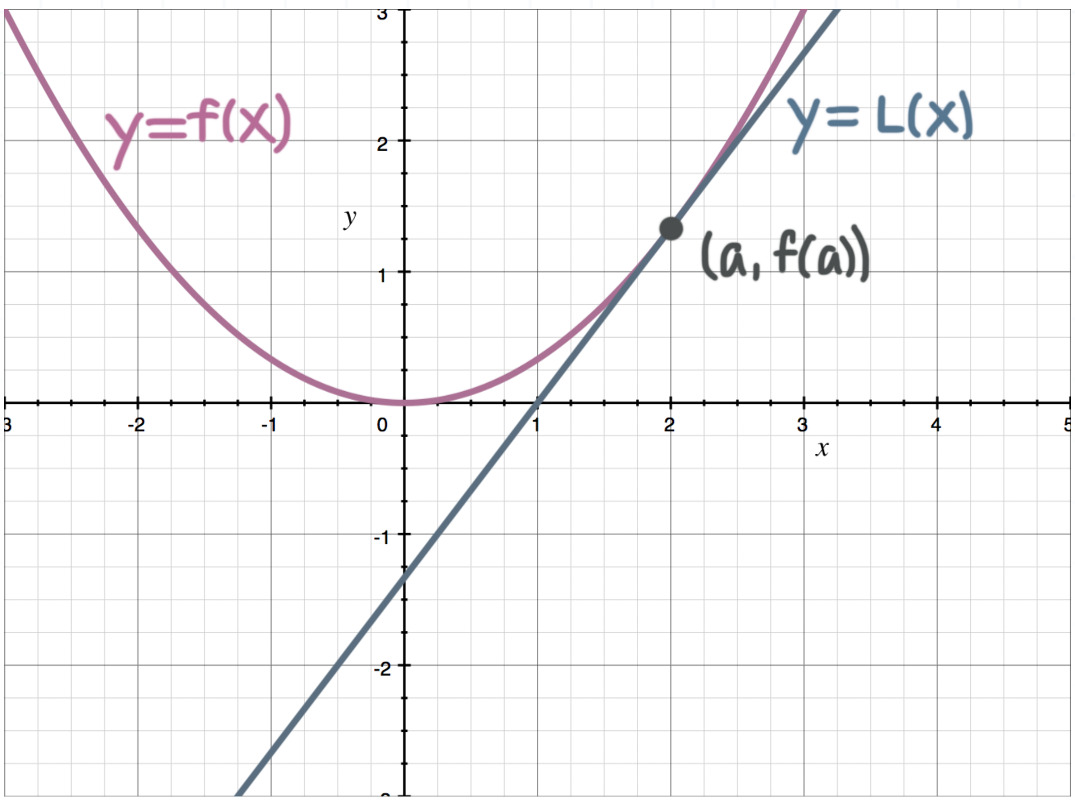
Linear approximation

Remember that the **tangent line** of a function $y = f(x)$ at a particular point is the line that skims along one side of a graph, intersecting the graph at exactly one point, called the **point of tangency** $(a, f(a))$, but never crossing the graph.



We can find the equation of the tangent line with only two pieces of information: 1) the point of tangency $(a, f(a))$, and 2) the slope of the tangent line $m = f'(a)$, which we find by evaluating the function's derivative at the point of tangency. Then the equation of the tangent line is

$$y = f(a) + f'(a)(x - a)$$

When we look at the graph of a curve and its tangent line at a particular point of tangency, what we see is that, not only do the curve and the tangent line have equal values at the point of tangency, but their values are very close to one another near the point of tangency. In other words,

© Krista King Math

around the point of tangency, the function's graph and the tangent line are really close to each other.

Because of that, we can use values along the tangent line, near the point of tangency, to approximate values along the curve at the same point.

When we use the tangent line equation as this kind of approximation tool, we call it the **linear approximation** (or linearization) equation, instead of the tangent line equation. So the tangent line and the linear approximation are the same thing, but using the term "linear approximation" specifically implies that we're using the line to approximate function values around the point of tangency. To indicate that we're doing linear approximation, we use $L(x)$ in the equation of the tangent line:

$$L(x) = f(a) + f'(a)(x - a)$$

Linear approximation is a useful tool because the function's value around the point of tangency isn't always easy to calculate, and finding the linear approximation's value might be much easier. Or, we might not have an equation for the function at all, in which case, linear approximation can help us to estimate a particular value.

Let's work through an example.

Example

Use linear approximation to estimate $f(1.9)$.

$$f(x) = \frac{2}{\sqrt{x - 1}}$$

© Krista King Math

The first thing we want to realize is that finding $f(1.9)$ gets pretty messy. If we substitute $x = 1.9$ into the function, we get $\sqrt{0.9}$ in the function's denominator. That's not necessarily an easy value to find. However, $\sqrt{0.9}$ is pretty close to $\sqrt{1}$, which is a very easy value to find.

Therefore, instead of trying to find $f(1.9)$, let's use a linear approximation equation and $a = 2$ to get an approximation for $f(1.9)$. First, evaluate the function at $a = 2$.

$$f(2) = \frac{2}{\sqrt{2-1}}$$

$$f(2) = \frac{2}{\sqrt{1}}$$

$$f(2) = 2$$

Find the derivative, $f'(x)$, by first rewriting the function.

$$f(x) = \frac{2}{(x-1)^{\frac{1}{2}}}$$

$$f(x) = 2(x-1)^{-\frac{1}{2}}$$

Then differentiate.

$$f'(x) = -\frac{1}{2}(2)(x-1)^{-\frac{3}{2}}$$

© Krista King Math

$$f'(x) = -\frac{1}{\sqrt{(x-1)^3}}$$

Evaluate the derivative at $a = 2$.

$$f'(2) = -\frac{1}{\sqrt{(2-1)^3}}$$

$$f'(2) = -\frac{1}{\sqrt{1}}$$

$$f'(2) = -1$$

Substituting the slope $f'(2) = -1$ and the point of tangency $(2,2)$ into the linear approximation gives

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = 2 + (-1)(x-2)$$

$$L(x) = 2 - (x-2)$$

$$L(x) = 2 - x + 2$$

$$L(x) = 4 - x$$

Now that we've built the linear approximation equation, we can substitute $x = 1.9$.

$$L(1.9) = 4 - 1.9$$

$$L(1.9) = 2.1$$

© Krista King Math

This value tells us that the value along the linear approximation line is 2.1 at $x = 1.9$. So 2.1 is a value that we can use as an estimate of the function's value at 1.9.

© Krista King Math