Trigonometric limits

Limit problems with trigonometric functions usually revolve around three key limit values.

$$\lim_{x \to 0} \sin x = 0$$

$$\lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

When we evaluate limits of trig functions, our goal is therefore to always try to reduce the function to some combination of these three limits, and maybe some other simple constants.

For these kinds of problems, in order to rework trig functions, we'll often use reciprocal identities,

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

or Pythagorean identities.

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Let's look at an example of how we go about reworking trig functions into the three limit values above.

Example

Evaluate the limit.

$$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

If we try substitution to evaluate the limit, we get

$$\frac{1 - \cos(0)}{0}$$

$$\frac{1 - 1}{0}$$

$$\frac{0}{0}$$

Substitution doesn't work, and there's nothing to factor, but since we have exactly two terms in the numerator, we can actually use the conjugate method for the first step of this problem.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \left(\frac{1 + \cos x}{1 + \cos x} \right)$$

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

Applying the Pythagorean identity $1 - \cos^2 x = \sin^2 x$ to the numerator gives

$$\lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

Notice now that we can factor out $(\sin x)/x$, which is one of our three key limits.

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \right)$$

$$\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1 + \cos x}$$

Since the first limit is one of the three key limits, we can replace it with its value.

$$1 \cdot \lim_{x \to 0} \frac{\sin x}{1 + \cos x}$$

$$\lim_{x \to 0} \frac{\sin x}{1 + \cos x}$$

And now we can evaluate the limit as $x \to 0$ using just simple substitution.

$$\frac{\sin 0}{1 + \cos 0}$$

$$\frac{0}{1 + 1}$$

$$\frac{0}{2}$$

$$0$$

So the limit is 0.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$