One-sided limits

The examples we've done so far have been straightforward, because we've always been able to evaluate the function at the value we were approaching. In other words, we have no trouble evaluating

$$\lim_{x \to 5}(x + 1)$$

because $f(x) = x + 1$ is defined at $x = 5$. Which means we can find the limit just by substituting $x = 5$ into the function.

Undefined values

But finding limits gets a little trickier when we start dealing with points of the function that are undefined. For instance, consider a different limit.

$$\lim_{x \to 0} \frac{1}{x}$$

In this problem, we're trying to find the value of the function as x approaches 0. But substituting $x = 0$ into the function gives $1/0$, and fractions are undefined when the denominator is 0.

So we can't evaluate this limit using only simple substitution. Instead, we need to look at what the function $f(x) = 1/x$ is doing just to the left of $x = 0$ and just to the right of $x = 0$. If we can understand the function's behavior on both sides of $x = 0$, then we might be able to draw a conclusion about what's happening to the function at exactly $x = 0$, despite the fact that we couldn't evaluate the function at that exact point.

General vs. one-sided limits

By looking at what the function is doing just to the left of $x = 0$ and just to the right of $x = 0$, we're investigating the function's **one-sided limits** at $x = 0$, or more specifically, its left-hand limit and right-hand limit at that point.

The **left-hand limit** is the limit of the function as we approach from the left side (or negative side), whereas the **right-hand limit** is the limit of the function as we approach from the right side (or positive side).

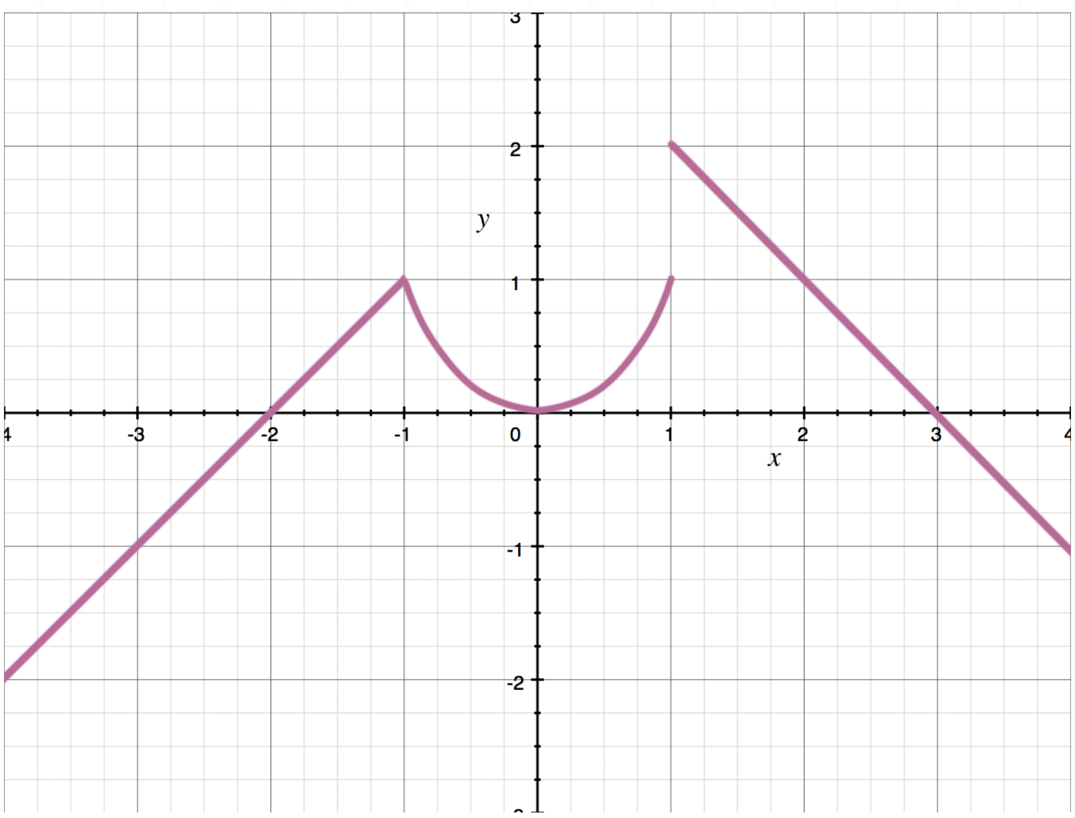
If the one-sided limits both exist for a function at a particular point, and if the one-sided limits at that point are equivalent, then the general limit of the function exists at that point.

In other words, the **general limit** exists at a point $x = c$ if

1. the left-hand limit exists at $x = c$, $\lim_{x \to c^-} f(x)$,
2. the right-hand limit exists at $x = c$, $\lim_{x \to c^+} f(x)$, and
3. those left- and right-hand limits are equal to one another, $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

These are the three conditions that must be met in order for the general limit to exist. Even though the left- and right-hand limits must exist in order for the general limit to exist, the left- and/or right-hand limits can exist even when the general limit does not exist.

Let's consider the graph below of a totally new function. The general limit exists for this graph at $x = -1$ because the left- and right- hand limits both approach the same value: 1. On the other hand, the general limit does not exist at $x = 1$ because the left- and right-hand limits are not equal; there, the left side of the graph is approaching 1, but the right side of the graph is approaching 2.



Therefore, the general limit of this function exists at $x = -1$ because the one-sided limits are equal there, but the general limit doesn't exist at $x = 1$ because the one-sided limits are unequal there.

Evaluating one-sided limits

The general limit for a function might look something like this:

$$\lim_{x\to 2} f(x) = b$$

We read this general limit equation as "The limit of the function $f(x)$ as x approaches 2 is equal to b."

Left-hand limits are written as

$$\lim_{x\to 2^-} f(x) = c$$

The negative sign after the 2 indicates that we're talking about the limit as we approach 2 from the negative, or left-hand side of the graph.

Right-hand limits are written as

$$\lim_{x\to 2^+} f(x) = d$$

The positive sign after the 2 indicates that we're talking about the limit as we approach 2 from the positive, or right-hand side of the graph.

To evaluate one-sided limits, we'll first try, just like general limits, to substitute the value the limit approaches into the function.

Example

Find the left- and right-hand limits of the function at $x = -3$, and say whether the general limit exists there.

$$f(x) = |x| + 3$$

This function includes $|x|$, which is the absolute value of x, which turns any value we plug in for x into a positive value. So the left-hand limit is

$$\lim_{x \to -3^-} (|x| + 3)$$

$$|-3| + 3$$

$$3 + 3$$

$$6$$

The right-hand limit is

$$\lim_{x \to -3^+} (|x| + 3)$$

$$|-3| + 3$$

$$3 + 3$$

$$6$$

The left- and right-hand limits both exist, and the left- and right-hand limits are equivalent, so the general limit exists, and the general limit is 6.

$$\lim_{x \to -3} (|x| + 3) = 6$$

If simple substitution doesn't work (because the function is undefined at the value being approached), then we can try to substitute values that are very close to the value being approached.

Example

Find the left- and right-hand limits of the function at $x = 0$, and say whether the general limit exists there.

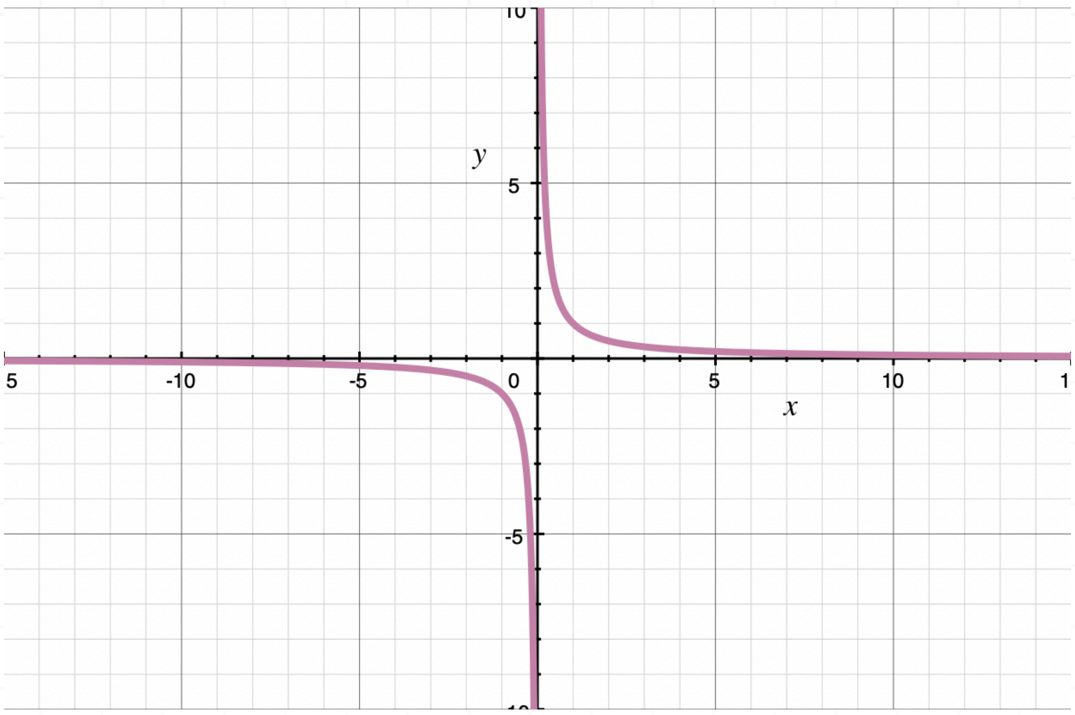
$$f(x) = \frac{1}{x}$$

We need to find the left- and right-hand limits at $x = 0$, but substituting $x = 0$ into the function gives a denominator of 0, which makes the fraction undefined. So instead of substitution, let's look at values on either side of $x = 0$, but which are very close to $x = 0$.

$$f(-0.0001) = \frac{1}{-0.0001} = -10{,}000$$

$$f(0.0001) = \frac{1}{0.0001} = 10{,}000$$

What these values tell us is that, as we get very close to $x = 0$ on the left side, the function's value is trending toward $-\infty$, but as we get very close to $x = 0$ on the right side, the function's value is trending toward ∞. If we graph the function, the graph confirms this behavior.



Therefore, the one-sided limits are

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Because these one-sided limits aren't equal, the general limit of the function $f(x) = 1/x$ doesn't exist at $x = 0$.