L'Hospital's Rule

Now that we've covered derivatives pretty extensively, we can introduce L'Hospital's Rule, which ties limits and derivatives together.

If you remember from the lessons on solving limits, we usually want to try solving for the limit using substitution, then factoring if substitution didn't work, then conjugate method if factoring didn't work, and then eventually we might start looking at other limit methods, if necessary.

But when other limit methods fail us, we do have a backup plan, which is L'Hospital's rule. If, no matter what other methods we try, substitution results in an **indeterminate form**, like these,

$$\frac{\pm\infty}{\pm\infty} \qquad \frac{0}{0} \qquad (0)(\pm\infty)$$

$$1^\infty \qquad 0^0 \qquad \infty^0 \qquad \infty-\infty$$

then we should try L'Hospital's rule to evaluate the limit.

Applying L'Hospital's Rule

To use L'Hospital's Rule, we need the function we're evaluating to be a fraction. To apply the rule, we replace both the numerator and denominator of the fraction with their own derivatives. In other words, we take the derivative of the numerator and make that the *new* numerator, and we take the derivative of the denominator and make that the *new* denominator.

Once we've replaced both, then we try substitution to evaluate the limit. If we get a real number answer, then L'Hospital's Rule was successful, we've found our answer, and we can stop there.

If, on the other hand, we still get an indeterminate form after using substitution, it's okay. We simply apply L'Hospital's Rule a second time. In fact, we can continue applying the rule over and over again, until eventually substitution gives a real number answer instead of an indeterminate form.

Not quotient rule

Because L'Hospital's Rule has us taking derivatives inside a fraction, people often confuse this process with the Quotient Rule, which was a derivative rule we learned earlier. These two rules are not related!

If we want to find the derivative of a fraction, we use Quotient Rule. But in this case, we're not trying to find the derivative of a fraction. Instead, we're trying to take the limit of a fraction, and that fraction is giving us an indeterminate form when we try to substitute. In this case, we apply L'Hospital's rule, which has us replacing the numerator and denominator with their own derivatives. And that's not at all related to the Quotient Rule.

Let's work through an example where we apply L'Hospital's rule to evaluate a limit as $x \to 0$.

Example

Evaluate the limit.

$$\lim_{x \to 0} \frac{e^x - 1}{\sin(2x)}$$

If we try substitution to evaluate at $x = 0$, we get an indeterminate form.

$$\frac{e^0 - 1}{\sin(2(0))}$$

$$\frac{1 - 1}{\sin(0)}$$

$$\frac{0}{0}$$

Because we get an indeterminate form, let's try applying L'Hospital's Rule. The derivative of the numerator $e^x - 1$ is e^x, and the derivative of the denominator $\sin(2x)$ is $2\cos(2x)$. Rewrite the function by replacing the numerator with its derivative and replacing the denominator with its derivative.

$$\lim_{x \to 0} \frac{e^x}{2\cos(2x)}$$

Try substitution again to evaluate the limit.

$$\frac{e^0}{2\cos(2(0))}$$

$$\frac{1}{2\cos(0)}$$

$$\frac{1}{2(1)}$$

$$\frac{1}{2}$$

In this last example, we were able to get to a real number answer, so we can stop, and this is the value of the limit.

But if evaluating the limit at 0 in this last step had again resulted in an indeterminate form, we would have reapplied L'Hospital's Rule, replacing the numerator e^x with its derivative e^x, and replacing the denominator $2\cos(2x)$ with its derivative $-4\sin(2x)$. Then we would have tried substitution again to evaluate the limit at $x=0$.

And we'd keep going, applying L'Hospital's Rule and then testing substitution, over and over, until we were able to evaluate the limit and get a real number answer, instead of an indeterminate form. Sometimes we'll have to apply L'Hospital's Rule three or even four times to get to the real number answer.

L'Hospital's Rule works really well on the two indeterminate forms $0/0$ and $\pm\infty/\pm\infty$. With other indeterminate forms, it may work better to write rewrite products as quotients and vice versa, using the rule

$$f(x)g(x)=\frac{g(x)}{\frac{1}{f(x)}}$$

Sometimes we'll need to use L'Hospital's rule on a fraction that's sitting inside some other function, so let's look at an example of a problem like that.

Example

Use L'Hospital's rule to evaluate the limit.

$$\lim_{x\to-3} 2x^{\frac{1}{x+3}}$$

If we try substitution to evaluate at $x = -3$, we get an indeterminate form.

$$2(-3)^{\frac{1}{-3+3}}$$

$$2(-3)^{\frac{1}{\infty}}$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x\to-3} 2x^{\frac{1}{x+3}}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x\to-3} \ln(2x^{\frac{1}{x+3}})$$

$$\ln y = \lim_{x\to-3} \frac{1}{x+3} \ln(2x)$$

$$\ln y = \lim_{x \to -3} \frac{\ln(2x)}{x+3}$$

With the limit rewritten, we'll apply L'Hospital's rule to the fraction.

$$\ln y = \lim_{x \to -3} \frac{\frac{1}{2x}(2)}{1}$$

$$\ln y = \lim_{x \to -3} \frac{\frac{1}{x}}{1}$$

$$\ln y = \lim_{x \to -3} \frac{1}{x}$$

Evaluate the limit,

$$\ln y = \frac{1}{-3}$$

$$\ln y = -\frac{1}{3}$$

then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^{-\frac{1}{3}}$$

$$y = e^{-\frac{1}{3}}$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to -3} 2x^{\frac{1}{x+3}}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x\to-3} 2x^{\frac{1}{x+3}} = e^{-\frac{1}{3}}$$