Making the function continuous

In this lesson, we build on our understanding of discontinuities and one-sided limits to explore how we can make a function continuous. Specifically, we will learn how to adjust functions to remove discontinuities and ensure a smooth, connected graph.

Point discontinuities

Previously, we learned that a point discontinuity was a single pinpoint of discontinuity in the graph. We saw that we could take a function like

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

and simplify it as

$$f(x) = \frac{(x + 4)(x + 7)}{x + 4}$$

$$f(x) = x + 7$$

Because we canceled the $x + 4$, we know the function has a point discontinuity at $x = -4$. We can "plug the hole" by redefining the function for $x = -4$, but doing so completely changes the function. So this defines a brand-new function, g, which is continuous at $x = -4$.

$$g(x) = \begin{cases} \frac{x^2 + 11x + 28}{x + 4} & x \neq -4 \\ 3 & x = -4 \end{cases}$$

Piecewise-defined functions

Remember that this kind of function, the function g we just built, is a piecewise function or piecewise-defined function, because it's defined "in pieces."

Sometimes we'll be given a piecewise-defined function and asked to find the value of an unknown constant that will make the function continuous. For instance, consider the function

$$f(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ 5 - x & 3 < x \le 5 \end{cases}$$

It's a piecewise-defined function where the first piece defines the function from $x = 0$ to $x = 3$ (including at $x = 3$), and the second piece defines the function for values greater than $x = 3$, all the way up to $x = 5$.

When $x = 3$, the first piece stops defining the function and the second piece takes over, so we can think of $x = 3$ as the "break point" between the pieces. If we can make the two pieces of the function meet each other at the break point, and if the left- and right-hand limits of the function are equal at the break point, then the function will be continuous there. Remember that a function $f(x)$ is continuous at $x = c$ if
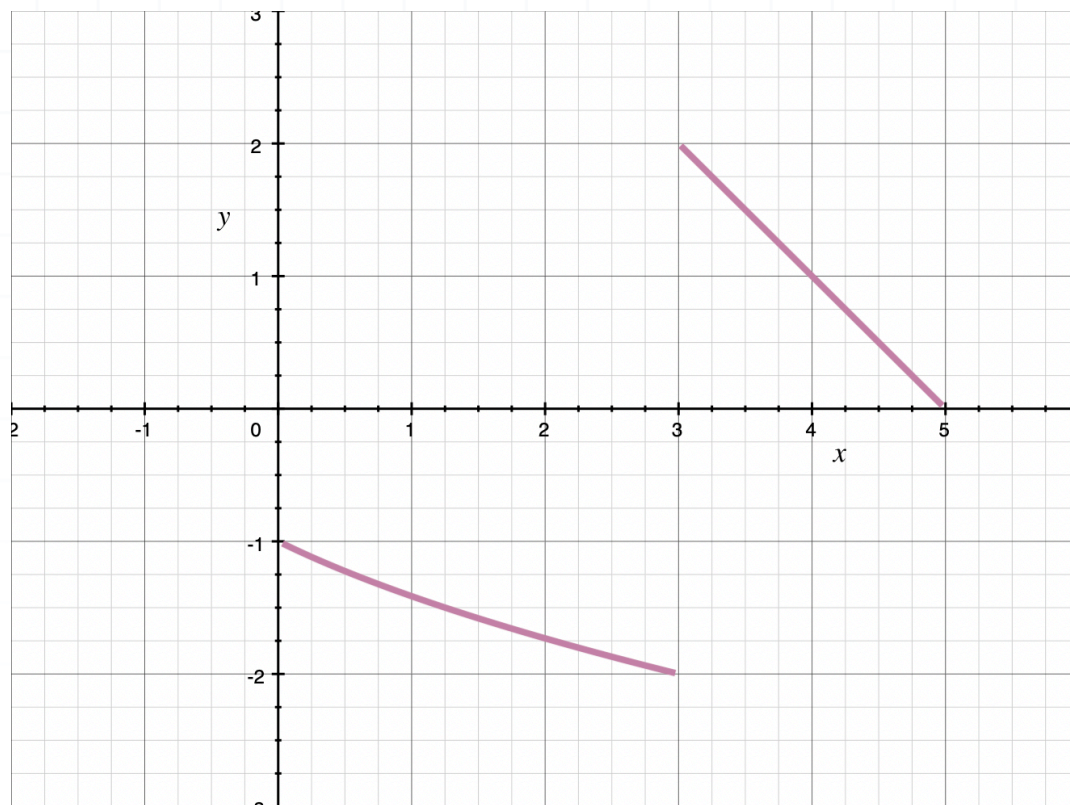
$$\lim_{x \to c} f(x) = f(c)$$

So for a problem like this one, we need to find the value of k that forces the two pieces of $f(x)$ to have the same value at $x = 3$.
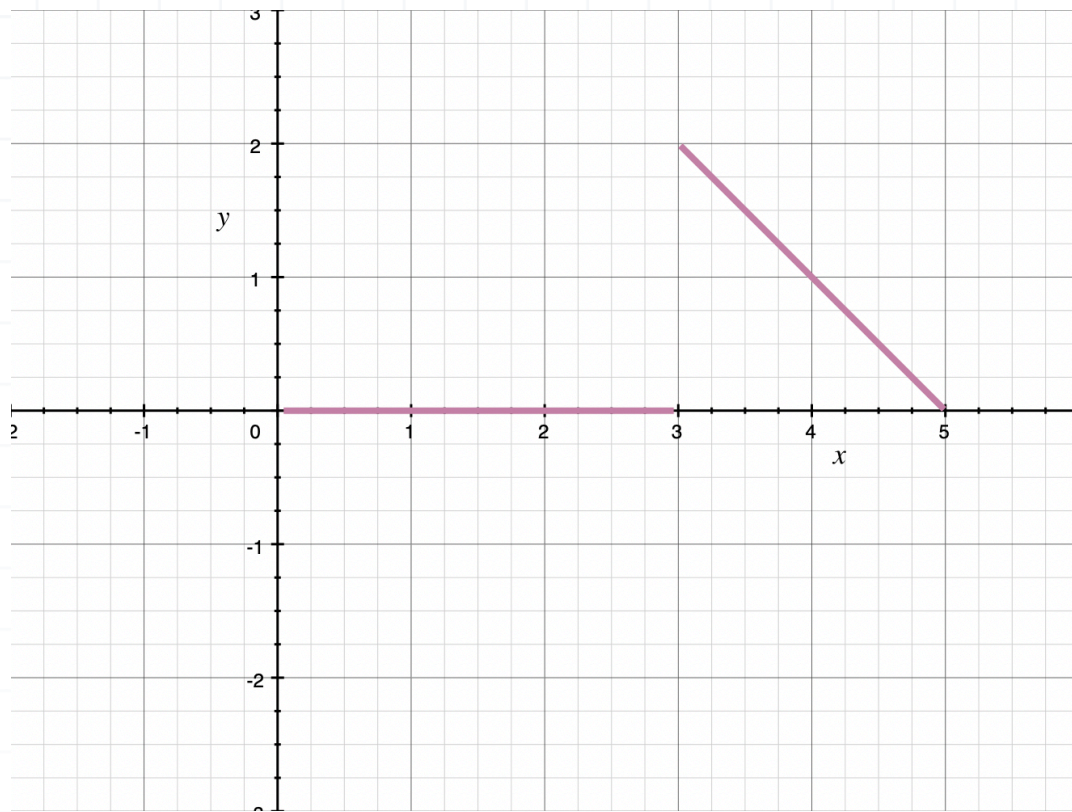
To visualize this, here's what the graph of $f(x)$ looks like with some different values of k. Notice how changing the value of k changes the shape of the $k\sqrt{x+1}$ piece, and also changes the value of the left piece of f at $x = 3$.

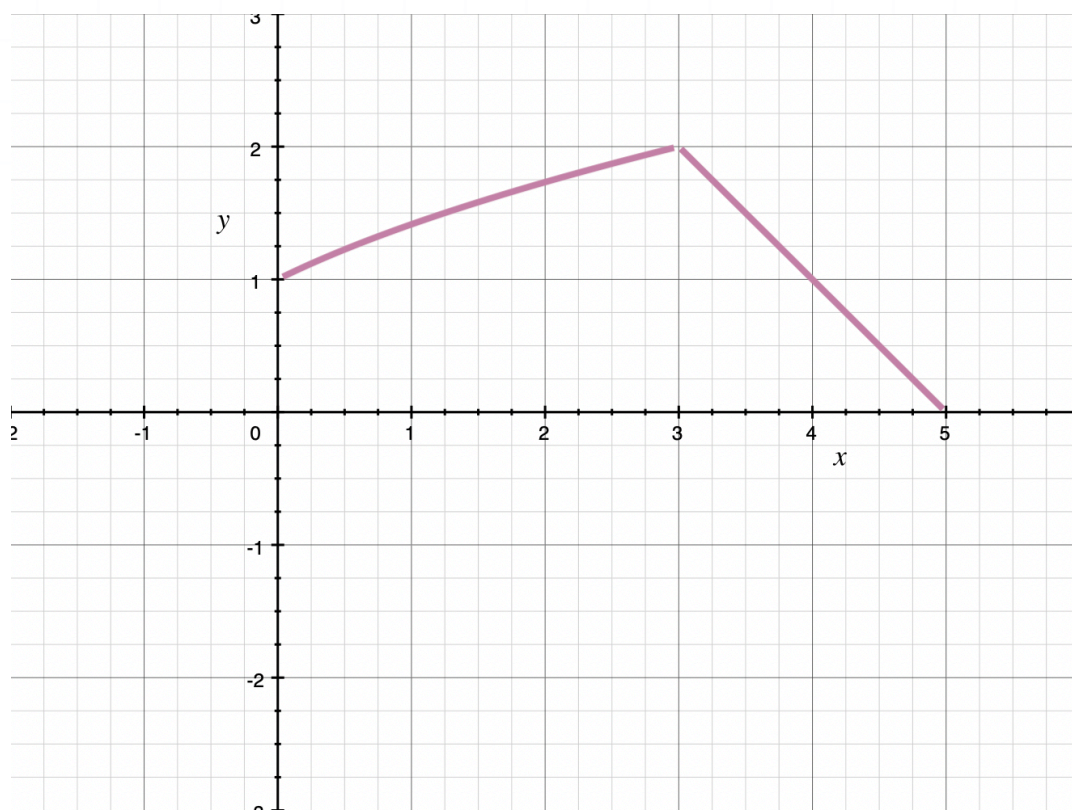If $k = -1$, the function has a jump discontinuity at $x = 3$ because the graph of $f(x)$ is

If $k = 0$, the function has a jump discontinuity at $x = 3$ because the graph of $f(x)$ is

If $k = 1$, the function is continuous at $x = 3$ because the graph of $f(x)$ is

We can see from the graphs that $k = 1$ will be the value that makes the function's pieces meet each other at $x = 3$. But how do we solve for the

value of k algebraically, so that we can avoid picking random values of the unknown constant and graphing the function with that value?

Well, we always want to start at the "break point" that we talked about earlier. For this function, $x = 3$ is the break point between the two pieces, so we need the pieces to have equal value at that point. Therefore, we'll set the pieces equal to one another, plug in $x = 3$, and then solve the equation for the unknown k.

$$k\sqrt{x + 1} = 5 - x$$

$$k\sqrt{3 + 1} = 5 - 3$$

$$k\sqrt{4} = 2$$

$$2k = 2$$

$$k = 1$$

So $k = 1$ is the value that forces the function's pieces to meet at the break point. For any other value of k, we'll get a jump discontinuity at the break point $x = 3$, but $k = 1$ makes the two pieces meet at the same point.

To find the value at which they meet, we plug $k = 1$ and the break point $x = 3$ back into the function.

$$f(3) = \begin{cases} 1\sqrt{3 + 1} & 0 \leq x \leq 3 \\ 5 - 3 & 3 < x \leq 5 \end{cases}$$

$$f(3) = \begin{cases} 2 & 0 \leq x \leq 3 \\ 2 & 3 < x \leq 5 \end{cases}$$

So when $k = 1$, both pieces of the function meet at $f(3) = 2$. Now to ensure continuity, all we have left to do is check that the function's left- and right-hand limits both approach 2.

Because $k\sqrt{x+1} = 1\sqrt{x+1} = \sqrt{x+1}$ models the function to the left of $x = 3$ and $5 - x$ models the function to the right of $x = 3$, so we'll check the left-hand limit of $\sqrt{x+1}$ and the right-hand limit of $5 - x$.

$$\lim_{x \to 3^-} \sqrt{x+1} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\lim_{x \to 3^+} 5 - x = 5 - 3 = 2$$

Because at $x = 3$ the left- and right-hand limits are equal,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

and because they both have the same value as the function itself at $x = 3$,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) = 2$$

we can say that the function is continuous at $x = 3$ when $k = 1$.

So to summarize, we'll always follow the same steps to find the value of an unknown constant in order to force the continuity of a piecewise function.

1. Set the function's pieces equal to one another and solve for the value of the unknown k.

2. Plug the value of k and the value of the break point into the function to ensure the function's pieces return the same value.

3. Find the left- and right-hand limits of the function at the break point.

4. Ensure that the left- and right-hand limits (from Step 3) are equal, and that they're both equal to the function's value (from Step 2) at the break point. If these three values are equal, the function is continuous at the break point.