Newton's Law of Cooling

Newton’s Law of Cooling models the way in which a warm object in a cooler environment cools down until it matches the temperature of its environment.

Therefore, this law is similar to the half-life equation we just learned, in the sense that they both model the rate at which something decays. A half-life equation models the rate at which a radioactive substance decays, whereas Newton’s Law models the rate at which the temperature is “decaying” from hotter to cooler.

The Law

The law tells us that the rate at which the object cools is proportional to the difference between the object and the environment around it.

In other words, if you put a boiling pot of soup in the freezer, it’ll cool down faster than if you simply leave the pot on the counter. That’s because the difference between the temperature of the soup and the freezer is greater than the difference between the temperature of the soup and the room-temperature countertop. The greater the temperature difference, the faster the object will cool.

The Newton’s Law of Cooling formula is

$$\frac{dT}{dt}=-k(T-T_a) \text{ with } T(0)=T_0$$

where T is the temperature over time t, k is the decay constant, T_a is the temperature of the environment ("ambient temperature"), and T_0 is the initial, or starting, temperature of the hot object.

The point of applying Newton's Law is to generate an equation that models the temperature of the hot, but cooling, object over time. That temperature function, and the solution to the Newton's Law equation, will be

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

The easiest way to understand how Newton's Law applies to a real-world scenario is to work through an example, so let's work through one.

Example

At a local restaurant, a big pot of soup, boiling at 100° C, has just been removed from the stove and set on the countertop, where the ambient temperature is 23° C. After 5 minutes, the soup cools to 98°. If the soup needs to be served to the restaurant's customers at 90° C, how long will it be before the soup is ready to serve?

For Newton's Law problems, it's especially helpful to list out what the question tells us.

$T_0 = 100^\circ$ Initial temperature of the soup

$T_a = 23^\circ$ Ambient temperature on the countertop

$$T(5) = 98^\circ \qquad \text{At time } t = 5 \text{ minutes, the soup has cooled to } 98^\circ$$

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 23 + (100 - 23)e^{-kt}$$

$$T(t) = 23 + 77e^{-kt}$$

Substitute the initial condition $T(5) = 98^\circ$,

$$T(5) = 23 + 77e^{-k(5)}$$

$$98 = 23 + 77e^{-5k}$$

in order to find a value for the decay constant k.

$$75 = 77e^{-5k}$$

$$\frac{75}{77} = e^{-5k}$$

$$\ln\frac{75}{77} = \ln(e^{-5k})$$

$$\ln\frac{75}{77} = -5k$$

$$k = -\frac{1}{5}\ln\frac{75}{77}$$

Substitute this value for k into the equation modeling temperature over time.

$$T(t) = 23 + 77e^{-\left(-\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$T(t) = 23 + 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

We want to find the time t at which the soup reaches 90°, so we'll substitute $T(t) = 90^\circ$.

$$90 = 23 + 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$67 = 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$\frac{67}{77} = e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

Apply the natural logarithm to both sides of the equation.

$$\ln\frac{67}{77} = \ln\left(e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}\right)$$

$$\ln\frac{67}{77} = \left(\frac{1}{5}\ln\frac{75}{77}\right)t$$

$$5\ln\frac{67}{77} = \left(\ln\frac{75}{77}\right)t$$

$$t = \frac{5\ln\frac{67}{77}}{\ln\frac{75}{77}}$$

$$t \approx 26.43$$

The conclusion then is that the pot of soup will cool from 100° C to 90° C in about 26.5 minutes, at which point it'll be ready to serve to the restaurant's customers.