Half-life

Growth and decay problems are another common application of derivatives.

We actually don't need to use derivatives in order to solve these problems, but derivatives are used to build the basic growth and decay formulas, which is why we study these applications in this part of calculus.

We won't work through how to prove these formulas, because in addition to derivatives, we also use integrals to build them, and we won't learn about integrals until later in calculus.

So, for now, we'll just state that the basic equation for exponential decay is

$$y = Ce^{-kt}$$

where C is the amount of a substance that we're starting with, k is the decay constant, and y is the amount of the substance we have remaining after time t. Since substances decay at different rates, k will vary depending on the substance.

Half-life equation

Every decaying substance has its own half-life, because **half-life** is the amount of time required for exactly half of our original substance to decay, leaving exactly half of what we started with. Because every substance decays at a different rate, each substance will have a different half-life. But regardless of the substance, when we're looking at half-life, we know that

$$y = \frac{C}{2}$$

Because y is the amount of substance that remains as the substance decays, and because C is the amount of substance we started with originally, when the substance has decayed to half of its original amount, y will be equivalent to $C/2$. So we can substitute this value in for y, and then simplify the decay formula.

$$\frac{C}{2} = Ce^{-kt}$$

$$\frac{1}{2} = e^{-kt}$$

So, when we're dealing with half-life specifically, instead of exponential decay in general, we can use this formula we got from substituting $y = C/2$.

Let's do an example problem.

Example

Fermium-253 has a half-life of 3 days. If we start with $1{,}200$ mg of Fermium-253, how much of the substance remains after 10 days?

Before doing anything else, we need to find the value of the decay constant k for Fermium-253. Substitute $C=1{,}200$ and $t=3$ into the half-life formula.

$$\frac{1}{2}=e^{-kt}$$

$$\frac{1}{2}=e^{-k(3)}$$

$$\frac{1}{2}=e^{-3k}$$

Apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2}=\ln(e^{-3k})$$

$$\ln\frac{1}{2}=-3k$$

$$k=-\frac{1}{3}\ln\frac{1}{2}$$

With a value for k, we can now solve for the amount of substance remaining, y, after $t=10$ days.

$$y=Ce^{-kt}$$

$$y=1{,}200e^{-\left(-\frac{1}{3}\ln\frac{1}{2}\right)(10)}$$

$$y=1{,}200e^{\frac{10}{3}\ln\frac{1}{2}}$$

$$y=119.06$$

Therefore, after $t = 10$ days, $y = 119.06$ mg of Fermium-253 remain.

Let's try another example.

Example

The half-life of Americium-243 is $7{,}370$ years. How long will it take a mass of Americium-243 to decay to 73% of its original size?

We haven't been told the exact amount in the original mass of Americium-243, but regardless of the size of the mass, we can say that we're starting with 100% of the mass.

To find the decay constant k, we'll substitute $t = 7{,}370$ years into the half-life equation.

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-k(7{,}370)}$$

$$\frac{1}{2} = e^{-7{,}370k}$$

Apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-7{,}370k})$$

$$\ln\frac{1}{2}=-7{,}370k$$

$$k=-\frac{1}{7{,}370}\ln\frac{1}{2}$$

With a value for k, we can now solve for the number of years it'll take for the substance to decay to $73\,\%$ of its original size.

If we started with $100\,\%$ of the original substance and $73\,\%$ of the substance remains, then we can substitute $C=1$ and $y=0.73$, along with the value we've just found for the decay constant k.

$$y=Ce^{-kt}$$

$$0.73=1e^{-\left(-\frac{1}{7{,}370}\ln\frac{1}{2}\right)t}$$

$$0.73=e^{\left(\frac{1}{7{,}370}\ln\frac{1}{2}\right)t}$$

Apply the natural logarithm to both sides in order to solve for t.

$$\ln 0.73=\ln\left(e^{\left(\frac{1}{7{,}370}\ln\frac{1}{2}\right)t}\right)$$

$$\ln 0.73=\left(\frac{1}{7{,}370}\ln\frac{1}{2}\right)t$$

$$7{,}370\ln 0.73=\left(\ln\frac{1}{2}\right)t$$

$$t=\frac{7{,}370\ln 0.73}{\ln\frac{1}{2}}$$

$$t \approx 3{,}346.21 \text{ years}$$

This answer tells us that it'll take about $3{,}346.21$ years for the mass of Americium-243 to decay to $73\,\%$ of its original size.