Calculus 1 Notes

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Idea of the limit

The **limit** of a function is the value the function approaches at a given value of x, regardless of whether the function actually reaches that value.

For an easy example, consider the function

$$f(x) = x + 1$$

To find the value of the function $f(x)$ when $x = 5$, we plug $x = 5$ into the function, and we get

$$f(5) = 5 + 1$$

$$f(5) = 6$$

So 6 is the limit of the function at $x = 5$, because 6 is the value that the function approaches as the value of x gets closer and closer to 5.

It's strange to talk about the value that a function "approaches," but if we look at some of the other values around $x = 5$, we start to get a better idea of what we mean. For instance,

if we plug $x = 4.9999$ into $f(x)$, then $f(x) = 5.9999$, or

if we plug $x = 5.0001$ into $f(x)$, then $f(x) = 6.0001$.

We start to see that, as we get closer to $x = 5$, whether we're approaching it from the 4.9999 side or the 5.0001 side, the value of $f(x)$ gets closer and closer to 6.

x	4.9998	4.9999	5	5.0001	5.0002
f(x)	5.9998	5.9999	6	6.0001	6.0002

In this simple example, the limit of the function is 6, because that's the actual value of the function at that point; the point is defined. In limit notation, here's how that looks:

$$\lim_{x \to 5}(x+1) = 6$$

This notation tells us that "the limit of the function $x+1$, as x approaches 5, is 6." If we generalize this, we say that the limit of the function $f(x)$ as x approaches a is L.

$$\lim_{x \to a} f(x) = L$$

Let's work through another example of how to find L.

Example

Find the limit.

$$\lim_{x \to 16} (\sqrt{x} + 2)$$

To find the limit, substitute the value that x approaches, $x = 16$, into the function.

$$\sqrt{16} + 2$$

$$4 + 2$$

$$6$$

So the value of the limit is 6.

$$\lim_{x \to 16} (\sqrt{x} + 2) = 6$$

One-sided limits

The examples we've done so far have been straightforward, because we've always been able to evaluate the function at the value we were approaching. In other words, we have no trouble evaluating

$$\lim_{x\to 5}(x+1)$$

because $f(x)=x+1$ is defined at $x=5$. Which means we can find the limit just by substituting $x=5$ into the function.

Undefined values

But finding limits gets a little trickier when we start dealing with points of the function that are undefined. For instance, consider a different limit.

$$\lim_{x\to 0}\frac{1}{x}$$

In this problem, we're trying to find the value of the function as x approaches 0. But substituting $x=0$ into the function gives $1/0$, and fractions are undefined when the denominator is 0.

So we can't evaluate this limit using only simple substitution. Instead, we need to look at what the function $f(x)=1/x$ is doing just to the left of $x=0$ and just to the right of $x=0$. If we can understand the function's behavior on both sides of $x=0$, then we might be able to draw a conclusion about what's happening to the function at exactly $x=0$, despite the fact that we couldn't evaluate the function at that exact point.

General vs. one-sided limits

By looking at what the function is doing just to the left of $x=0$ and just to the right of $x=0$, we're investigating the function's **one-sided limits** at $x=0$, or more specifically, its left-hand limit and right-hand limit at that point.

The **left-hand limit** is the limit of the function as we approach from the left side (or negative side), whereas the **right-hand limit** is the limit of the function as we approach from the right side (or positive side).

If the one-sided limits both exist for a function at a particular point, and if the one-sided limits at that point are equivalent, then the general limit of the function exists at that point.

In other words, the **general limit** exists at a point $x=c$ if

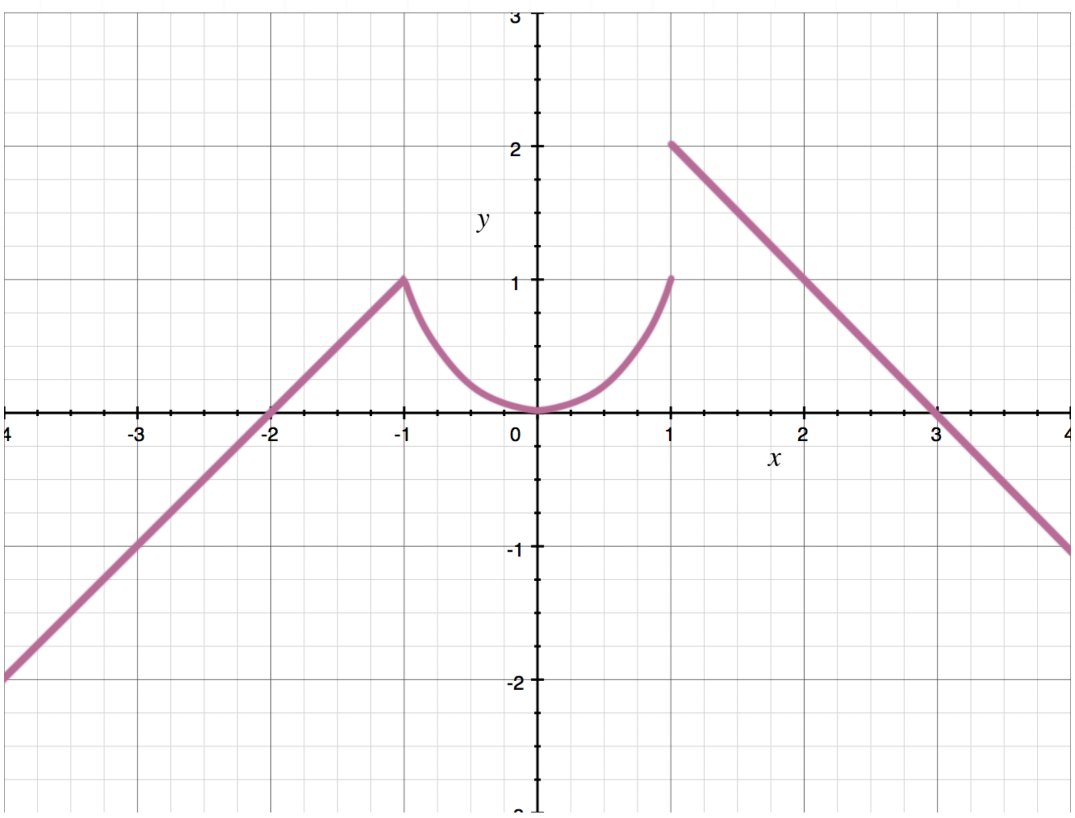
1. the left-hand limit exists at $x=c$, $\lim_{x\to c^-} f(x)$,

2. the right-hand limit exists at $x=c$, $\lim_{x\to c^+} f(x)$, and

3. those left- and right-hand limits are equal to one another, $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)$.

These are the three conditions that must be met in order for the general limit to exist. Even though the left- and right-hand limits must exist in order for the general limit to exist, the left- and/or right-hand limits can exist even when the general limit does not exist.

Let's consider the graph below of a totally new function. The general limit exists for this graph at $x = -1$ because the left- and right- hand limits both approach the same value: 1. On the other hand, the general limit does not exist at $x = 1$ because the left- and right-hand limits are not equal; there, the left side of the graph is approaching 1, but the right side of the graph is approaching 2.



Therefore, the general limit of this function exists at $x = -1$ because the one-sided limits are equal there, but the general limit doesn't exist at $x = 1$ because the one-sided limits are unequal there.

Evaluating one-sided limits

The general limit for a function might look something like this:

$$\lim_{x \to 2} f(x) = b$$

We read this general limit equation as "The limit of the function $f(x)$ as x approaches 2 is equal to b."

Left-hand limits are written as

$$\lim_{x \to 2^-} f(x) = c$$

The negative sign after the 2 indicates that we're talking about the limit as we approach 2 from the negative, or left-hand side of the graph.

Right-hand limits are written as

$$\lim_{x \to 2^+} f(x) = d$$

The positive sign after the 2 indicates that we're talking about the limit as we approach 2 from the positive, or right-hand side of the graph.

To evaluate one-sided limits, we'll first try, just like general limits, to substitute the value the limit approaches into the function.

Example

Find the left- and right-hand limits of the function at $x = -3$, and say whether the general limit exists there.

$$f(x) = |x| + 3$$

This function includes $|x|$, which is the absolute value of x, which turns any value we plug in for x into a positive value. So the left-hand limit is

$$\lim_{x \to -3^-} (|x| + 3)$$

$$|-3| + 3$$

$$3 + 3$$

$$6$$

The right-hand limit is

$$\lim_{x \to -3^+} (|x| + 3)$$

$$|-3| + 3$$

$$3 + 3$$

$$6$$

The left- and right-hand limits both exist, and the left- and right-hand limits are equivalent, so the general limit exists, and the general limit is 6.

$$\lim_{x \to -3} (|x| + 3) = 6$$

If simple substitution doesn't work (because the function is undefined at the value being approached), then we can try to substitute values that are very close to the value being approached.

Example

Find the left- and right-hand limits of the function at $x = 0$, and say whether the general limit exists there.

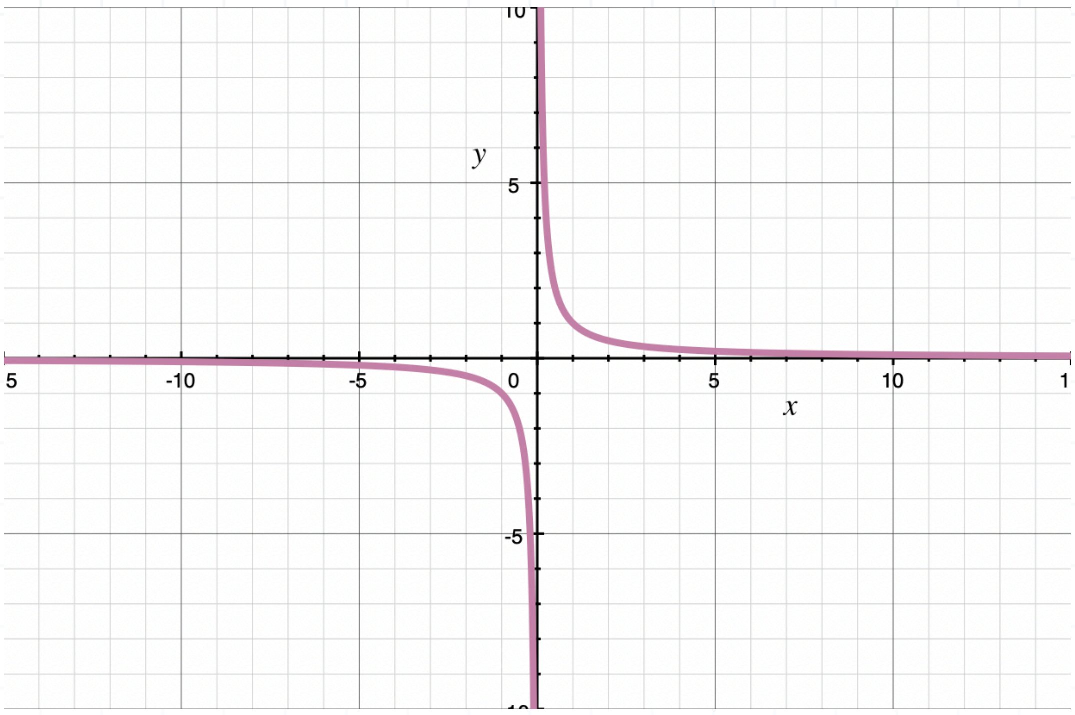
$$f(x) = \frac{1}{x}$$

We need to find the left- and right-hand limits at $x = 0$, but substituting $x = 0$ into the function gives a denominator of 0, which makes the fraction undefined. So instead of substitution, let's look at values on either side of $x = 0$, but which are very close to $x = 0$.

$$f(-0.0001) = \frac{1}{-0.0001} = -\,10{,}000$$

$$f(0.0001) = \frac{1}{0.0001} = 10{,}000$$

What these values tell us is that, as we get very close to $x = 0$ on the left side, the function's value is trending toward $-\infty$, but as we get very close to $x = 0$ on the right side, the function's value is trending toward ∞. If we graph the function, the graph confirms this behavior.



Therefore, the one-sided limits are

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Because these one-sided limits aren't equal, the general limit of the function $f(x) = 1/x$ doesn't exist at $x = 0$.

Proving that the limit does not exist

Given what we now know about how the existence of the one-sided limits dictates the existence of the general limit, we can use this relationship as a test for showing whether or not the general limit exists.

Remember we said before that the general limit exists at a point $x = c$ if

1. the left-hand limit exists at $x = c$,
2. the right-hand limit exists at $x = c$, and
3. those left- and right-hand limits are equal to one another.

Therefore, the general limit **does not exist (DNE)** at $x = c$ if

1. the left-hand limit does not exist at $x = c$, and/or
2. the right-hand limit does not exist at $x = c$, and/or
3. the left- and right-hand limits both exist, but aren't equal to one another.

Let's do an example where we show algebraically that the limit does not exist.

Example

Prove that the limit does not exist.

$$\lim_{x \to 2} \frac{1}{x-2}$$

If we try substitution, we get an undefined value, because the denominator of the fraction becomes 0.

$$\frac{1}{2-2}$$

$$\frac{1}{0}$$

Because we can't use substitution, we'll instead use values on either side of $x = 2$, very close to $x = 2$, to determine how the function is behaving as $x \to 2$.

$$f(1.9999) = \frac{1}{1.9999 - 2} = \frac{1}{-0.0001} = -10{,}000$$

$$f(2.0001) = \frac{1}{2.0001 - 2} = \frac{1}{0.0001} = 10{,}000$$

From the function's values around $x = 2$, we can tell that the function is tending toward $-\infty$ to the left of $x = 2$, and toward ∞ to the right of $x = 2$.

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$

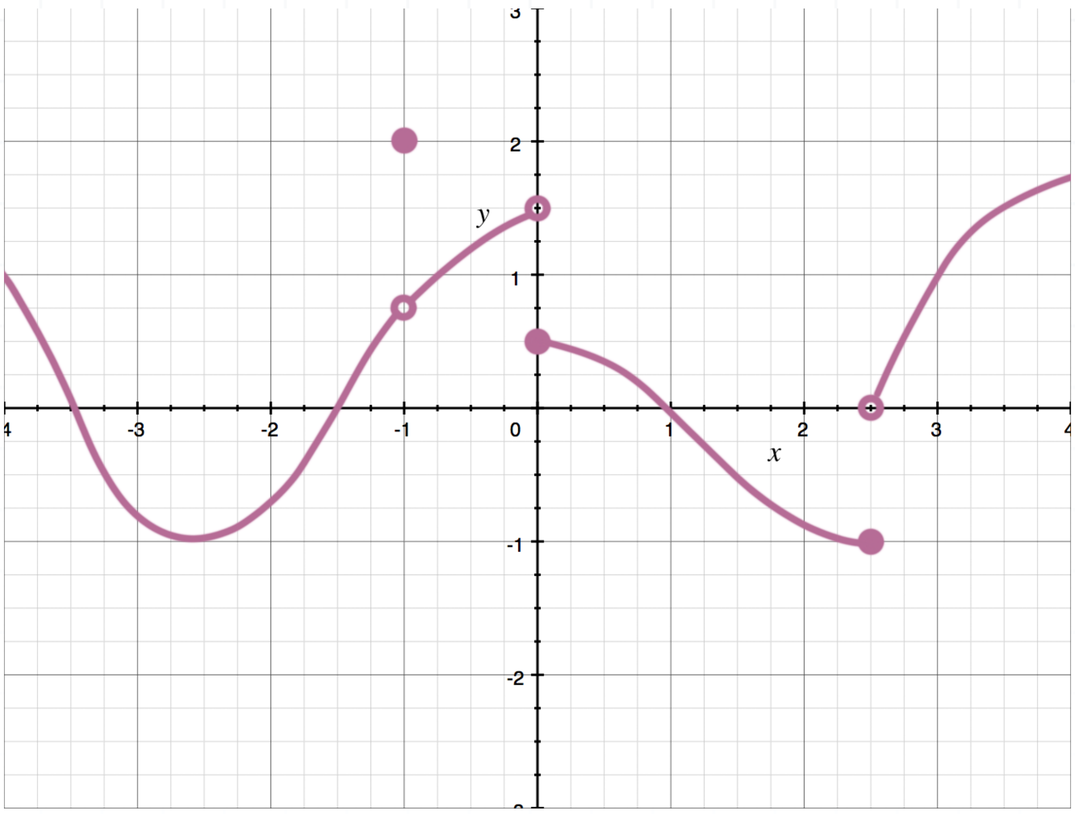
$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty$$

Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at $x = 2$.

We can also determine graphically that the limit does not exist.

Example

Use the graph to determine whether or not the limit exists at $x = 0$.



At $x = 0$, the function is approaching $3/2$ from the left side. But from the right side, the function is approaching $1/2$. So if we say that the graph represents the function $f(x)$, then the one-sided limits are

$$\lim_{x \to 0^-} f(x) = \frac{3}{2}$$

$$\lim_{x \to 0^+} f(x) = \frac{1}{2}$$

Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at $x = 0$.

Infinite one-sided limits

We also want to look at what happens to the general limit when both one-sided limits are infinite.

1. If both one-sided limits are ∞, then the general limit exists and is equal to ∞.

2. If both one-sided limits are $-\infty$, then the general limit exists and is equal to $-\infty$.

3. If one of the one-sided limits is ∞ while the other one-sided limit at the same point is $-\infty$, then the general limit doesn't exist.

Let's do an example where the limit is infinite.

Example

Find the limit.

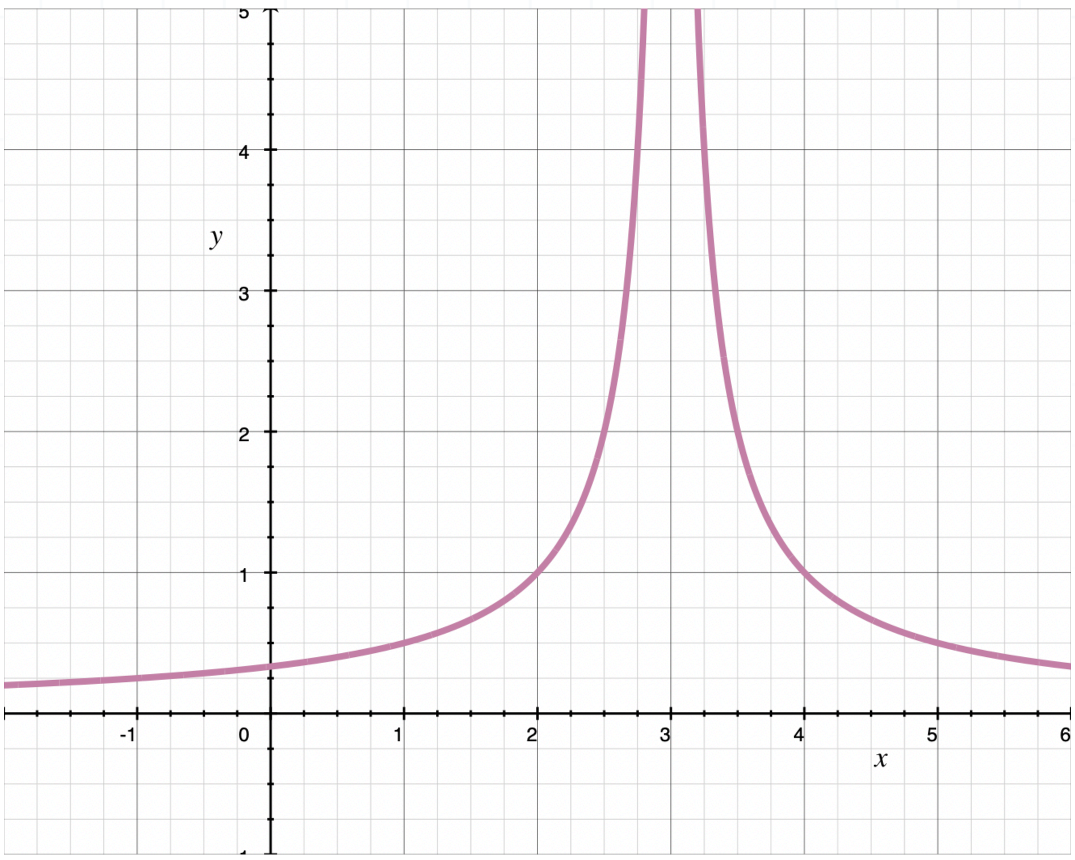
$$\lim_{x\to 3}\frac{1}{|x-3|}$$

We can get a sense of the one-sided limits if we evaluate the function at values close to $x=3$.

$$\lim_{x\to3^-}\frac{1}{|x-3|}\approx\frac{1}{|2.999999-3|}\approx\frac{1}{0.000001}\approx 1{,}000{,}000$$

$$\lim_{x\to3^+}\frac{1}{|x-3|}\approx\frac{1}{|3.000001-3|}\approx\frac{1}{0.000001}\approx 1{,}000{,}000$$

When we evaluate the function at values close to $x=3$, we see that both one-sided limits are headed toward ∞. If we use the graph to confirm this hunch,



we see that at $x=3$, the function is approaching ∞ from the left side and ∞ from the right side.

$$\lim_{x\to3^-}\frac{1}{|x-3|}=\infty$$

$$\lim_{x\to3^+}\frac{1}{|x-3|}=\infty$$

Because the one-sided limits are equal, the general limit exists and is equal to that same value.

$$\lim_{x \to 3} \frac{1}{|x-3|} = \infty$$

Precise definition of the limit

The precise definition of the limit is something we use as a proof for the existence of a limit.

The precise definition

Let's start by stating that $f(x)$ is a function on an open interval that contains $x = a$, but that the function doesn't necessarily exist at $x = a$. The **precise definition of the limit** of the function tells us that, at $x = a$, the limit is L,

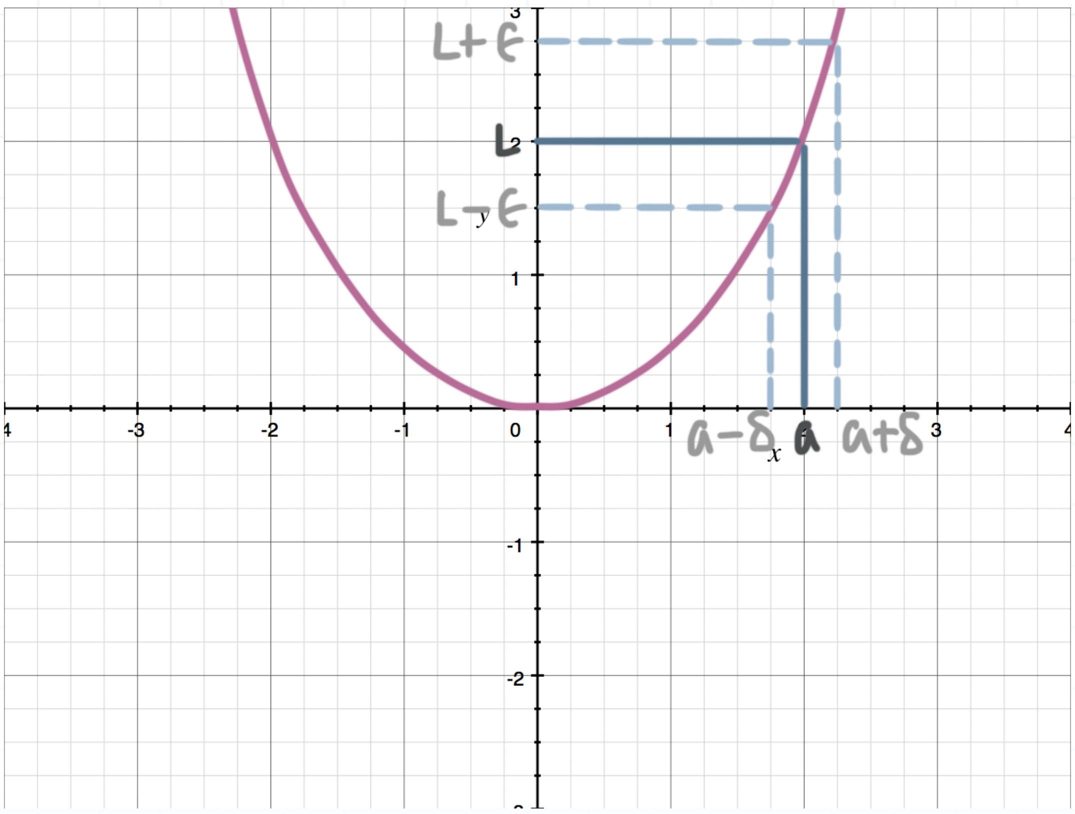
$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

What does all this mean? Well, since the open interval includes a but doesn't necessarily exist at a, we'll have to look at how the function behaves as it approaches a. L just represents the value of the limit.

When we're evaluating a limit, we're looking at the function as it approaches a specific point. In the graph,



that point is (a, L). The precise definition of the limit proves that the limit exists and is L, as long as any number we pick between $a - \delta$ and $a + \delta$ will always return a value between $L - \epsilon$ and $L + \epsilon$.

If this is true, then we know that if we pick a value that's closer and closer to a, the value we get back will be closer and closer to L. And that's the definition of the limit: as x approaches a, the value of the function gets closer to L.

Example

Using the precise definition of the limit, prove the following limit.

$$\lim_{x \to 4} (2x - 3) = 5$$

Substituting $2x - 3$ for $f(x)$, 5 for L, and 4 for a into the definition, we get

$$|(2x-3)-5| < \epsilon \text{ whenever } 0 < |x-4| < \delta$$

If we simplify $|(2x-3)-5| < \epsilon$, we get

$$|2x-8| < \epsilon$$

$$2|x-4| < \epsilon$$

$$|x-4| < \frac{\epsilon}{2}$$

Notice now that the left side of this inequality looks just like the middle part of the inequality above that contains δ. When this happens, we set δ equal to the right-hand side of the last inequality, and we get

$$\delta = \frac{\epsilon}{2}$$

$$0 < |x-4| < \delta = \frac{\epsilon}{2}$$

Going back to the beginning,

$$|(2x-3)-5| = |2x-8|$$

$$|(2x-3)-5| = 2|x-4|$$

and using the assumption that $\delta = \epsilon/2$ and that $0 < |x-4| < \delta$, by substitution, we get

$$|(2x-3)-5| < 2\left|\frac{\epsilon}{2}\right|$$

$$|(2x-3)-5| < \epsilon$$

Since we started with $0 < |x - 4| < \frac{\epsilon}{2}$ and ended with $|(2x - 3) - 5| < \epsilon$, we've shown that for all $\epsilon > 0$, if $\delta = \frac{\epsilon}{2}$ then

$$|(2x - 3) - 5| < \epsilon \text{ whenever } 0 < |x - 4| < \delta$$

Therefore,

$$\lim_{x \to 4}(2x - 3) = 5$$

Solving for delta

Sometimes we'll want to find δ, given other values in the precise definition of the limit. When this is the case, we'll follow a specific set of steps in order to find the value of δ.

Example

Find δ when $f(x) = x^2$, such that, if $|x - 2| < \delta$ then $|x^2 - 4| < 0.5$.

We want to use the value for ϵ to determine the δ value by remembering that

$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

from the precise definition of the limit.

To solve for δ, we'll take the epsilon value $\epsilon = 0.5$ and the value of L to find the two y-values. This means we have $4 + 0.5 = 4.5$ and $4 - 0.5 = 3.5$. Then we can plug these values into the function to get the associated x-values.

$$4.5 = x^2$$

$$x = 2.12$$

and

$$3.5 = x^2$$

$$x = 1.87$$

We'll find $|x - a|$ with these two x-values and $a = 2$.

For $x = 2.12$: $|x - a| = |2.12 - 2| = |0.12| = 0.12$

For $x = 1.87$: $|x - a| = |1.87 - 2| = |-0.13| = 0.13$

If the two values are different, the smaller value will be the value we need to pick for δ. Which means that for the function $f(x) = x^2$, such that if $|x - 2| < \delta$ then $|x^2 - 4| < 0.5$, we know that $\delta = 0.12$.

Limits of combinations

Up to this point, we've taken the limit of just one function at a time. But we can find the limit of combinations of different functions.

For instance, assume that $f(x) = x + 1$ and that $g(x) = x^2 - 4$. We could find the limit for the combination like

$$\lim_{x \to 1}[f(x) + g(x)]$$

Here, we're finding the limit of the sum of the functions $f(x)$ and $g(x)$. The sum $f(x) + g(x)$ is the combination, so we're finding the limit of the combination.

Properties of limits

But the sum isn't the only kind of combination we can take. If the limits of two functions at the same value both exist,

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

then we can define five kinds of combinations. These are the **properties of limits**, and they apply to any number of functions.

$$\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x) \qquad \text{for any real number } c$$

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ $\lim_{x\to a} g(x) \neq 0$

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$ for any real number n

Two ways to evaluate

There are two ways to find the limit of the combination.

1. Find the combination of the functions, then take the limit of the combination.

2. Take the limit of each function, then find the combination of the limits.

We'll get to the same result, regardless of which method we use. Let's work through the example given above, and use both methods to find the limit of the combination.

Example

Evaluate the limit, given $f(x) = x + 1$ and $g(x) = x^2 - 4$.

$$\lim_{x\to 1}[f(x) + g(x)]$$

Using the first method, we'll find the combination of the functions.

$$f(x) + g(x)$$

$$x + 1 + x^2 - 4$$

$$x^2 + x - 3$$

Now we'll take the limit of the combination.

$$\lim_{x \to 1} [f(x) + g(x)]$$

$$\lim_{x \to 1} (x^2 + x - 3)$$

$$1^2 + 1 - 3$$

$$-1$$

Let's use the second method to double-check the result. We'll take the limit of each function individually.

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x^2 - 4) = 1^2 - 4 = -3$$

Now we'll take the combination of the limits. Because we're using the combination $f(x) + g(x)$, we'll sum the limits.

$$\lim_{x \to 1} f(x) + \lim_{x \to 1} g(x)$$

$$2 + (-3)$$

$$2-3$$

$$-1$$

Using both methods, we got a result of -1.

Limits of composites

Think of a **composite function** as a “function of a function.”

For instance, assume that $f(x)=x+1$ and $g(x)=x^2-4$. If we find the composite $f(g(x))$, it means we’re plugging $g(x)=x^2-4$ into $f(x)=x+1$. That means we replace every x in $f(x)$ with x^2-4.

$$f(x)=x+1$$

$$f(g(x))=x^2-4+1$$

$$f(g(x))=x^2-3$$

Alternatively, we could find the composite $g(f(x))$, in which case, we’d be plugging $f(x)=x+1$ into $g(x)=x^2-4$. That means we replace every x in $g(x)$ with $x+1$.

$$g(x)=x^2-4$$

$$g(f(x))=(x+1)^2-4$$

$$g(f(x))=(x+1)(x+1)-4$$

$$g(f(x))=x^2+x+x+1-4$$

$$g(f(x))=x^2+2x-3$$

To find the limit of a composite function, we’ll find the composite first, and then take the limit of the composite. Let’s finish this example so that we can see how to find the limits of both composites.

Example

If $f(x) = x + 1$ and $g(x) = x^2 - 4$, find each limit.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} g(f(x))$$

First, find the composite $f(g(x))$.

$$f(x) = x + 1$$

$$f(g(x)) = x^2 - 4 + 1$$

$$f(g(x)) = x^2 - 3$$

Next, find the limit of $f(g(x))$.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} (x^2 - 3)$$

$$(-1)^2 - 3$$

$$-2$$

Now find the composite $g(f(x))$.

$$g(x) = x^2 - 4$$

$$g(f(x)) = (x+1)^2 - 4$$

$$g(f(x)) = (x+1)(x+1) - 4$$

$$g(f(x)) = x^2 + x + x + 1 - 4$$

$$g(f(x)) = x^2 + 2x - 3$$

Next, find the limit of $g(f(x))$.

$$\lim_{x \to -1} g(f(x))$$

$$\lim_{x \to -1} (x^2 + 2x - 3)$$

$$(-1)^2 + 2(-1) - 3$$

$$1 - 2 - 3$$

$$-4$$

So the limits of the composite functions are

$$\lim_{x \to -1} f(g(x)) = -2$$

$$\lim_{x \to -1} g(f(x)) = -4$$

Two ways to evaluate the limit of the composite

Formally, the theorem for the limit of a composite tells us that, if f is continuous at b, and the limit as $x \to a$ of $g(x)$ is b,

$$\lim_{x \to a} g(x) = b$$

then the limit as $x \to a$ of $f(g(x))$ will be $f(b)$

$$\lim_{x \to a} f(g(x)) = f(b)$$

Therefore, we have the following property for the limit of composite functions,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

as long as f is continuous at b. This equation shows us that we actually have two options when it comes to evaluating the limit of a composite:

1. The left side of the equation shows us taking the composite function $f(g(x))$ first, and then finding the limit as $x \to a$ of the composite.

2. The right side of the equation shows us taking the limit as $x \to a$ of the inner function first, and then plugging that resulting value into the outer function.

Let's continue with the previous example so that we can compare the process for evaluating the limit in both ways.

Example (cont'd)

If $f(x) = x + 1$ and $g(x) = x^2 - 4$, use the theorem for the limit of a composite to evaluate the limit in two ways.

$$\lim_{x \to -1} f(g(x))$$

This is the first part of the last example, and we already solved the limit one way. Previously, we found the composite $f(g(x))$

$$f(x) = x + 1$$

$$f(g(x)) = x^2 - 4 + 1$$

$$f(g(x)) = x^2 - 3$$

and then we took the limit of $f(g(x))$.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} (x^2 - 3)$$

$$(-1)^2 - 3$$

$$-2$$

Now we want to show that we can solve it a second way. This time, we'll take the limit of the inner function first,

$$\lim_{x \to -1} (x^2 - 4)$$

$$(-1)^2 - 4$$

$$1 - 4$$

$$-3$$

and then we'll evaluate the outer function at this resulting value.

$$f(x) = x + 1$$

$$f(-3) = -3 + 1$$

$$f(-3) = -2$$

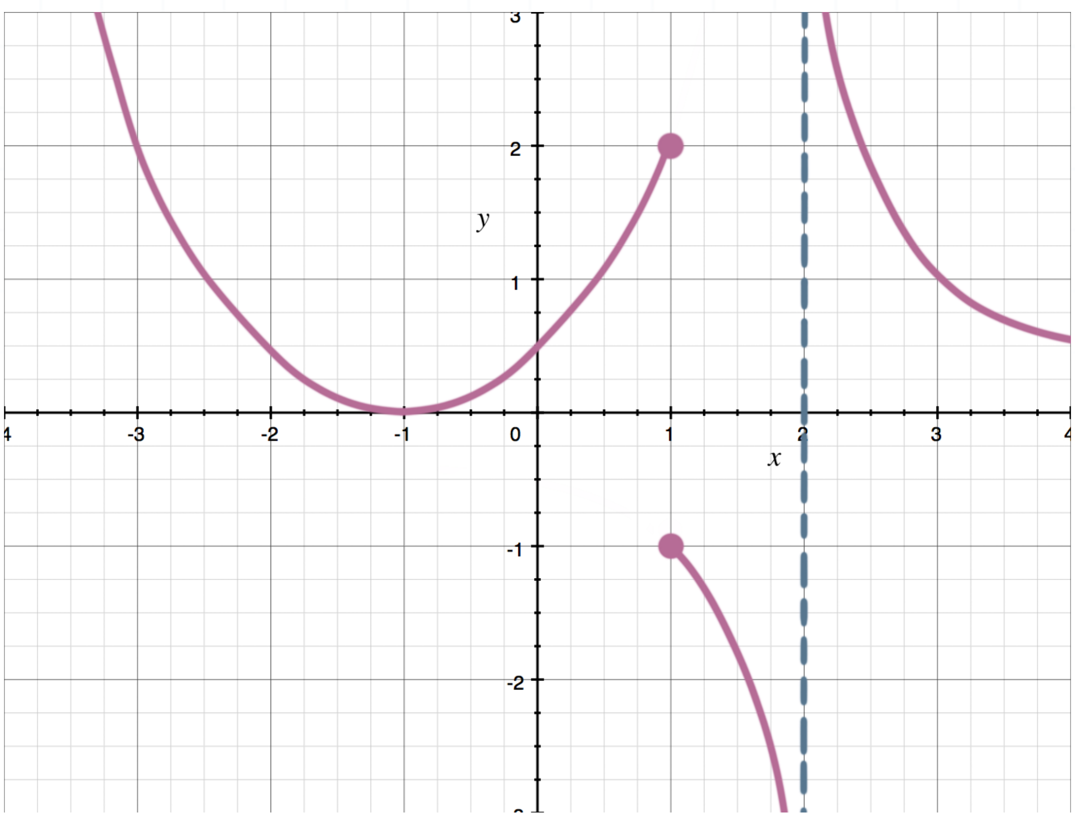
We end up with -2 using both methods, whether we find the composite first and then take the limit of the composite function, or whether we take the limit of the inner function and then evaluate the outer function at that resulting value.

Point discontinuities

We already know that the general limit doesn't exist wherever the left- and right-hand limits are not equal.

The idea of continuity is exactly what it sounds like. If a function has continuity at a particular point, it means the function is continuous at that point, meaning that there are no holes, jumps, or asymptotes in the graph there.

For instance, this graph



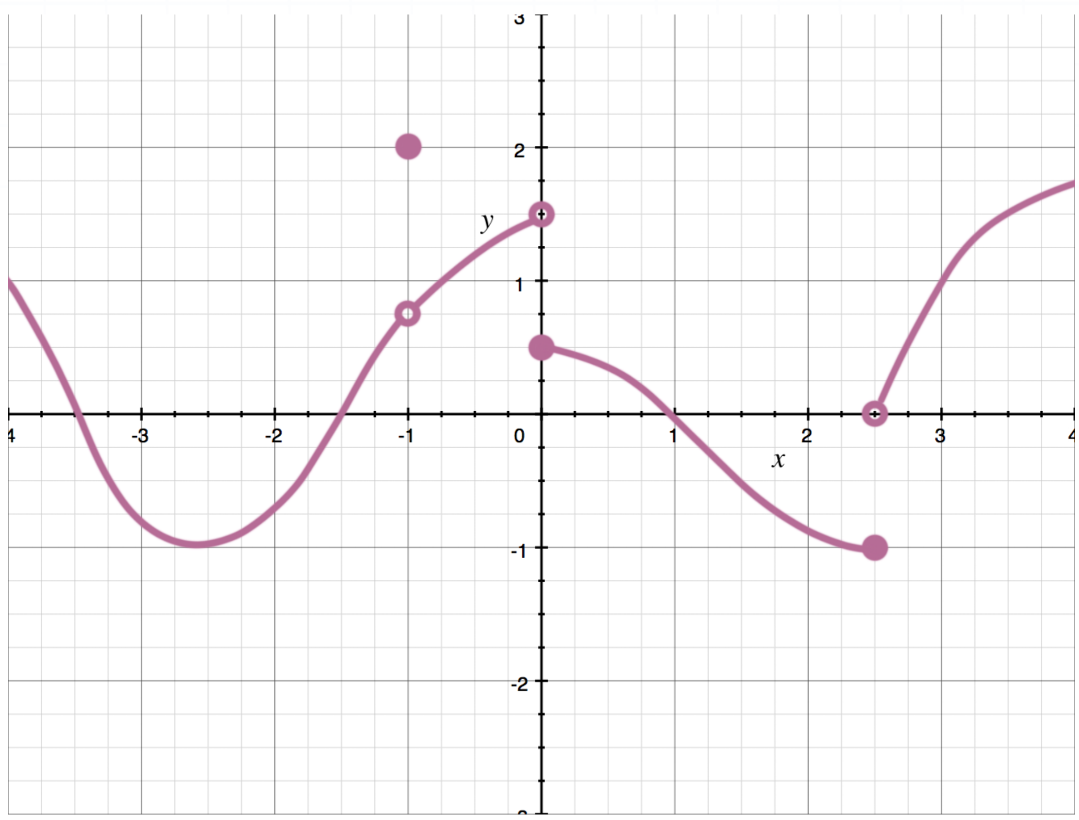
has a discontinuity at $x = 1$, because there's a jump there. The left piece of the graph has a value of 2 at $x = 1$, whereas the right piece of the graph has a value of -1 at $x = 1$. Because $2 \neq -1$, there's a jump in the graph at that point. The graph also has a discontinuity at $x = 2$, because there's a vertical asymptote there. It isn't continuous at the asymptote because the asymptote breaks the graph into two pieces.

If we can draw the graph without ever lifting our pencil off the paper as we sketch it out from left to right, then it's continuous everywhere. At any point where we have to lift our pencil off the paper in order to continue sketching it, the graph will have a discontinuity at that point.

There are different types of discontinuities, all of which mean different things for the value of the limit at the discontinuity.

Point (removable) discontinuities

A **point discontinuity** exists wherever there's a hole in the graph at one specific point. In this graph,



there's a point discontinuity at $x = -1$, which is shown by the empty hole in the graph there. When there's a point discontinuity, the function will look continuous and smooth around that point, but then have an empty hole in the graph at that exact spot.

We get this kind of discontinuity with rational functions (a rational function is a fraction in which the numerator and denominator are both polynomials) like this one:

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

It looks like this function has a vertical asymptote at $x = -4$, because $x = -4$ makes the denominator 0. But if we factor the numerator,

$$f(x) = \frac{(x + 4)(x + 7)}{x + 4}$$

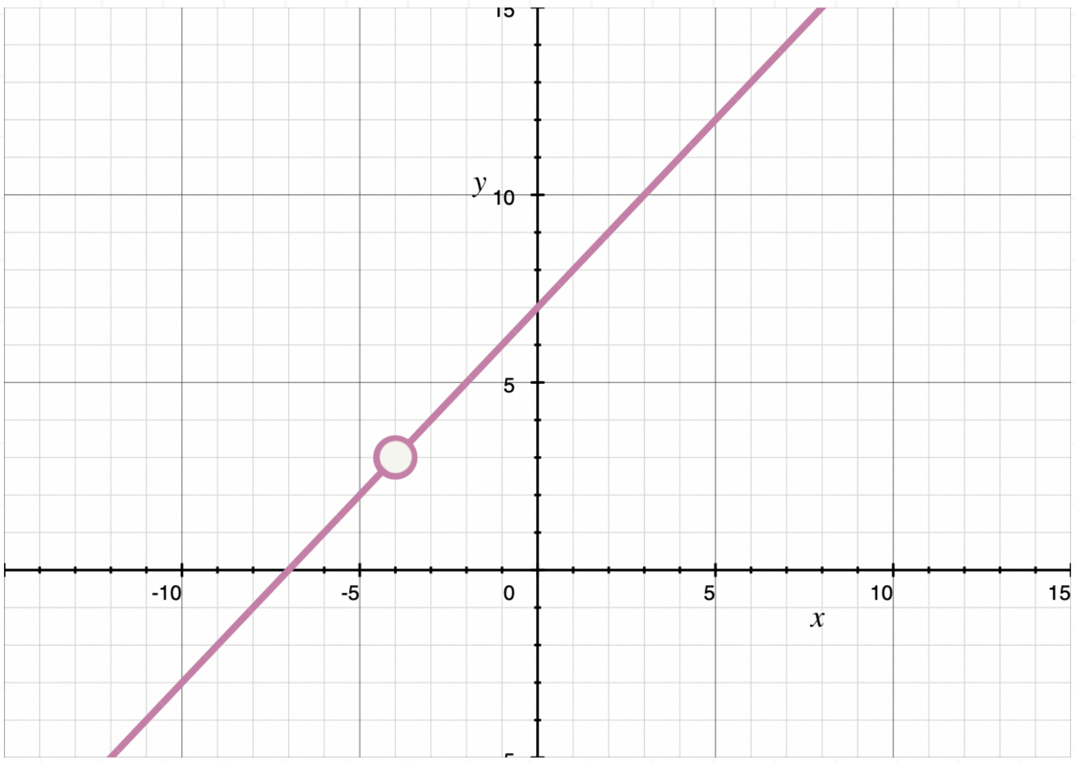
we can see that a factor of $x + 4$ will cancel from both the numerator and denominator.

$$f(x) = \frac{x + 4}{x + 4}(x + 7)$$

$$f(x) = 1(x + 7)$$

$$f(x) = x + 7$$

Because we were able to eliminate the denominator, we know the function actually does <u>not</u> have a vertical asymptote at $x = -4$. Instead, the original function $f(x)$ follows the curve $f(x) = x + 7$, but it has a removable discontinuity at $x = -4$, like this:



Notice how, if we changed the function to

$$f(x) = \begin{cases} \dfrac{x^2 + 11x + 28}{x + 4} & x \neq -4 \\ 3 & x = -4 \end{cases}$$

we would remove the hole in the graph at $(-4,3)$. Functions written this way are called "piecewise functions" or "piecewise-defined functions," because they're defined in pieces.

The first "piece" of this piecewise function defines the function everywhere except at $x = -4$. The second "piece" steps in to plug the hole and define the function as having a value of 3 when $x = -4$ specifically.

That's why point discontinuities are also called **removable discontinuities**: because we can "remove" the discontinuity just by redefining the function as a piecewise function.

Let's look at an example where we find the point discontinuities of a function.

Example

Find any point discontinuities in the graph of the function.

$$f(x) = \frac{x-2}{x^2+x-6}$$

If we factor the denominator of the function,

$$f(x) = \frac{x-2}{(x-2)(x+3)}$$

it looks as if there are discontinuities in the function at $x = 2$ and $x = -3$, because those values both make the denominator equal to 0. But we realize that we can cancel a factor of $x - 2$, leaving

$$f(x) = \frac{1}{x+3}$$

Therefore, we can say there's a point (removable) discontinuity at $x = 2$. As a side note, because $x = -3$ still makes the denominator equal to 0, even after we've simplified the function, the function has a vertical asymptote at $x = -3$.

Let's work through another example of how to remove a removable discontinuity by redefining the function as a piecewise function.

Example

Redefine the function as a piecewise function in order to remove the discontinuity.

$$f(x) = \frac{x^2 - 16}{x - 4}$$

We can see that the function is discontinuous at $x = 4$, because $x = 4$ makes the denominator equal to 0. But we can factor and then simplify the function.

$$f(x) = \frac{(x-4)(x+4)}{x-4}$$

$$f(x) = x + 4$$

Evaluate $f(x)$ at $x = 4$.

$$f(4) = 4 + 4$$

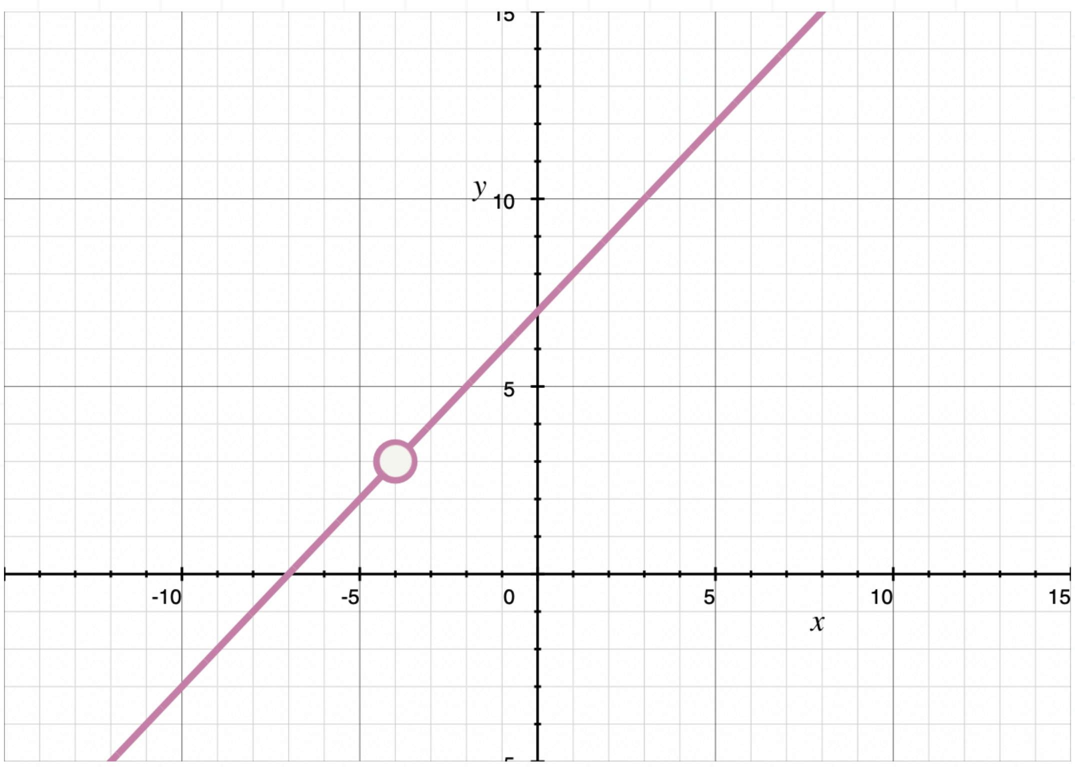
$$f(4) = 8$$

Therefore, we can make the function continuous if we redefine it as

$$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & x \neq 4 \\ 8 & x = 4 \end{cases}$$

Does the limit exist at a point discontinuity?

Keep in mind that the general limit always exists at a point/removable discontinuity. That's because the left-hand limit and right-hand limit both exist, and those one-sided limits are equal. For instance, in the graph



the left-hand limit as $x \to -4$ is 3, and the right-hand limit as $x \to -4$ is 3. Because both one-sided limits are 3, the general limit exists and is equal to 3. Even though the function is discontinuous at $x = -4$, the graph is approaching a value of 3 from both sides of $x = -4$ as we get really close to $x = -4$, so the general limit there is 3.

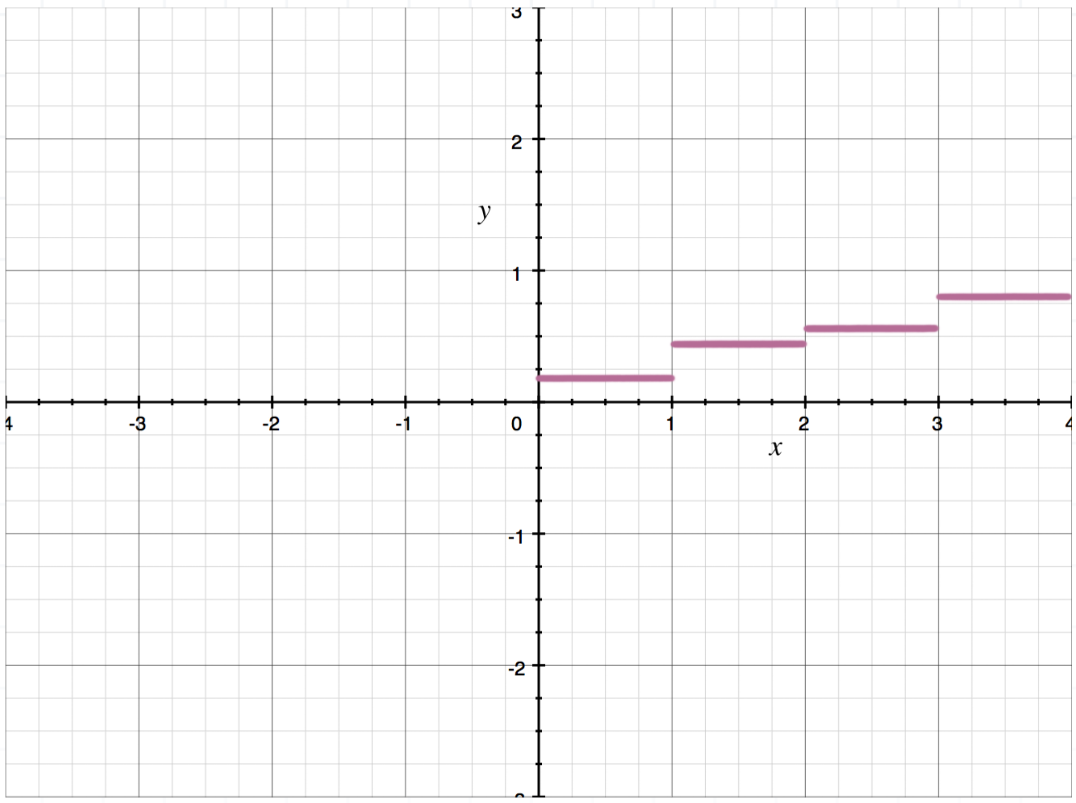
Jump discontinuities

We'll usually encounter **jump discontinuities** with piecewise-defined functions, which are functions for which different parts of the domain are defined by different expressions.

For instance, we might define the cost of postage as a function. If the cost per ounce of any package lighter than 1 pound is 20 cents per ounce, the cost of every ounce from 1 pound to anything less than 2 pounds is 40 cents per ounce, etc., then the piecewise function that defines the cost of postage might be

$$f(x) = \begin{cases} 0.2 & 0 < x < 1 \\ 0.4 & 1 \le x < 2 \\ 0.6 & 2 \le x < 3 \\ 0.8 & 3 \le x < 4 \\ 1.0 & 4 \le x \end{cases}$$

The graph of this piecewise function would be

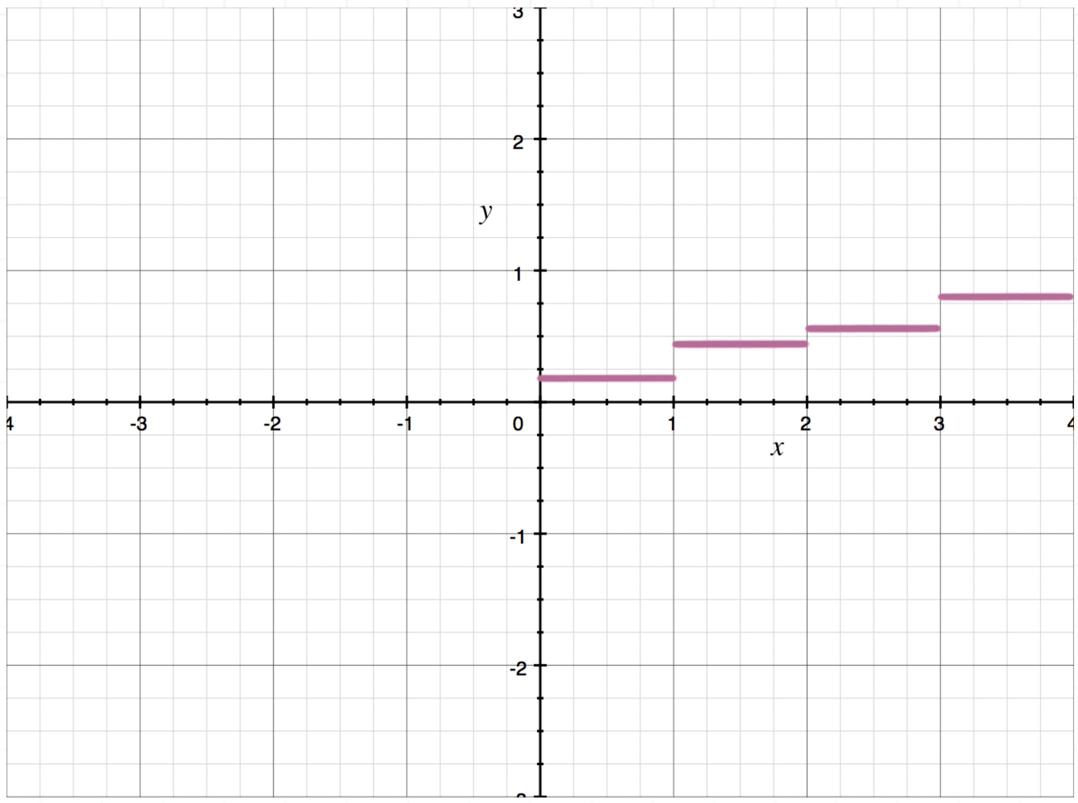


Every break in this graph is a jump discontinuity. We can remember that these are jump discontinuities if we imagine walking along on top of the first segment of the graph. In order to continue, we'd have to jump up to the second segment, then to the third, and so on.

The general limit never exists at a jump discontinuity because, while the left- and right-hand limits both exist, they're not equal to one another.

Does the limit exist at a jump discontinuity?

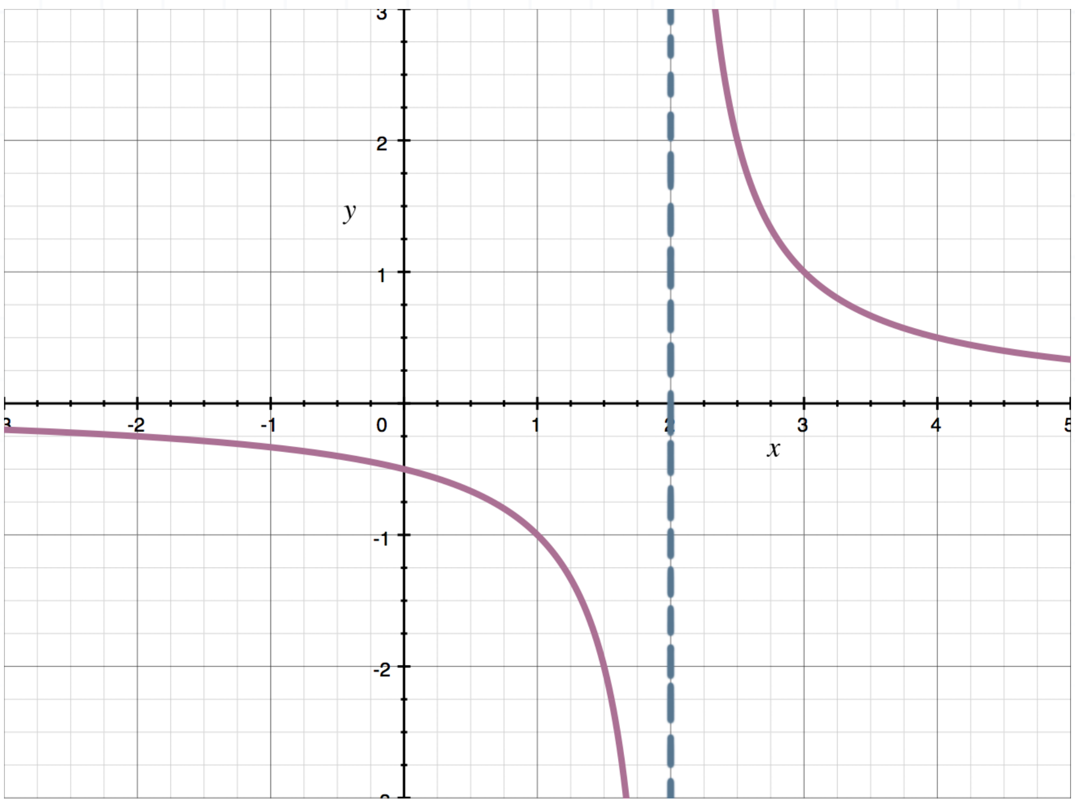
Keep in mind that the general limit never exists at a jump discontinuity. That's because the left-hand limit and right-hand limit both exist, but those one-sided limits aren't equal. For instance, in the graph



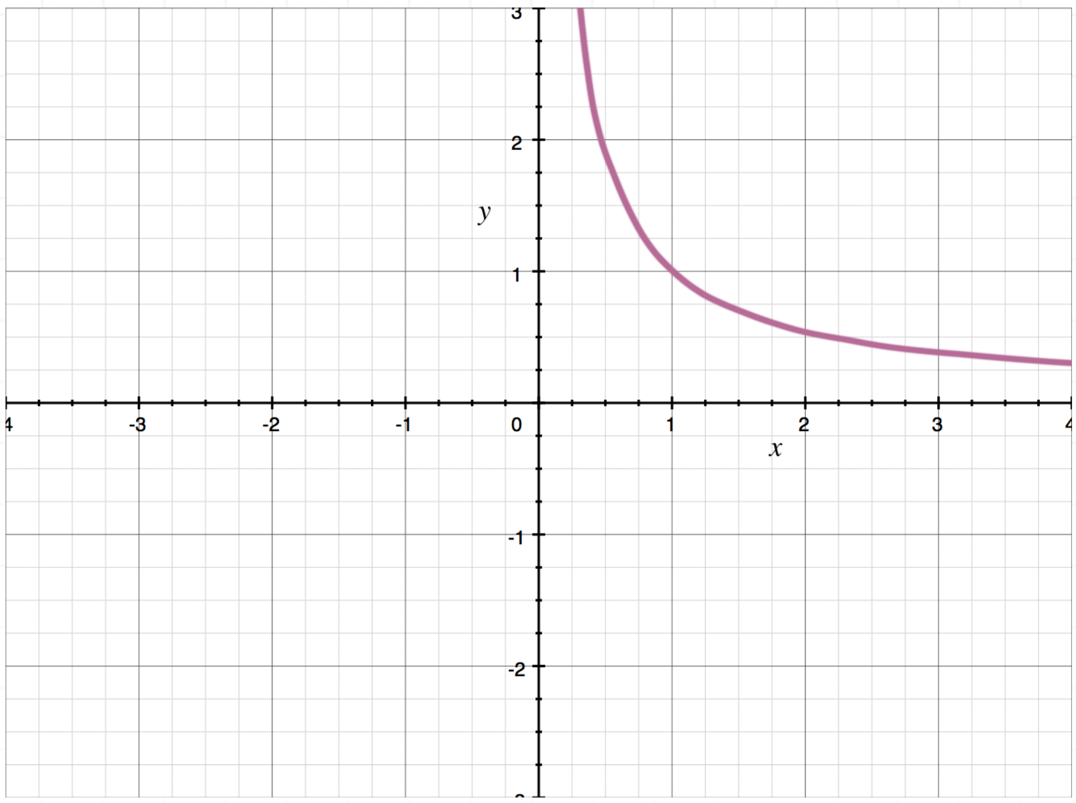
the left-hand limit as $x \to 1$ is 0.2, and the right-hand limit as $x \to 1$ is 0.4. Both one-sided limits exist, but they aren't equal, which means the general limit doesn't exist.

Infinite discontinuities

An **infinite discontinuity**, or essential discontinuity, is the kind of discontinuity that occurs at an asymptote. When the graph exists on both sides of a vertical asymptote, the graph has an infinite discontinuity at the asymptote.



The vertical asymptote in the graph below at $x=0$ is not a discontinuity, because the graph doesn't exist on both sides of the asymptote, which means the asymptote doesn't break up any part of the graph.



We often find both removable (point) and nonremovable (infinite) discontinuities within rational functions. Going back to the example we used previously for point discontinuities,

$$f(x) = \frac{x-2}{x^2+x-6}$$

we factored the denominator to get

$$f(x) = \frac{x-2}{(x-2)(x+3)}$$

In this form, we can see that the denominator is 0 at both $x = 2$ and $x = -3$. Because the factor of $x - 2$ can be canceled,

$$f(x) = \frac{1}{x+3}$$

there's a point (removable) discontinuity at $x = 2$. Since the factor of $x + 3$ can't be canceled, and therefore $x = -3$ will always make the denominator 0, there's a vertical asymptote and an infinite discontinuity at $x = -3$.

Does the limit exist at an infinite discontinuity?

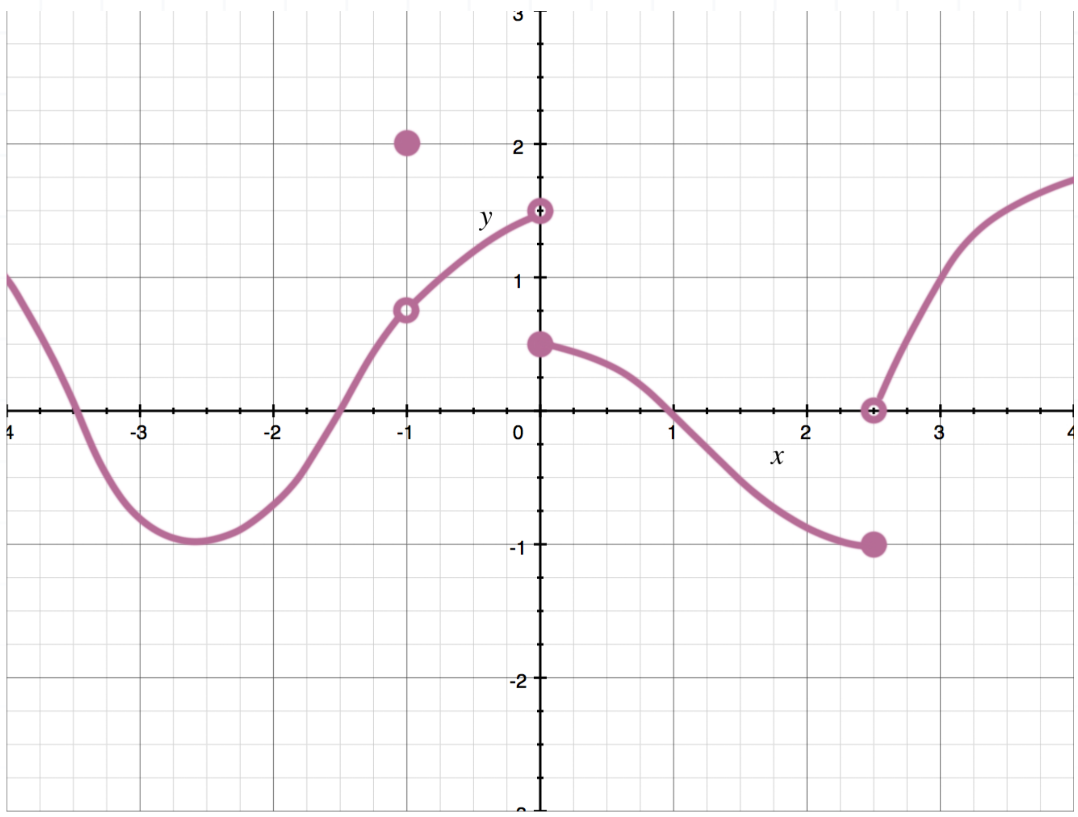
The general limit may or may not exist at an infinite discontinuity. If the function approaches $-\infty$ on one side and ∞ on the other side, then the general limit doesn't exist, because the left- and right-hand limits aren't equal.

The story gets a little more complicated when the left- and right-hand limits both approach $-\infty$ or both approach ∞. Technically, the function must approach a finite value for the limit to be defined, and neither $-\infty$ nor ∞ are finite values, so the general limit would never exist at a vertical asymptote.

But oftentimes, for the sake of providing more information about how the function behaves, we'll say that the value of the general limit is ∞ when both the left- and right-hand limits are ∞, or that the general limit is $-\infty$ when both the left- and right-hand limits are $-\infty$.

Endpoint discontinuities

Sometimes we'll define a function on a particular interval, such that the function does not extend beyond that defined interval. For instance, let's say we've been given this graph,



and told that the graph does not extend beyond the boundaries of what's shown. In other words, we're saying that the graph ends at $x = -4$ on the left, and isn't defined anywhere left of that value, and also that the graph ends at $x = 4$ on the right, and isn't defined anywhere right of that value.

In this case, we'd say that the graph has **endpoints** at $x = -4$ and $x = 4$. Regular functions don't typically have endpoints; endpoints are something that we usually artificially impose upon the function.

The limit at an endpoint

Either way, when we have defined endpoints for a function, the general limit will not exist at those endpoints. To see why, think about the endpoint $x=-4$ from the graph above. The right-hand limit at that point is 1, but because the graph doesn't extend to the left of $x=-4$, the left-hand limit does not exist there.

$$\lim_{x\to -4^-} f(x) = \text{DNE}$$

$$\lim_{x\to -4^+} f(x) = 1$$

Because the left-hand limit does not exist, that means the general limit does not exist. In the same way, think about the endpoint $x=4$ from the same graph. The left-hand limit at that point is about $7/4$, but because the graph doesn't extend to the right of $x=4$, the right-hand limit does not exist there.

$$\lim_{x\to 4^-} f(x) = \frac{7}{4}$$

$$\lim_{x\to 4^+} f(x) = \text{DNE}$$

Again, the takeaway here is that the general limit does not exist at an endpoint. The left-hand limit might exist, or the right-hand limit might exist, but the general limit won't exist because one of those one-sided limits won't exist.

Intermediate Value Theorem with an interval

The Intermediate Value Theorem is a theorem we use to prove that a function has a root inside a particular interval.

The **root** of a function, graphically, is a point where the graph of the function crosses the x-axis. Algebraically, the root of a function is the point where the function's value is equal to 0.

Formally, the **Intermediate Value Theorem** states:

> Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and let y_0 be a real number lying between $f(a)$ and $f(b)$. If $f(a) < y_0 < f(b)$ or $f(b) < y_0 < f(a)$, then there will be at least one c on the interval (a, b) where $y_0 = f(c)$.

The formal theorem is technical, but it's actually saying something simple, which is that, assuming $f(x)$ is continuous on $[a, b]$, if we can show that the function's value is negative at one side of the interval and positive at the other side, then that fact alone proves that the graph of the function must cross the x-axis at some point inside the interval.

That being said, in order to use the Intermediate Value Theorem, we need to 1) have a closed interval $[a, b]$ (which means that a and b are included as part of the interval), and 2) show that the function is continuous everywhere in the interval.

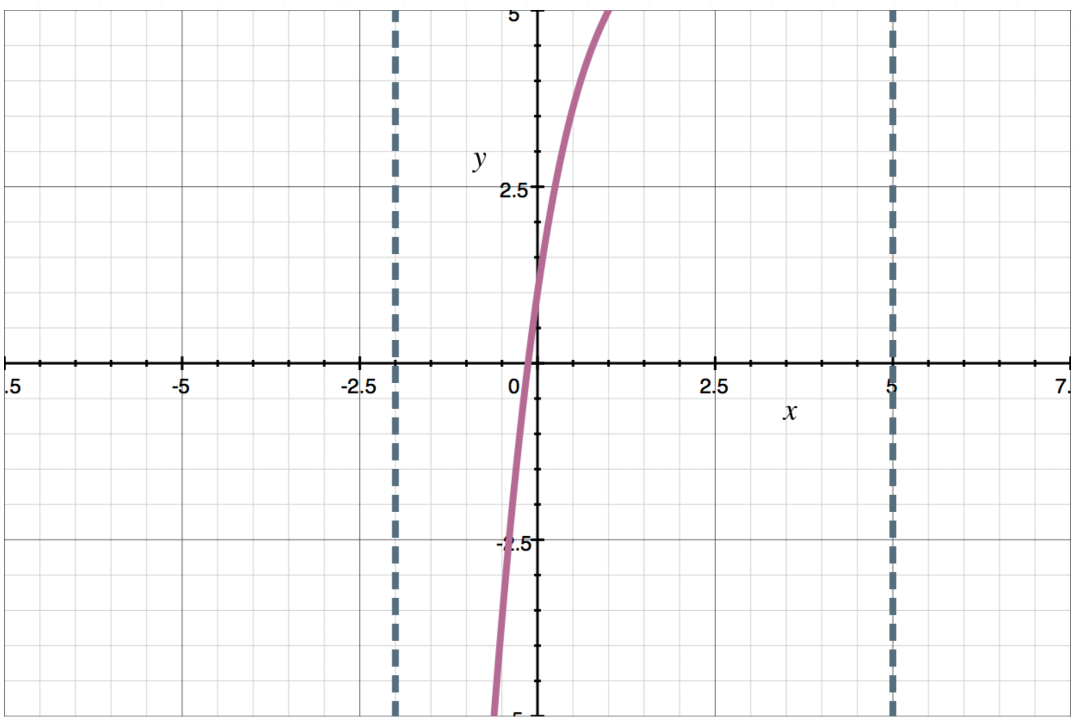
Example

Show that the function has at least one root in the interval $[-2,5]$.

$$f(x) = x^3 - 4x^2 + 7x + 1$$

First, we should confirm that this function is continuous over the given interval.

Continuity breaks typically occur in functions that have variables in the denominator of fractions, trigonometric functions, logarithmic functions, or functions with square roots. This function has none of these elements, so we know that it's continuous. If we want to double-check ourselves, we can graph the function.



Next, we substitute the endpoints into the function. We get

$$f(-2) = (-2)^3 - 4(-2)^2 + 7(-2) + 1$$

$$f(-2) = -8 - 16 - 14 + 1$$

$$f(-2) = -37$$

and

$$f(5) = (5)^3 - 4(5)^2 + 7(5) + 1$$

$$f(5) = 125 - 100 + 35 + 1$$

$$f(5) = 61$$

Since we know that $f(x) = x^3 - 4x^2 + 7x + 1$ is continuous, and since the function's value at the left edge of the interval is negative (below the x -axis), and the value of the function at the right side of the interval is positive (above the x-axis), we know that there's at least one point c at which the function will cross the x-axis.

$$-37 < f(c) < 61$$

Therefore, the function has at least one solution (root) in the interval $[-2,5]$.

Intermediate Value Theorem without an interval

When we're asked to use the Intermediate Value Theorem to prove that the function has a root, but we're not given the interval in which to find the root, then we're forced to come up with our own interval.

Because the Intermediate Value Theorem can only be used on closed intervals $[a, b]$, we'll need to find a closed interval to investigate.

There are different ways we can go about finding the interval, the first of which is to blindly guess. We can try picking two random values to be the ends of the interval, and we might get lucky and find that a negative value at one end and a positive value at the other.

If that doesn't work, we might be able to consider some other aspect of the function. For instance, if we're dealing with the trigonometric function $\sin x$ or $\cos x$, we know that those functions oscillate back and forth between -1 and 1. So we could choose a value where the trig function is -1, and another value where it's 1, and use those as the endpoints of the interval, since the rest of the graph will simply be an identical section to that one.

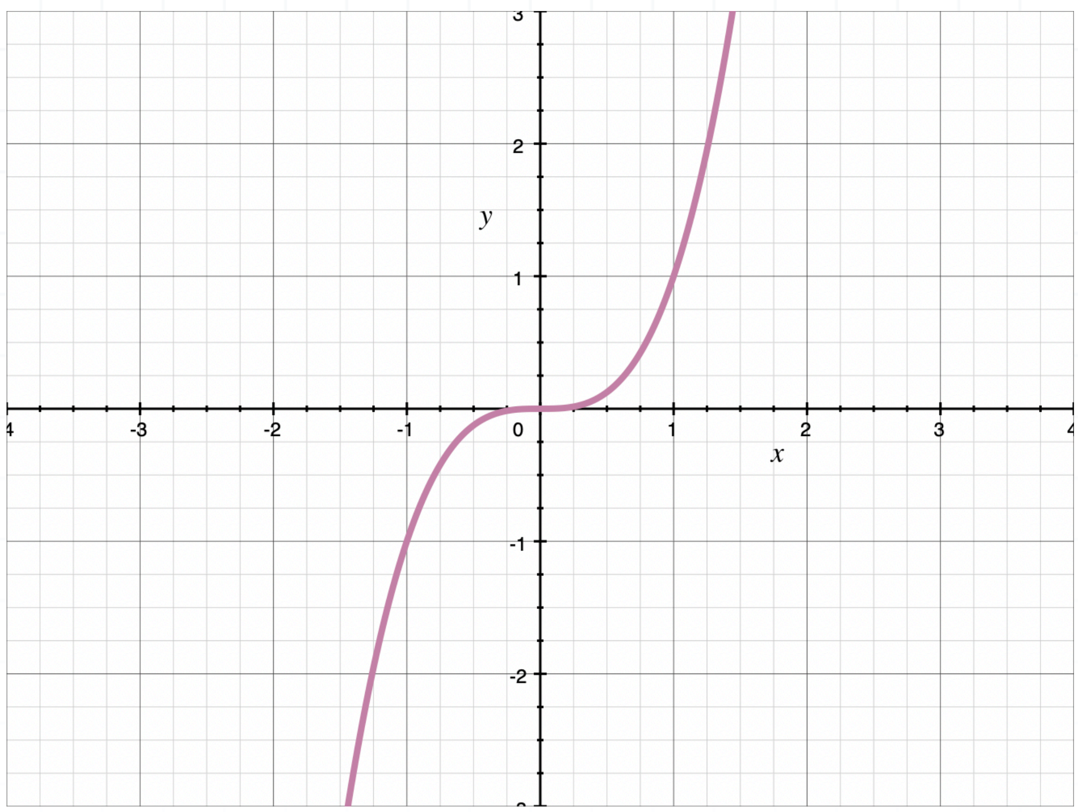
Example

Use the Intermediate Value Theorem to prove that the function has at least one real root.

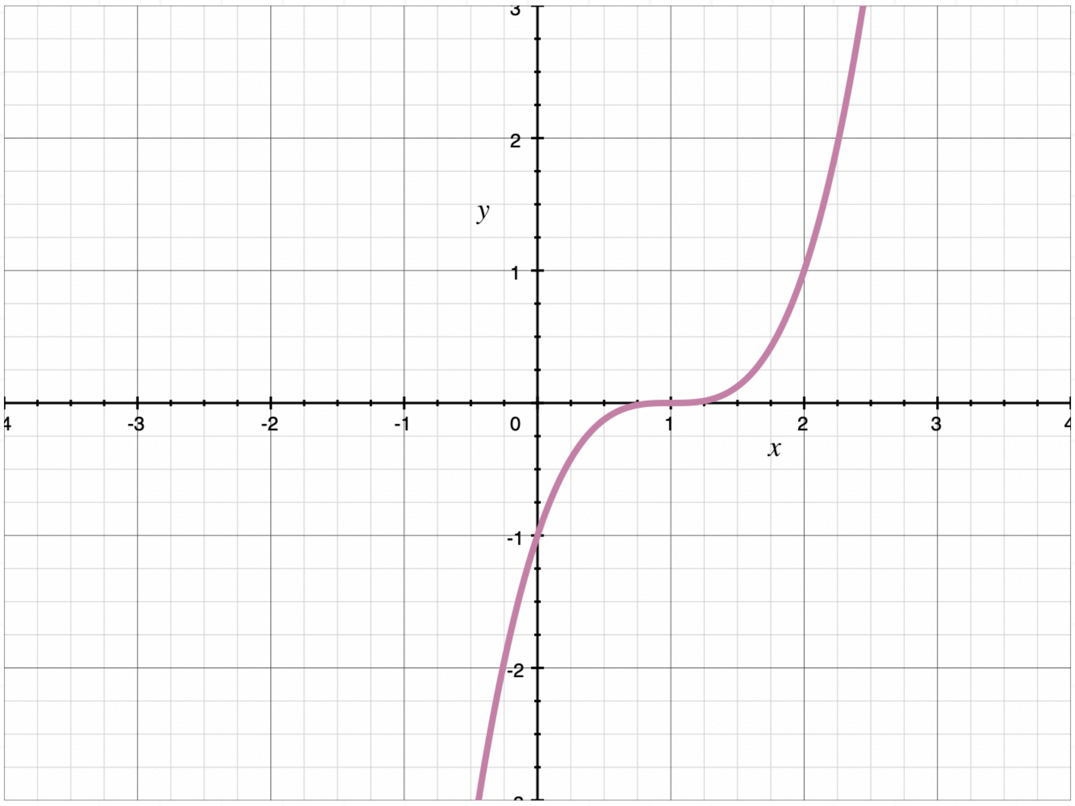
$$f(x) = (x-1)^3$$

This is a fairly simple function, so we can actually sketch it, and then pick an interval around the root.

If we think first about the function $f(x) = x^3$, we can sketch it.



When we transform $f(x) = x^3$ into $f(x) = (x-1)^3$, we're simply replacing x with $x-1$, which means the graph shifts 1 unit to the right, and the graph of $f(x) = (x-1)^3$ is



From the graph, we expect the root of the function to be at $x = 1$, so let's choose the interval $[0,2]$, and test it. For the endpoints, we get

$$f(0) = (0 - 1)^3$$

$$f(0) = (-1)^3$$

$$f(0) = -1$$

and

$$f(2) = (2 - 1)^3$$

$$f(2) = (1)^3$$

$$f(2) = 1$$

Because we get a negative value at the left edge of the interval, and a positive value at the right edge of the interval, we've proven with the

Intermediate Value Theorem that the function $f(x) = (x-1)^3$ has a root in the interval $[0,2]$.

Solving with substitution

As we've seen in previous lessons, the simplest way to evaluate a limit is to substitute the value we're approaching into the function.

For instance, given the function $f(x) = x + 1$, finding the limit as $x \to 5$ is as easy as substituting $x = 5$ into $f(x)$.

$$\lim_{x \to 5}(x + 1)$$

$$5 + 1$$

$$6$$

If $f(x)$ is an expression that contains only polynomials, roots, absolute values, exponentials, logarithms, trig or inverse trig functions, then we may be able to evaluate using substitution, and we'll have

$$\lim_{x \to a} f(x) = f(a)$$

But if the function is undefined at $x = a$, or if $x = a$ is the transition point between two pieces of a piecewise-defined function, then we can't apply the substitution rule.

Nevertheless, when we evaluate a limit we should always try substitution first before any other technique, because it's the easiest and fastest method. If substitution doesn't work, then we can try evaluating the limit by a different method.

Let's look at another example where we use substitution to evaluate the limit.

Example

Evaluate the limit.

$$\lim_{x \to -2} (x^2 + 2x + 6)$$

Since we're approaching $x = -2$, we'll substitute $x = -2$ into the function.

$$(-2)^2 + 2(-2) + 6$$

$$4 - 4 + 6$$

$$6$$

So the limit of the function as $x \to -2$ is 6.

Solving with factoring

When we can't use substitution to evaluate a limit, because plugging in the value we're approaching gives an undefined value, factoring is the next approach we should try.

We usually use factoring when we're finding the limit of a rational function. Our goal will be to factor both the numerator and denominator as completely as possible, and then cancel any common factors.

Our hope is that, once we've canceled common factors, we'll be left with a function that can then be evaluated using substitution.

Example

Evaluate the limit.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

Using substitution to evaluate the limit means we'll plug $x = 4$ into the function.

$$\frac{4^2 - 16}{4 - 4}$$

$$\frac{16 - 16}{4 - 4}$$

$$\frac{0}{0}$$

When we use substitution, we get an undefined result, so we should try factoring, instead. There's no factoring to be done in the denominator, but the numerator can be factored as the difference of squares.

$$\lim_{x \to 4} \frac{(x-4)(x+4)}{x-4}$$

Once the numerator is factored, we can see that there's a common factor of $(x-4)$ in both the numerator and denominator. We can cancel it.

$$\lim_{x \to 4} \frac{x-4}{x-4}(x+4)$$

$$\lim_{x \to 4} 1(x+4)$$

$$\lim_{x \to 4} x+4$$

Now that the function has been factored and simplified, we'll try substitution with the simplified function. Substituting $x=4$ into the function gives

$$4+4$$

$$8$$

Therefore, the limit of the function as $x \to 4$ is 8.

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = 8$$

Solving with conjugate method

If substitution and factoring don't work or don't apply, the next technique we should consider to evaluate a limit is the conjugate method.

Most often, we'll use conjugate method when our function is a fraction, and either the numerator and/or denominator contains exactly two terms. For instance, given the limit problem

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$$

we notice right away that substitution leads to a 0 denominator, and there's no factoring to be done in either the numerator or denominator. However, our function is a fraction, and the numerator contains exactly two terms, $\sqrt{4+h}$ and -2, so conjugate method might be a good technique for evaluating this limit.

Because the numerator contains two terms, we want to find the conjugate of the numerator. The **conjugate** of an expression is an expression with the same two terms, but with the opposite sign between the terms. For instance, the conjugate of $\sqrt{4+h}-2$ is $\sqrt{4+h}+2$. It's the same two terms, but the sign in between the terms has been flipped from $-$ to $+$. As another example, the conjugate of $-3+\sqrt{x}$ would be $-3-\sqrt{x}$.

Once we've found the appropriate conjugate, we multiply both the numerator and denominator by the conjugate we've found. The reason we do this is because it should simplify the function and, hopefully, allow us to evaluate the limit with substitution.

Example

Evaluate the limit.

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$$

As we've said, substitution and factoring both lead to dead-ends when we try to use them to evaluate this limit, so we'll try conjugate method, instead.

The conjugate of $\sqrt{4+h}-2$ is $\sqrt{4+h}+2$, so we'll multiply both the numerator and denominator by that conjugate.

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$$

It's okay to multiply by the conjugate like this, because, since we're multiplying both the numerator and denominator by the same value, it's as if we're multiplying by 1, which doesn't change the value of the function. We're just rewriting the function in a fancy way, but we're not actually changing its value.

$$\lim_{h\to 0}\frac{\sqrt{4+h}\sqrt{4+h}+2\sqrt{4+h}-2\sqrt{4+h}-4}{h(\sqrt{4+h}+2)}$$

$$\lim_{h\to 0}\frac{4+h+2\sqrt{4+h}-2\sqrt{4+h}-4}{h(\sqrt{4+h}+2)}$$

The two middle terms in the numerator cancel each other. This is why the conjugate method is helpful in simplifying certain functions.

$$\lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$\lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)}$$

The common factor of h can be canceled from both the numerator and denominator.

$$\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}$$

Now that we've simplified the function using conjugate method, we'll try substitution to evaluate the limit as $h \to 0$.

$$\frac{1}{\sqrt{4+0}+2}$$

$$\frac{1}{2+2}$$

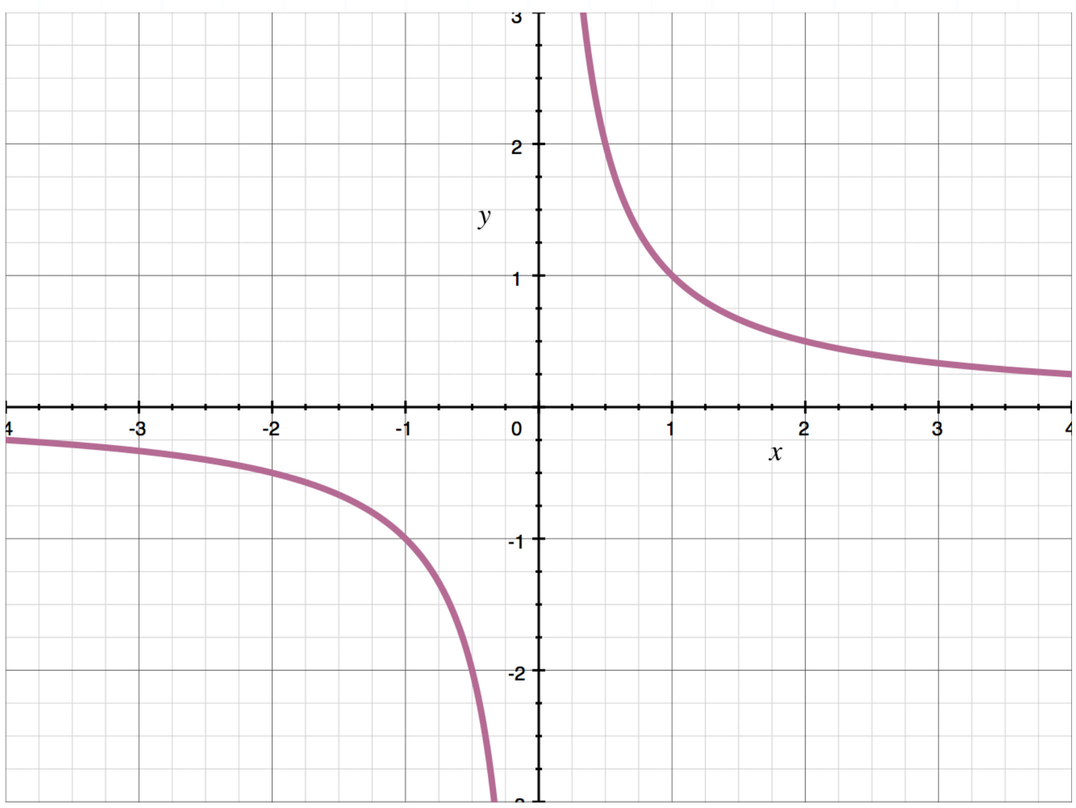
$$\frac{1}{4}$$

Infinite limits and vertical asymptotes

There's a difference between "limits at infinity" and "infinite limits." When we see *limits at infinity*, it means we're talking about the limit of the function as we approach ∞ or $-\infty$. Contrast that with *infinite limits*, which means that the value of the limit is ∞ or $-\infty$ as we approach a particular point.

Limits at infinity, infinite limits

In the graph of $f(x) = 1/x$,



the function has infinite, one-sided limits at $x = 0$. There's a vertical asymptote there, and we can see that the function approaches $-\infty$ from the left, and ∞ from the right.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Talking about limits at infinity for this function, we can see that the function approaches 0 as we approach either ∞ or $-\infty$.

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

How to find infinite limits

Infinite limits exist around vertical asymptotes in the function. Of course, we get a vertical asymptote whenever the denominator of a rational function in lowest terms is equal to 0.

Here's an example of a rational function in lowest terms, meaning that we can't factor and cancel anything from the fraction.

$$\lim_{x \to 1} \frac{1}{(x-1)^2}$$

We can see that setting $x = 1$ gives 0 in the denominator, which means that we have a vertical asymptote at $x = 1$. Therefore, we know we'll have infinite limits on either side of $x = 1$.

Once we've established that this is a rational function in lowest terms and that a vertical asymptote exists, all that's left to determine is whether the one-sided limits around $x = 1$ approach ∞ or $-\infty$.

In order to do that, we can substitute values very close to $x = 1$. If the result is positive, the limit will be ∞; if the result is negative, the limit will be $-\infty$.

$$f(0.99) = \frac{1}{(0.99 - 1)^2} = \frac{1}{(-0.01)^2} = \frac{1}{0.0001} = 10{,}000 = \infty$$

$$f(1.01) = \frac{1}{(1.01 - 1)^2} = \frac{1}{(0.01)^2} = \frac{1}{0.0001} = 10{,}000 = \infty$$

Because the value of the function tends toward ∞ on both sides of the vertical asymptote, we can say that the general limit of the function as $x \to 1$ is ∞.

$$\lim_{x \to 1} \frac{1}{(x - 1)^2} = \infty$$

Limits at infinity and horizontal asymptotes

We've seen how to find the infinite limits around a vertical asymptote, but now we want to focus on finding the limits of a function at infinity. More specifically, we're interested in the limit of the function as $x \to -\infty$ and as $x \to \infty$.

If the function approaches a finite value as $x \to -\infty$ or $x \to \infty$, it means the function has a horizontal asymptote at that value. Technically, a limit at infinity is a limit of the form

$$\lim_{x \to \pm\infty} f(x) = L$$

which means the function has a horizontal asymptote $y = L$, as long as L is a finite number. For instance, if the limit of the function as $x \to -\infty$ and as $x \to \infty$ is 0, then the function has a horizontal asymptote at $y = 0$.

Let's look at an example where we work through how to find the limits of a function at $\pm\infty$, and use those limits to draw a conclusion about any horizontal asymptote(s) of the function.

Example

Find the limits of the function as $x \to -\infty$ and $x \to \infty$, then say whether the function has a horizontal asymptote.

$$f(x) = \frac{1}{x - 3}$$

We can see right away that the vertical asymptotes exists at $x = 3$, since that's the value that makes the denominator 0.

To find any horizontal asymptotes, we'll look at the limits as $x \to -\infty$ and $x \to \infty$. Even though, technically, we can't plug $-\infty$ or ∞ into a function, we can imagine how the function behaves when we plug in infinitely large positive values or infinitely large negative values.

$$\lim_{x \to \infty} \frac{1}{x-3} = \frac{1}{\infty - 3} = \frac{1}{\infty} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x-3} = \frac{1}{-\infty - 3} = \frac{1}{-\infty} = 0$$

In both cases, as $x \to -\infty$ and $x \to \infty$, we have fractions with extremely large denominators, and numerators that are, in comparison, incredibly small.

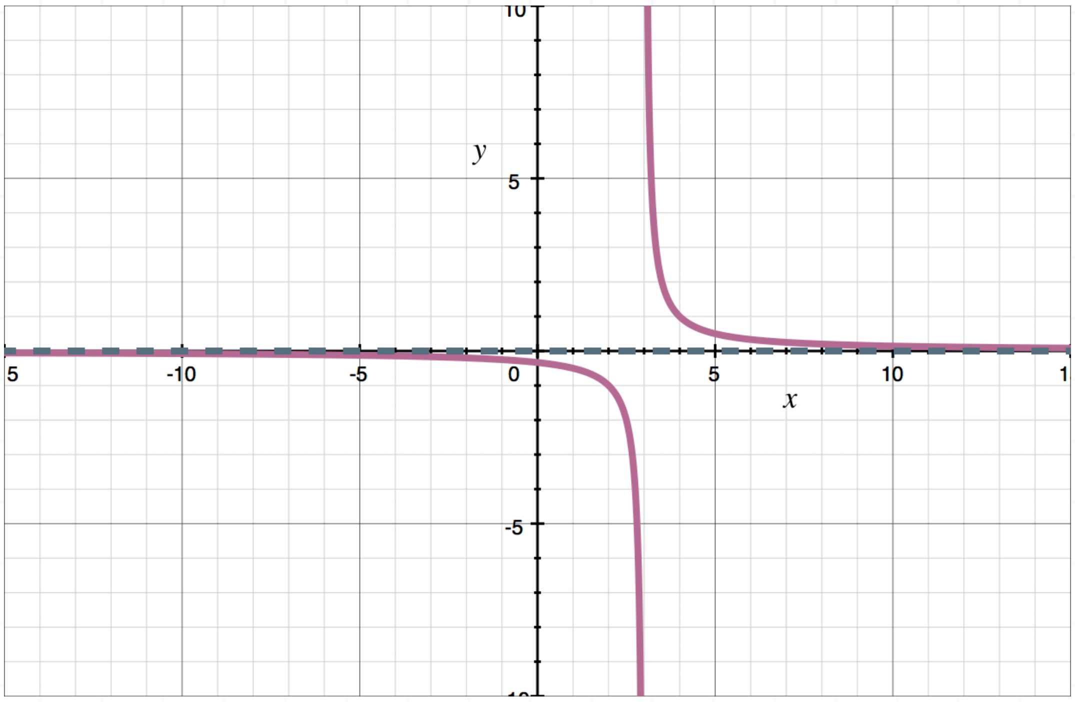
It's critical to understand here that any fraction with a constant value in the numerator and an infinitely large value in the denominator (whether that infinitely large value is positive or negative), will tend toward 0.

In other words, these results tell us that the limit of the function is 0 as $x \to -\infty$, and as $x \to \infty$.

$$\lim_{x \to \infty} \frac{1}{x-3} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x-3} = 0$$

Therefore, the function has a horizontal asymptote at $y = 0$. If we sketch the graph of the function, we can confirm our findings.



Rules for horizontal asymptotes

Given a rational function (a fraction in which the numerator and denominator are polynomials), we can determine the equation of the horizontal asymptote simply by comparing the numerator to the denominator.

We need to start by identifying the degree of the numerator and denominator. The **degree** is the exponent on the term with the largest exponent. For instance, $x^3 - 2x^2 + 6x + 1$ is a third-degree polynomial, because x^3 is the largest-degree term.

The largest-degree term isn't necessarily the first term. If the same polynomial is written as $6x - 2x^2 + x^3 + 1$, it's still a third-degree polynomial, because x^3 is the largest-degree term.

Once we've identified the degree of the numerator (N) and denominator (D), then we compare them.

N < D: If the degree of the numerator is less than the degree of the denominator, then the horizontal asymptote is given by $y = 0$.

$$\lim_{x \to \pm\infty} f(x) = 0$$

N > D: If the degree of the numerator is greater than the degree of the denominator, then the function doesn't have a horizontal asymptote.

$$\lim_{x \to \pm\infty} f(x) \text{ does not exist}$$

N = D: If the degree of the numerator is equal to the degree of the denominator, then the horizontal asymptote is given by the ratio of the coefficients on the highest-degree terms.

$$\lim_{x \to \pm\infty} f(x) = \text{N/D}$$

Let's do an example where we see these rules in action.

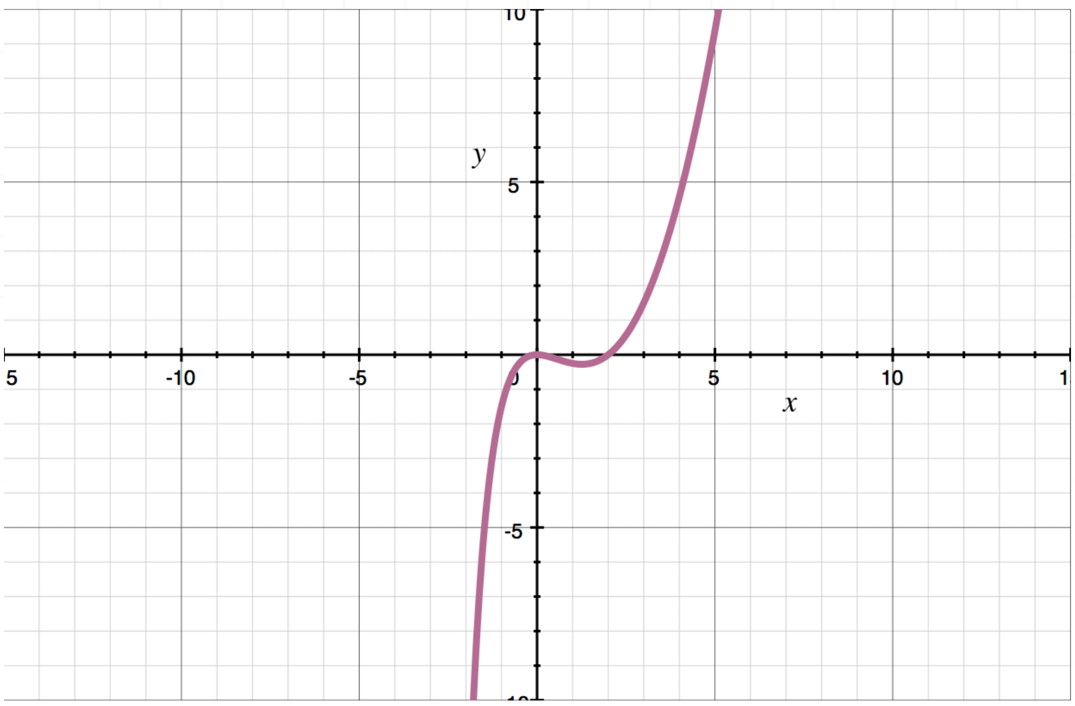
Example

Find any horizontal asymptote that exists for each function.

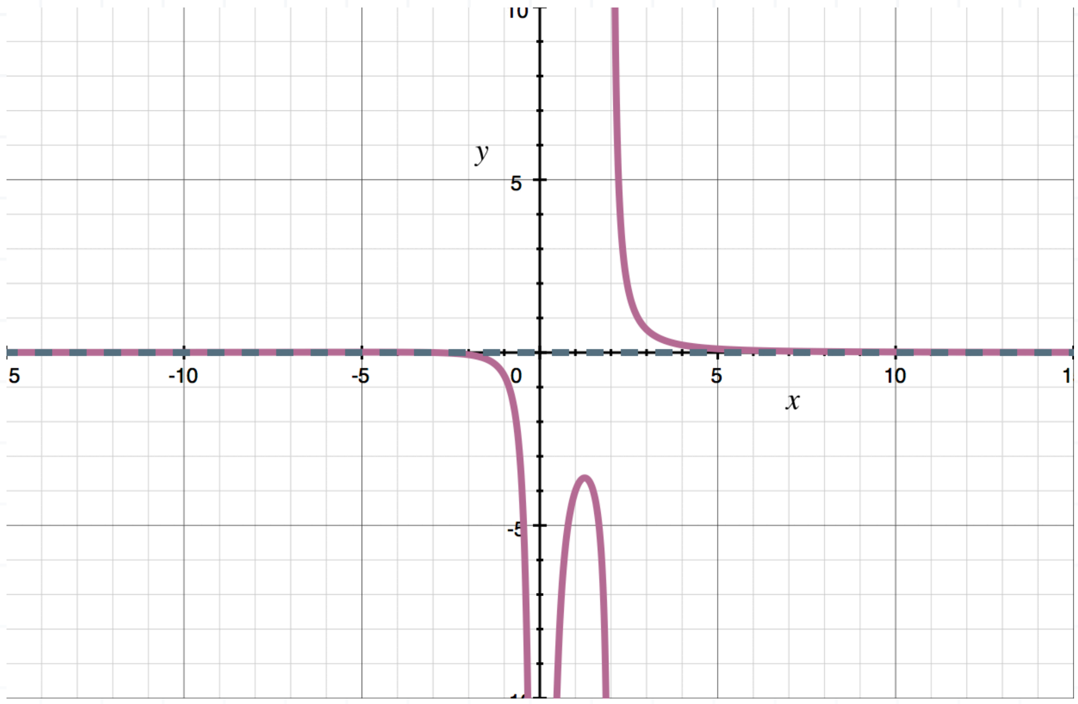
$$f(x) = \frac{x^3 - 2x^2}{x + 3} \qquad g(x) = \frac{x + 3}{x^3 - 2x^2} \qquad h(x) = \frac{x^3 + 3}{x^3 - 2x^2}$$

To determine any horizontal asymptotes that the functions may have, we need to compare the degree of the numerator to the degree of the denominator.

For the function $f(x)$, the degree of the numerator is 3, and the degree of the denominator is 1, so N > D, which means the function doesn't have a horizontal asymptote. The graph of $f(x)$ confirms this.



For the function $g(x)$, the degree of the numerator is 1, and the degree of the denominator is 3, so N < D, which means the function has a horizontal asymptote at $y = 0$. The graph of $g(x)$ confirms this.

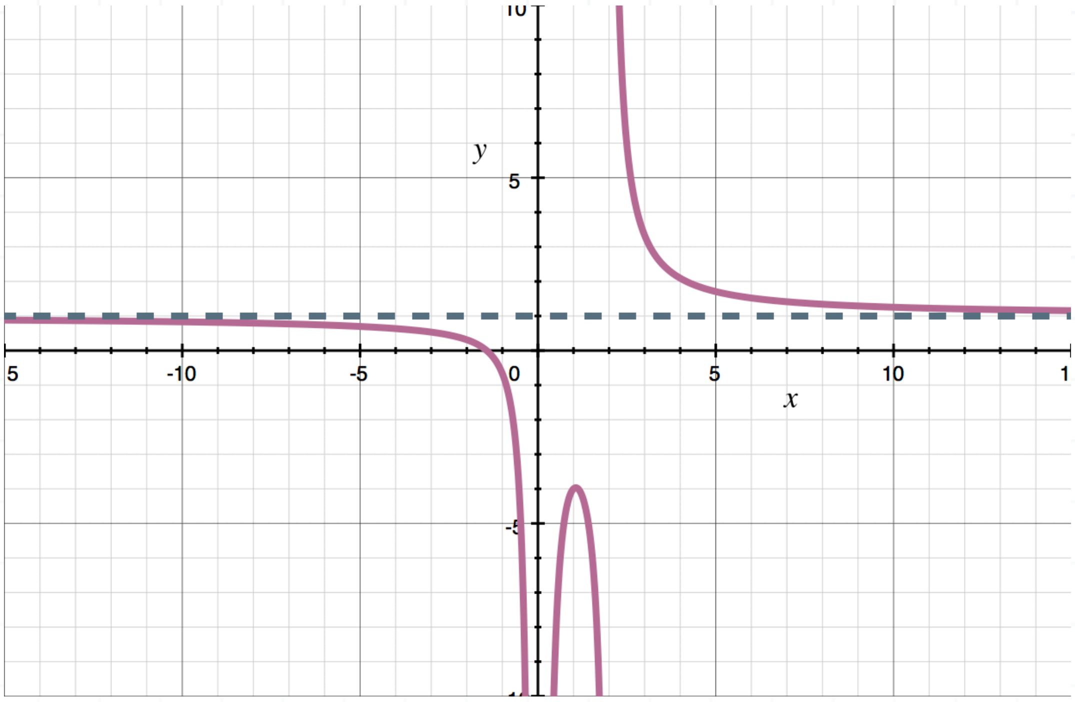


For the function $h(x)$, the degree of the numerator is 3, and the degree of the denominator is 3, so N = D, which means the equation of the horizontal asymptote is given by the ratio of the coefficients on the highest-degree terms. In $h(x)$, the highest-degree term in the numerator is x^3, and its coefficient is 1; the highest-degree term in the denominator is x^3, and its coefficient is 1. So the horizontal asymptote is

$$y = \frac{1}{1}$$

$$y = 1$$

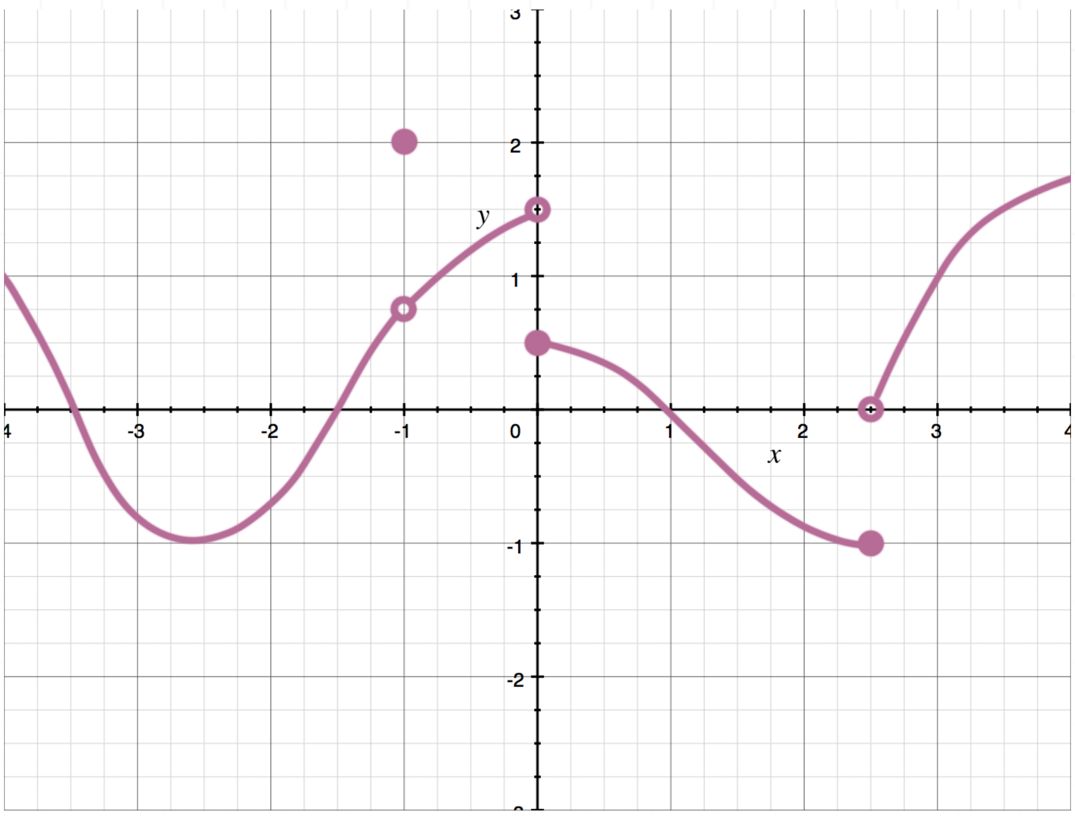
The graph of $h(x)$ confirms this.



y
5
10
-5
-10
0
5
10
x

Crazy graphs

"Crazy graphs" is a fabricated topic of math textbooks and professors. They're graphs like this one:



This kind of graph would almost never exist in real life. So, almost certainly, this graph is totally made up.

The only reason we create these kinds of "weird" or "crazy" graphs, is to have a graph that illustrates lots of different limit laws, all in one place. For instance, for the graph above, there are lots of things we can say:

1. There's a point discontinuity at $x = -1$.

2. There are jump discontinuities at $x = 0$ and at $x = 5/2$.

3. The limits at $x = -1$ are

$$\lim_{x \to -1^-} f(x) = 3/4$$

$$\lim_{x \to -1^+} f(x) = 3/4$$

$$\lim_{x \to -1} f(x) = 3/4$$

4. The limits at $x = 0$ are

$$\lim_{x \to -0^-} f(x) = 3/2$$

$$\lim_{x \to -0^+} f(x) = 1/2$$

$$\lim_{x \to 0} f(x) = \text{DNE}$$

5. The limits at $x = 5/2$ are

$$\lim_{x \to 5/2^-} f(x) = -1$$

$$\lim_{x \to 5/2^+} f(x) = 0$$

$$\lim_{x \to 5/2} f(x) = \text{DNE}$$

In other words, graphs like these, even though we're creating a totally unrealistic function, give us lots of easy practice with discontinuity and limit problems, and that's why we use them.

Trigonometric limits

Limit problems with trigonometric functions usually revolve around three key limit values.

$$\lim_{x \to 0} \sin x = 0$$

$$\lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

When we evaluate limits of trig functions, our goal is therefore to always try to reduce the function to some combination of these three limits, and maybe some other simple constants.

For these kinds of problems, in order to rework trig functions, we'll often use reciprocal identities,

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

or Pythagorean identities.

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Let's look at an example of how we go about reworking trig functions into the three limit values above.

Example

Evaluate the limit.

$$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

If we try substitution to evaluate the limit, we get

$$\frac{1 - \cos(0)}{0}$$

$$\frac{1 - 1}{0}$$

$$\frac{0}{0}$$

Substitution doesn't work, and there's nothing to factor, but since we have exactly two terms in the numerator, we can actually use the conjugate method for the first step of this problem.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \left(\frac{1 + \cos x}{1 + \cos x} \right)$$

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

Applying the Pythagorean identity $1 - \cos^2 x = \sin^2 x$ to the numerator gives

$$\lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

Notice now that we can factor out $(\sin x)/x$, which is one of our three key limits.

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \right)$$

$$\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1 + \cos x}$$

Since the first limit is one of the three key limits, we can replace it with its value.

$$1 \cdot \lim_{x \to 0} \frac{\sin x}{1 + \cos x}$$

$$\lim_{x \to 0} \frac{\sin x}{1 + \cos x}$$

And now we can evaluate the limit as $x \to 0$ using just simple substitution.

$$\frac{\sin 0}{1 + \cos 0}$$

$$\frac{0}{1 + 1}$$

$$\frac{0}{2}$$

$$0$$

So the limit is 0.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Making the function continuous

In this lesson, we build on our understanding of discontinuities and one-sided limits to explore how we can make a function continuous. Specifically, we will learn how to adjust functions to remove discontinuities and ensure a smooth, connected graph.

Point discontinuities

Previously, we learned that a point discontinuity was a single pinpoint of discontinuity in the graph. We saw that we could take a function like

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

and simplify it as

$$f(x) = \frac{(x + 4)(x + 7)}{x + 4}$$

$$f(x) = x + 7$$

Because we canceled the $x + 4$, we know the function has a point discontinuity at $x = -4$. We can "plug the hole" by redefining the function for $x = -4$, but doing so completely changes the function. So this defines a brand-new function, g, which is continuous at $x = -4$.

$$g(x) = \begin{cases} \frac{x^2 + 11x + 28}{x + 4} & x \neq -4 \\ 3 & x = -4 \end{cases}$$

Piecewise-defined functions

Remember that this kind of function, the function g we just built, is a piecewise function or piecewise-defined function, because it's defined "in pieces."

Sometimes we'll be given a piecewise-defined function and asked to find the value of an unknown constant that will make the function continuous. For instance, consider the function

$$f(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ 5 - x & 3 < x \le 5 \end{cases}$$

It's a piecewise-defined function where the first piece defines the function from $x = 0$ to $x = 3$ (including at $x = 3$), and the second piece defines the function for values greater than $x = 3$, all the way up to $x = 5$.

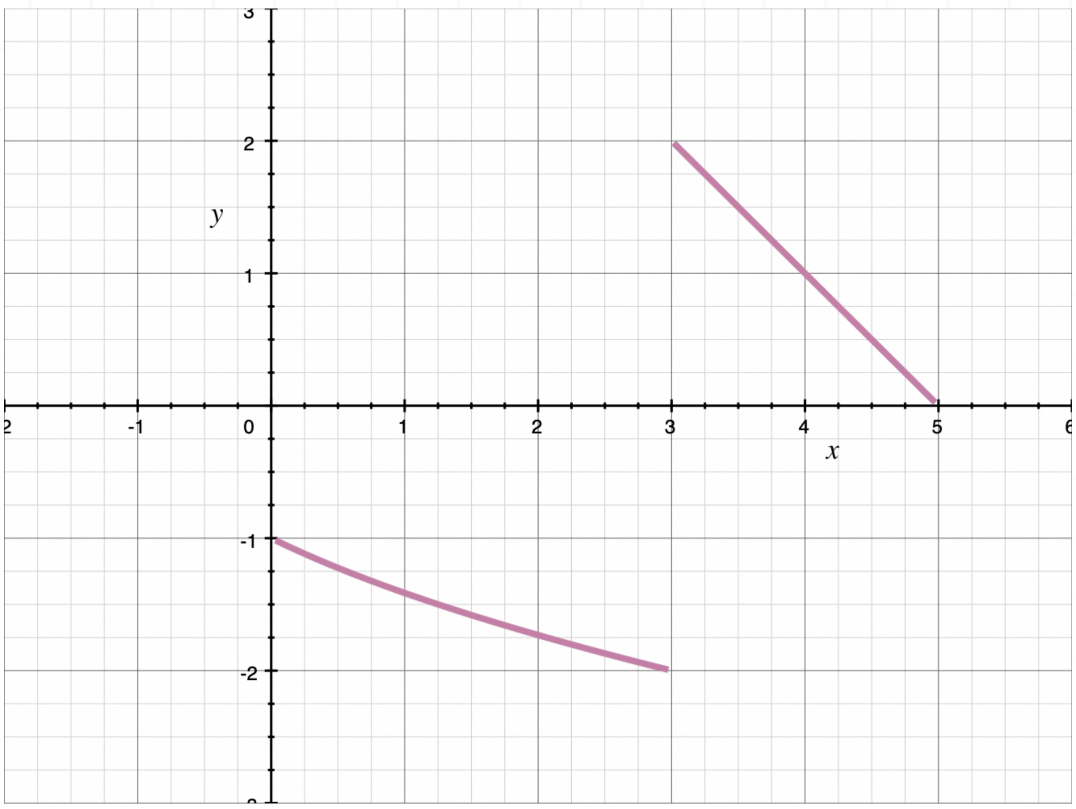
When $x = 3$, the first piece stops defining the function and the second piece takes over, so we can think of $x = 3$ as the "break point" between the pieces. If we can make the two pieces of the function meet each other at the break point, and if the left- and right-hand limits of the function are equal at the break point, then the function will be continuous there. Remember that a function $f(x)$ is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

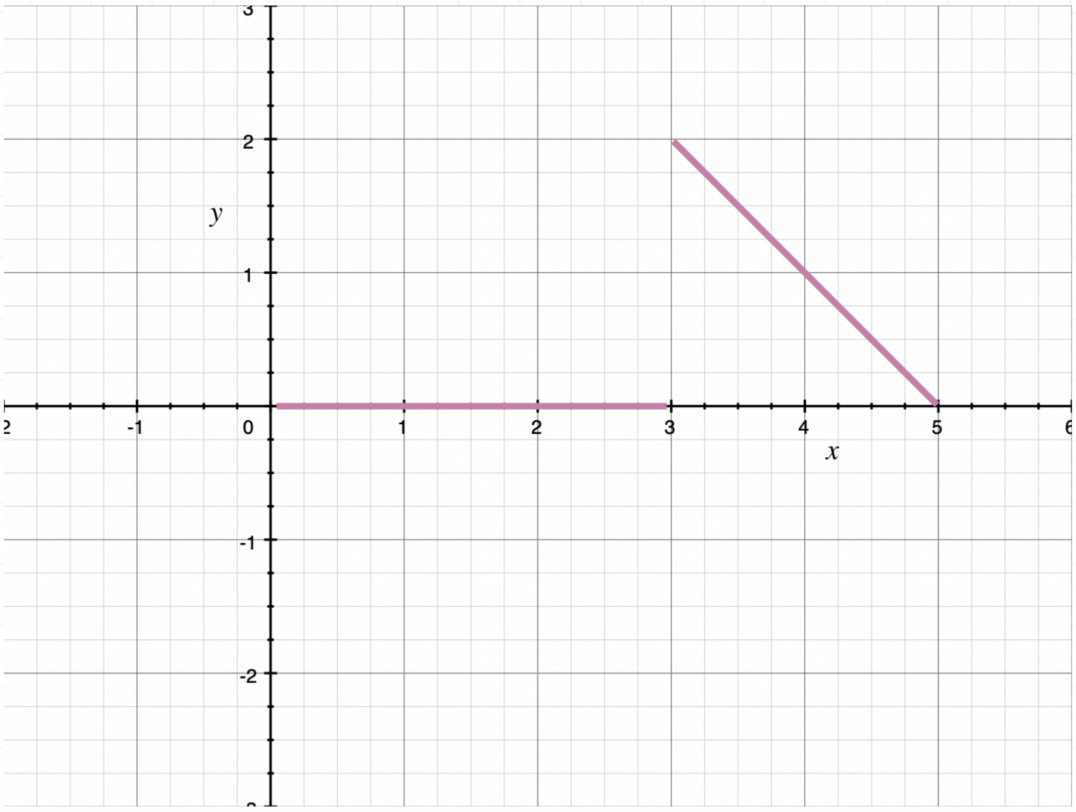
So for a problem like this one, we need to find the value of k that forces the two pieces of $f(x)$ to have the same value at $x = 3$.

To visualize this, here's what the graph of $f(x)$ looks like with some different values of k. Notice how changing the value of k changes the shape of the $k\sqrt{x+1}$ piece, and also changes the value of the left piece of f at $x=3$.

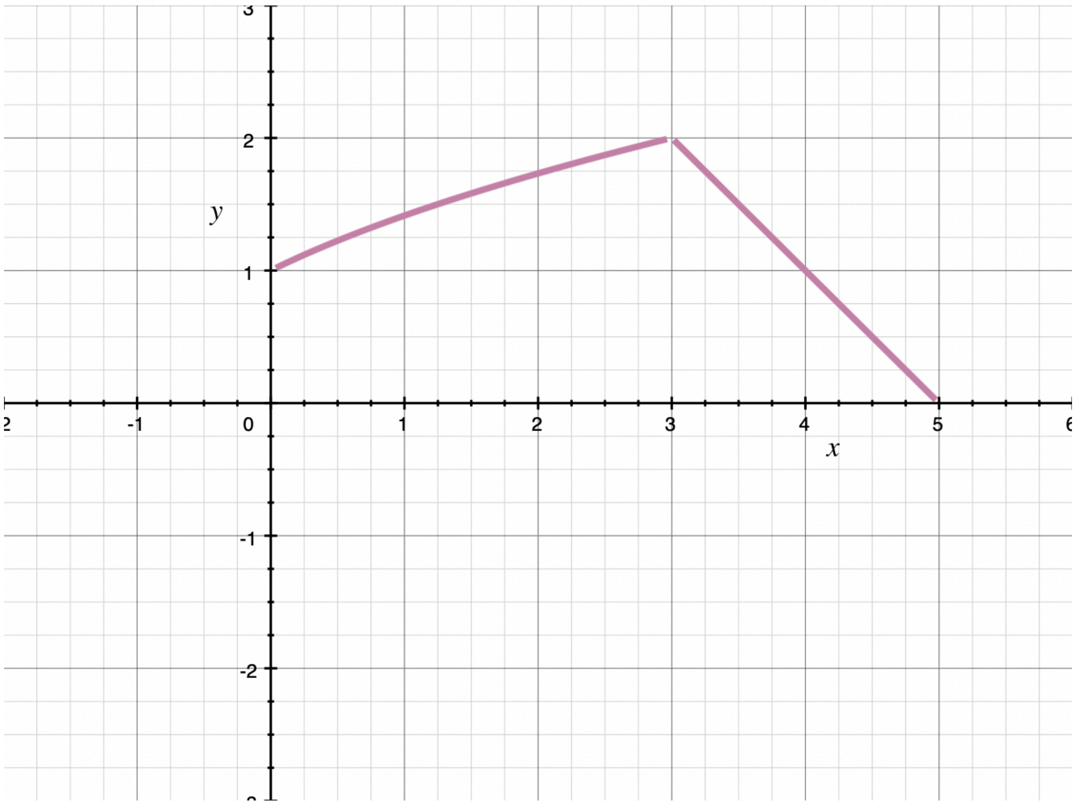
If $k=-1$, the function has a jump discontinuity at $x=3$ because the graph of $f(x)$ is



If $k=0$, the function has a jump discontinuity at $x=3$ because the graph of $f(x)$ is



If $k = 1$, the function is continuous at $x = 3$ because the graph of $f(x)$ is



We can see from the graphs that $k = 1$ will be the value that makes the function's pieces meet each other at $x = 3$. But how do we solve for the value of k algebraically, so that we can avoid picking random values of the unknown constant and graphing the function with that value?

Well, we always want to start at the "break point" that we talked about earlier. For this function, $x = 3$ is the break point between the two pieces, so we need the pieces to have equal value at that point. Therefore, we'll set the pieces equal to one another, plug in $x = 3$, and then solve the equation for the unknown k.

$$k\sqrt{x+1} = 5 - x$$

$$k\sqrt{3+1} = 5 - 3$$

$$k\sqrt{4} = 2$$

$$2k = 2$$

$$k = 1$$

So $k = 1$ is the value that forces the function's pieces to meet at the break point. For any other value of k, we'll get a jump discontinuity at the break point $x = 3$, but $k = 1$ makes the two pieces meet at the same point.

To find the value at which they meet, we plug $k = 1$ and the break point $x = 3$ back into the function.

$$f(3) = \begin{cases} 1\sqrt{3+1} & 0 \le x \le 3 \\ 5 - 3 & 3 < x \le 5 \end{cases}$$

$$f(3) = \begin{cases} 2 & 0 \le x \le 3 \\ 2 & 3 < x \le 5 \end{cases}$$

So when $k = 1$, both pieces of the function meet at $f(3) = 2$. Now to ensure continuity, all we have left to do is check that the function's left- and right-hand limits both approach 2.

Because $k\sqrt{x+1} = 1\sqrt{x+1} = \sqrt{x+1}$ models the function to the left of $x = 3$ and $5 - x$ models the function to the right of $x = 3$, so we'll check the left-hand limit of $\sqrt{x+1}$ and the right-hand limit of $5 - x$.

$$\lim_{x\to 3^-} \sqrt{x+1} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\lim_{x\to 3^+} 5 - x = 5 - 3 = 2$$

Because at $x = 3$ the left- and right-hand limits are equal,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

and because they both have the same value as the function itself at $x = 3$,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 2$$

we can say that the function is continuous at $x = 3$ when $k = 1$.

So to summarize, we'll always follow the same steps to find the value of an unknown constant in order to force the continuity of a piecewise function.

1. Set the function's pieces equal to one another and solve for the value of the unknown k.

2. Plug the value of k and the value of the break point into the function to ensure the function's pieces return the same value.

3. Find the left- and right-hand limits of the function at the break point.

4. Ensure that the left- and right-hand limits (from Step 3) are equal, and that they're both equal to the function's value (from Step 2) at the break point. If these three values are equal, the function is continuous at the break point.

Squeeze Theorem

The **Squeeze Theorem** allows us to find the limit of a function at a particular point, even when the function is undefined at that point. The way that we do it is by showing that our function can be "squeezed" between two other functions at the given point, and proving that the limits of these other functions are equal.

If we can show that two functions have the same value at a particular point, and we know that the original function has to run through the other two (be squeezed, or pinched, or sandwiched between them), then the original function can't take on any possible value other than the value of the other two.

Let's get a little more technical and take a look at the actual theorem.

We assume that our original function is $h(x)$, and that it's squeezed between two other functions, $f(x)$ and $g(x)$, so

$$f(x) \leq h(x) \leq g(x)$$

We also assume that the limits of our other two functions are equal as we approach the point c we're interested in, so

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

If we can show that both of the above statements are true, then we can say

$$L \leq \lim_{x \to c} h(x) \leq L$$

and we know that the original function $h(x)$ must have the same limit as the other two functions.

$$\lim_{x \to c} h(x) = L$$

We don't need to know what's actually happening to $h(x)$ at $x = c$. We're only concerned with the limit, so we just need to know what's happening *around* $x = c$.

Let's do an example where we use Squeeze Theorem to find the limit.

Example

Evaluate the limit.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

If we try to evaluate the limit using substitution, we get a 0 value in the denominator of the fraction, which is undefined. So we can't use substitution, and there's nothing we can really do with factoring or conjugate method. So we'll try Squeeze Theorem.

We know that the sine function oscillates back and forth between -1 and 1, so no matter what value we use for x, we know that this inequality will always be true:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

The goal now is to manipulate the inequality until the function in the center is identical to the original function. To get there, we'll multiply through by x^2.

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

We've now "squeezed" our original function between two other functions, $-x^2$ and x^2. Both of these squeezing functions have a well-defined limit as $x \to 0$.

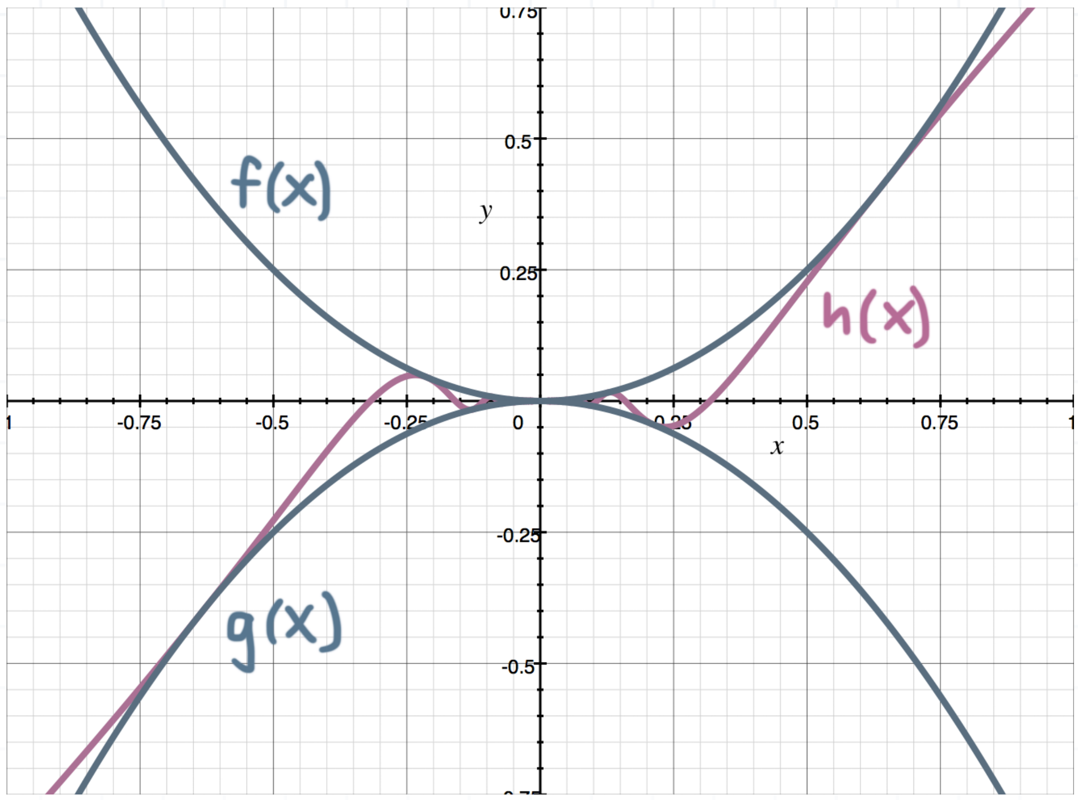
$$\lim_{x\to 0} -x^2 = -0^2 = 0$$

$$\lim_{x\to 0} x^2 = 0^2 = 0$$

Since these two functions have the same limit as $x \to 0$, and we know that the original function is squeezed between these other two, there's no possible value of the limit of the original function other than the value of the limit of the squeezing functions at the same point. Therefore, it must be true that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

The graph below confirms that the three equations all exist as they approach $x = 0$ from both the left- and right-hand sides, and that they all have the same value at that point.



Some Squeeze Theorem problems will be presented as an inequality already, in which case finding the limit will be fairly straightforward.

Let's walk through an example of how to solve Squeeze Theorem problems given to us in this form.

Example

Find the limit of $f(x)$ as $x \to -1$.

$$x \le f(x) \le x^2 - 2$$

A problem like this one tells us already that the function $f(x)$ is squeezed between x and $x^2 - 2$. And we've been asked to find

$$\lim_{x \to -1} f(x)$$

We can apply this same limit throughout the inequality to get

$$\lim_{x \to -1} x \le \lim_{x \to -1} f(x) \le \lim_{x \to -1} (x^2 - 2)$$

The limit in the center of the inequality is the limit we want to find. If we can show that the limits on the left and right of this inequality are equal, then we'll be able to use squeeze theorem to prove the value of the limit in the center.

We'll start by evaluating the limits on the left and right.

$$-1 \le \lim_{x \to -1} f(x) \le (-1)^2 - 2$$

$$-1 \le \lim_{x \to -1} f(x) \le 1 - 2$$

$$-1 \le \lim_{x \to -1} f(x) \le -1$$

Because we get equivalent values on the left and right of the inequality, we can conclude that the value in center, the limit of $f(x)$ as $x \to -1$, can only be equal to -1.

$$\lim_{x \to -1} f(x) = -1$$

Definition of the derivative

The derivative is the answer to the single most important question in Calculus 1, which is “How do we find the slope of a function at a particular point?”

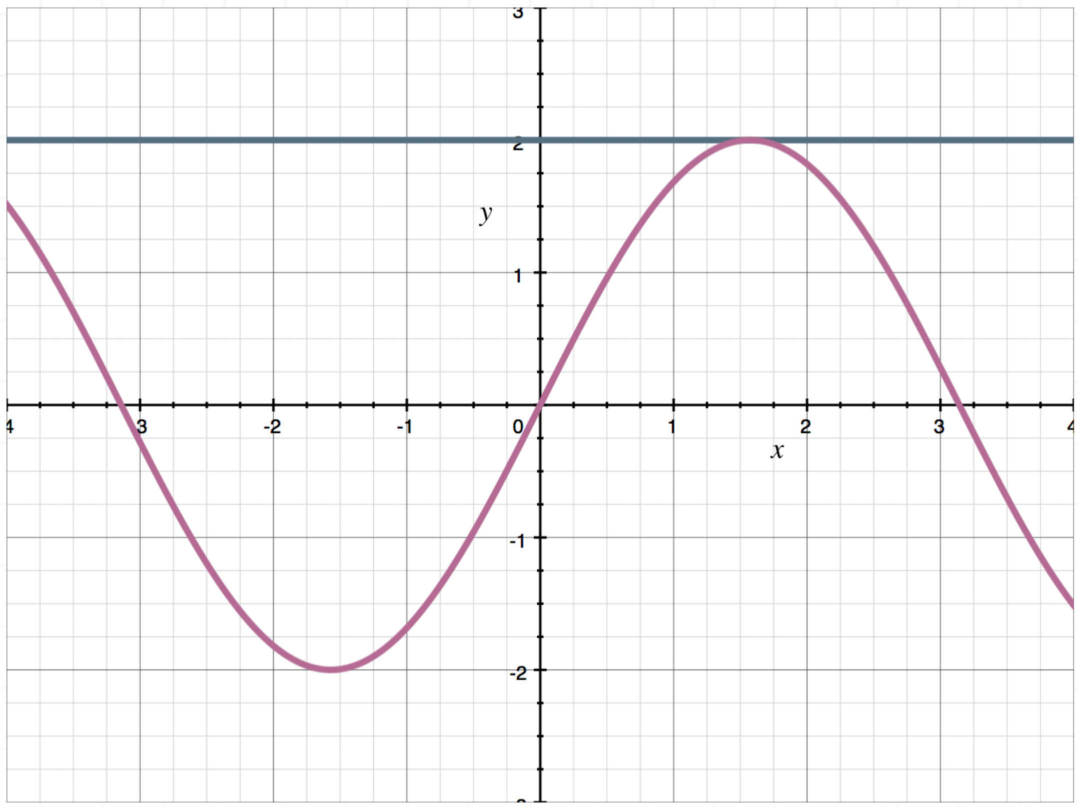
We already know, from Algebra, how to find the slope of a straight line. But in Calculus, we want to figure out how to find the slope of a function, even when it’s curved.

To find that slope, we 1) calculate the derivative of the function in general, and then we 2) evaluate the derivative at the point we’re interested in.

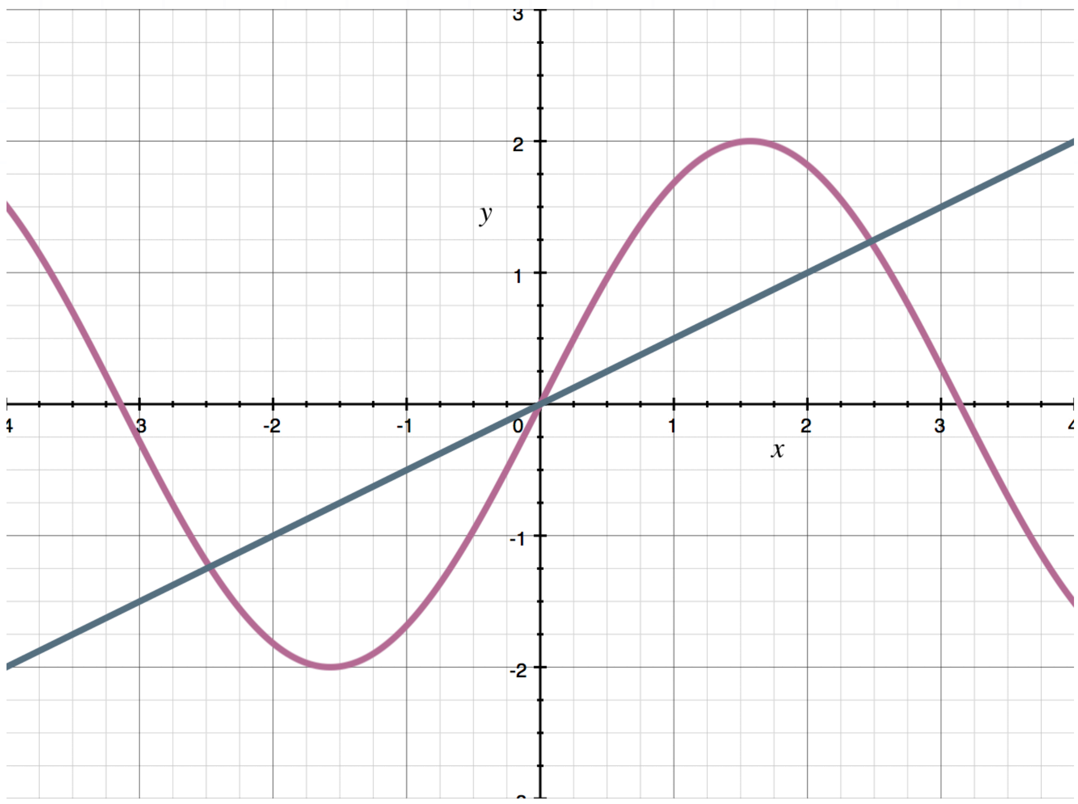
To better understand the idea of the derivative in general, we want to start by thinking about the difference between a tangent line and a secant line.

Secant and tangent lines

A tangent line is a line that *just* barely touches the edge of a graph, intersecting it at exactly one specific point. The line doesn’t cross the graph, it skims along the graph and stays along the same side of the graph.



A secant line, on the other hand, is a line that runs right through the graph, crossing it at a point.



In both graphs here, we're showing the same curve. Let's call it the function $f(x)$. Theoretically, we could use the secant line to approximate the rate of change of the curve between $x = 0$ and $x \approx 2.5$. Those are two of the points where the secant line intersects $f(x)$.

That would give us an idea about the **average rate of change** of the function in that interval, $[0,2.5]$. But as we can see from the graph, the slope of $f(x)$ is changing constantly throughout that interval. So, if we're actually interested in the slope of the function at $x \approx 1.5$, for example, the average rate of change over $[0,2.5]$ would give us somewhat of an estimate, but it wouldn't give us an exact rate of change at $x \approx 1.5$.

On the other hand, if we use the tangent line instead, we can see that it intersects the graph at one single point, about $x \approx 1.5$. So if we use the slope of the tangent instead of the slope of the secant line, we could get the **instantaneous rate of change** of the function at that exact point.

That's what the derivative allows us to do. Instead of using the secant line to settle for an average rate of change over an interval, it lets us use the tangent line to find the instantaneous (exact) rate of change at a specific point.

Building the derivative

We've said that slope of the secant line is the average rate of change over the interval between the points where the secant line intersects the graph, and that the slope of the tangent line instead indicates an instantaneous rate of change at the single point where it intersects the graph.

To express this mathematically, we start with a point $(c, f(c))$ on a graph, and then move a certain distance Δx to the right of that point, and call that new point $(c + \Delta x, f(c + \Delta x))$.

Connecting those points together gives us a secant line. Remember that the slope of any line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and if we plug the values from our two points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$ into this slope equation, we get

$$m = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}$$

$$m = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c}$$

$$m = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

This is the slope of any generic secant line that intersects the curve at two points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$.

What we want to do now is turn this secant line into a tangent line, which we can do by moving the two intersection points closer and closer together. As we move the two points closer to each other, the secant line will start to look more and more like a tangent line.

Eventually, if we move the points so close together that we reduce the distance between them to 0, then the secant line will literally become the tangent line. Mathematically, this means that we're reducing the value of Δx to 0, since Δx represents the horizontal distance between the points.

Therefore, we can say that the **definition of the derivative** at $x = c$ is

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

We'll also see this same definition of the derivative formula written as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

and we can use either one to find the derivative of any function at a particular point.

Calculating the derivative

When it comes to actually applying the definition to calculate the derivative of a function at a particular point x, we'll

1. substitute $x + \Delta x$ for every x in the original function, then plug this result into the definition for $f(x + \Delta x)$, then

2. plug the original function $f(x)$ into the definition.

That fills out the definition of the derivative formula. From there, we'll simplify the **difference quotient**, which is the fraction that makes up the definition of the derivative, and then evaluate the limit.

Example

Use the definition of the derivative to differentiate the function.

$$f(x) = x^2 - 5x + 6$$

After replacing x with $(x + \Delta x)$ in $f(x)$,

$$f(x + \Delta x) = (x + \Delta x)^2 - 5(x + \Delta x) + 6$$

we'll substitute for $f(x + \Delta x)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5(x + \Delta x) + 6 - f(x)}{\Delta x}$$

Then plug $f(x)$ into the definition.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5(x + \Delta x) + 6 - (x^2 - 5x + 6)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 5x - 5\Delta x + 6 - x^2 + 5x - 6}{\Delta x}$$

Collect like terms,

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 5x - 5\Delta x + 6 + 5x - 6}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 5\Delta x + 6 - 6}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x - 5\Delta x}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 2x - 5)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (\Delta x + 2x - 5)$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = 0 + 2x - 5$$

$$f'(x) = 2x - 5$$

The answer we just got from this example is the derivative of the original function $f(x) = x^2 - 5x + 6$.

The amazing thing is that, once we have the derivative, we can find the slope of the function at any point we'd like!

For instance, if we want to know the slope of the function at $x = 1$, we plug $x = 1$ into the derivative.

$$f'(1) = 2(1) - 5$$

$$f'(1) = 2 - 5$$

$$f'(1) = -3$$

Then we can say that the slope of the function at $x = 1$ is -3.

Power rule

At this point, we understand the idea of the derivative, and we know how to find it using the definition.

While the definition of the derivative can always be used to find the derivative of a function, it's not usually the most efficient way of finding the derivative.

It'll be faster for us to use the derivative rules we're about to learn. In this lesson, we'll look at the first of those derivative rules, which is the power rule.

The power rule

The **power rule** lets us take the derivative of power functions. Power functions are things like x^2, $3x^4$, $6x^5$, etc.

Power rule tells us that, to take the derivative of a function like these ones, we just multiply the exponent by the coefficient, and then subtract 1 from the exponent.

Formally, power rule says that, for any function of the form ax^n, the derivative will be

$$\frac{d}{dx}(ax^n) = (a \cdot n)x^{n-1}$$

For instance, to find the derivative of $3x^4$, we'll bring down the exponent of 4 to multiply it by the coefficient of 3, and we'll subtract 1 from the exponent of 4. So the derivative would be

$$3(4)x^{4-1}$$

$$12x^3$$

We can also use power rule to find the derivative of polynomials, which are combinations of power functions.

Example

Find the derivative of the function.

$$f(x) = 7x^3 + 2x^2 - 3x$$

We can use power rule to take the derivative of the function one term at a time. We'll apply the power rule to each term.

$$f'(x) = 7(3)x^{3-1} + 2(2)x^{2-1} - 3(1)x^{1-1}$$

$$f'(x) = 21x^{3-1} + 4x^{2-1} - 3x^{1-1}$$

$$f'(x) = 21x^2 + 4x^1 - 3x^0$$

$$f'(x) = 21x^2 + 4x - 3(1)$$

$$f'(x) = 21x^2 + 4x - 3$$

We want to notice a couple of things about this last example. First, after applying power rule, we ended up with $4x^1$ for the second term of the derivative. It's not necessary to write an exponent when the exponent is 1; it's implied. So $4x^1$ can be written more simply as just $4x$.

Second, we used power rule to take the derivative of the third term, $-3x$. To apply power rule, we had to realize that $-3x$ is equivalent to $-3x^1$, so that we could use the exponent of 1. After applying power rule, like normal, to $-3x^1$, we got $-3x^0$. Anything raised to the 0 power is equal to 1, so x^0 turns into 1.

The takeaway here is that the derivative of any term where the exponent is 1, will be equal to the coefficient. So the derivative of $-3x$ is -3, the derivative of $7x$ is 7, and the derivative of x is 1.

The derivative of a constant

Similarly, power rule tells us that the derivative of any constant will always be 0. In other words, the derivative of -3 is 0, the derivative of 7 is 0, and the derivative of 1 is 0.

As an example, take the constant -7. We can rewrite -7 as $-7x^0$, since x^0 is equivalent to 1, and multiplying 1 doesn't change the value of the constant. If we then use the power rule to take the derivative of the constant $-7x^0$, we get $-7(0)x^{0-1}$. But because we now have 0 multiplying the constant, we get 0 for the value of the derivative.

Let's do one more example where we apply the power rule to terms with different exponents.

Example

Find the derivative of the function.

$$f(x) = -2x^3 - 3x^2 + 6x - 5$$

We'll take the derivative one term at a time. We already know the derivative of $6x$ will be 6, and that the derivative of the constant -5 will be 0.

$$f'(x) = -2(3)x^{3-1} - 3(2)x^{2-1} + 6 - 0$$

$$f'(x) = -6x^{3-1} - 6x^{2-1} + 6$$

$$f'(x) = -6x^2 - 6x^1 + 6$$

$$f'(x) = -6x^2 - 6x + 6$$

Derivatives of combinations

Now that we've defined the power rule, and the rule for the derivative of a constant, let's summarize the set of basic derivative rules.

Constant rule	$\frac{d}{dx}(a) = 0$
Constant multiple rule	$\frac{d}{dx}(af(x)) = af'(x)$
Power rule	$\frac{d}{dx}(ax^n) = (a \cdot n)x^{n-1}$
Sum rule	$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
Difference rule	$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

We actually see all five of these rules used in the last example, where we differentiated $f(x) = -2x^3 - 3x^2 + 6x - 5$. We use the sum and difference rules to take the derivative of the entire polynomial, one term at a time. We use the constant multiple rule and power rule to differentiate the first three terms, $-2x^3 - 3x^2 + 6x$, and the constant rule to differentiate the last term, -5.

Power rule for negative powers

We also use power rule to differentiate power functions with negative powers. In order to use it, we need to remember a key exponent rule.

$$x^{-a} = \frac{1}{x^a}$$

This rule tells us that, when we have a power function with a negative exponent, x^{-a}, we can turn that exponent from negative to positive by moving the power function to the denominator.

Similarly, if we find ourselves with a negative exponent already in the denominator,

$$\frac{1}{x^{-a}} = x^a$$

we can turn that exponent from negative to positive by moving the power function to the numerator.

So regardless of where the negative exponent exists, we can change it to a positive exponent by moving it to the opposite location (from the numerator to the denominator, or from the denominator to the numerator).

And regardless of whether the exponent is positive or negative, we can apply power rule in the same way we've used it up to this point.

Let's do an example where we use power rule to differentiate a power function with negative exponents.

Example

Use power rule to take the derivative.

$$y = x^{-3} + 2x^{-6}$$

We'll apply power rule, one term at a time, in the same way we applied it previously. In other words, we'll bring the exponent down to multiply it by the coefficient, and we'll subtract 1 from the exponent.

$$y' = -3x^{-3-1} + 2(-6)x^{-6-1}$$

$$y' = -3x^{-4} - 12x^{-7}$$

We've found the derivative, so we can stop here. Or, if we want to make the exponents positive, we can move those terms into their denominators.

$$y' = -\frac{3}{x^4} - \frac{12}{x^7}$$

Now let's work through an example in which we need to move a power function from the denominator into the numerator, before we can apply the power rule.

Example

Find the derivative of the function.

$$y = \frac{1}{x^2} - \frac{3}{x^{-3}}$$

In order to be able to apply power rule, we want to move the power functions into the numerator. When we do, the signs of the exponents will flip.

$$y = 1(x^{-2}) - 3(x^3)$$

$$y = x^{-2} - 3x^3$$

Now that we've moved the power functions into the numerator, we can apply power rule in order to differentiate y.

$$y' = -2x^{-2-1} - 3(3)x^{3-1}$$

$$y' = -2x^{-3} - 9x^2$$

We can leave the derivative this way, or we can rewrite it so that all the exponents are positive.

$$y' = -\frac{2}{x^3} - 9x^2$$

Power rule for fractional powers

Power rule also applies in the same way when the exponent is a fraction. In other words, we bring the exponent down to multiply it by the coefficient, and subtract 1 from the exponent.

Example

Find the derivative of the function.

$$y = x^{\frac{3}{2}} + 6x^{-\frac{1}{4}}$$

Apply power rule to differentiate the function.

$$y' = \frac{3}{2}x^{\frac{3}{2}-1} + 6\left(-\frac{1}{4}\right)x^{-\frac{1}{4}-1}$$

$$y' = \frac{3}{2}x^{\frac{3}{2}-\frac{2}{2}} - \frac{3}{2}x^{-\frac{1}{4}-\frac{4}{4}}$$

$$y' = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{5}{4}}$$

We could leave the derivative this way, or we could rewrite it so that all the exponents are positive.

$$y' = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2x^{\frac{5}{4}}}$$

This example brings up the other point we want to make in this lesson, which is about the connection between fractional exponents and roots.

Every fractional exponent can be broken apart as a root. The numerator of the fraction becomes the power on the radicand (the value beneath the root), and the denominator of the fraction becomes the root. Here are a few examples:

$$x^{\frac{5}{4}} = \sqrt[4]{x^5}$$

$$x^{\frac{2}{3}} = \sqrt[3]{x^2}$$

$$x^{\frac{2}{5}} = \sqrt[5]{x^2}$$

In the first example we're taking the "fourth root of x^5," in the second example we're taking the "third root of x^2," and in the third example we're taking the "fifth root of x^2."

When the denominator of the fractional exponent is 2, it means we're taking the second root of something, which is the same as taking the square root, so we don't include the 2 with the root. Having the square root symbol on its own implies the 2. Therefore,

$$x^{\frac{1}{2}} = \sqrt{x}$$

This example follows the same pattern, because the numerator of 1 becomes the exponent on x, but an exponent of 1 is implied, so we don't need to write it in the radicand, and the denominator of 2 becomes the root, but a root of 2 is implied, so we don't need to write it with the root.

The takeaway here is that, when we're asked to differentiate something with a root, we want to convert that root into a power function with a fractional exponent, and then use power rule to take its derivative.

Here's an example of how that works.

Example

Find the derivative of the function.

$$y = 3\sqrt[4]{x^5} - 4\sqrt[3]{x^7}$$

Before we take the derivative, we'll need to convert both roots into power functions with fractional exponents.

$$y = 3x^{\frac{5}{4}} - 4x^{\frac{7}{3}}$$

Now we'll apply power rule to differentiate.

$$y' = 3\left(\frac{5}{4}\right)x^{\frac{5}{4}-1} - 4\left(\frac{7}{3}\right)x^{\frac{7}{3}-1}$$

$$y' = \frac{15}{4}x^{\frac{5}{4}-\frac{4}{4}} - \frac{28}{3}x^{\frac{7}{3}-\frac{3}{3}}$$

$$y' = \frac{15}{4}x^{\frac{1}{4}} - \frac{28}{3}x^{\frac{4}{3}}$$

We can leave the derivative this way, or we can rewrite it by converting the fractional exponents back into roots.

$$y' = \frac{15}{4}\sqrt[4]{x} - \frac{28}{3}\sqrt[3]{x^4}$$

Product rule with two functions

We know how to use power rule to take the derivative of a power function, and now we'll learn how to use product rule to take the derivative of a product.

If a function is, itself, the product of two functions, then we have to use product rule to find the derivative of that function. Given a function

$$y = f(x)g(x)$$

the **product rule** says that its derivative will be

$$y' = f(x)g'(x) + f'(x)g(x)$$

In other words, to use the product rule, we'll multiply the first function by the derivative of the second function, then add the derivative of the first function times the second function to that result.

Let's do an example where the function is the product of two power functions.

Example

Find the derivative of the function.

$$y = (x^2)(6x^3)$$

The two functions in this problem are x^2 and $6x^3$. It doesn't matter which one we choose for $f(x)$ and which one we choose for $g(x)$, since the product rule just has us adding $f(x)g'(x)$ and $f'(x)g(x)$. We'll get the correct answer either way, as long as we stay consistent.

Let's choose $f(x) = x^2$ and $g(x) = 6x^3$, and then list out these two functions, along with their derivatives.

$$f(x) = x^2$$

$$f'(x) = 2x$$

and

$$g(x) = 6x^3$$

$$g'(x) = 18x^2$$

Once we've got these all listed out, we can plug them directly into the product rule formula.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x^2)(18x^2) + (2x)(6x^3)$$

$$y' = 18x^4 + 12x^4$$

$$y' = 30x^4$$

We can use the power rule to double-check the answer that we got in this last example. If, instead of using product rule, we had instead started the problem by multiplying out the function, we'd get

$$h(x) = (x^2)(6x^3)$$

$$h(x) = 6x^5$$

Then, once the function is multiplied out, we can apply power rule to take the derivative.

$$h'(x) = 6(5)x^{5-1}$$

$$h'(x) = 30x^4$$

So we get the same answer both ways.

Product rule with three or more functions

As we learned in the last lesson, the product rule is what we use to take the derivative of a product. So we already know that the product rule works well for differentiating the product of two functions.

Product rule with three functions

In this lesson, we want to show that the product rule can be extended to more than two functions. For instance, given the function

$$y = f(x)g(x)h(x)$$

then the product rule tells us that the derivative is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

Notice how each term in the derivative is just the derivative of one of the functions from the original equation, multiplied by the rest of the functions from the original equation. And in the product rule formula, we include every possible combination.

Extending the product rule

That being said, we can use this same method to extend the product rule to as many terms as we have multiplied together in the original function.

So for an equation that's the product of four functions, $y = ABCD$, the derivative will include four terms: each possible combination.

$$y' = A'BCD + AB'CD + ABC'D + ABCD'$$

Let's do an example where we apply the product rule to the product of three functions.

Example

Use product rule to find the derivative.

$$y = (4x^6)(-2x)(-x^3)$$

First, let's list out each of the functions in the product, as well as their derivatives.

$$f(x) = 4x^6$$

$$f'(x) = 24x^5$$

and

$$g(x) = -2x$$

$$g'(x) = -2$$

and

$$h(x) = -x^3$$

$$h'(x) = -3x^2$$

Now we can plug everything we've found directly into the product rule formula.

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = (24x^5)(-2x)(-x^3) + (4x^6)(-2)(-x^3) + (4x^6)(-2x)(-3x^2)$$

$$y' = 48(x^5)(x)(x^3) + 8(x^6)(x^3) + 24(x^6)(x)(x^2)$$

$$y' = 48x^9 + 8x^9 + 24x^9$$

$$y' = 80x^9$$

We can confirm this answer if we instead start the problem by multiplying out the functions.

$$y = (4x^6)(-2x)(-x^3)$$

$$y = 8(x^6)(x)(x^3)$$

$$y = 8x^{10}$$

Once we've simplified the function this way, we can take the derivative using power rule.

$$y' = 8(10)x^{10-1}$$

$$y' = 80x^9$$

We get the same answer both ways, so we know that we applied the product rule correctly.

Let's do another example with more functions.

Example

Use product rule to find the derivative.

$$y = (8x^{12})\left(\frac{6}{7}x^2\right)(x)(-3)(2x^3)$$

Let's apply product rule.

$$y' = (96x^{11})\left(\frac{6}{7}x^2\right)(x)(-3)(2x^3)$$

$$+(8x^{12})\left(\frac{12}{7}x\right)(x)(-3)(2x^3)$$

$$+(8x^{12})\left(\frac{6}{7}x^2\right)(1)(-3)(2x^3)$$

$$+(8x^{12})\left(\frac{6}{7}x^2\right)(x)(0)(2x^3)$$

$$+(8x^{12})\left(\frac{6}{7}x^2\right)(x)(-3)(6x^2)$$

$$y' = -\frac{3{,}456}{7}(x^{11})(x^2)(x)(x^3)$$

$$-\frac{576}{7}(x^{12})(x)(x)(x^3)$$

$$-\frac{288}{7}(x^{12})(x^2)(x^3)$$

$$+0$$

$$-\frac{864}{7}(x^{12})(x^2)(x)(x^2)$$

$$y' = -\frac{3,456}{7}x^{17} - \frac{576}{7}x^{17} - \frac{288}{7}x^{17} - \frac{864}{7}x^{17}$$

$$y' = -\frac{5,184}{7}x^{17}$$

We can confirm this result by first multiplying out the functions,

$$y = (8x^{12})\left(\frac{6}{7}x^2\right)(x)(-3)(2x^3)$$

$$y = -\frac{288}{7}(x^{12})(x^2)(x)(x^3)$$

$$y = -\frac{288}{7}x^{18}$$

and then applying power rule, instead.

$$y = -\frac{288}{7}(18)x^{18-1}$$

$$y = -\frac{5,184}{7}x^{17}$$

Quotient rule

In the same way that we use product rule to find the derivative of a product, we use quotient rule to find the derivative of a quotient.

Quotient rule

The **quotient rule** says that the derivative of a quotient is given by

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Just like the other derivative rules so far (power rule and product rule), the derivative here is just a combination of the $f(x)$ and its derivative and $g(x)$ and its derivative.

But this time, $f(x)$ is the numerator of the original function, and $g(x)$ is the denominator of the original function.

Let's do an example where we use quotient rule to find the derivative.

Example

Use quotient rule to find the derivative.

$$y=\frac{x^4}{6x^2}$$

First, let's list out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

and

$$g(x) = 6x^2$$

$$g'(x) = 12x$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(4x^3)(6x^2) - (x^4)(12x)}{(6x^2)^2}$$

$$y' = \frac{24(x^3)(x^2) - 12(x^4)(x)}{(6x^2)(6x^2)}$$

$$y' = \frac{24x^5 - 12x^5}{36x^4}$$

$$y' = \frac{12x^5}{36x^4}$$

Simplify the function as much as possible by canceling common factors.

$$y' = \frac{x}{3}$$

$$y' = \frac{1}{3}x$$

We can verify the result from this last example by first simplifying the quotient,

$$y = \frac{x^4}{6x^2}$$

$$y = \frac{x^2}{6}$$

$$y = \frac{1}{6}x^2$$

and then applying power rule.

$$y' = \frac{2}{6}x^1$$

$$y' = \frac{1}{3}x$$

Reciprocal rule

When the numerator of the quotient we want to differentiate is a constant, the quotient rule formula gets simpler.

That's because we already know that the derivative of a constant is 0. So, assuming the numerator of the quotient is a constant,

$$y = \frac{a}{g(x)}$$

the derivative of the quotient would be

$$y' = \frac{(0)g(x) - ag'(x)}{[g(x)]^2}$$

$$y' = \frac{-ag'(x)}{[g(x)]^2}$$

This is the **reciprocal rule** formula. Again, it's just the simplified version of the quotient rule that applies only to quotients in which the numerator is a constant.

Because this rule comes directly from the quotient rule, there's really no reason to know the reciprocal rule (because we can always use the quotient rule, instead), other than to save ourselves a tiny bit of time when we're differentiating a quotient with a constant numerator.

That being said, let's still do an example where we apply the quotient rule directly.

Example

Use the reciprocal rule to find the derivative.

$$y = \frac{1}{2x+1} + \frac{5}{3x-1}$$

The reciprocal rule formula just needs the negative numerator, the denominator, and the denominator's derivative.

- For the first fraction, $a = 1$ so $-a = -1$, the denominator is $2x + 1$, and the denominator's derivative is 2.

- For the second fraction, $a = 5$ so $-a = -5$, the denominator is $3x - 1$, and the denominator's derivative is 3.

We'll set up the reciprocal rule formula for each fraction, and then plug in these values.

$$y' = \frac{-ag'(x)}{[g(x)]^2} + \frac{-ag'(x)}{[g(x)]^2}$$

$$y' = \frac{-1(2)}{(2x+1)^2} + \frac{-5(3)}{(3x-1)^2}$$

$$y' = -\frac{2}{(2x+1)^2} - \frac{15}{(3x-1)^2}$$

Trigonometric derivatives

We've learned about the basic derivative rules, and now we want to shift our attention toward the derivatives of specific kinds of functions. In this lesson we'll be looking at the derivatives of trigonometric functions, and later on we'll look at the derivatives of exponential and logarithmic functions.

Trigonometric derivatives

There are six basic trig functions, and we should know the derivative of each one.

Trigonometric function	Its derivative
$y = \sin x$	$y' = \cos x$
$y = \cos x$	$y' = -\sin x$
$y = \tan x$	$y' = \sec^2 x$
$y = \cot x$	$y' = -\csc^2 x$
$y = \sec x$	$y' = \sec x \tan x$
$y = \csc x$	$y' = -\csc x \cot x$

Remember that, in a trig function like $y = \sin x$, the $\sin$ and x are not multiplied together. So y is not the product of $\sin$ and x. Instead, x is the argument of the sine function, so we read $\sin x$ as "sine of x."

Let's look at an example of what it looks like to take the derivative of a trig function.

Example

Find the derivative.

$$y = 4\cos x$$

The derivative of $y = \cos x$ is $y' = -\sin x$. So the derivative will be

$$y' = 4(-\sin x)$$

$$y' = -4\sin x$$

Let's do another example with a little more going on.

Example

Find the derivative.

$$y = 8x^5 - 9\cot x$$

Dealing with one term at a time, we get

$$y' = 8(5)x^{5-1} - 9(-\csc^2 x)$$

$$y' = 40x^4 + 9\csc^2 x$$

Let's try one more.

Example

Find the derivative.

$$y = \sec x - 7x^5 \sin x + 3\csc x \cos x$$

Let's look at one term at a time. The derivative of $\sec x$ is $\sec x \tan x$.

To find the derivative of $-7x^5 \sin x$, we'll need to use product rule. If $f(x) = -7x^5$ and $f'(x) = -35x^4$, and $g(x) = \sin x$ and $g'(x) = \cos x$, then we can plug directly into the product rule formula.

$$f(x)g'(x) + f'(x)g(x)$$

$$(-7x^5)(\cos x) + (-35x^4)(\sin x)$$

$$-7x^5 \cos x - 35x^4 \sin x$$

The find the derivative of $3\csc x \cos x$, we'll need to use product rule. If $f(x) = 3\csc x$ and $f'(x) = -3\csc x \cot x$, and $g(x) = \cos x$ and $g'(x) = -\sin x$, then we can plug directly into the product rule formula.

$$f(x)g'(x) + f'(x)g(x)$$

$$3\csc x(-\sin x) + (-3\csc x \cot x)\cos x$$

$$-3\csc x\left(\frac{1}{\csc x}\right)-3\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)\cos x$$

$$-3-3\frac{\cos^2 x}{\sin^2 x}$$

$$-3-3\cot^2 x$$

Putting these derivatives together, we get

$$y'=\sec x\tan x-7x^5\cos x-35x^4\sin x-3-3\cot^2 x$$

Exponential derivatives

Exponential and logarithmic functions have their own set of derivative rules.

The exponential function we'll work with most often is $y=e^x$. The number e is a special constant with a value equal to $e \approx 2.718281828459045...$ It's similar to how we use π to represent the constant $\pi \approx 3.141592653589793...$

These kinds of exponential functions look a lot like the power functions we've seen before (like x^3), but there's one critical difference.

In power functions, the base is a variable, and the exponent is a constant. But in exponential functions, the base is the constant, and the exponent is the variable. So they're opposite situations.

Exponential derivative rules

Here are the derivatives of simple exponential functions:

Exponential functions	**Their derivatives**
$y=e^x$	$y'=e^x$
$y=a^x$	$y'=a^x(\ln a)$

There are two important things to say about these derivative formulas.

First, we have $\ln a$ as part of the formula for the derivative of $y=a^x$. That $\ln$ value represents the **natural log function**. Similar to the way we used $\sin x$

to represent the sine function, we use $\ln x$ to represent the natural log function. So whenever we see $\ln$, we know that we're dealing with the natural log. Whatever value of x we plug into $\ln x$, the natural log function will output a corresponding value. We'll talk more about logs and natural logs and their derivatives in the next lesson.

Second, there's actually no difference between the derivative formulas for $y = e^x$ and $y = a^x$. When we take the derivative of any $y = a^x$, we always need to multiply by $\ln a$. When we do that for the derivative of $y = e^x$, we get

$$y' = e^x(\ln e)$$

But $\ln e = 1$, so the derivative simplifies to

$$y' = e^x(1)$$

$$y' = e^x$$

So when the base of the exponential function is e, that $\ln a$ essentially disappears. But for any base other than e, that $\ln a$ will remain as part of the derivative.

Let's try an example where we differentiate an exponential function with base a.

Example

Find the derivative of the exponential function.

$$y = 42^x$$

In this function, $a = 42$ and the exponent is x. We'll differentiate by applying the formula for exponential derivatives.

$$y' = a^x(\ln a)$$

$$y' = 42^x(\ln(42))$$

Now let's try an example with base e.

Example

Find the derivative of the exponential function.

$$y = 3e^x$$

The base is e and the exponent is x, so the derivative is

$$y' = 3e^x$$

Let's try one final, more complex example.

Example

Find the derivative of the exponential function.

$$y = 8x^3 - 4^x e^x + 6^x$$

We have to take the derivative one term at a time, remembering to apply product rule when we get to $4^x e^x$.

$$y' = 24x^2 - [4^x(\ln 4)(e^x) + (4^x)(e^x)] + 6^x(\ln 6)$$

$$y' = 24x^2 - 4^x e^x \ln 4 - 4^x e^x + 6^x \ln 6$$

Logarithmic derivatives

We briefly introduced the natural logarithm in the last lesson. Now we want to dig deeper into log and natural log functions and their derivatives.

The log functions and their derivatives are given as

Type	**Log function**	**Its derivative**
Logarithm	$y = \log_a x$	$y' = \dfrac{1}{x \ln a}$
Natural logarithm	$y = \ln x$	$y' = \dfrac{1}{x}$

There's actually no difference between the derivative formulas for $y = \log_a x$ and $y = \ln x$. When we take the derivative of any $y = \log_a x$, we always need to include that $\ln a$ that's in the denominator of its derivative.

But the natural log function $y = \ln x$ is actually just the standard logarithm, but with a base of e instead of a. In other words, $y = \ln x$ is the same thing as $y = \log_e x$. With that in mind, we can say that the derivative of $y = \log_e x$ is

$$y' = \frac{1}{x \ln e}$$

And, like we learned in the last lesson, $\ln e = 1$, which means the derivative will simplify to just

$$y' = \frac{1}{x(1)}$$

$$y' = \frac{1}{x}$$

which is the derivative formula for $y = \ln x$.

So when the base of the logarithm is e, which is another way of saying that the logarithm $y = \log_a x$ becomes the natural logarithm $y = \ln x$, then the derivative formula simplifies from

$$y' = \frac{1}{x \ln a}$$

to

$$y' = \frac{1}{x}$$

Let's try an example where we differentiate a function in the form $y = \log_a x$.

Example

Find the derivative of the logarithmic function.

$$y = 6 \log_8 x$$

Differentiate by applying the derivative formula for $y = \log_a x$.

$$y' = \frac{1}{x \ln a}$$

$$y' = 6\left(\frac{1}{x\ln 8}\right)$$

$$y' = \frac{6}{x\ln 8}$$

Now let's try an example with a natural log function.

Example

Find the derivative of the logarithmic function.

$$y = 5\ln x$$

To find the derivative we need to apply the derivative formula for natural logs.

$$y' = \frac{1}{x}$$

$$y' = 5\left(\frac{1}{x}\right)$$

$$y' = \frac{5}{x}$$

Let's try one final example that's a little more complex.

Example

Find the derivative of the logarithmic function.

$$y = 9x\ln x + 4x^7\log_3 x - 2x^{12}$$

We need to take the derivative one term at a time, applying the derivative formulas for the log and natural log. We'll also need to apply product rule to the first and second terms.

$$y' = 9\ln x + 9x\left(\frac{1}{x}\right) + \left[(28x^6)(\log_3 x) + (4x^7)\left(\frac{1}{x\ln 3}\right)\right] - 24x^{11}$$

$$y' = 9\ln x + 9 + 28x^6\log_3 x + \frac{4x^6}{\ln 3} - 24x^{11}$$

Chain rule with power rule

Up to now, we've essentially only been calculating derivatives of power functions. Even when we looked at product rule and quotient rule, we used functions that were the products of power functions, or the quotients of power functions.

In other words, we're comfortable differentiating things like $3x^2$, $-2x$, or $6x^{-3}$. But what do we do when we want to differentiate a function like $3(2x+1)^2$? This is like a power function, but having $2x+1$ inside the parentheses makes this derivative a little more complicated.

The chain rule

That's where the chain rule comes in. The **chain rule** lets us calculate derivatives of nested functions, where one function is the "outside" function and one function is the "inside function. If we want to differentiate a nested function like

$$y = g[f(x)]$$

then $g[f(x)]$ is the outside function and $f(x)$ is the inside function. The derivative is

$$y' = g'[f(x)]f'(x)$$

Notice here that we took the derivative first of the outside function, $g[f(x)]$, leaving the inside function, $f(x)$, completely untouched, and then we multiplied that result by the derivative of the inside function.

So applying the chain rule requires just two simple steps:

1. Take the derivative of the "outside" function, leaving the "inside" function untouched.

2. Multiply that result by the derivative of the "inside" function.

Sometimes, depending on the complexity of the inside function, it can be helpful to use substitution to make it easier to think about $g[f(x)]$. If we decide to use substitution, we just replace the inside function with u, and the function simplifies from $g[f(x)]$ to $y = g[u]$. Then the derivative of this simplified version is

$$y' = g'[u]u'$$

If we're going to use substitution, we need to make sure we back-substitute at the end of the problem, in order to get the final answer back in terms of the original variable.

Example

Use chain rule to find the derivative.

$$y = (4x^8 - 6)^6$$

The outside function is the power function $(4x^8 - 6)^6$, and the inside function is $4x^8 - 6$.

Let's use the substitution method, and say that $u = 4x^8 - 6$ and $u' = 32x^7$. Then we can rewrite the original equation $y = (4x^8 - 6)^6$ as

$$y = (u)^6$$

We'll apply power rule and chain rule to find the derivative, and we'll get

$$y' = 6(u)^5(u')$$

Then we'll back-substitute for u and u'.

$$y' = 6(4x^8 - 6)^5(32x^7)$$

$$y' = 192x^7(4x^8 - 6)^5$$

We just worked an example where we used chain rule in conjunction with power rule.

We'll also need to know how to use the chain rule in combination with product rule and quotient rule, and with trigonometric functions, all of which we'll tackle in the next few lessons.

Chain rule with trig, log, and exponential functions

We've looked at derivatives of trigonometric, exponential, and logarithmic functions, but so far we've always kept the arguments equal to x. In other words, we've found the derivatives of $y = \sin x$, $y = e^x$, and $y = \ln x$.

But we want to be able to differentiate these kinds of functions, even when the argument is something other than x. For instance, $y = \sin(3x)$, $y = e^{-x^2}$, and $y = \ln(x^2 + 4x + 2)^3$.

A new argument

We'll differentiate all of these functions by applying chain rule. In fact, we've been using chain rule all along, even when the argument was simply x, but the effect of chain rule was invisible to us.

For instance, with a basic trig function like $y = \cos x$, the argument is x, so x acts like an "inside function." When we differentiate $y = \cos x$, we have to take the derivative of the $\cos$ part to get $-\sin$, but then we have to multiply by the derivative of the inside function. The derivative of the inside function x is 1, so the derivative of $y = \cos x$ actually looks like this:

$$y = \cos x$$

$$y' = (-\sin x)(1)$$

$$y' = -\sin x$$

In this case, applying chain rule and multiplying by 1 doesn't change the value of the derivative, which is what we mean when we say that the effect of chain rule was invisible to us.

But when the argument is anything other than x, the derivative of the argument will be something other than 1, and therefore applying chain rule will of course have an actual effect on the value of the derivative. For instance, the derivative of $y = \sin(2x)$ will be $y' = 2\cos(2x)$, and the derivative of $y = \sec(3x^2)$ will be $y' = 6x\sec(3x^2)\tan(3x^2)$.

So if we modify our trig derivative rules to account for chain rule, they now look like this:

Trigonometric function	**Its derivative**
$y = \sin(g(x))$	$y' = g'(x)(\cos(g(x)))$
$y = \cos(g(x))$	$y' = g'(x)(-\sin(g(x)))$
$y = \tan(g(x))$	$y' = g'(x)(\sec^2(g(x)))$
$y = \cot(g(x))$	$y' = g'(x)(-\csc^2(g(x)))$
$y = \sec(g(x))$	$y' = g'(x)(\sec(g(x))\tan(g(x)))$
$y = \csc(g(x))$	$y' = g'(x)(-\csc(g(x))\cot(g(x)))$

And just like with trig functions, we need to apply chain rule every time we take the derivative of an exponential or logarithmic function. The "inside function" of a trig function is its argument; the "inside function" of an exponential function is its exponent, and the "inside function" of a log function is its argument.

If we want to show chain rule as part of the exponential derivative formulas, we get

Exponential function	**Its derivative**
$y = e^{g(x)}$	$y' = e^{g(x)}g'(x)$
$y = a^{g(x)}$	$y' = a^{g(x)}(\ln a)g'(x)$

And the logarithmic derivative formulas, after adding in chain rule, become

Log function	**Its derivative**
$y = \log_a g(x)$	$y' = \dfrac{1}{g(x)\ln a}g'(x)$
$y = \ln g(x)$	$y' = \dfrac{1}{g(x)}g'(x)$

Let's look at an example of a trig function in which the argument is something other than x.

Example

Use chain rule to find the derivative.

$$y = 8x^5 - 9\cot(7x^4)$$

Dealing with one term at a time, and remembering to use chain rule to handle the derivative of $-9\cot(7x^4)$, we get

$$y' = 8(5)x^{5-1} - 9(-\csc^2(7x^4))(7(4)x^{4-1})$$

$$y' = 40x^4 - 9(-\csc^2(7x^4))(28x^3)$$

$$y' = 40x^4 + 252x^3 \csc^2(7x^4)$$

Now let's do one with an exponential function.

Example

Find the derivative of the exponential function.

$$y = 42^{6x}$$

In this function, $a = 42$ and the exponent is $6x$. We'll differentiate by applying the formula for exponential derivatives.

$$y' = a^{g(x)}(\ln a)g'(x)$$

$$y' = 42^{6x}(\ln(42))(6)$$

$$y' = 6(42)^{6x}\ln(42)$$

And let's do one more with a log function, just so that we can see the log derivative formulas in action.

Example

Find the derivative of the logarithmic function.

$$g(x) = \ln\sqrt{x^3 + x}$$

Take the derivative, remembering to apply chain rule.

$$g'(x) = \frac{1}{\sqrt{x^3 + x}} \cdot \frac{1}{2}(x^3 + x)^{-\frac{1}{2}} \cdot (3x^2 + 1)$$

$$g'(x) = \frac{(3x^2 + 1)(x^3 + x)^{-\frac{1}{2}}}{2\sqrt{x^3 + x}}$$

$$g'(x) = \frac{3x^2 + 1}{2\sqrt{x^3 + x}\sqrt{x^3 + x}}$$

$$g'(x) = \frac{3x^2 + 1}{2(x^3 + x)}$$

$$g'(x) = \frac{3x^2 + 1}{2x(x^2 + 1)}$$

Chain rule with product rule

We can tell by now that these derivative rules are very often used together. We've seen power rule used together with both product rule and quotient rule, and we've seen chain rule used with power rule.

In this lesson, we want to focus on using chain rule with product rule. But these chain rule/product rule problems are going to require power rule, too.

Let's look at an example of how we might see the chain rule and product rule applied together to differentiate the same function.

Example

Find the derivative of the function.

$$y = 8((6x)(3x^2))^{-4}$$

If we use substitution with $u = (6x)(3x^2)$, then we can rewrite the function as

$$y = 8u^{-4}$$

We'll differentiate using power rule. We also have to apply chain rule, and multiply by the derivative of the inside function.

$$y' = 8(-4)u^{-4-1}u'$$

$$y' = -32u^{-5}u'$$

We need to plug in for u and u', so let's find u' using product rule.

$$u' = f(x)g'(x) + f'(x)g(x)$$

$$u' = (6x)(6x) + (6)(3x^2)$$

Now we can back-substitute into $y' = -32u^{-5}u'$, and then simplify.

$$y' = -32((6x)(3x^2))^{-5}((6)(3x^2) + (6x)(6x))$$

$$y' = -32(18x^3)^{-5}(18x^2 + 36x^2)$$

$$y' = -32(18x^3)^{-5}(54x^2)$$

We could leave the derivative like this, or we could rewrite it to make all the exponents positive.

$$y' = -\frac{32(54x^2)}{(18x^3)^5}$$

$$y' = -\frac{54x^2}{9^5x^{15}}$$

$$y' = -\frac{2}{2{,}187x^{13}}$$

In this last example, the product was nested inside the power function. Let's do another example, but this time where the power functions are nested inside the product.

Example

Find the derivative of the function.

$$y = (x^2 + 1)^7(9x^4)$$

We have the product of $(x^2 + 1)^7$ and $9x^4$, so we need to use product rule. Let's start by listing out both of these functions, and their derivatives.

$$f(x) = (x^2 + 1)^7$$

$$f'(x) = 14x(x^2 + 1)^6$$

and

$$g(x) = 9x^4$$

$$g'(x) = 36x^3$$

We only needed power rule to find $g'(x)$, but we had to use power rule with chain rule in order to find $f'(x)$. If we'd found that derivative using substitution, we would have set $u = x^2 + 1$, and then found $u' = 2x + 0$, or $u' = 2x$. So $f'(x)$ was found to be

$$f'(x) = 7(u)^{7-1}u'$$

$$f'(x) = 7(x^2 + 1)^6(2x)$$

$$f'(x) = 14x(x^2 + 1)^6$$

Now we can plug everything we've found into the product rule formula.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = ((x^2 + 1)^7)(36x^3) + (14x(x^2 + 1)^6)(9x^4)$$

$$y' = 36x^3(x^2 + 1)^7 + 126x^5(x^2 + 1)^6$$

Chain rule with quotient rule

Of course, we'll use chain rule with quotient rule, as well. Sometimes we'll have a quotient nested inside some other power function or product function, and sometimes we'll have power or product functions nested inside quotients.

Let's look at an example in which we differentiate a power function that has a quotient as its "inside function."

Example

Find the derivative of the function.

$$y = \left(\frac{6x^4}{3x}\right)^8$$

First, we'll substitute for the quotient and set

$$u = \frac{6x^4}{3x}$$

We'll need to find u', and since u is a quotient, we'll apply quotient rule to take its derivative.

$$u' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$u' = \frac{(24x^3)(3x) - (6x^4)(3)}{(3x)^2}$$

$$u' = \frac{72x^4 - 18x^4}{9x^2}$$

Going back to the original function, we used substitution to simplify it to

$$y = (u)^8$$

Use power rule to take the derivative, making sure to apply chain rule and multiplying by u'.

$$y = 8(u)^7 u'$$

$$y' = 8\left(\frac{6x^4}{3x}\right)^7 \left(\frac{72x^4 - 18x^4}{9x^2}\right)$$

$$y' = 8\left(\frac{6x^4}{3x}\right)^7 \left(\frac{54x^4}{9x^2}\right)$$

We could leave the derivative this way, but let's simplify the fractions by factoring and canceling like terms.

$$y' = 8\left(\frac{2x^4}{x}\right)^7 \left(\frac{6x^4}{x^2}\right)$$

$$y' = 8(2x^3)^7(6x^2)$$

$$y' = 48x^2(2x^3)^7$$

We can double-check the answer we got in this last example by first simplifying the original function,

$$y = \left(\frac{6x^4}{3x}\right)^8$$

$$y = (2x^3)^8$$

and then differentiating the simplified function.

$$y' = 8(2x^3)^{8-1}(6x^2)$$

$$y' = 48x^2(2x^3)^7$$

Let's look at another example, this time where we have power functions nested inside a quotient.

Example

Find the derivative of the function.

$$y = \frac{(6x^4 - 5)^2}{(7x^2 + 3)^3}$$

Because we have a quotient, we'll need to apply the quotient rule. We know that we'll need the numerator $f(x)$ and denominator $g(x)$, as well as the derivatives of both the numerator and denominator.

$$f(x) = (6x^4 - 5)^2$$

$$f'(x) = 2(6x^4 - 5)^{2-1}(24x^3) = 48x^3(6x^4 - 5)$$

and

$$g(x) = (7x^2 + 3)^3$$

$$g'(x) = 3(7x^2 + 3)^{3-1}(14x) = 42x(7x^2 + 3)^2$$

Plugging these values into the quotient rule gives

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(48x^3(6x^4 - 5))((7x^2 + 3)^3) - ((6x^4 - 5)^2)(42x(7x^2 + 3)^2)}{((7x^2 + 3)^3)^2}$$

$$y' = \frac{48x^3(6x^4 - 5)(7x^2 + 3)^3 - 42x(6x^4 - 5)^2(7x^2 + 3)^2}{(7x^2 + 3)^6}$$

There's a common factor of $(7x^2 + 3)^2$ between the numerator and denominator, so we'll cancel it.

$$y' = \frac{48x^3(6x^4 - 5)(7x^2 + 3) - 42x(6x^4 - 5)^2}{(7x^2 + 3)^4}$$

Inverse trigonometric derivatives

Now that we've covered the derivatives of the six basic trig functions, we want to look at the derivatives of the six inverse trig functions.

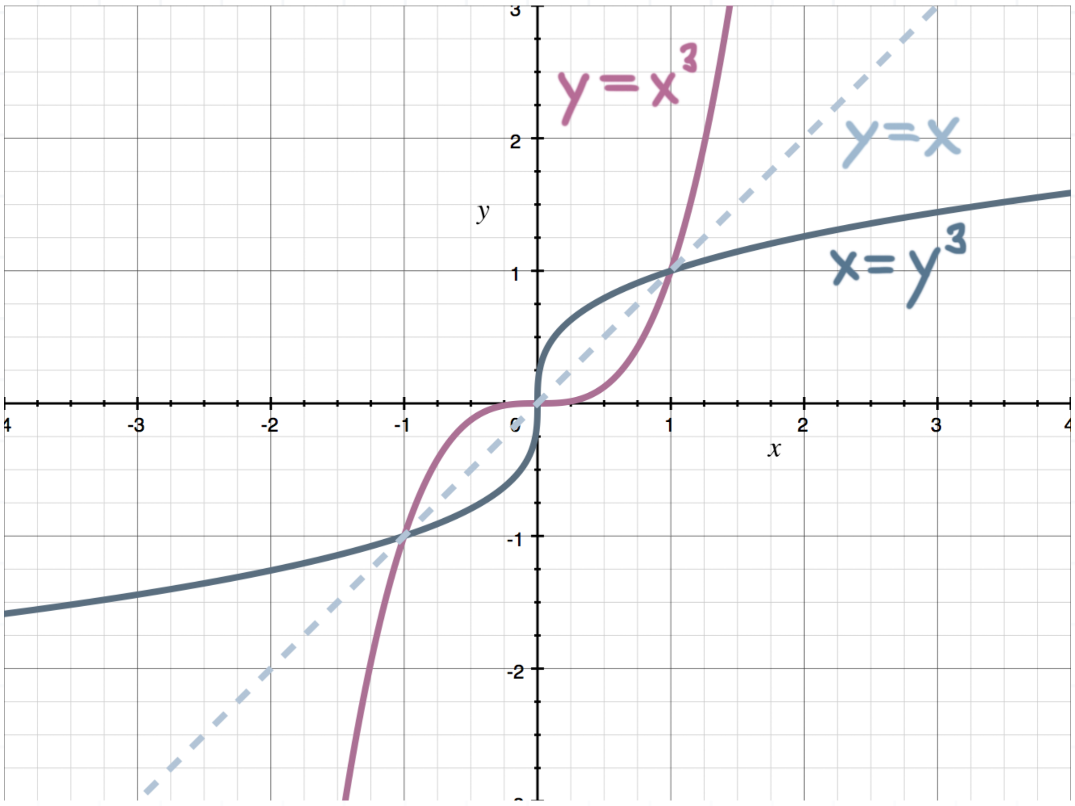
Before we get to the derivatives though, let's first define the six inverse trig functions, themselves. And before we get to the inverse trig functions, let's remind ourselves about inverse functions in general.

Inverse functions

Inverse functions are usually covered in Algebra, but let's do a quick refresher here.

To find a function's inverse, all we need to do is change the places of the x and y variables in the function's equation. For instance, given the function $y=x^3$, its inverse is $x=y^3$. Or given the function $y=3x^2-6x+2$, its inverse is $x=3y^2-6y+2$. We can see how we're just swapping out x variables for y variables and vice versa in order to get the inverse.

A function and its inverse will always be reflections of each other over the line $y=x$. As an example, if we graph $y=x^3$ and $x=y^3$ on the same set of axes, we can see that they are a perfect reflection of each other across $y=x$.



We can use the same "flip x and y" method to find the inverse of a trig function. The inverse of $y = \sin x$ is therefore $x = \sin y$. To solve this equation for y, we take inverse sine of both sides, which will cancel the $\sin$ and get y by itself.

$$x = \sin y$$

$$\sin^{-1}(x) = \sin^{-1}(\sin y)$$

$$\sin^{-1} x = y$$

$$y = \sin^{-1} x$$

So we can see that $y = \sin x$ and $y = \sin^{-1} x$ are inverses of one another. We can use the notation $\sin^{-1}$ to indicate "inverse sine," or we can use $\arcsin$ instead. Both indicate the inverse of the $\sin$ function and can be used interchangeably. If we're going to use $\sin^{-1}$ notation, we just need to remember that the -1 is **not** an exponent. Remember that the negative exponent rule tells us that

$$x^{-1} = \frac{1}{x}$$

but inverse trigonometric functions don't not follow this rule. So

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

Because we have to be so careful with the $\sin^{-1}$ notation to remember that the negative exponent rule doesn't apply, many people prefer to use the $\arcsin$ notation instead of the $\sin^{-1}$ notation.

Inverse trig derivatives

Below is a table for the derivatives of the inverse trig functions.

Inverse trig function	**Derivative**
$y = \sin^{-1} x$	$y' = \dfrac{1}{\sqrt{1-x^2}}$
$y = \cos^{-1} x$	$y' = -\dfrac{1}{\sqrt{1-x^2}}$
$y = \tan^{-1} x$	$y' = \dfrac{1}{1+x^2}$
$y = \sec^{-1} x$	$y' = \dfrac{1}{\lvert x \rvert \sqrt{x^2-1}}$

$y = \csc^{-1} x$ $\qquad$ $y' = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

$y = \cot^{-1} x$ $\qquad$ $y' = -\dfrac{1}{1 + x^2}$

Just like with trig functions, when we differentiate an inverse trig function, we always have to apply chain rule and multiply by the derivative of the argument.

Because all the formulas above are for trig functions with an argument of x. Because the argument is x, and the derivative of x is 1, applying chain rule just means that we multiply by 1, which doesn't affect the value of the derivative.

But if the argument of the inverse trig function is anything other than x, then applying chain rule means we'll be multiplying by something other than 1, which means that applying chain rule will affect the value of the derivative.

Here's a table of formulas for the inverse trig functions when the argument is something other than x.

Inverse trig function	**Derivative**
$y = \sin^{-1}[g(x)]$	$y' = \dfrac{g'(x)}{\sqrt{1 - [g(x)]^2}}$
$y = \cos^{-1}[g(x)]$	$y' = -\dfrac{g'(x)}{\sqrt{1 - [g(x)]^2}}$

$$y = \tan^{-1}[g(x)] \qquad y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y = \csc^{-1}[g(x)] \qquad y' = -\frac{g'(x)}{|g(x)|\sqrt{[g(x)]^2 - 1}}$$

$$y = \sec^{-1}[g(x)] \qquad y' = \frac{g'(x)}{|g(x)|\sqrt{[g(x)]^2 - 1}}$$

$$y = \cot^{-1}[g(x)] \qquad y' = -\frac{g'(x)}{1 + [g(x)]^2}$$

With these formulas in hand, let's try an example where we find the derivative of an inverse trig function.

Example

Find the derivative of the inverse trig function.

$$y = 7\tan^{-1}(4x^3)$$

Apply the formula for the derivative of inverse tangent with $g(x) = 4x^3$.

$$y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y' = 7\left(\frac{12x^2}{1 + (4x^3)^2}\right)$$

$$y' = \frac{84x^2}{1 + 16x^6}$$

It's common to see inverse trigonometric functions mixed into more elaborate functions, so let's try an example with some other things going on.

Example

Find the derivative of the function.

$$y = 2\sec^{-1}(x^3) - 54x^7$$

We differentiate one term at a time, which means we can differentiate $-54x^7$ using power rule. We only have to worry about the formula for the derivative of inverse secant with $g(x) = x^3$,

$$y' = \frac{g'(x)}{|g(x)|\sqrt{[g(x)]^2 - 1}}$$

when we differentiate the first term.

$$y' = 2\left(\frac{3x^2}{|x^3|\sqrt{(x^3)^2 - 1}}\right) - 378x^6$$

$$y' = \frac{6x^2}{|x^3|\sqrt{x^6 - 1}} - 378x^6$$

Let's try one more example.

Example

Find the derivative of the function.

$$y = 2x^6 - x^3\cos^{-1}(2x) + 8\sin(3x^5)$$

We'll need to use product rule for the second term, $-x^3\cos^{-1}(2x)$. Taking the derivative term by term, using the formula for the derivative of inverse cosine with $g(x) = 2x$,

$$y' = -\frac{g'(x)}{\sqrt{1 - [g(x)]^2}}$$

and the formula from the previous lesson for the derivative of sine,

$$y' = \cos x$$

we get

$$y' = 12x^5 - \left[(3x^2)(\cos^{-1}(2x)) + (x^3)\left(-\frac{2}{\sqrt{1-(2x)^2}}\right)\right] + 8\cos(3x^5)(15x^4)$$

$$y' = 12x^5 - \left(3x^2\cos^{-1}(2x) - \frac{2x^3}{\sqrt{1-4x^2}}\right) + 120x^4\cos(3x^5)$$

$$y' = 12x^5 - 3x^2 \cos^{-1}(2x) + \frac{2x^3}{\sqrt{1 - 4x^2}} + 120x^4 \cos(3x^5)$$

Hyperbolic derivatives

We've looked at trig and inverse trig functions and their derivatives, and now we'll look at hyperbolic and inverse hyperbolic trig functions and their derivatives in order to round out our lessons on derivatives with trig functions.

Fortunately, the derivatives of the hyperbolic functions are really similar to the derivatives of trig functions, so they'll be pretty easy for us to remember. We only see a difference between the two when it comes to the derivative of cosine vs. the derivative of hyperbolic cosine.

The derivative of $y = \cos x$ is $y' = -\sin x$, but the derivative of $y = \cosh x$ is $y' = \sinh x$. Because the rest of the hyperbolic derivatives table follows the same pattern as the normal trig derivatives table, we'd expect the derivative of hyperbolic cosine to be $y' = -\sinh x$, with the negative sign included. But its actual derivative doesn't include the negative sign.

Hyperbolic function	**Derivative**
$y = \sinh x$	$y' = \cosh x$
$y = \cosh x$	$y' = \sinh x$
$y = \tanh x$	$y' = \text{sech}^2 x$
$y = \text{csch} x$	$y' = -\text{csch} x \coth x$
$y = \text{sech} x$	$y' = -\text{sech} x \tanh x$
$y = \coth x$	$y' = -\text{csch}^2 x$

Just like with trig derivatives and inverse trig derivatives, we always apply chain rule and multiply by the derivative of the argument. In the table above, the argument is always x, and since the derivative of x is 1. Of course, multiplying by 1 doesn't affect the value of the derivative.

But if the argument is anything other than x, then the derivative of the argument will be something other than 1, which means that applying the chain rule will affect the value of the derivative. When that's the case, we can think of the hyperbolic derivative formulas as

Hyperbolic function	**Derivative**
$y = \sinh[g(x)]$	$y' = \cosh[g(x)][g'(x)]$
$y = \cosh[g(x)]$	$y' = \sinh[g(x)][g'(x)]$
$y = \tanh[g(x)]$	$y' = \text{sech}^2[g(x)][g'(x)]$
$y = \text{csch}[g(x)]$	$y' = -\text{csch}[g(x)]\coth[g(x)][g'(x)]$
$y = \text{sech}[g(x)]$	$y' = -\text{sech}[g(x)]\tanh[g(x)][g'(x)]$
$y = \coth[g(x)]$	$y' = -\text{csch}^2[g(x)][g'(x)]$

Let's work through a few examples so that we can get comfortable applying these derivative formulas.

Example

Find the derivative of the hyperbolic function.

$$y = 9\tanh(x^5)$$

Apply the formula for the derivative of hyperbolic tangent.

$$y' = \text{sech}^2[g(x)][g'(x)]$$

$$y' = 9\text{sech}^2(x^5)(5x^4)$$

$$y' = 45x^4\text{sech}^2(x^5)$$

Let's try another example where the hyperbolic function occurs as part of a larger function.

Example

Find the function's derivative.

$$y = 2\sinh(4x^6) - 26x^2$$

Take the derivative one term at a time, applying the formula for the derivative of hyperbolic sine to the first term,

$$y' = \cosh[g(x)][g'(x)]$$

and power rule to the second term.

$$y' = 2\cosh(4x^6)(24x^5) - 52x$$

$$y' = 48x^5\cosh(4x^6) - 52x$$

Let's try one more, slightly more complex example.

Example

Find the derivative.

$$y = 8x^{-4} + \text{csch}(3x^2) + \frac{\sinh(x^3)}{\tanh(2x^7)}$$

Let's work on one term at a time. The derivative of the first term can be found using power rule.

$$y' = 8(-4)x^{-4-1}$$

$$y' = -32x^{-5}$$

To find the derivative of the second term, we'll apply the formula for the derivative of hyperbolic cosecant.

$$y' = -\text{csch}[g(x)]\coth[g(x)][g'(x)]$$

$$y' = -\text{csch}(3x^2)\coth(3x^2)(6x)$$

$$y' = -6x\text{csch}(3x^2)\coth(3x^2)$$

To find the derivative of the third term, we'll use quotient rule, and apply the formulas for the derivatives of hyperbolic sine and hyperbolic tangent.

$$y' = \frac{\cosh(x^3)(3x^2)\tanh(2x^7) - \sinh(x^3)\text{sech}^2(2x^7)(14x^6)}{(\tanh(2x^7))^2}$$

$$y' = \frac{3x^2\cosh(x^3)\tanh(2x^7) - 14x^6\sinh(x^3)\text{sech}^2(2x^7)}{\tanh^2(2x^7)}$$

Putting these derivatives together gives the derivative for the original function.

$$y' = -32x^{-5} - 6x\text{csch}(3x^2)\coth(3x^2)$$

$$+\frac{3x^2\cosh(x^3)\tanh(2x^7) - 14x^6\sinh(x^3)\text{sech}^2(2x^7)}{\tanh^2(2x^7)}$$

Inverse hyperbolic derivatives

Lastly, we want to be able to find the derivatives of the inverse hyperbolic functions.

Just like with the inverse trig functions, we can express inverse hyperbolic functions in a couple of different ways. For example, inverse hyperbolic sine can be written as $\sinh^{-1} x$, or as $\text{arcsinh}x$.

The derivatives of the inverse hyperbolic functions are given in the table.

Inverse function	**Derivative**	**Restriction**
$y = \sinh^{-1} x$	$y' = \dfrac{1}{\sqrt{x^2+1}}$	
$y = \cosh^{-1} x$	$y' = \dfrac{1}{\sqrt{x^2-1}}$	$x > 1$
$y = \tanh^{-1} x$	$y' = \dfrac{1}{1-x^2}$	$\lvert x \rvert < 1$
$y = \text{csch}^{-1} x$	$y' = -\dfrac{1}{\lvert x \rvert \sqrt{x^2+1}}$	$x \neq 0$
$y = \text{sech}^{-1} x$	$y' = -\dfrac{1}{x\sqrt{1-x^2}}$	$0 < x < 1$
$y = \coth^{-1} x$	$y' = \dfrac{1}{1-x^2}$	$\lvert x \rvert > 1$

As always, if the argument of the inverse hyperbolic function is anything other than x, then the derivative of the argument will be something other than 1, which means that applying the chain rule will actually have an affect on the value of the derivative.

So, if the argument of the hyperbolic function is something other than x, then the derivatives will be given by the formulas in this table:

Inverse function	**Derivative**	**Restrictions**
$y = \sinh^{-1}[g(x)]$	$y' = \dfrac{g'(x)}{\sqrt{[g(x)]^2 + 1}}$	
$y = \cosh^{-1}[g(x)]$	$y' = \dfrac{g'(x)}{\sqrt{[g(x)]^2 - 1}}$	$g(x) > 1$
$y = \tanh^{-1}[g(x)]$	$y' = \dfrac{g'(x)}{1 - [g(x)]^2}$	$\lvert g(x) \rvert < 1$
$y = \operatorname{csch}^{-1}[g(x)]$	$y' = -\dfrac{g'(x)}{\lvert g(x) \rvert \sqrt{[g(x)]^2 + 1}}$	$g(x) \neq 0$
$y = \operatorname{sech}^{-1}[g(x)]$	$y' = -\dfrac{g'(x)}{g(x)\sqrt{1 - [g(x)]^2}}$	$0 < g(x) < 1$
$y = \coth^{-1}[g(x)]$	$y' = \dfrac{g'(x)}{1 - [g(x)]^2}$	$\lvert g(x) \rvert > 1$

With these formulas in mind, let's try an example where we find the derivative of an inverse hyperbolic function.

Example

Find the derivative of the inverse hyperbolic function.

$$y = -8\coth^{-1}(21x^3)$$

Apply the formula for the derivative of inverse hyperbolic cotangent with $g(x) = 21x^3$, remembering to apply chain rule.

$$y' = \frac{g'(x)}{1 - [g(x)]^2}$$

$$y' = -8\left(\frac{1}{1 - (21x^3)^2}\right)(63x^2)$$

$$y' = -\frac{504x^2}{1 - 441x^6}$$

Now let's try an example with an inverse hyperbolic function occurring as part of a larger function.

Example

Find the derivative of the function.

$$y = 6x^{-4} - \cosh^{-1}(4x^7)$$

Take the derivative of one term at a time, applying the formula for the derivative of inverse hyperbolic cosine with $g(x) = 4x^7$,

$$y' = \frac{g'(x)}{\sqrt{[g(x)]^2 - 1}}$$

and remembering to apply chain rule.

$$y' = -24x^{-5} - \left(\frac{1}{\sqrt{(4x^7)^2 - 1}}\right)(28x^6)$$

$$y' = -\frac{24}{x^5} - \frac{28x^6}{\sqrt{16x^{14} - 1}}$$

Let's try one more example that's a little more complex.

Example

Find the derivative of the function.

$$y = \text{sech}^{-1}(81x^4) - 5x^{-9}\sinh^{-1}(6x^7) + 103x^8$$

Apply the formulas for the derivative of inverse hyperbolic secant with $g(x) = 81x^4$ and inverse hyperbolic sine with $g(x) = 6x^7$. We'll also need to use product rule for the second term.

$$y' = \left(-\frac{1}{81x^4\sqrt{1-(81x^4)^2}}\right)(324x^3)$$

$$-\left[(-45x^{-10})(\sinh^{-1}(6x^7)) + (5x^{-9})\left(\frac{1}{\sqrt{(6x^7)^2+1}}\right)(42x^6)\right] + 824x^7$$

$$y' = -\frac{324x^3}{81x^4\sqrt{1-(81x^4)^2}} - \left(-45x^{-10}\sinh^{-1}(6x^7) + \frac{210x^{-9}x^6}{\sqrt{(6x^7)^2+1}}\right) + 824x^7$$

$$y' = -\frac{324x^3}{81x^4\sqrt{1-(81x^4)^2}} + 45x^{-10}\sinh^{-1}(6x^7) - \frac{210x^{-9}x^6}{\sqrt{(6x^7)^2+1}} + 824x^7$$

$$y' = \frac{45\sinh^{-1}(6x^7)}{x^{10}} - \frac{4}{x\sqrt{1-6{,}561x^8}} - \frac{210}{x^3\sqrt{36x^{14}+1}} + 824x^7$$

Logarithmic differentiation

Some problems are easiest to solve using logarithmic differentiation. **Logarithmic differentiation** is a problem-solving method in which we start by applying the natural log function to both sides of the equation. We use logarithmic differentiation when it's easier to differentiate the logarithm of a function than the function itself.

If we let $y = f(x)$, then we take the natural log of both sides, differentiate both sides using chain rule, and work toward rewriting the equation so that it's solved for y'.

$$\ln y = \ln f(x)$$

$$(\ln y)' = (\ln f(x))'$$

$$\frac{1}{y}y'(x) = (\ln(f(x)))'$$

$$y'(x) = y(\ln f(x))'$$

$$y'(x) = f(x)(\ln f(x))'$$

Oftentimes, we'll utilize laws of logarithms in order to simplify one or both sides of the equation. As a reminder, these are the laws of logs we'll want to use:

Laws of logs	**Laws of natural logs**
$\log_a a^x = x$	$\ln(e^x) = x$
$a^{\log_a x} = x$	$e^{\ln x} = x$

$$\log_a x^r = r\log_a x \qquad \ln(x^a) = a\ln x$$

$$\log_a(xy) = \log_a x + \log_a y \qquad \ln(xy) = \ln x + \ln y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y \qquad \ln\frac{x}{y} = \ln x - \ln y$$

The easiest way to illustrate this method is to work through an example.

Example

Use logarithmic differentiation to find the derivative of the function.

$$y = \frac{(\ln x)^x}{2^{3x+1}}$$

To start, we'll apply the natural log to both sides of the equation.

$$\ln y = \ln\left(\frac{(\ln x)^x}{2^{3x+1}}\right)$$

Use laws of logs to rewrite the right-hand side.

$$\ln y = \ln((\ln x)^x) - \ln(2^{3x+1})$$

$$\ln y = x\ln(\ln x) - (3x+1)\ln 2$$

$$\ln y = x\ln(\ln x) - 3x\ln 2 - \ln 2$$

Now we'll take the derivative of both sides. We'll need to use product rule to differentiate $x\ln(\ln x)$.

$$\frac{1}{y}y' = \left[(1)(\ln(\ln x)) + (x)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)\right] - 3\ln 2 - 0$$

$$\frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x} - 3\ln 2$$

$$\frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x} - \ln(2^3)$$

We want to solve for y', so we'll multiply both sides by y in order to get y' by itself.

$$y' = y\left[\ln(\ln x) + \frac{1}{\ln x} - \ln 8\right]$$

Now we'll use the original equation to substitute for y.

$$y' = \frac{(\ln x)^x}{2^{3x+1}}\left[\ln(\ln x) + \frac{1}{\ln x} - \ln 8\right]$$

Since these are a little tricky, let's do one more example.

Example

Use logarithmic differentiation to find the derivative of the function.

$$y = x^{\left(x^{\left(x^4\right)}\right)}$$

To start, we'll apply the natural log to both sides of the equation.

$$\ln y = \ln(x^{(x^{(x^4)})})$$

Use laws of logs to rewrite the right-hand side.

$$\ln y = (x^{(x^4)})\ln x$$

Apply the natural log to both sides again.

$$\ln(\ln y) = \ln((x^{(x^4)})\ln x)$$

Use laws of logs to rewrite the right-hand side.

$$\ln(\ln y) = \ln(x^{(x^4)}) + \ln(\ln x)$$

$$\ln(\ln y) = x^4 \ln x + \ln(\ln x)$$

Now we'll take the derivative of both sides. We'll need to use product rule to differentiate $x^4 \ln x$.

$$\left(\frac{1}{\ln y}\right)\left(\frac{1}{y}\right)(y') = \left[(x^4)\left(\frac{1}{x}\right) + (4x^3)(\ln x)\right] + \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$$

$$\frac{1}{y \ln y}y' = x^3 + 4x^3 \ln x + \frac{1}{x \ln x}$$

We want to solve for y', so we'll multiply both sides by $y \ln y$ in order to get y' by itself.

$$y' = y \ln y \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x}\right)$$

Now we'll use the original equation to substitute for y.

$$y' = x^{(x^{(x^4)})} \ln(x^{(x^{(x^4)})}) \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x} \right)$$

$$y' = x^{(x^{(x^4)})}(x^{(x^4)}) \ln x \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x} \right)$$

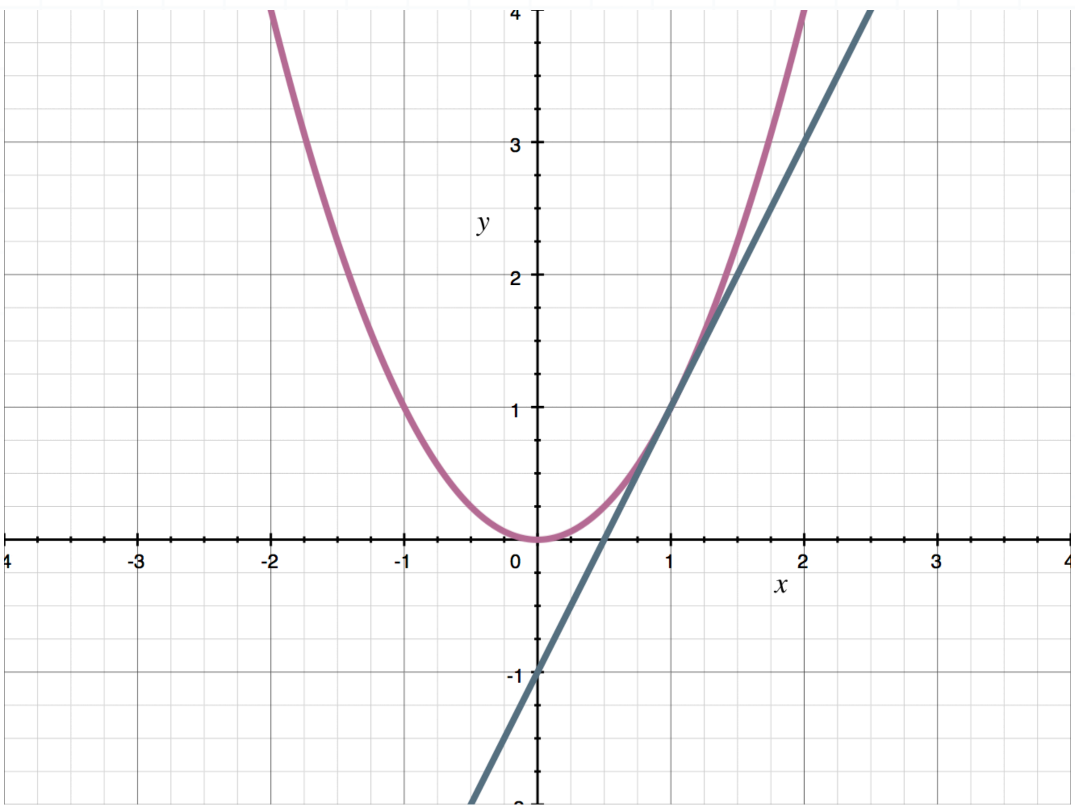
$$y' = x^{(x^{(x^4)}+x^4)} \ln x \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x} \right)$$

$$y' = x^{(x^{(x^4)}+x^4)} \left(x^3 \ln x + 4x^3 \ln^2 x + \frac{1}{x} \right)$$

Tangent lines

We briefly talked about the tangent line when we first introduced the derivative, but now we want to spend more time talking about how to find its equation.

Remember that a **tangent line** is a line that touches the graph of a function at exactly one point. For instance, a tangent line might look like



The line comes alongside the graph, and touches the curve at exactly $x = 1$. The tangent line doesn't cross the graph, but stays on the same side of the curve.

As long as it's defined, we can find the equation of the tangent line to any curve, at any point on the curve. No matter which curve we're using, or where along that curve we're finding the tangent line, the equation of the tangent line will be

$$y = f(a) + f'(a)(x - a)$$

In this equation, $x = a$ is the x-value at which the tangent line intersects the curve. So $f(a)$ is the y-value where the tangent line intersects the curve, and $f'(a)$ is the value of the function's derivative at $x = a$, or the slope of the tangent line to $f(x)$ at $x = a$.

Vertical and horizontal tangent lines

If $f'(a)$ is undefined, then the tangent line is vertical. If $f'(a) = 0$, then the tangent line is horizontal.

- To find the equation of a vertical tangent line, first find the derivative of $f(x)$, $f'(x)$. If $f'(a)$ is undefined, then the equation of the vertical tangent line will be $x = a$.

- To find the equation of a horizontal tangent line, first find the derivative of $f(x)$, $f'(x)$, then solve $f'(a) = 0$. Find the y-value where the tangent line intersects the curve, $f(a)$, then the equation of the horizontal tangent line will be $y = f(a)$.

Let's walk through a full example, so that we can see step-by-step how to find the equation of a tangent line.

Example

Find the equation of the tangent line to $f(x)$ at $x = 4$.

$$f(x) = 6x^2 - 2x + 5$$

First, plug $x = 4$ into the original function.

$$f(4) = 6(4)^2 - 2(4) + 5$$

$$f(4) = 96 - 8 + 5$$

$$f(4) = 93$$

Next, take the derivative and then evaluate the derivative at $x = 4$.

$$f'(x) = 12x - 2$$

$$f'(4) = 12(4) - 2$$

$$f'(4) = 46$$

Finally, substitute both $f(4)$ and $f'(4)$ into the tangent line formula, along with $a = 4$, since this is the value at which we're finding the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = 93 + 46(x - 4)$$

We can either leave the equation in this form, or we can simplify it further.

$$y = 93 + 46x - 184$$

$$y = 46x - 91$$

Let's do one more example, but with a different type of function.

Example

Find the equation of the tangent line to $f(x)$ at $x = 0$.

$$f(x) = 3\sin x$$

First, plug $x = 0$ into the original function.

$$f(0) = 3\sin(0)$$

$$f(0) = 3(0)$$

$$f(0) = 0$$

Next, take the derivative and then evaluate the derivative at $x = 0$.

$$f'(x) = 3\cos x$$

$$f'(0) = 3\cos(0)$$

$$f'(0) = 3(1)$$

$$f'(0) = 3$$

Finally, substitute both $f(0)$ and $f'(0)$ into the tangent line formula, along with $a = 0$, since this is the value at which we're finding the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = 0 + 3(x - 0)$$

$$y = 3x$$

Value that makes two tangent lines parallel

Now that we know how to find the equation of a tangent line, we can think about the relationship between multiple tangent lines of the same function.

For instance, we might be interested in the points at which two tangent lines of the function are parallel to one another. Remember that parallel lines have the same slope, so two parallel tangent lines will have the same slope.

To find the slope of a tangent line of a function, we differentiate the function, and then evaluate it at the point of tangency.

Example

Find the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 1$ are parallel.

$$f(x) = x^3 + 2x^2 - 3x + 1$$

If the tangent lines are parallel, they must have the same slope. So, if two tangent lines at $x = a$ and $x = a + 1$ are parallel, it means their slopes are equal, which means the value of the function's derivative will be equal at $x = a$ and $x = a + 1$.

So we'll start by finding the derivative of $f(x)$.

$$f'(x) = 3x^2 + 4x - 3$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 3a^2 + 4a - 3$$

$$f'(a+1) = 3(a+1)^2 + 4(a+1) - 3$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 4a - 3 = 3(a+1)^2 + 4(a+1) - 3$$

$$3a^2 + 4a - 3 = 3(a^2 + 2a + 1) + 4a + 4 - 3$$

$$3a^2 + 4a - 3 = 3a^2 + 6a + 3 + 4a + 4 - 3$$

Collect like terms and solve for a.

$$4a - 3 = 6a + 3 + 4a + 4 - 3$$

$$4a - 3 = 6a + 4a + 4$$

$$4a - 3 = 10a + 4$$

$$-3 = 6a + 4$$

$$-7 = 6a$$

$$a = -\frac{7}{6}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = -\frac{7}{6} + 1$$

$$a + 1 = -\frac{7}{6} + \frac{6}{6}$$

$$a + 1 = -\frac{1}{6}$$

Therefore, the function has parallel tangent lines one unit apart at $x = -7/6$ and $x = -1/6$.

Let's do another example where we find both tangent line equations, instead of just the points at which the tangent lines are parallel.

Example

Find the equations of the tangent lines to $f(x)$ at $x = a$ and $x = a + 2$, if those tangent lines are parallel.

$$f(x) = 2x^3 + 8x - 2$$

We'll start by finding the derivative of $f(x)$.

$$f'(x) = 6x^2 + 8$$

Now we'll plug both $x = a$ and $x = a + 2$ into the derivative.

$$f'(a) = 6a^2 + 8$$

$$f'(a + 2) = 6(a + 2)^2 + 8$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$6a^2 + 8 = 6(a+2)^2 + 8$$

$$6a^2 + 8 = 6(a^2 + 4a + 4) + 8$$

$$6a^2 + 8 = 6a^2 + 24a + 24 + 8$$

Collect like terms and solve for a.

$$8 = 24a + 24 + 8$$

$$0 = 24a + 24$$

$$24a = -24$$

$$a = -1$$

If this is the value of a, then $a + 2$ is

$$a + 2 = -1 + 2$$

$$a + 2 = 1$$

Therefore, the function has parallel tangent lines two units apart at $x = -1$ and $x = 1$.

Now that we know where the tangent lines are located, we can find their equations. For the tangent line at $a = -1$, we'll need $f(a)$ and $f'(a)$.

$$f(x) = 2x^3 + 8x - 2$$

$$f(-1) = 2(-1)^3 + 8(-1) - 2$$

$$f(-1) = 2(-1) - 8 - 2$$

$$f(-1) = -2 - 8 - 2$$

$$f(-1) = -12$$

and

$$f'(a) = 6a^2 + 8$$

$$f'(-1) = 6(-1)^2 + 8$$

$$f'(-1) = 6 + 8$$

$$f'(-1) = 14$$

Then the equation of the tangent line at $a = -1$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = -12 + 14(x - (-1))$$

$$y = -12 + 14(x + 1)$$

$$y = -12 + 14x + 14$$

$$y = 14x + 2$$

For the tangent line at $a = 1$, we'll need $f(a)$ and $f'(a)$.

$$f(x) = 2x^3 + 8x - 2$$

$$f(1) = 2(1)^3 + 8(1) - 2$$

$$f(1) = 2 + 8 - 2$$

$$f(1) = 8$$

and

$$f'(a) = 6a^2 + 8$$

$$f'(1) = 6(1)^2 + 8$$

$$f'(1) = 6 + 8$$

$$f'(1) = 14$$

Then the equation of the tangent line at $a = 1$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = 8 + 14(x - 1)$$

$$y = 8 + 14x - 14$$

$$y = 14x - 6$$

So the equations of the parallel tangent lines at $x = -1$ and $x = 1$ are

$$y = 14x + 2$$

$$y = 14x - 6$$

Remember that you can always double-check your answers to problems like these by graphing the given function and the two tangent line equations, to verify visually that the tangent lines look parallel.

Values that make the function differentiable

In this lesson, we want to look at another common derivative problem, similar to the way that we tackled parallel tangent lines in the last lesson.

In these kinds of problems, we're given a piecewise-defined function that includes some unknown constant(s), and asked to find the values of those constants that will force the differentiability of the piecewise function.

A function is differentiable at a particular point if it's continuous and has a well-defined derivative at that point.

1. A piecewise function will be continuous at its break point if the one-sided limits of the function's value at the break point are equal.

2. A piecewise function will have a well-defined derivative at the break point if the one-sided limits of the derivatives from each piece at the break point are equal.

So, in order to solve problems like these, we'll find the condition that forces the continuity of the piecewise function at its breakpoint, we'll find the condition that forces the derivative to be well-defined at its breakpoint, and then we'll use these conditions as a system of equations to solve for the unknown constant(s) that satisfy both conditions.

Let's work through an example so that we can actually break down the process we've just described.

Example

Find the values of a and b that make the piecewise-defined function differentiable.

$$f(x) = \begin{cases} 2x^2 - ax + b & x \le 1 \\ 3x^3 - x^2 - 6 & x > 1 \end{cases}$$

The break point of the function is at $x = 1$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 1$ equal to one another.

$$\lim_{x\to 1^-} (2x^2 - ax + b) = \lim_{x\to 1^+} (3x^3 - x^2 - 6)$$

$$2(1)^2 - a(1) + b = 3(1)^3 - 1^2 - 6$$

$$2 - a + b = 3 - 1 - 6$$

$$2 - a + b = -4$$

$$-a + b = -6$$

Now we'll work on the well-defined derivative by setting the one-sided limits of the derivatives of each piece at the break point $x = 1$ equal to one another.

$$\lim_{x\to 1^-} (4x - a) = \lim_{x\to 1^+} (9x^2 - 2x)$$

$$4(1) - a = 9(1)^2 - 2(1)$$

$$4 - a = 9 - 2$$

$$4 - a = 7$$

$$-a = 3$$

$$a = -3$$

Pull together these two equations into a system of equations.

$$a = -3$$

$$-a + b = -6$$

We need to solve the system, which we can do by substituting the first equation $a = -3$ into the second equation.

$$-(-3) + b = -6$$

$$3 + b = -6$$

$$b = -9$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = -3$ and $b = -9$.

Normal lines

We know how to find the equation of the tangent line, but in this lesson we want to turn toward the equation of the normal line.

Equation of the normal line

For every tangent line, we can find a corresponding normal line, because the **normal line** to a function at a particular point is the line that's perpendicular to the tangent line to the function at that same point.

So if the slope of the tangent line is m, then the slope of the normal line is the negative reciprocal of m, or $-1/m$.

We can find the equation of the normal line by following these steps:

1. Take the derivative of the original function, and evaluate it at the given point. This is the slope of the tangent line, which we'll call m.

2. Find the negative reciprocal of m, which will be $-1/m$. This is the slope of the normal line, which we'll call n. So $n = -1/m$.

3. Plug n and the given point into the point-slope formula for the equation of the line, $(y - y_1) = n(x - x_1)$.

4. Simplify the normal line equation by solving for y.

Let's do an example where we walk through these steps in order to find the equation of the normal line.

Example

Find the equation of the normal line to the function $f(x)$ at $(1,9)$.

$$f(x) = 6x^2 + 3$$

Let's follow the steps we just outlined. First, we'll take the derivative of the function, and then evaluate it at $(1,9)$.

$$f'(x) = 12x$$

$$f'(1) = 12(1)$$

$$f'(1) = 12$$

This is the slope of the tangent line at $(1,9)$. Since $m = 12$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{1}{12}$$

We'll plug $n = -1/12$ and the point $(1,9)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(1,9)$.

$$y - y_1 = n(x - x_1)$$

$$y - 9 = -\frac{1}{12}(x - 1)$$

$$12y - 108 = -(x - 1)$$

$$12y - 108 = -x + 1$$

$$12y = -x + 109$$

$$y = -\frac{1}{12}x + \frac{109}{12}$$

Average rate of change

So far, we've been calculating the slope of a function at a specific point, by finding the slope of the tangent line there. When we look at the slope of the function at an exact point, we're looking at the rate of change at that exact point, so we call that the **instantaneous rate of change**.

In contrast, we can look at the rate of change over a larger interval, instead of at one specific point. When we look at rate of change over an interval, we call that the **average rate of change**.

Average rate of change

When we calculate average rate of change of a function over a given interval, we're calculating the average number of units that the function moves up or down along the y-axis, per unit along the x-axis.

We could also say that we're measuring how much change occurs in the function's value per unit of the x-axis.

To find the average rate of change of a function $f(x)$ over an interval $[a, b]$, we'll first calculate the value of the function at both ends of the interval. Then we plug those values and the ends of the interval into the formula for average rate of change,

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Let's do an example where we find the average rate of change.

Example

Find the average rate of change of $f(x)$ on the interval $[0,4]$.

$$f(x) = 2x^2 - 2$$

From the interval, we know $x_1 = 0$ and $x_2 = 4$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into $f(x) = 2x^2 - 2$. We get

$$f(0) = 2(0)^2 - 2$$

$$f(0) = 0 - 2$$

$$f(0) = -2$$

and

$$f(4) = 2(4)^2 - 2$$

$$f(4) = 2(16) - 2$$

$$f(4) = 30$$

Now we can plug the values we've found into the formula for average rate of change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(4) - f(0)}{4 - 0}$$

$$\frac{\Delta f}{\Delta x}=\frac{30-(-2)}{4}$$

$$\frac{\Delta f}{\Delta x}=\frac{32}{4}$$

$$\frac{\Delta f}{\Delta x}=8$$

The average rate of change of $f(x)$ on the interval $[0,4]$ is 8.

If we describe what that means, we can say that the function increases by 32 units, from $f(0)=-2$ up to $f(4)=30$, between $x=0$ and $x=4$. If the function increases by 32 units over a four-unit span, $[0,4]$, it means the function increases by an average of 8 units, for each unit we move along the x-axis between $x=0$ and $x=4$.

Implicit differentiation

Up to now, we've been differentiating functions defined for $f(x)$ in terms of x, or equations defined for y in terms of x. In other words, every equation we've differentiated has had the variables separated on either side of the equal sign.

For instance, the equation $y = 3x^2 + 2x + 1$ has the y variable on the left side, and the x variable on the right side. We don't have x and y variables mixed together on the left, and they aren't mixed together on the right, either.

Separable or not

We sometimes have equations where the variables are mixed together, but they can be easily separated. If we're given the equation $y - 2x = 3x^2 + 1$, we do have x and y variables mixed together on the left side, but we can easily separate the variables by simply adding $2x$ to both sides of the equation to get $y = 3x^2 + 2x + 1$.

Other times, we'll have the variables mixed together, and it's actually impossible to separate them. For instance, the equation $xy = 3(x - y)^2 + 2x + 1$ can't be rewritten with all the y variables on the left and all the x variables on the right. When the variables can't be separated, we can use implicit differentiation to find the function's derivative.

In other words, **implicit differentiation** allows us to take the derivative of a function that contains both x and y on the same side of the equation.

How to use implicit differentiation

When we use implicit differentiation, we have to treat y differently than we have in the past. With implicit differentiation, we treat y as a function and not just as a variable. We treat x just as we have before, as a variable, but we treat y as a function.

Practically, this means that each time we take the derivative of y, we multiply the result by the derivative of y. We can write the derivative of y as either y' or as dy/dx.

To use implicit differentiation, we'll follow these steps:

1. Differentiate both sides with respect to x.

2. Whenever we encounter y, we differentiate it like we would x, but we multiply that term by the derivative of y, which we write as y' or as dy/dx.

3. Move all terms involving dy/dx to the left side and everything else to the right.

4. Factor out dy/dx on the left and divide both sides by the other left-side factor so that dy/dx is the only thing remaining on the left.

Once we get dy/dx (or y') alone on the left, we've solved for the derivative of y', which was our goal when we started differentiating.

Let's walk through an example so that we can see how this set of steps gets us to the derivative.

Example

Use implicit differentiation to find the derivative.

$$x^3 + y^3 = 9xy$$

We'll differentiate both sides with respect to x. When we do, we'll treat x as we normally do, but we'll treat y as a function.

Working term-by-term, we take the derivative of x^3 like we normally would and we get $3x^2$. When we take the derivative of y^3, we get $3y^2$, but since we took the derivative of y, we have to multiply by dy/dx.

To take the derivative of the right side of the equation, we need to use product rule, since x and y are both variables and therefore need to be treated as separate functions. So we'll say that one function is $9x$ and that the other is y. The derivative of $9x$ will be 9, like normal. The derivative of y would be 1, but since we're taking the derivative of y, we have to multiply by dy/dx.

Therefore, after implicit differentiation, the derivative looks like this:

$$3x^2 + 3y^2\frac{dy}{dx} = (9)(y) + (9x)(1)\frac{dy}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

Move all terms that include dy/dx to the left side, and move everything else to the right side.

$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

Factor out dy/dx on the left,

$$\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2$$

and then divide both sides by $(3y^2 - 9x)$ in order to get dy/dx by itself.

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

We can leave the derivative this way, or we can replace dy/dx with y' if we prefer that notation.

$$y' = \frac{3y - x^2}{y^2 - 3x}$$

Notice how this answer looks similar to all the derivatives we've found before. It's a derivative equation that's solved for y' on the left side, which is exactly what we're used to seeing.

In the past, we were able to easily get the derivative in this form, because the equations we were differentiating were already solved for y. As we've seen from this example, when we get an equation that isn't already solved for y, we can still get to the derivative equation we want, we just have to use implicit differentiation to get there.

Equation of the tangent line with implicit differentiation

In the same way we learned how to find the equation of the tangent line previously, we can find the equation of a tangent line for a function, even when we have to use implicit differentiation to find the function's derivative.

For instance, in the last lesson, we used implicit differentiation to find that the derivative of $x^3 + y^3 = 9xy$ was

$$y' = \frac{3y - x^2}{y^2 - 3x}$$

If we wanted to find the equation of the tangent line to $x^3 + y^3 = 9xy$ at the point $(2,4)$, we'd find the slope of the tangent line by substituting the point into the derivative, y'.

$$y'(2,4) = \frac{3(4) - 2^2}{4^2 - 3(2)}$$

$$y'(2,4) = \frac{12 - 4}{16 - 6}$$

$$y'(2,4) = \frac{8}{10}$$

$$y'(2,4) = \frac{4}{5}$$

Once we have the slope at the point of tangency, we plug the slope $m = 4/5$ and the point $(2,4)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{4}{5}(x - 2)$$

$$y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x - \frac{8}{5} + \frac{20}{5}$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

Finding the tangent line equation

From this, we can generalize the set up steps we can use every time in order to find the equation of the tangent line to an implicitly-defined function at a particular point.

1. Find the derivative using implicit differentiation.

2. Evaluate the derivative at the given point to find the slope of the tangent line.

3. Plug the slope of the tangent line and the given point into the point-slope formula for the equation of a line, $y - y_1 = m(x - x_1)$.

4. Simplify the tangent line equation.

Let's do another full example, starting from the beginning with a new function.

Example

Find the equation of the tangent line at $(1,2)$.

$$3y^2 - 2x^5 = 10$$

We could actually solve this equation for y, and then differentiate using traditional techniques, but it's a little tedious to do it that way. It's probably faster for us if we just leave the equation the way it's written, and use implicit differentiation.

Remember that whenever we use implicit differentiation to take the derivative of a term involving y, we have to multiply the result by the derivative of y, which we can write as dy/dx or as y'. The derivative is

$$6y\frac{dy}{dx} - 10x^4 = 0$$

Now we'll simplify and solve for dy/dx.

$$6y\frac{dy}{dx} = 10x^4$$

$$\frac{dy}{dx} = \frac{10x^4}{6y}$$

$$\frac{dy}{dx} = \frac{5x^4}{3y}$$

Keep in mind that, when we're finding the equation of the tangent line, we actually don't need to solve explicitly for dy/dx. Once we use implicit differentiation, we can move directly to plugging in the point of tangency, without first solving for dy/dx. Either way, whether we solve for dy/dx before or after plugging in the point of tangency, our next step is to plug in $(1,2)$.

$$\frac{dy}{dx}(1,2)=\frac{5(1)^4}{3(2)}$$

$$\frac{dy}{dx}(1,2)=\frac{5}{6}$$

Now that we have the slope of the tangent line $m=5/6$, and the point of tangency, we can plug directly into the point-slope formula for the equation of the tangent line.

$$y-y_1=m(x-x_1)$$

$$y-2=\frac{5}{6}(x-1)$$

$$y-2=\frac{5}{6}x-\frac{5}{6}$$

$$y=\frac{5}{6}x-\frac{5}{6}+\frac{12}{6}$$

$$y=\frac{5}{6}x+\frac{7}{6}$$

Higher-order derivatives

Up to now, we've only dealt with first derivatives. The **first derivative** is exactly the value we've been finding all along; it's what you get when you take the derivative.

In this lesson, we want to turn to **higher-order derivatives**, which are the second derivative, third derivative, fourth derivative, etc.

The **second derivative** is the derivative of the derivative; it's the derivative of the first derivative. The **third derivative** is the derivative of the second derivative, the **fourth derivative** is the derivative of the third derivative, etc.

As long as each derivatives continues to be differentiable, theoretically, there's no limit to the number of derivatives we can find. Each derivative models the slope of the derivative before it. So, in the same way that the derivative models the slope of the original function, the second derivative models the slope of the first derivative, the third derivative models the slope of the second derivative, etc.

Function	**What it models**
$f(x)$	
$f'(x)$	Models the slope of $f(x)$
$f''(x)$	Models the slope of $f'(x)$
$f'''(x)$	Models the slope of $f''(x)$

$f^{(4)}(x)$	Models the slope of $f'''(x)$
...	...

Rules and notation

We use all the same rules for finding higher-order derivatives as we did to find the first derivative. In other words, the power, product, quotient, and chain rules still apply in the same way. And taking derivatives of trigonometric, exponential, logarithmic functions, etc., will be the same in higher-order derivative functions as it was in the original function.

It's nice to keep our derivative notation consistent.

Function	**1st derivative**	**2nd derivative**	**3rd derivative**
y	y'	y''	y'''
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$

Applications

Higher-order derivatives are important in calculus and in real-life applications. Later we'll look in depth at how to use derivatives to sketch the graph of a function. The first derivative gives us information about where the function is increasing and decreasing, and the second derivative gives us information about where the function is concave up and down.

We'll also look at position functions. When an equation models the position of an object, its first derivative models the object's velocity, and its second derivative models the object's acceleration.

Let's work through an example so that we can see how to get to the second derivative of a function.

Example

Find the second derivative of the function.

$$y = x^3 + 3x^2 - 4x + 5$$

Start by using power rule to find the first derivative.

$$y = x^3 + 3x^2 - 4x + 5$$

$$y' = 3x^2 + 6x - 4$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (3x^2 + 6x - 4)'$$

$$y'' = 6x + 6$$

For practice, let's try one more example. This time, we'll use the dy/dx notation for all of our derivatives.

Example

Find the function's second and the third derivatives.

$$y = \cos^2 x$$

Use chain rule to find the first derivative.

$$y = \cos^2 x$$

$$\frac{dy}{dx} = 2\cos x(-\sin x)$$

$$\frac{dy}{dx} = -2\sin x\cos x$$

Now we can use product rule to find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2\sin x\cos x)$$

$$\frac{d^2y}{dx^2} = -2\cos x\cos x + (-2\sin x(-\sin x))$$

$$\frac{d^2y}{dx^2} = -2\cos^2 x + 2\sin^2 x$$

$$\frac{d^2y}{dx^2} = 2\sin^2 x - 2\cos^2 x$$

Find the third derivative by taking the derivative of the second derivative.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(2\sin^2 x - 2\cos^2 x)$$

$$\frac{d^3y}{dx^3} = 2(2\sin x)(\cos x) - 2(2\cos x)(-\sin x)$$

$$\frac{d^3y}{dx^3} = 4\sin x\cos x + 4\sin x\cos x$$

$$\frac{d^3y}{dx^3} = 8\sin x\cos x$$

Second derivatives with implicit differentiation

We use the same rules for finding the second derivative as we did to find the first derivative. That means that, when we're using implicit differentiation to take the derivative of an implicitly-defined function, we'll use implicit differentiation to find the first derivative, and then we'll apply implicit differentiation again to find the second derivative.

If the second derivative includes any value of y', we can substitute the value of the first derivative into the second derivative.

Let's work through an example so that we can see how to get to the second derivative of an implicitly-defined function.

Example

Use implicit differentiation to find the second derivative.

$$2y^2 + 6x^2 = 76$$

This is a function we could solve explicitly for y, but instead of taking those extra steps up front, let's instead use implicit differentiation to find the first and second derivatives. Using implicit differentiation, the first derivative is

$$4yy' + 12x = 0$$

$$4yy' = -12x$$

Solve for y'.

$$y' = \frac{-12x}{4y}$$

$$y' = -\frac{3x}{y}$$

To find the second derivative, we need to differentiate this first derivative we've just found. Since we have a quotient, we'll use quotient rule. We still need to follow our rules for implicit differentiation, multiplying by y' each time we take the derivative of y. The second derivative will be

$$y'' = -\frac{(3)(y) - (3x)(1)(y')}{(y)^2}$$

$$y'' = -\frac{3y - 3xy'}{y^2}$$

This is the second derivative, but it includes y', which we can substitute for using the equation we found earlier for the first derivative.

$$y'' = -\frac{3y - 3x\left(-\frac{3x}{y}\right)}{y^2}$$

$$y'' = -\frac{3y + \frac{9x^2}{y}}{y^2}$$

Find a common denominator within the numerator, then combine the fractions in the numerator into one fraction.

$$y'' = -\frac{\frac{3y^2}{y} + \frac{9x^2}{y}}{y^2}$$

$$y'' = -\frac{\frac{9x^2+3y^2}{y}}{y^2}$$

$$y'' = -\frac{9x^2+3y^2}{y}\left(\frac{1}{y^2}\right)$$

$$y'' = -\frac{9x^2+3y^2}{y^3}$$

For practice, let's try one more example. This time, we'll use the dy/dx notation for the derivative.

Example

Use implicit differentiation to find the second derivative.

$$xy + 24x = 6y^2$$

Using implicit differentiation (and product rule on the xy term), the first derivative is

$$\left[(1)(y) + (x)(1)\left(\frac{dy}{dx}\right)\right] + 24 = 12y\frac{dy}{dx}$$

$$y + x\frac{dy}{dx} + 24 = 12y\frac{dy}{dx}$$

Solve for dy/dx.

$$x\frac{dy}{dx} - 12y\frac{dy}{dx} = -y - 24$$

$$\frac{dy}{dx}(x - 12y) = -y - 24$$

$$\frac{dy}{dx} = \frac{-y - 24}{x - 12y}$$

To find the second derivative, we need to differentiate this first derivative we've just found. Since we have a quotient, we'll use quotient rule. We still need to follow our rules for implicit differentiation, multiplying by dy/dx each time we take the derivative of y. The second derivative will be

$$\frac{d^2y}{dx^2} = \frac{\left(-1\frac{dy}{dx}\right)(x - 12y) - (-y - 24)\left(1 - 12\frac{dy}{dx}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{dy}{dx}(x - 12y) + (y + 24)\left(1 - 12\frac{dy}{dx}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(-x\frac{dy}{dx} + 12y\frac{dy}{dx}\right) + \left(y - 12y\frac{dy}{dx} + 24 - 288\frac{dy}{dx}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + 12y\frac{dy}{dx} + y - 12y\frac{dy}{dx} + 24 - 288\frac{dy}{dx}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y + 24 - 288\frac{dy}{dx}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y + 24 - (x + 288)\frac{dy}{dx}}{(x - 12y)^2}$$

This is the second derivative, but it includes dy/dx, which we can substitute for using the equation we found earlier for the first derivative.

$$\frac{d^2y}{dx^2} = \frac{y + 24 - (x + 288)\left(\frac{-y - 24}{x - 12y}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y + 24 + (x + 288)\left(\frac{y + 24}{x - 12y}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y + 24 + \frac{(y + 24)(x + 288)}{x - 12y}}{(x - 12y)^2}$$

Find a common denominator within the numerator, then combine the fractions in the numerator into one fraction.

$$\frac{d^2y}{dx^2} = \frac{\frac{(y + 24)(x - 12y)}{x - 12y} + \frac{(y + 24)(x + 288)}{x - 12y}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{(y + 24)(x - 12y) + (y + 24)(x + 288)}{x - 12y}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(y + 24)(x - 12y) + (y + 24)(x + 288)}{x - 12y}\left(\frac{1}{(x - 12y)^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(y + 24)(x - 12y) + (y + 24)(x + 288)}{(x - 12y)^3}$$

We can factor a $y + 24$ out to the front of the numerator in order to simplify further.

$$\frac{d^2y}{dx^2} = \frac{(y+24)\left[(x-12y)+(x+288)\right]}{(x-12y)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(y+24)(x-12y+x+288)}{(x-12y)^3}$$

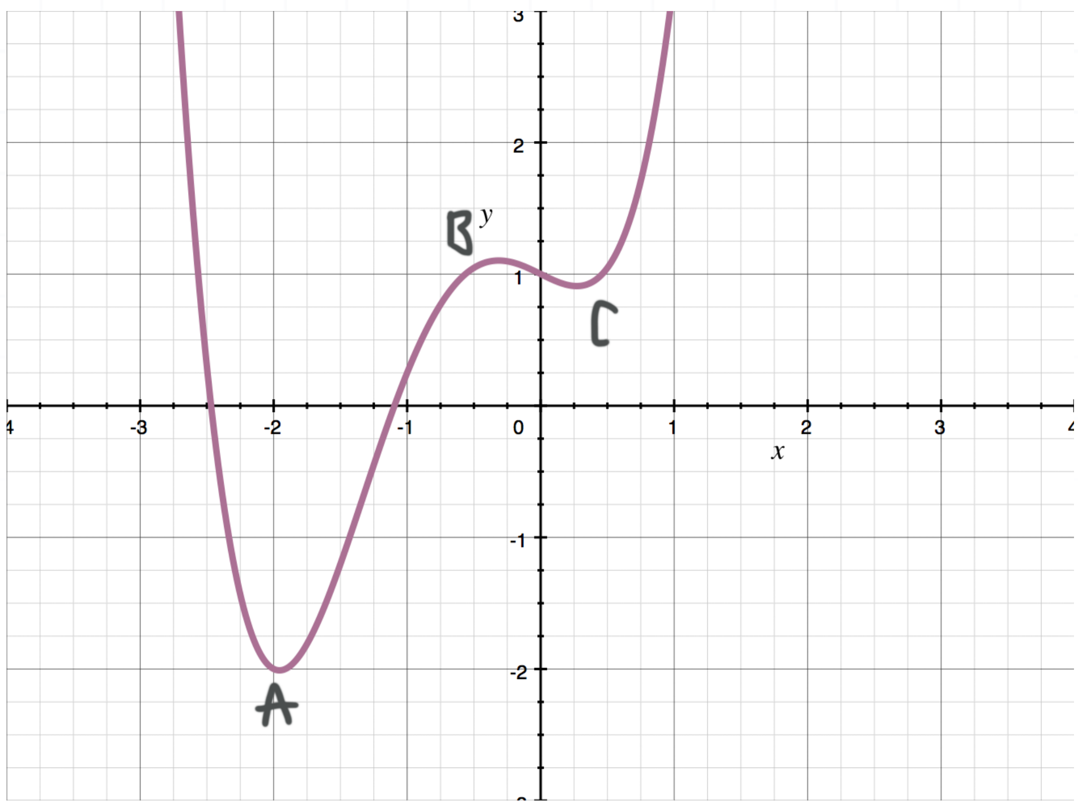
$$\frac{d^2y}{dx^2} = \frac{(y+24)(2x-12y+288)}{(x-12y)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2(y+24)(x-6y+144)}{(x-12y)^3}$$

Critical points and the first derivative test

The optimization process is all about finding a function's least and greatest values. If we use a calculator to sketch the graph of a function, we can usually spot the least and greatest values.

For example, for the function sketched out below, we can see just by looking at the graph that the function reaches its lowest value at A. This lowest possible point is the function's **global minimum**.



There's a reason why it's important that we be able to find this global minimum. Let's pretend for a moment that the function shown in the graph actually models the likelihood that food will spoil in a restaurant freezer at varying temperatures.

If I'm the business owner of this restaurant, and I want to minimize the likelihood that my food will spoil, then I'm very interested in finding this global minimum. If I know how to do the math to calculate this value, then

I'll know the exact temperature at which I should set the freezer, in order to minimize the chance that my food will spoil.

And that's really valuable! Getting this right can save me time and money, and help make sure that my restaurant runs smoothly and is successful in the long term.

That's what optimization is all about. It let's us calculate the point at which a function is maximized or minimized, and that has all kinds of real-world applications, which we'll talk about in depth later in the course.

In this section, we're going to talk all about the optimization process, starting with how to find a function's least and greatest values. We'll finish the section by translating these optimization steps into learning how to sketch the function's graph.

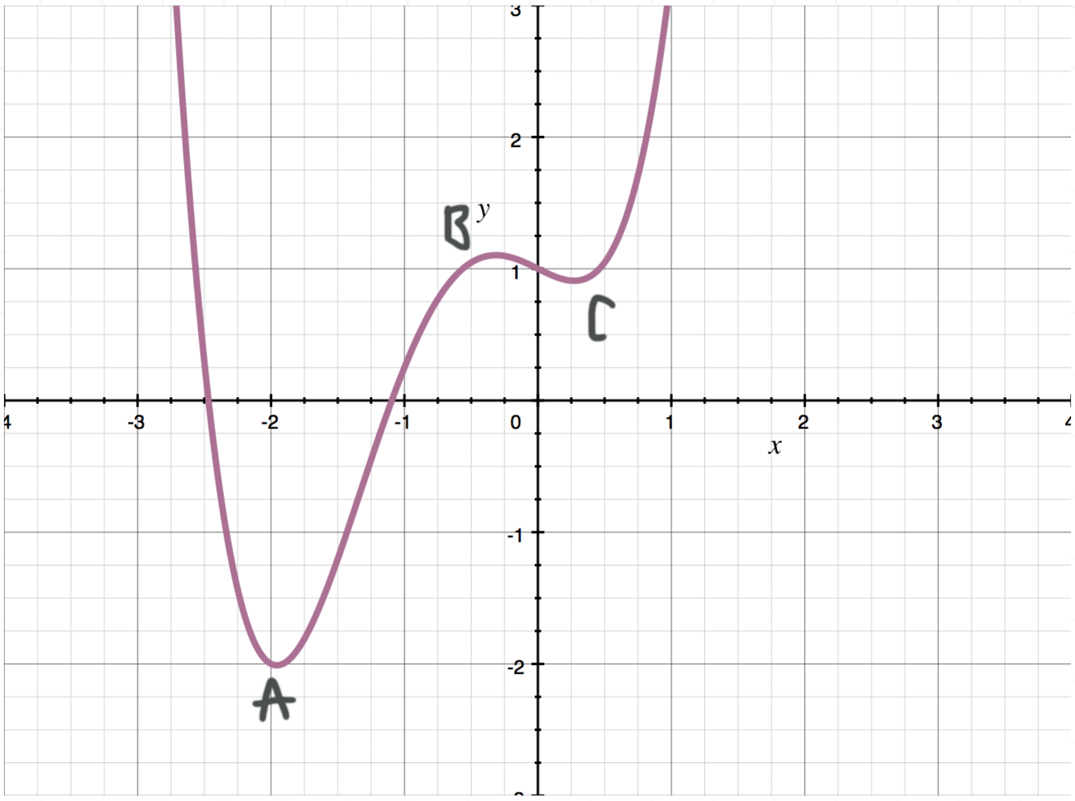
Local and global extrema

In general, a function's least and greatest values are its **extrema**. Think about the extrema as being the function's "extreme points."

The function's extrema are made up of its least points, which we call the **minima**, and its greatest points, which we call the **maxima**.

Within the function's minima, we distinguish between local (relative) minima and global (absolute) minima. And within the function's maxima, we distinguish between local (relative) maxima and global (absolute) maxima.

As an example, let's look again at the same graph as before.



We already said that the function's absolute/global minimum is at A, and that's because A is the point where the function has the lowest value over its entire domain. But the function also has a local minimum at C, because C is the function's lowest point in the area near C.

The function also has a local maximum at B, because B is the function's highest point in the area near B.

We wouldn't define an absolute maximum for the function, because it shoots up toward ∞ both to the left of A and to the right of C, so there's no finite point we could name that would describe the highest value that the function ever reaches.

So, in general,

- A **local/relative maximum** exists wherever the function changes direction from increasing to decreasing. If a local maximum also happens to be the function's highest point anywhere in its domain, then it's also the **global/absolute maximum**. A function can have infinitely many local/relative maxima, but it'll only one (or no) global/absolute maximum.

- A **local/relative minimum** exists wherever the function changes direction from decreasing to increasing. If a local minimum also happens to be the function's lowest point anywhere in its domain, then it's also the **global/absolute minimum**. A function can have infinitely many local/relative minima, but it'll only have one (or no) global/absolute minimum.

Keep in mind that not all of these extrema will exist for every function. In the graph we looked at before, the function has

- a local minimum at A and a local minimum at C
- a local maximum at B
- a global minimum at A
- no finite global maximum

But we could imagine a different function, like $y = x^2$, which is a parabola with its vertex at the origin that opens up. That parabola has only one local minimum at $x = 0$, that local minimum also happens to be the global minimum, but the parabola has no local maximum and no global maximum.

To take another example, the line $y = x$ is the line that runs through the origin with slope $m = 1$, and it has no extrema at all (no local maxima or minima, and no global maximum or minimum) because it never changes direction.

So which extrema we're able to classify will always depend on the particular function we're working with.

Critical points

The first step in any optimization process is always to find the function's critical points.

Critical points exist where the derivative is equal to 0 (or possibly where the derivative is undefined), and they represent points at which the graph is neither increasing nor decreasing.

Whenever the function changes direction, it occurs at a critical point. When this happens, the function will have at least a local maximum or minimum at the critical point, if not a global maximum or minimum there.

To find critical points, we simply take the derivative, set it equal to 0, and then solve for the variable.

Let's work through an example where we find the critical points of a function.

Example

Find the critical points of the function.

$$f(x) = x + \frac{4}{x}$$

Rewrite the function using negative exponent rules.

$$f(x) = x + 4x^{-1}$$

Use power rule to take the derivative.

$$f'(x) = 1 - 4x^{-2}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

Set the derivative equal to 0 and solve for x.

$$1 - \frac{4}{x^2} = 0$$

$$1 = \frac{4}{x^2}$$

$$x^2 = 4$$

$$x = \pm 2$$

These are the critical points of $f(x)$.

Increasing and decreasing

Because the function will only change direction, from increasing to decreasing or from decreasing to increasing at critical points, the next step is to investigate the behavior in between the critical points.

- Where the derivative is positive, the function is increasing. A function is **increasing** when it moves up as we move from left to right.

- Where the derivative is negative, the function is decreasing. A function is **decreasing** when it moves down as we move from left to right.

To test the sign of the derivative, we'll simply pick a value between each pair of critical points, and plug that test value into the derivative to see whether we get a positive result or a negative result. If the test value gives a positive result, it means the function is increasing on that interval, and if the test value gives a negative result, it means the function is decreasing on that interval.

If we find one critical point for the function, then we just need to look at the derivative's sign on the left side and right side of that one critical point.

But if we find multiple critical points, then we need to find the derivative's sign to the left of the left-most critical point, to the right of the right-most critical point, and between each critical point.

Let's continue with one of the previous examples, looking at the sign of the derivative between each critical point.

Example

The critical points of the function are $x = \pm 2$. Where is the function increasing and where is it decreasing?

$$f(x) = x + \frac{4}{x}$$

Previously, we used the derivative to find that the function had critical points at $x = \pm 2$. Once we have the critical points, it's helpful to plot them along a number line from least to greatest, left to right.

From this diagram, we can see that we have to test three intervals.

$$-\infty < x < -2$$

$$-2 < x < 2$$

$$2 < x < \infty$$

To test $-\infty < x < -2$, we'll plug a test value of $x = -3$ into the derivative, since $x = -3$ is a value in that interval. We could have picked any other value to use instead, as long as it fell in the interval $-\infty < x < -2$.

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(-3) = 1 - \frac{4}{(-3)^2}$$

$$f'(-3) = 1 - \frac{4}{9}$$

$$f'(-3) = \frac{9}{9} - \frac{4}{9}$$

$$f'(-3) = \frac{5}{9} > 0$$

To test $-2 < x < 2$, we'll plug $x = -1$ into the derivative. Even though we could have used $x = 0$ as a test value for this interval, it's nice to avoid 0 as a test value, since not all function's evaluate nicely at 0, this one included.

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(-1) = 1 - \frac{4}{(-1)^2}$$

$$f'(-1) = 1 - 4$$

$$f'(-1) = -3 < 0$$

To test $2 < x < \infty$, we'll plug $x = 3$ into the derivative.

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(3) = 1 - \frac{4}{3^2}$$

$$f'(3) = 1 - \frac{4}{9}$$

$$f'(3) = \frac{9}{9} - \frac{4}{9}$$

$$f'(3) = \frac{5}{9} > 0$$

The derivative was positive on the first interval, negative on the second interval, and positive on the third interval. Plot these signs on the critical point diagram we drew earlier.

Remember that the original function $f(x)$ is increasing where we found a positive result, and decreasing where we found a negative result. So we can say

- $f(x)$ is increasing on $-\infty < x < -2$
- $f(x)$ is decreasing on $-2 < x < 2$
- $f(x)$ is increasing on $2 < x < \infty$

Since critical points represent points at which the graph of the function will change direction, we'll test the sign of the derivative to the left of the left-most critical point, to the right of the right-most critical point, and then in between each critical point.

First derivative test

Once we've found the intervals on which the function is increasing and decreasing, we've really already completed the first derivative test, other than explicitly stating conclusions about the function's maximum and minimum values.

Because the **first derivative test** is just the test for finding the function's maxima and minima.

In other words, once we have the critical point diagram filled in with the signs of the derivative,

the first derivative test lets us state the following conclusions:

- If the derivative is negative to the left of the critical point and positive to the right of it, the graph has a local minimum at that point (and it's possible this local minimum *might* be a global minimum).

- If the derivative is positive to the left of the critical point and negative to the right of it, the graph has a local maximum at that point (and it's possible this local maximum *might* be a global maximum).

So for the previous example, the first derivative test allows us to conclude that the function has a local maximum at $x = -2$ and a local minimum at $x = 2$.

Inflection points and the second derivative test

In the last lesson, we saw that the first derivative allowed us to determine

- where the function has critical points, which told us
- where the function changed direction from increasing to decreasing, or vice versa, and therefore
- where the function had local maxima and/or local minima.

As it turns out, the second derivative can be used in a similar way, except that the second derivative allows us to determine

- where the function has inflection points, which tell us
- where the graph changes concavity from concave up to concave down, or vice versa.

Inflection points

In other words, an **inflection point** is a point at which the function changes from concave up to concave down, or from concave down to concave up.

In the same way that we found critical points by setting the first derivative equal to 0, we'll find inflection points by setting the second derivative equal to 0. Inflection points will exist wherever the second derivative is equal to 0 (or possibly at points where the second derivative is undefined) and the function changes from concave up to concave down.

Let's continue on with the same example we were using in the last lesson, and walk through how to find the function's inflection points.

Example

Find the inflection points of the function whose first derivative is given.

$$f'(x) = 1 - \frac{4}{x^2}$$

In the previous lesson, we were working with the function

$$f(x) = x + \frac{4}{x}$$

and we'd already found that its first derivative was

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(x) = 1 - 4x^{-2}$$

So we'll take the derivative of this first derivative in order to find the function's second derivative.

$$f''(x) = 0 + 8x^{-3}$$

$$f''(x) = \frac{8}{x^3}$$

To find inflection points, we set this second derivative equal to 0.

$$0 = \frac{8}{x^3}$$

There's no solution to this equation. If we multiply both sides by x^3, we'll get $0 = 8$, which is nonsensical. We can also see that the second derivative is undefined at $x = 0$, since an $x = 0$ value would make the denominator of the fraction 0, which is always undefined. So $x = 0$ is the only possible inflection point.

Let's do another example.

Example

Find the inflection points of the function whose first derivative is given.

$$f'(x) = 3x^2 + x - 10$$

The first and second derivatives are

$$f'(x) = 3x^2 + x - 10$$

$$f''(x) = 6x + 1$$

To find inflection points, set the second derivative equal to 0.

$$6x + 1 = 0$$

$$6x = -1$$

$$x = -\frac{1}{6}$$

So $x = -1/6$ is the only possible inflection point.

It's possible that the function will have no inflection points. Lines (first-degree functions) and parabolas (second-degree) functions, will never have inflection points, because there concavity never changes.

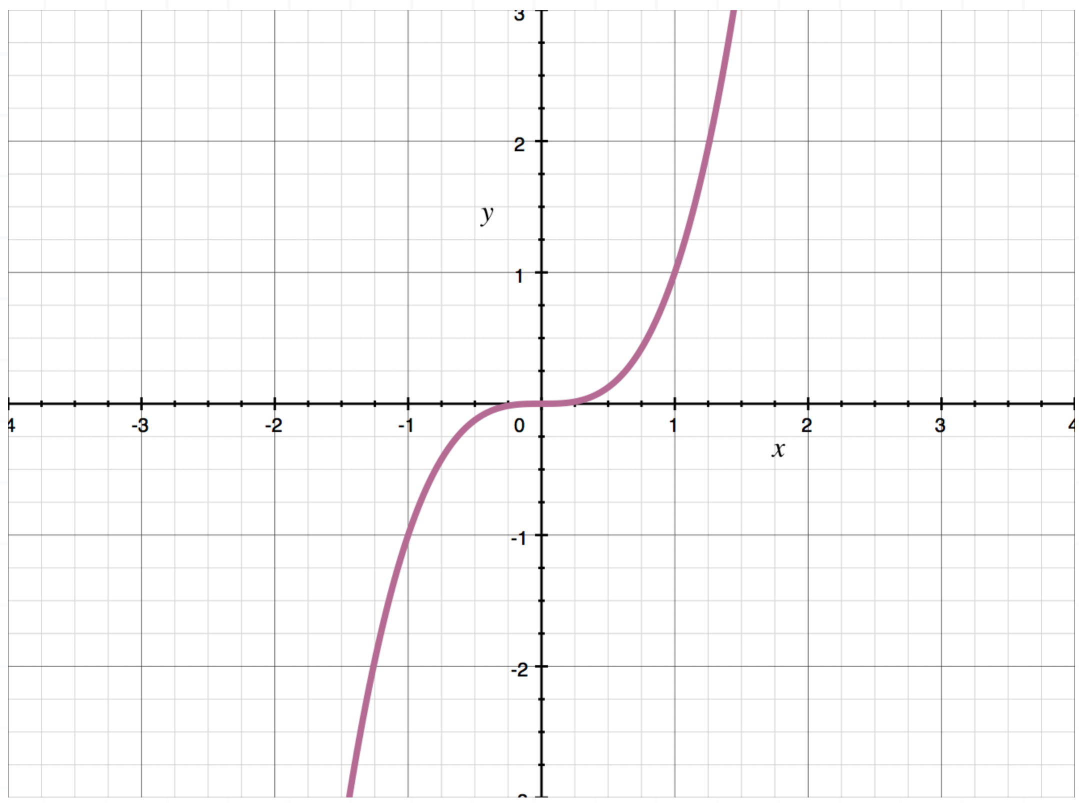
Concave up and concave down

Because the inflection points are the points at which the function changes concavity, from concave up to concave down or from concave down to concave up, the next step is to investigate the behavior in between the inflection points.

- Where the second derivative is positive, the function is concave up. A function is **concave up** when it's "scooping" upwards, like a bowl or a cup.

- Where the second derivative is negative, the function is concave down. A function is **concave down** when it's "scooping" downwards, like a hat or a dome.

As an example, let's look at the graph below. From $-\infty < x < 0$, the graph is concave down. We can think "concave down looks like a frown." The inflection point at which the graph changes concavity is at $x = 0$. On the interval $0 < x < \infty$, the graph is concave up, and it looks like a smile.



To test the sign of the second derivative, we'll simply pick a value between each pair of inflection points, and plug that test value into the second derivative to see whether we get a positive result or a negative result. If the test value gives a positive result, it means the function is concave up on that interval, and if the test value gives a negative result, it means the function is concave down on that interval.

If we find one inflection point for the function, then we just need to look at the sign of the second derivative on the left side and right side of that one inflection point.

But if we find multiple inflection points, then we need to find the sign of the second derivative to the left of the left-most inflection point, to the right of the right-most inflection point, and between each inflection point.

Let's continue with the same example we used to find inflection points, looking at the sign of the second derivative around that point.

Example

The only potential inflection point of the function is $x = 0$. Where is the function concave up and where is it concave down?

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

Previously, we used the second derivative to find that the function had a potential inflection point at $x = 0$. Once we have the inflection point(s), it's helpful to plot it along a number line from least to greatest, left to right.

From this diagram, we can see that we have to test two intervals.

$$-\infty < x < 0$$

$$0 < x < \infty$$

To test $-\infty < x < 0$, we'll plug a test value of $x = -1$ into the second derivative, since $x = -1$ is a value in that interval. We could have picked any other value to use instead, as long as it fell in the interval $-\infty < x < 0$.

$$f''(x) = \frac{8}{x^3}$$

$$f''(-1) = \frac{8}{(-1)^3}$$

$$f''(-1) = \frac{8}{-1}$$

$$f''(-1) = -8 < 0$$

To test $0 < x < \infty$, we'll plug $x = 1$ into the derivative.

$$f''(x) = \frac{8}{x^3}$$

$$f''(1) = \frac{8}{(1)^3}$$

$$f''(1) = \frac{8}{1}$$

$$f''(1) = 8 > 0$$

The second derivative was negative on the first interval and positive on the second interval. Plot these signs on the inflection point diagram we drew earlier.

Remember that the original function $f(x)$ is concave up where we found a positive result, and concave down where we found a negative result. So we can say

- $f(x)$ is concave down on $-\infty < x < 0$

- $f(x)$ is concave up on $0 < x < \infty$

Let's look at the concavity of the function we were looking at before, $f'(x) = 3x^2 + x - 10$.

Example

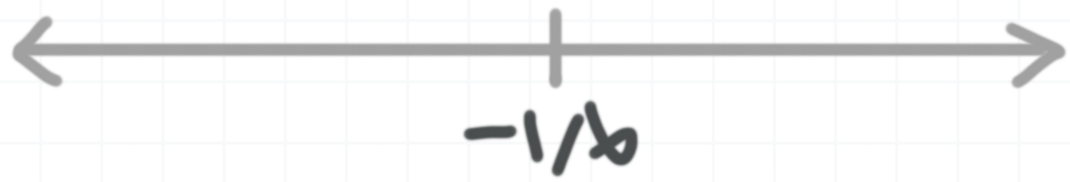
If the only potential inflection point of the function is $x = -1/6$, where is the function concave up and where is it concave down?

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

$$f'(x) = 3x^2 + x - 10$$

$$f''(x) = 6x + 1$$

Previously, we used the second derivative to find that the function had a potential inflection point at $x = -1/6$. Once we have the inflection point(s), it's helpful to plot it along a number line from least to greatest, left to right.



From this diagram, we can see that we have to test two intervals.

$$-\infty < x < -\frac{1}{6}$$

$$-\frac{1}{6} < x < \infty$$

To test $-\infty < x < -1/6$, we'll plug a test value of $x = -1$ into the second derivative, since $x = -1$ is a value in that interval. We could have picked any other value to use instead, as long as it falls in the interval $-\infty < x < -1/6$.

$$f''(x) = 6x + 1$$

$$f''(-1) = 6(-1) + 1$$

$$f''(-1) = -6 + 1$$

$$f''(-1) = -5 < 0$$

To test $-1/6 < x < \infty$, we'll plug $x = 1$ into the derivative.

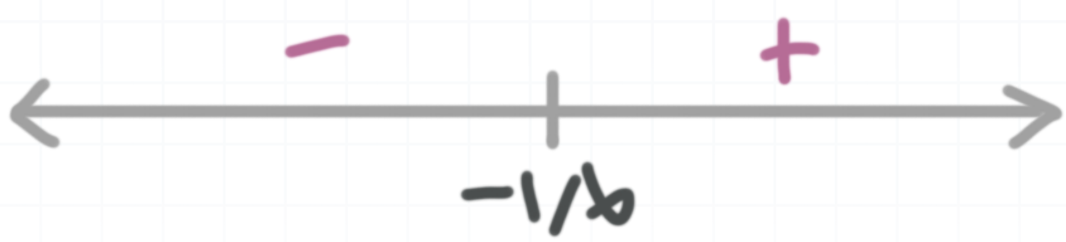
$$f''(x) = 6x + 1$$

$$f''(1) = 6(1) + 1$$

$$f''(1) = 6 + 1$$

$$f''(1) = 7 > 0$$

The second derivative was negative on the first interval and positive on the second interval. Plot these signs on the inflection point diagram we drew earlier.



Remember that the original function $f(x)$ is concave up when we find a positive result, and concave down when we find a negative result. So we can say

- $f(x)$ is concave down on $-\infty < x < -1/6$
- $f(x)$ is concave up on $-1/6 < x < \infty$

Second derivative test

Surprisingly, the second derivative test actually isn't related to inflection points or concavity. Like the first derivative test, the **second derivative test** is just another way of classifying the function's local extrema.

To use the second derivative test, we just plug any critical points we've found (not the inflection points) into the second derivative. If the result is negative, there's a local maximum at that critical point. If the result is positive, there's a local minimum at that critical point. If the result is zero, then the second derivative test is inconclusive at that critical point.

Be careful! Notice here how the results are opposite what we might think they would be.

- If $f''(x) > 0$ at a critical point, there's a local minimum there.
- If $f''(x) < 0$ at a critical point, there's a local maximum there.

Let's keep going with the same example we've been working through to see how to use the second derivative test to classify critical points.

Example

The function has critical points at $x = \pm 2$. Use the second derivative test to find local extrema.

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

We already found in the last example that the second derivative of the original function $f(x)$ was

$$f''(x) = \frac{8}{x^3}$$

To find the function's local maxima and minima, we'll plug both critical points, $x = \pm 2$, into the second derivative.

$$f''(-2) = \frac{8}{(-2)^3}$$

$$f''(-2) = \frac{8}{-8}$$

$$f''(-2) = -1 < 0$$

and

$$f''(2) = \frac{8}{2^3}$$

$$f''(2) = \frac{8}{8}$$

$$f''(2) = 1 > 0$$

Since the second derivative is negative at $x = -2$, the function has a local maximum at that point, and since the second derivative is positive at $x = 2$, the function has a local minimum at that point.

Notice how these are the same results we got from using the first derivative test in the last lesson.

Let's do another example.

Example

The function has critical points at $x = -2, 5/3$. Use the second derivative test to find local extrema.

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

$$f'(x) = 3x^2 + x - 10$$

We already found that the second derivative was

$$f''(x) = 6x + 1$$

To find the function's local maxima and minima, we'll plug both critical points, $x = -2$ and $x = 5/3$, into the second derivative.

$$f''(-2) = 6(-2) + 1$$

$$f''(-2) = -12 + 1$$

$$f''(-2) = -11 < 0$$

and

$$f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) + 1$$

$$f''\left(\frac{5}{3}\right) = 10 + 1$$

$$f''\left(\frac{5}{3}\right) = 11 > 0$$

Since the second derivative is negative at $x = -2$, the function has a local maximum at that point, and since the second derivative is positive at $x = 5/3$, the function has a local minimum at that point.

Notice how these are the same results we got from using the first derivative test previously.

Intercepts and vertical asymptotes

At this point, we understand the optimization process:

- We know how to find the critical points of a function.
- We know how to determine where the function is increasing and where it's decreasing.
- We know how to classify critical points as local and/or global maxima and minima.
- We know how to determine where the function is concave up and where it's concave down.

Interestingly, all of these optimization steps have actually produced a lot of information about the function's graph. Without having done anything else, we should already be starting to get a pretty clear picture of what the function's graph might look like.

For the rest of this section, we'll look at other pieces of information we can gather about the function, and then we'll bring everything together to sketch its graph.

In this lesson specifically, we'll look at the function's intercepts and vertical asymptotes.

Intercepts

One quick thing we can do to get a clearer picture of the graph of a function is to find its intercepts, which are the points at which the function crosses the x- and y-axes.

To find the points where the graph intersects the x-axis, we'll substitute $y = f(x) = 0$. And to find the points where the graph intersects the y-axis, we'll substitute $x = 0$.

Throughout the notes in this section, we've been working through the same example, and we'll continue with that example here.

Example

Find the function's intercepts.

$$f(x) = x + \frac{4}{x}$$

First, let's find the y-intercepts by substituting $x = 0$.

$$y = 0 + \frac{4}{0}$$

$$y = \frac{4}{0}$$

Because the fraction is undefined, this tells us that the function has no y-intercepts, which means it never crosses the y-axis. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x + \frac{4}{x}$$

$$0 = x^2 + 4$$

$$x^2 = -4$$

Since there are no real solutions to this equation, which tells us that the function also has no x-intercepts.

Let's do another example where we find the function's intercepts.

Example

Find the function's intercepts.

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

First, let's find the y-intercept by substituting $x = 0$.

$$y = 0^3 + \frac{1}{2}(0)^2 - 10(0) - 5$$

$$y = -5$$

So the function has a y-intercept at $(0, -5)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x^3 + \frac{1}{2}x^2 - 10x - 5$$

$$0 = \frac{1}{2}x^2(2x + 1) - 5(2x + 1)$$

$$0 = (2x + 1)\left(\frac{1}{2}x^2 - 5\right)$$

$$x = -\frac{1}{2}, \ -\sqrt{10}, \sqrt{10}$$

So the function has x-intercepts at $(-1/2,0)$, $(-\sqrt{10},0)$ and $(\sqrt{10},0)$.

Vertical asymptotes

An **asymptote** is a line which a function's graph approaches, but never crosses. The graph will get closer and closer to the asymptote, but no matter how far you go out, the graph will never touch or cross the line of the asymptote.

Asymptotes can be perfectly vertical, perfectly horizontal, or neither, in which case we say that the asymptote is slanted.

For now, we'll look just at vertical asymptotes, and how to find them. They're actually the easiest kinds of asymptotes to test for, because **vertical asymptotes** only exist where the function is undefined.

Remember, a function is undefined whenever we have a value of 0 as the denominator of a fraction, or whenever we have a negative value inside a square root sign, or whenever the argument of the logarithmic function is equal to 0. Also, four of the standard trig functions, $\tan x$, $\cot x$, $\sec x$, and $\csc x$, have vertical asymptotes.

Let's continue with the same example to find the function's vertical asymptotes, if it has any.

Example

Find any vertical asymptotes of the function.

$$f(x) = x + \frac{4}{x}$$

We can see right away that this function includes a fraction. We know that a fraction will be undefined whenever its denominator is 0, which means this fraction will be undefined at $x = 0$.

Therefore, the entire function $f(x)$ will be undefined when $x = 0$, and we can say that the function has a vertical asymptote there.

That means the function's graph will come alongside the vertical line $x = 0$, skimming it very closely, but will never touch or cross it.

Realize that our conclusions about the vertical asymptote match the answer we got earlier when we looked at the function's intercepts.

In the intercepts example, we found that the graph never crossed the y-axis, because we got an undefined value when we substituted $x = 0$. Because we got a 0 in the denominator of the fraction in that example, we could have concluded right away that there was also a vertical asymptote at that point.

Let's do one more example with vertical asymptotes.

Example

Find any vertical asymptotes of the function.

$$f(x) = x^3 + \frac{1}{2}x^2 - 10x - 5$$

We can see right away that this function doesn't include a fraction, square root, log function, or trig function. We know that the cubic polynomials are defined for all real numbers, so there's no vertical asymptote.

Horizontal and slant asymptotes

Now that we know how to find a function's vertical asymptotes, if it has any, let's turn our attention toward finding any horizontal and slant asymptotes for the function.

Then in the next lesson, we'll finally put this all together to sketch the function's graph.

Horizontal asymptotes

We primarily find horizontal asymptotes in the graphs of rational functions (but we do see them in other functions as well, like exponential functions).

When we're looking for horizontal asymptotes of rational functions, we only care about the highest-degree term in the numerator and the highest-degree term in the denominator.

Remember that the "degree" is the power of the term. So the degree of x^4 is four, the degree of $3x^2$ is two, and the degree of $7x$ is 1, since the x variable is raised to the first power. The degree of a constant term, like -8, is zero, because -8 can be rewritten as $-8x^0$.

Here's how we test for horizontal asymptotes:

1. If the degree of the numerator is less than the degree of the denominator, then the x-axis is a horizontal asymptote.

2. If the degree of the numerator is equal to the degree of the denominator, then the ratio of the coefficients on these highest-degree terms is the equation of the horizontal asymptote.

3. If the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

As an example, let's say we have a rational function written as

$$f(x) = \frac{x^3 + \text{lower-degree terms}}{x^2 + \text{lower-degree terms}}$$

The x^3 term is the highest-degree term in the numerator, and all the numerator's other terms have a degree lower than three. The x^2 term is the highest-degree term in the denominator, and all the denominator's other terms have a degree lower than two.

Then the degree of the numerator is 3 and the degree of the denominator is 2. Because the degree of the numerator is greater than the degree of the denominator, that function has no horizontal asymptote.

A rational function will always have zero or one horizontal asymptote; it'll never have more than one.

Let's find any horizontal asymptote for the function we've been working with throughout this section.

Example

Find the function's horizontal asymptote, if it has one.

$$f(x) = x + \frac{4}{x}$$

Before we can use our rule for finding horizontal asymptotes of rational functions, we need to rewrite $f(x)$ so that the function is just one fraction. We'll multiply the first term by x/x, which will give us a common denominator.

$$f(x) = x\left(\frac{x}{x}\right) + \frac{4}{x}$$

$$f(x) = \frac{x^2}{x} + \frac{4}{x}$$

With a common denominator, we can add the fractions.

$$f(x) = \frac{x^2 + 4}{x}$$

In this form, we can see that the degree of the numerator is 2, and the degree of the denominator is 1. So the degree of the numerator is greater than the degree of the denominator, which means the function doesn't have a horizontal asymptote.

Slant asymptotes

A slant asymptote exists when the degree of the numerator is exactly one greater than the degree of the denominator. So the hypothetical function we mentioned earlier,

$$f(x) = \frac{x^3 + \text{lower-degree terms}}{x^2 + \text{lower-degree terms}}$$

would have a slant asymptote, because the degree of the numerator is exactly one greater than the degree of the denominator, $3 > 2$.

If we've determined that the function has a slant asymptote, then in order to find its equation, we divide the denominator into the numerator using polynomial long division.

Let's see whether our ongoing example problem has a slant asymptote.

Example

Show that the function has a slant asymptote, then find its equation.

$$f(x) = x + \frac{4}{x}$$

In the previous example, remember that we've already rewritten this function as

$$f(x) = \frac{x^2 + 4}{x}$$

The degree of the numerator is exactly one greater than the degree of the denominator, $2 > 1$, so the function has a slant asymptote.

To find it equation, we divide the numerator by the denominator using polynomial long division. When we do, we get

$$f(x) = x + \frac{4}{x}$$

We get right back to the original function. That won't always happen, it's just that this particular function happened to be the composition of the quotient and remainder.

Whenever we use long division in this way to find the slant asymptote, the quotient gives the equation of the slant asymptote, and the denominator of the fraction gives the equation of the vertical asymptote.

Therefore, in this case, the slant asymptote is the line $y = x$, and the vertical asymptote is the line $x = 0$.

Sketching graphs

At this point, we have plenty of information about the graph of the function, and we can pull that information together in order to create a sketch.

Throughout this section, we've been working with the same function.

$$f(x) = x + \frac{4}{x}$$

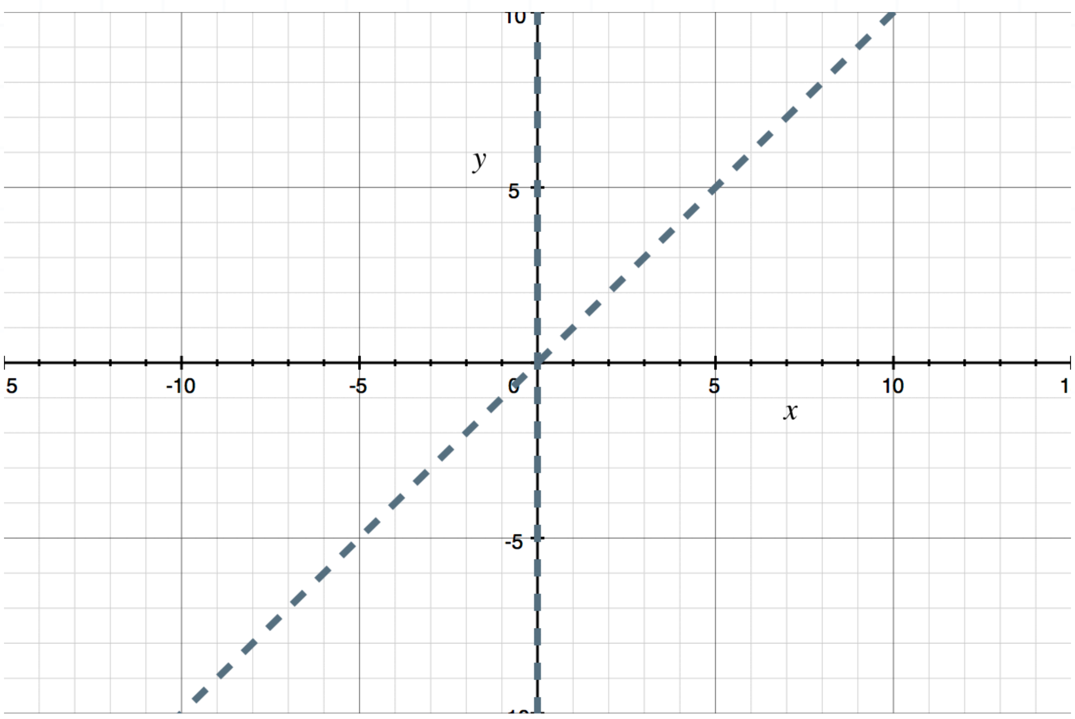
And we've calculated the following values for it:

- The critical points are $x = \pm 2$.
- The function is increasing on $-\infty < x < -2$, decreasing on $-2 < x < 2$, and increasing again on $2 < x < \infty$.
- The function has a local maximum at $x = -2$ and a local minimum at $x = 2$.
- The function has an inflection point at $x = 0$.
- The function is concave down on $-\infty < x < 0$ and concave up on $0 < x < \infty$.
- The function has no intercepts, so it doesn't cross the x- or y-axis.
- The function has a vertical asymptote at $x = 0$, no horizontal asymptote, and a slant asymptote at $y = x$.

From this information, we can get a pretty good sketch of the graph. We can also use information about the domain and range of the function and/or its symmetry.

Sketching the graph

A great starting point is to sketch in any asymptotes, because we know their exact equations, and they'll serve as guiding lines when we trace out the graph. Draw in the lines $x=0$ and $y=x$.



The function has a local maximum at $x=-2$ and a local minimum at $x=2$, so let's plug those values into the function in order to find those particular points along the graph. We get

$$f(-2)=-2+\frac{4}{-2}$$

$$f(-2)=-2-2$$

$$f(-2) = -4$$

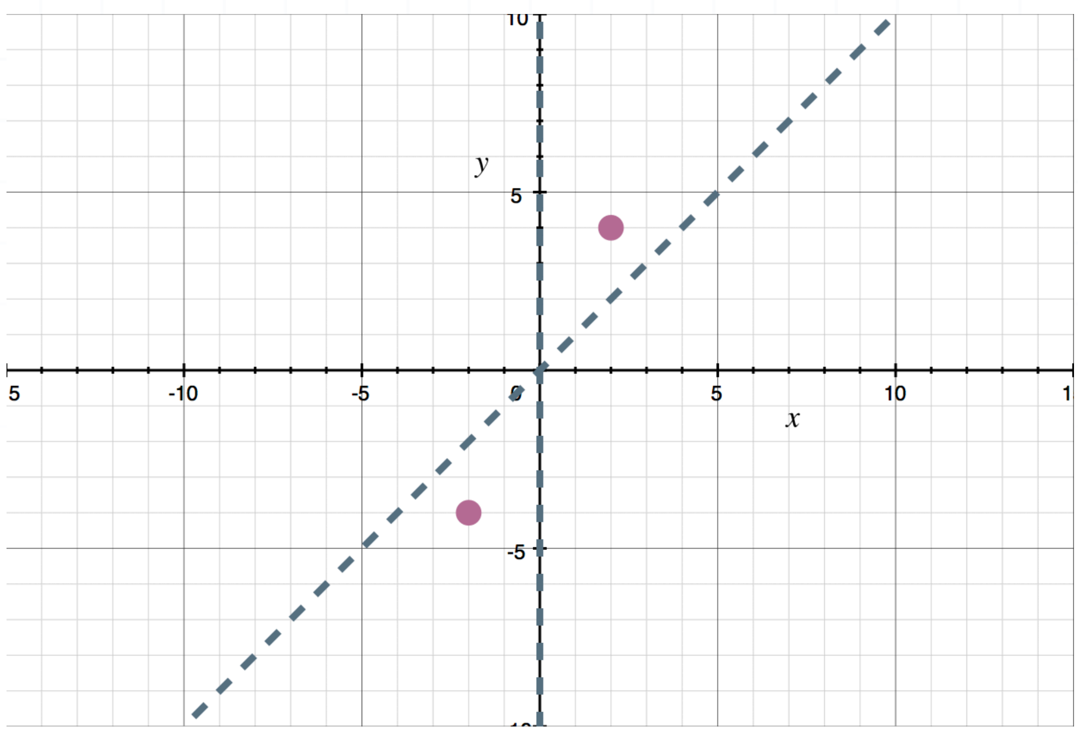
and

$$f(2) = 2 + \frac{4}{2}$$

$$f(2) = 2 + 2$$

$$f(2) = 4$$

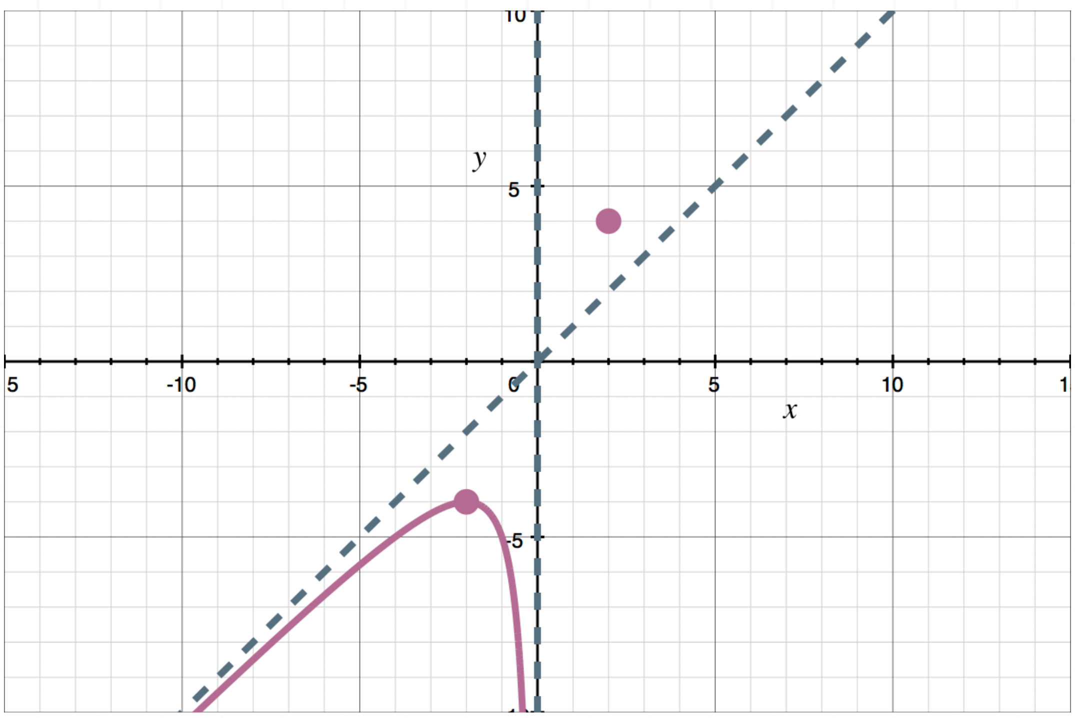
Add those critical points, $(-2,-4)$ and $(2,4)$, to the graph.



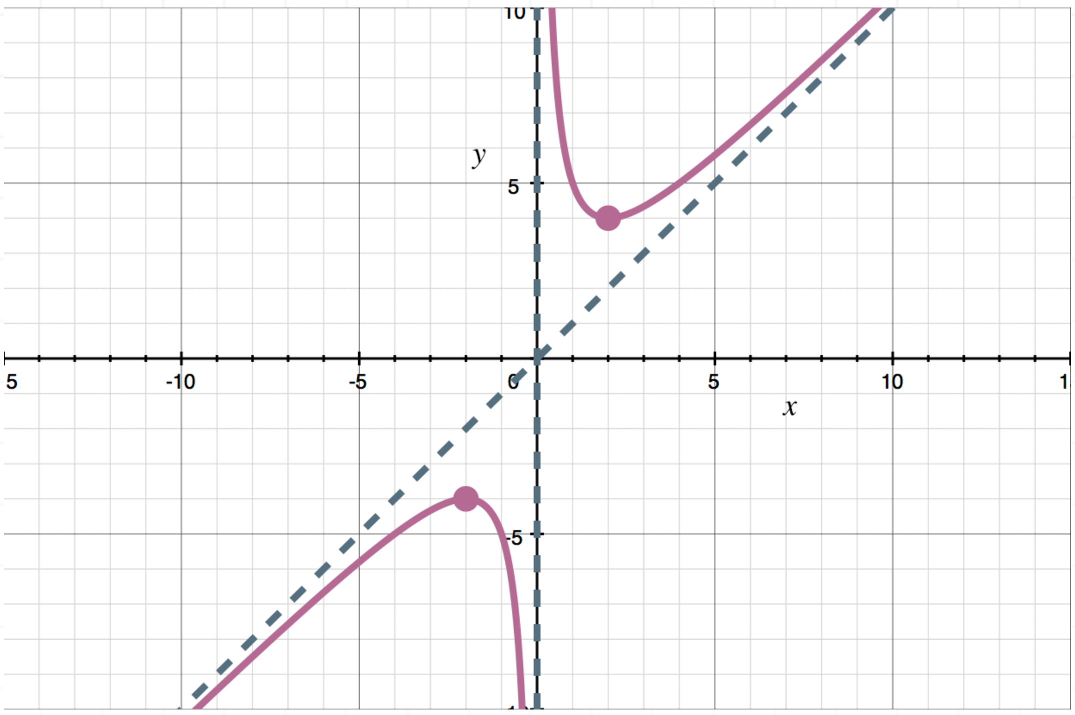
Now we need to hold a few things in our heads at once to sketch the rest of the graph.

First, we know the graph doesn't cross the y-axis, but we have points that are on the graph on either side of that line, which means the graph will have two separate, unattached pieces.

Starting with the piece to the left of the y-axis, we know that the curve can't cross $y = x$ or $x = 0$, so we'll have to stay inside that lower-left triangle that's created by those lines. We know the function is increasing to the left of that critical point on $-\infty < x < -2$, and decreasing to the right of it. And we know that $x = -2$ represents a local maximum. We also know that the function is concave down on $-\infty < x < 0$. If we put all that together, we can say that the left piece of the graph looks something like this:



For the piece to the right of the y-axis, we know that the curve can't cross $y = x$ or $x = 0$, so we'll have to stay inside that upper-right triangle that's created by those lines. We know the function is decreasing to the left of that critical point, and increasing to the right of it on $2 < x < \infty$. And we know that $x = 2$ represents a local minimum. We also know that the function is concave up on $0 < x < \infty$. If we put all that together, we can add the right piece of the graph:



This is the graph of the function.

Keep in mind, this kind of graph sketching takes lots of practice. First, we have to work through the entire optimization process, then find all the intercepts and asymptotes. Then, we have to put all of this information together to sketch the graph.

To make this graph sketching as manageable as possible, we want to make sure we put everything that's easy onto the graph. This would include asymptotes, coordinate points that represent the intercepts, and the coordinate points that represent the critical and inflection points.

This way, hopefully, all we have left to really "hold in our head" at one time is the increasing/decreasing and concave down/concave up behavior. But we'll have the asymptotes and some points already there to guide us.

All that being said, sketching graphs this way takes practice, so don't be discouraged if it doesn't come easily when you first start.

Here is a summary of steps we can use to sketch the graph of a function:

- Calculate the first derivative, find critical points, and use the first derivative test to determine where the function is increasing and decreasing, then classify those critical points as maxima or minima.

- Calculate the second derivative, find inflection points, and determine where the function is concave up and concave down.

- Find any vertical, horizontal, and slant asymptotes, and any x- and y- intercepts.

- Consider the domain and range of the function and determine any points of discontinuity.

Extrema on a closed interval

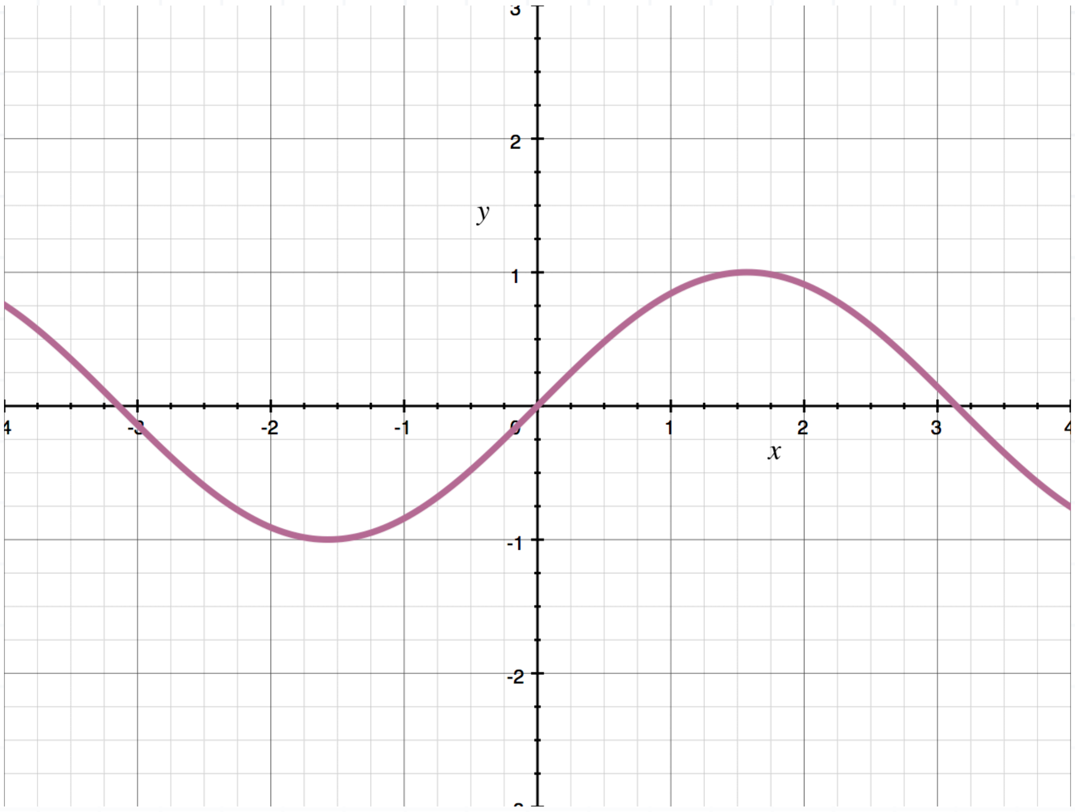
We already know how to find the extrema of a function by finding critical points, and then using either the first derivative test or second derivative test in order to classify them.

And we know how to classify those critical points as local maxima or minima, or global maxima or minima.

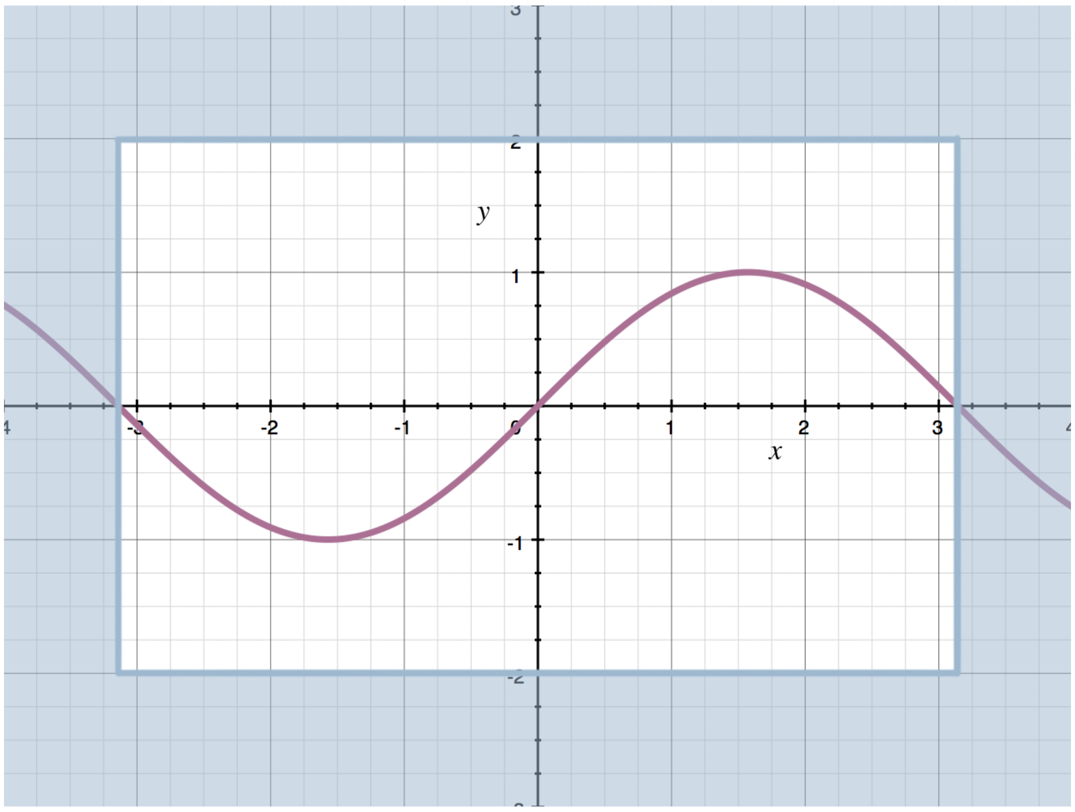
But the process of classifying extrema has always been based on the function's entire domain. In other words, we've never limited the interval in which we were looking for critical points, but that's what we want to do now.

Extrema in closed intervals

Sometimes we'll only be interested in extrema within a particular interval. For instance, instead of looking at the entire domain of $y = \sin x$,



maybe instead we just want to focus on the interval $[-\pi, \pi]$, and look for extrema there.



When we want to find and classify extrema on a closed interval like this, all of our steps we'll be the same as what we've done in the past, except for one key difference: we'll need to consider the values of the function at the endpoints of the interval.

So, in the case of this $y = \sin x$ function on $[-\pi, \pi]$, we'd need to look for critical points like normal, but then we'd also need to look at the values $y = \sin(-\pi)$ and $y = \sin \pi$.

The reason we consider those values is because the function may have its least or greatest value, within the interval, at the edge of the interval. So when we're determining which maxima will be local maxima and global maximum, and which minima will be the local minima and global minimum, we'll consider the values of the function at every critical point in the interval, and at both endpoints of the interval.

Let's work through an example so that we can see how to classify the extrema of a function when we've restricted its domain with a closed interval.

Example

Calculate the extrema of the function on $[-2,2]$. Distinguish between absolute and relative extrema.

$$y = x^3 - 2x + 1$$

First, take the derivative.

$$y' = 3x^2 - 2$$

Find critical points.

$$3x^2 - 2 = 0$$

$$3x^2 = 2$$

$$x^2 = \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}}$$

We'll use the first derivative test to classify these critical points, using test values of $x = -1$, $x = 0$, and $x = 1$.

$$y' = 3(-1)^2 - 2$$

$$y' = 3 - 2$$

$$y' = 1 > 0$$

and

$$y' = 3(0)^2 - 2$$

$$y' = 0 - 2$$

$$y' = -2 < 0$$

and

$$y' = 3(1)^2 - 2$$

$$y' = 3 - 2$$

$$y' = 1 > 0$$

If we summarize these findings, we can say that the function is

- increasing on $-\infty < x < -\sqrt{2/3}$
- decreasing on $-\sqrt{2/3} < x < \sqrt{2/3}$
- increasing on $\sqrt{2/3} < x < \infty$

Based on the increasing/decreasing behavior of the function, it has a local maximum at $x = -\sqrt{2/3}$ and a local minimum at $x = \sqrt{2/3}$.

Now that we've tested the critical points, we need to plug each of them, and both endpoints of the interval, into the original function, $y = x^3 - 2x + 1$.

For $x = -\sqrt{2/3} \approx -0.82$,

$$y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 1$$

$$y = -\frac{2\sqrt{2}}{3\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{3}} + 1$$

$$y \approx 2.09$$

For $x = \sqrt{2/3} \approx 0.82$,

$$y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right) + 1$$

$$y = \frac{2\sqrt{2}}{3\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}} + 1$$

$$y \approx -0.09$$

For $x = -2$,

$$y = (-2)^3 - 2(-2) + 1$$

$$y = -8 + 4 + 1$$

$$y = -3$$

For $x = 2$,

$$y = 2^3 - 2(2) + 1$$

$$y = 8 - 4 + 1$$

$$y = 5$$

We have four points. If we arrange them in order of y-values from least to greatest, we get

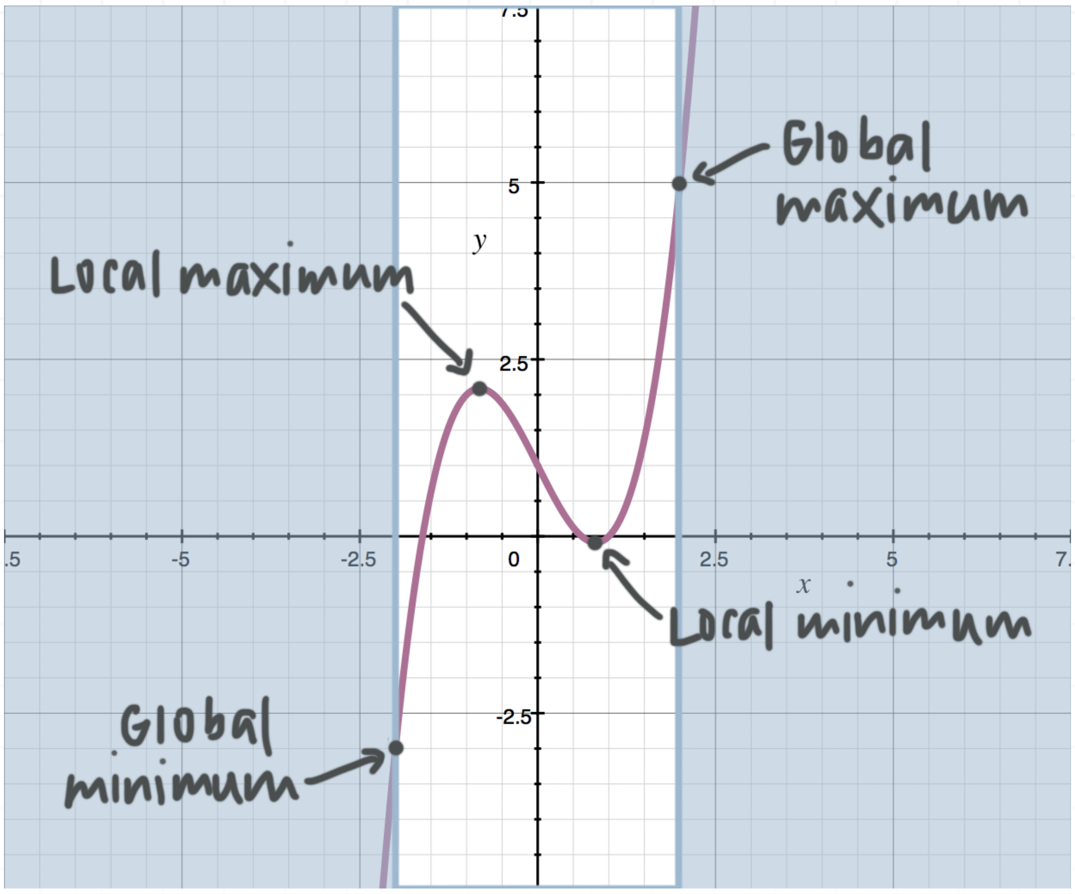
	$(-2, -3)$
Local minimum	$\left(\sqrt{\frac{2}{3}}, -0.09\right)$
Local maximum	$\left(-\sqrt{\frac{2}{3}}, 2.09\right)$

$$(2,5)$$

The second point is a local minimum, but the function has a lower value at the endpoint $(-2,-3)$, so the function's global minimum in the interval is at $(-2,-3)$. The third point is a local maximum, but the function has a higher value at the endpoint $(2,5)$, so the function's global maximum in the interval is at $(2,5)$.

Global minimum	$(-2,-3)$
Local minimum	$\left(\sqrt{\frac{2}{3}},-0.09\right)$
Local maximum	$\left(-\sqrt{\frac{2}{3}},2.09\right)$
Global maximum	$(2,5)$

If we sketch the function, and the interval $[-2,2]$ that we're interested in, we see these results.


Global maximum
Local maximum
Local minimum
Global minimum
y
x
5
2.5
-2.5
7.5
-5
-2.5
0
2.5
5

Sketching f(x) from f'(x)

We talked earlier about graph sketching, and looked at how to pull together all of the optimization, intercept, and asymptote information together in order to get a picture of what the graph looks like.

Now we want to apply that graph sketching process to the relationship between the graphs of a function, its derivative, and its second derivative. The goal here is to be able to start with either $f(x)$ or $f'(x)$ or $f''(x)$, and use information about only one of those functions in order to sketch graphs of the other two.

This process can get pretty confusing, but we can organize what we know about the relationships between these functions in a chart, which will help us navigate as we go. Let's start building that chart now.

The chart

Let's make a chart with the three functions, $f(x)$, $f'(x)$, and $f''(x)$, as three columns. When we talked about critical points and the first derivative test, we learned that when the derivative is equal to 0, the original function has a critical point.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	

When a function is equal to 0, it has an x-intercept, because it's crossing the x-axis.

And we learned that when the derivative is positive, the original function is increasing, but when the derivative is negative, the original function is decreasing.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	

If the value of a function is positive, the graph of the function is of course above the x-axis, and if the value of a function is negative, its graph is below the x-axis at that point.

When we talked about inflection points and the second derivative test, we learned that the original function has an inflection point when its second derivative is 0. And we learned that the original function is concave up when the second derivative is positive, and that the original function is concave down when the second derivative is negative.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	

Inflection point		0 (x-intercept)
Concave up		Positive (above the x-axis)
Concave down		Negative (below the x-axis)

At this point, we want to realize that the first row of our chart tells us that $f(x)$ has a critical point when $f'(x) = 0$. This relationship exists because $f'(x)$ is the derivative of $f(x)$.

But the same relationship exists between $f'(x)$ and $f''(x)$: the second derivative $f''(x)$ is the derivative of the first derivative $f'(x)$. So the same relationship will exist. Which means we can use the information from the first three rows of our chart, to fill in the empty second column in the last three rows of the chart.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	
Inflection point	Critical point	0 (x-intercept)
Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)

Now let's imagine for a moment that we're done with the first column of this chart, and we're looking just at the second and third columns. Again, we have the same relationship between $f'(x)$ and $f''(x)$ as we do between

$f(x)$ and $f'(x)$. So we could, for instance, say that the first derivative $f'(x)$ will have an inflection point whenever the second derivative $f''(x)$ has a critical point.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	
Inflection point	Critical point	0 (x-intercept)
Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)
	Inflection point	Critical point

And, in the same way, we know that the first derivative will be concave up when the second derivative is increasing, and concave down when the second derivative is decreasing. So we get a final version of this chart.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	**0 (x-intercept)**	
Increasing	**Positive (above the x-axis)**	
Decreasing	**Negative (below the x-axis)**	
Inflection point	**Critical point**	**0 (x-intercept)**

Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)
	Inflection point	Critical point
	Concave up	Increasing
	Concave down	Decreasing

By color-coding the chart, we can really see the patterns. Furthermore, we realize that we only need to remember the relationships in the middle three rows, because they capture all the information from the first three rows and the last three rows.

So we could really simplify the chart, and say that for any three functions, where you have an original function, its derivative, and its second derivative, the relationships between them look like this:

$f(x)$	$f'(x)$	$f''(x)$
Inflection point	Critical point	0 (x-intercept)
Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)

It may be easier for some people to use the expanded version of the chart, but easier for others to use this condensed version of the chart, so feel free to work with whichever one works better for you.

A "possible" graph

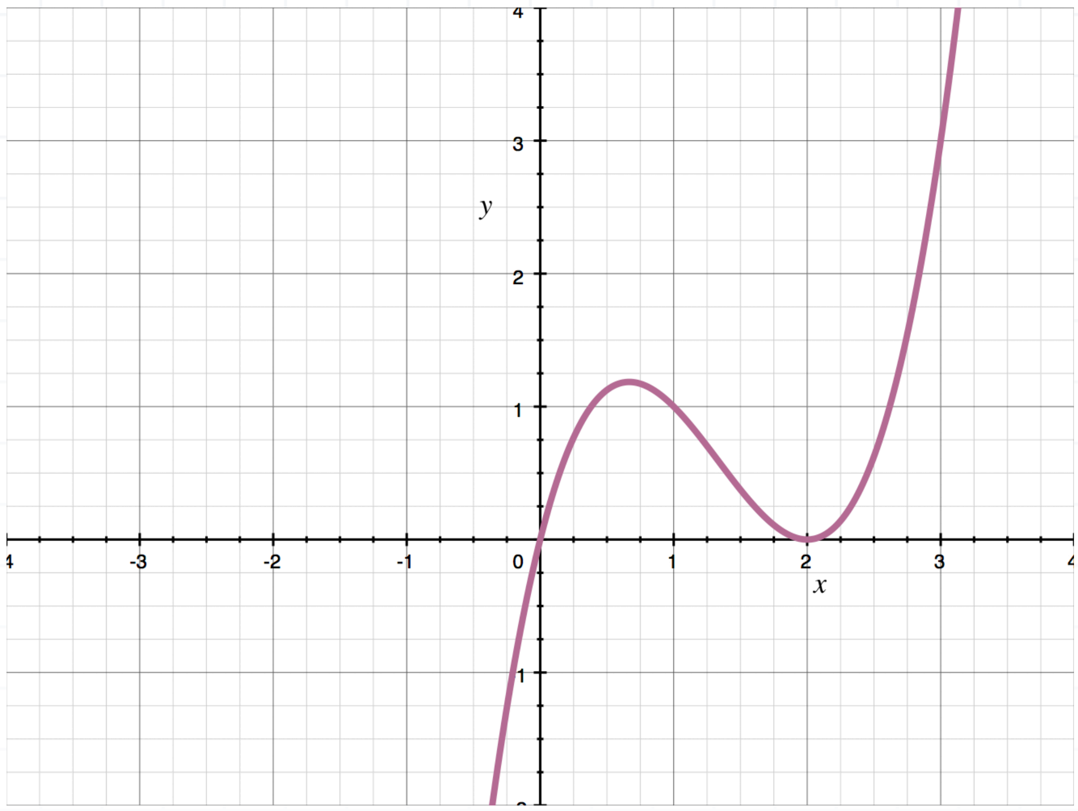
Oftentimes when we're doing these kinds of sketching problems, we'll be asked to sketch a "possible" graph. What that means is that we're just supposed to sketch a graph that doesn't contradict any of the information we've been given.

For instance, we might be given some information about $f''(x)$ and asked to sketch a possible $f'(x)$. From the graph of $f''(x)$, we may not have enough information to sketch $f'(x)$ exactly as it actually is, point for point. But we can make a rough sketch of it that at least doesn't contradict any of the relationships that we outlined in the chart we made.

So that'll be our goal, and now that we have everything in place, let's put this into practice with an example.

Example

Given the graph of $f'(x)$, sketch a possible $f(x)$ and a possible $f''(x)$.



First, let's work on collecting information from this graph of $f'(x)$. We'll work right down the $f'(x)$ column in the center of the chart, listing in order the pieces of information there. From the graph we have of $f'(x)$, it may be hard to tell the exact value of some of these pieces, but an estimate is all we need.

$f'(x)$	
0	$x = 0$ and $x = 2$
Positive	$0 < x < \infty$
Negative	$-\infty < x < 0$
Critical point	$x = 2/3$ and $x = 2$
Increasing	$-\infty < x < 2/3$ and $2 < x < \infty$

Decreasing	$2/3 < x < 2$
Inflection point	$x = 4/3$
Concave up	$4/3 < x < \infty$
Concave down	$-\infty < x < 4/3$

To translate this information about the graph of $f'(x)$ into a graph of $f(x)$, all we need to do is swap out the labels from the second column of the chart for their corresponding labels from the first column of the chart:

$f(x)$

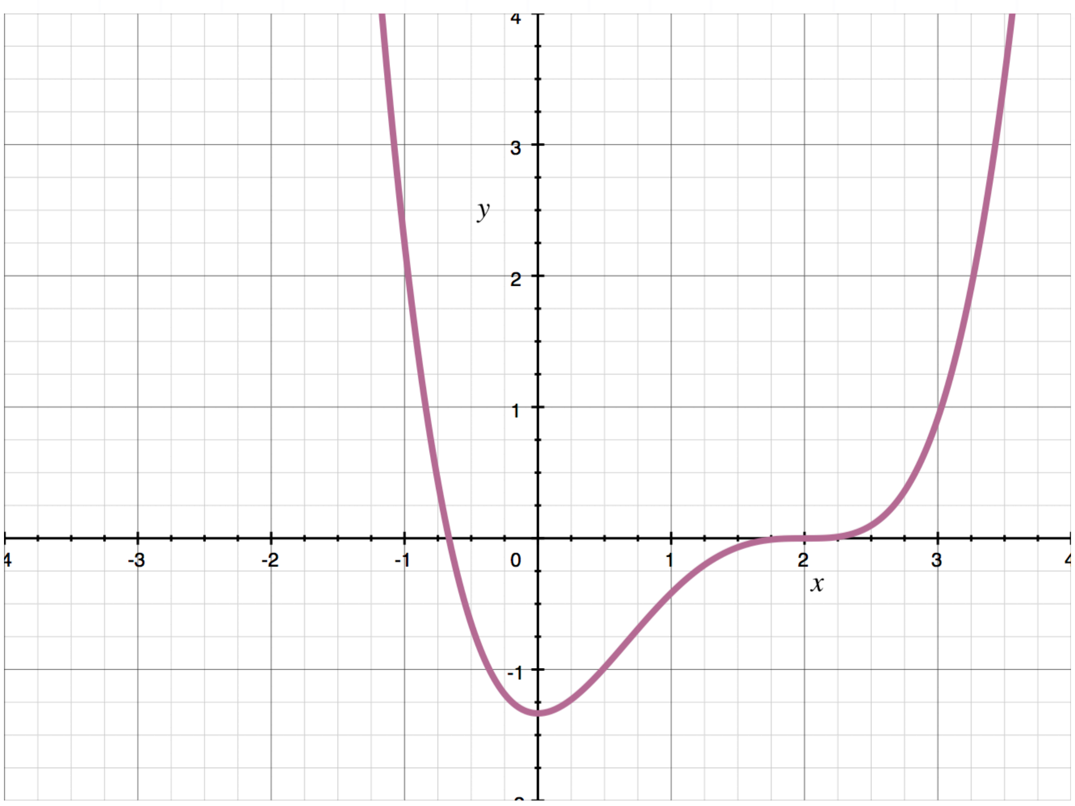
Critical point	$x = 0$ and $x = 2$
Increasing	$0 < x < \infty$
Decreasing	$-\infty < x < 0$
Inflection point	$x = 2/3$ and $x = 2$
Concave up	$-\infty < x < 2/3$ and $2 < x < \infty$
Concave down	$2/3 < x < 2$
	$x = 4/3$
	$4/3 < x < \infty$
	$-\infty < x < 4/3$

So when it comes to sketching $f(x)$, we can ignore the last three pieces of information, and simplify the table.

$f(x)$

Critical point	$x = 0$ and $x = 2$
Increasing	$0 < x < \infty$
Decreasing	$-\infty < x < 0$
Inflection point	$x = 2/3$ and $x = 2$
Concave up	$-\infty < x < 2/3$ and $2 < x < \infty$
Concave down	$2/3 < x < 2$

If we put all of this information together, one possible graph of $f(x)$ might look like this:



Now we'll go back to the original information we collected about $f'(x)$, but this time swap out the labels from the second column of the chart for their corresponding labels from the first column of the chart:

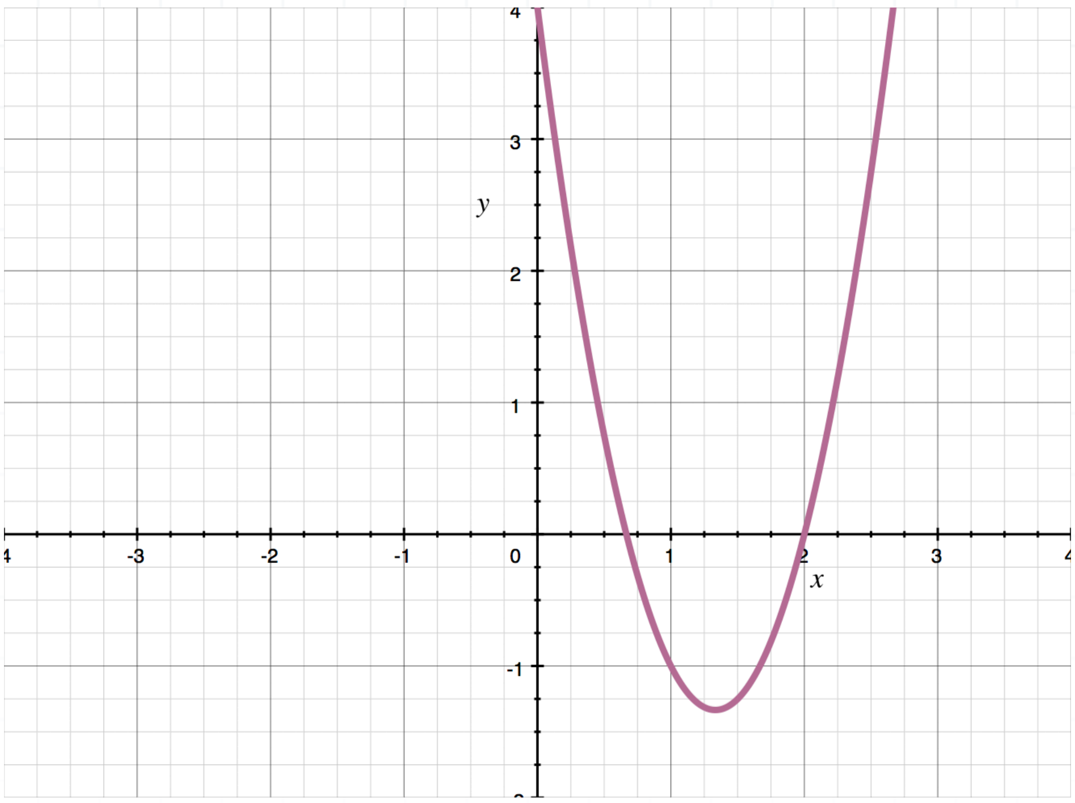
$f''(x)$	
	$x = 0$ and $x = 2$
	$0 < x < \infty$
	$-\infty < x < 0$
0	$x = 2/3$ and $x = 2$
Positive	$-\infty < x < 2/3$ and $2 < x < \infty$
Negative	$2/3 < x < 2$
Critical point	$x = 4/3$
Increasing	$4/3 < x < \infty$
Decreasing	$-\infty < x < 4/3$

So when it comes to sketching $f''(x)$, we can ignore the first three pieces of information, and simplify the table.

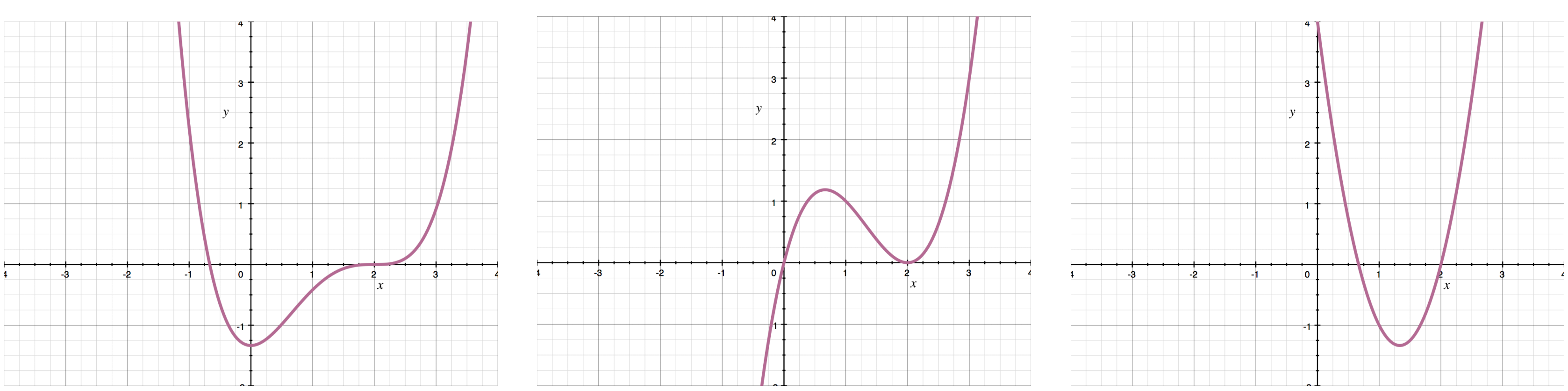
$f''(x)$	
0	$x = 2/3$ and $x = 2$
Positive	$-\infty < x < 2/3$ and $2 < x < \infty$
Negative	$2/3 < x < 2$
Critical point	$x = 4/3$
Increasing	$4/3 < x < \infty$

Decreasing $-\infty < x < 4/3$

If we put all of this information together, one possible graph of $f''(x)$ might look like this:



There's one last thing we should notice about this last example. The graphs of $f(x)$, $f'(x)$, and $f''(x)$, in that order, are



Here's what we see:

- The original function $f(x)$ is quartic (a fourth-degree function)

- The derivative $f'(x)$ is cubic (a third-degree function)

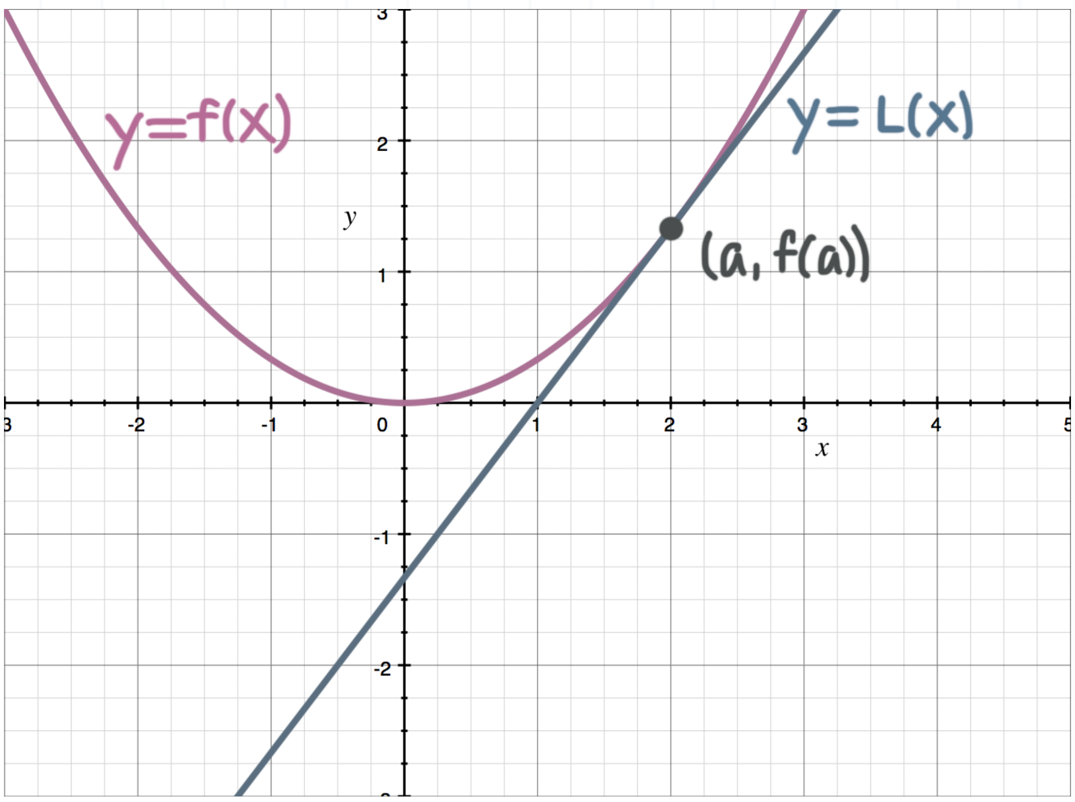
- The second derivative $f''(x)$ is parabolic (a second-degree function)

This should make sense! If the original function is quartic, then it'll include an x^4 term. When we differentiate that term, we'll get an x^3 term in the derivative function. And when we differentiate that term, we'll get an x^2 term in the second derivative function.

Understanding how $f(x) \to f'(x) \to f''(x)$ can follow a pattern like $x^4 \to x^3 \to x^2$, and how $f''(x) \to f'(x) \to f(x)$ can follow the opposite $x^2 \to x^3 \to x^4$ pattern, can really help us to know ahead of time what our graph will generally look like, since we know the approximate shape of parabolic, cubic, and quartic functions.

Linear approximation

Remember that the **tangent line** of a function $y = f(x)$ at a particular point is the line that skims along one side of a graph, intersecting the graph at exactly one point, called the **point of tangency** $(a, f(a))$, but never crossing the graph.



We can find the equation of the tangent line with only two pieces of information: 1) the point of tangency $(a, f(a))$, and 2) the slope of the tangent line $m = f'(a)$, which we find by evaluating the function's derivative at the point of tangency. Then the equation of the tangent line is

$$y = f(a) + f'(a)(x - a)$$

When we look at the graph of a curve and its tangent line at a particular point of tangency, what we see is that, not only do the curve and the tangent line have equal values at the point of tangency, but their values are very close to one another near the point of tangency. In other words, around the point of tangency, the function's graph and the tangent line are really close to each other.

Because of that, we can use values along the tangent line, near the point of tangency, to approximate values along the curve at the same point.

When we use the tangent line equation as this kind of approximation tool, we call it the **linear approximation** (or linearization) equation, instead of the tangent line equation. So the tangent line and the linear approximation are the same thing, but using the term "linear approximation" specifically implies that we're using the line to approximate function values around the point of tangency. To indicate that we're doing linear approximation, we use $L(x)$ in the equation of the tangent line:

$$L(x) = f(a) + f'(a)(x - a)$$

Linear approximation is a useful tool because the function's value around the point of tangency isn't always easy to calculate, and finding the linear approximation's value might be much easier. Or, we might not have an equation for the function at all, in which case, linear approximation can help us to estimate a particular value.

Let's work through an example.

Example

Use linear approximation to estimate $f(1.9)$.

$$f(x) = \frac{2}{\sqrt{x - 1}}$$

The first thing we want to realize is that finding $f(1.9)$ gets pretty messy. If we substitute $x = 1.9$ into the function, we get $\sqrt{0.9}$ in the function's denominator. That's not necessarily an easy value to find. However, $\sqrt{0.9}$ is pretty close to $\sqrt{1}$, which is a very easy value to find.

Therefore, instead of trying to find $f(1.9)$, let's use a linear approximation equation and $a = 2$ to get an approximation for $f(1.9)$. First, evaluate the function at $a = 2$.

$$f(2) = \frac{2}{\sqrt{2-1}}$$

$$f(2) = \frac{2}{\sqrt{1}}$$

$$f(2) = 2$$

Find the derivative, $f'(x)$, by first rewriting the function.

$$f(x) = \frac{2}{(x-1)^{\frac{1}{2}}}$$

$$f(x) = 2(x-1)^{-\frac{1}{2}}$$

Then differentiate.

$$f'(x) = -\frac{1}{2}(2)(x-1)^{-\frac{3}{2}}$$

$$f'(x) = -\frac{1}{\sqrt{(x-1)^3}}$$

Evaluate the derivative at $a = 2$.

$$f'(2) = -\frac{1}{\sqrt{(2-1)^3}}$$

$$f'(2) = -\frac{1}{\sqrt{1}}$$

$$f'(2) = -1$$

Substituting the slope $f'(2) = -1$ and the point of tangency $(2,2)$ into the linear approximation gives

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = 2 + (-1)(x-2)$$

$$L(x) = 2 - (x-2)$$

$$L(x) = 2 - x + 2$$

$$L(x) = 4 - x$$

Now that we've built the linear approximation equation, we can substitute $x = 1.9$.

$$L(1.9) = 4 - 1.9$$

$$L(1.9) = 2.1$$

This value tells us that the value along the linear approximation line is 2.1 at $x = 1.9$. So 2.1 is a value that we can use as an estimate of the function's value at 1.9.

Estimating a root

Very commonly, we'll use linear approximation to estimate the value of a root.

Linear approximation is particularly good at this. For instance, without a calculator, it's extremely difficult to find $\sqrt{82}$. At the same time, we know immediately that $\sqrt{81}=9$.

So to estimate $\sqrt{82}$, we'll instead consider the function $f(x)=\sqrt{x}$, and use $(a,f(a))=(81,9)$ as the point of tangency along $f(x)=\sqrt{81}$, in order to get an approximation for $\sqrt{82}$.

As always with linear approximations, we'll differentiate the function and evaluate the derivative at the point of tangency, and then substitute the slope and the point of tangency into the linear approximation equation.

Let's do an example with a fourth root.

Example

Use linear approximation to estimate $\sqrt[4]{17}$.

We certainly don't know the value of $\sqrt[4]{17}$, but we know that $\sqrt[4]{16}=2$. So instead of trying to calculate $\sqrt[4]{17}$ directly, let's use the function $f(x)=\sqrt[4]{x}$.

Differentiate the function.

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$$

$$f'(x) = \frac{1}{4x^{\frac{3}{4}}}$$

and then evaluate it at $a = 16$.

$$f'(16) = \frac{1}{4(16)^{\frac{3}{4}}}$$

$$f'(16) = \frac{1}{4(16^{\frac{1}{4}})^3}$$

$$f'(16) = \frac{1}{4(2)^3}$$

$$f'(16) = \frac{1}{4(8)}$$

$$f'(16) = \frac{1}{32}$$

So along the function $f(x) = \sqrt[4]{x}$, we have the point of tangency $(16,2)$ and the slope $m = 1/32$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 2 + \frac{1}{32}(x - 16)$$

$$L(x) = 2 + \frac{1}{32}x - \frac{16}{32}$$

$$L(x) = \frac{1}{32}x - \frac{1}{2} + 2$$

$$L(x) = \frac{1}{32}x + \frac{3}{2}$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[4]{17}$. Substitute $x = 17$.

$$L(17) = \frac{1}{32}(17) + \frac{3}{2}$$

$$L(17) = \frac{17}{32} + \frac{3}{2}$$

$$L(17) = \frac{17}{32} + \frac{48}{32}$$

$$L(17) = \frac{65}{32}$$

Absolute, relative, and percentage error

Once we've used a linear approximation equation to estimate the function's value at a particular point, we may want to calculate how good or bad the estimate was.

Depending on the point of tangency we chose, and the behavior of the function near that point, the linear approximation we found might be a pretty good estimate of the function's actual value, or it might not be.

To determine whether or not we've found a good estimate, we want to look at error.

Absolute error

The **absolute error** for an estimation at a particular point a is the absolute value of the difference between the function's actual value and the linear approximation at that point.

$$E_A(a) = |f(a) - L(a)|$$

We can use E_A to indicate absolute error. We take the absolute value of the difference to ensure that we always get a positive value for absolute error. That's because absolute error is truly just the distance between the value of the linear approximation line $L(x)$ at $x = a$ and the value of the function $f(x)$ at $x = a$.

If we didn't use absolute value in the absolute error formula, we'd get a positive value for absolute error when the linear approximation line was below the function's curve, but a negative value for absolute error when the linear approximation line was above the function's curve. We're just interested in the distance between the two values, so we take the absolute value.

For instance, if we're trying to find a function's value at $x = 1.9$, the actual value might be $f(1.9) = 2.1081$, while the linear approximation might be $L(1.9) = 2.1$.

So the absolute error at $x = 1.9$ would be

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(1.9) = |f(1.9) - L(1.9)|$$

$$E_A(1.9) = |2.1081 - 2.1|$$

$$E_A(1.9) = |0.0081|$$

$$E_A(1.9) = 0.0081$$

This is the absolute error, which is the distance between the function's actual value at $x = 1.9$ and the linear approximation at $x = 1.9$.

Relative error

Once we've found absolute error, we can use it to find relative error. Think of **relative error** as the amount of error, compared to the function's actual value. As the name suggests, it tells us the amount of error in the approximation, *relative* to the function's actual value.

We find it by dividing absolute error by the function's actual value.

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

Notice how, if we wanted to skip the absolute error step and go straight to relative error, we could have written the relative error formula to include the absolute error calculation.

$$E_R(a) = \frac{|f(a) - L(a)|}{f(a)}$$

In the previous example, we found that the absolute error was $E_A = 0.0081$, and that the function's actual value was $f(1.9) = 2.1081$. So the relative error at $x = 1.9$ would be

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(1.9) = \frac{E_A(1.9)}{f(1.9)}$$

$$E_R(1.9) = \frac{0.0081}{2.1081}$$

$$E_R(1.9) \approx 0.003842$$

The easiest way to conceptualize relative error is to compare different examples.

Let's say that, in one problem, you know that the function's actual value at a point is 100, and the absolute error at that point is 10. Then relative error at that point is $10/100 = 0.10$. Compared to the value of the actual function, 100, the absolute error of 10 doesn't seem too bad, which is reflected in the somewhat small value of the relative error, $E_R = 0.10$.

But if, in another problem, you know that the function's actual value at a point is 12, and the absolute error at that point is still 10 (like in the previous problem), then the relative error at that point is $10/12 \approx 0.83$. Compared to the value of the actual function, 12, the absolute error of 10 is pretty bad, which is reflected in the somewhat large value of the relative error, $E_R \approx 0.83$.

Percentage error

The **percentage error** is simply the relative error expressed as a percentage. So to find percentage error, we just have to multiply the relative error by $100\,\%$. So we could write the formula for percentage error in three ways:

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(a) = 100\% \cdot \frac{E_A(a)}{f(a)}$$

$$E_P(a) = 100\% \cdot \frac{|\,f(a) - L(a)\,|}{f(a)}$$

In the previous example, we found relative error at $x = 1.9$ to be $E_R(1.9) \approx 0.003842$. Converting that to percentage error, we get

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(a) = 100\% \cdot 0.003842$$

$$E_P(a) = 0.3842\,\%$$

Now that we understand how to find each of these error values, let's work through a full example.

Example

Use a linear approximation to estimate the value of $\sqrt{50}$, then find the absolute, relative, and percentage error of the estimate.

We need to realize here that $\sqrt{50}$ is a difficult value to find. But it's very close to $\sqrt{49}$, which we already know is 7. So instead of thinking specifically about $\sqrt{50}$, let's think about $\sqrt{x}$, and therefore use the function $f(x) = \sqrt{x}$.

Differentiate the function,

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f'(x) = \frac{1}{2x^{\frac{1}{2}}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

then evaluate the derivative at $x = 49$.

$$f'(49) = \frac{1}{2\sqrt{49}}$$

$$f'(49) = \frac{1}{2(7)}$$

$$f'(49) = \frac{1}{14}$$

So the linear approximation line intersects $f(x) = \sqrt{x}$ at the point of tangency $(49,7)$, and has a slope of $m = 1/14$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 7 + \frac{1}{14}(x - 49)$$

$$L(x) = 7 + \frac{1}{14}x - \frac{49}{14}$$

$$L(x) = \frac{1}{14}x - \frac{7}{2} + \frac{14}{2}$$

$$L(x) = \frac{1}{14}x + \frac{7}{2}$$

Then the linear approximation at $x = 50$ is

$$L(50) = \frac{1}{14}(50) + \frac{7}{2}$$

$$L(50) = \frac{50}{14} + \frac{7}{2}$$

$$L(50) = \frac{50}{14} + \frac{49}{14}$$

$$L(50) = \frac{99}{14}$$

$$L(50) \approx 7.07142857$$

In comparison, the actual value of $\sqrt{50}$ is

$$f(x) = \sqrt{x}$$

$$f(50) = \sqrt{50}$$

$$f(50) \approx 7.07106781$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(50) = |f(50) - L(50)|$$

$$E_A(50) \approx |7.07106781 - 7.07142857|$$

$$E_A(50) \approx |-0.00036076|$$

$$E_A(50) \approx 0.00036076$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(50) = \frac{E_A(50)}{f(50)}$$

$$E_R(50) \approx \frac{0.00036076}{7.07106781}$$

$$E_R(50) \approx 0.00005102$$

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(50) = 100\% \cdot E_R(50)$$

$$E_P(50) \approx 100\% \cdot 0.00005102$$

$$E_P(50) \approx 0.005102\,\%$$

Given the extremely small percentage error, we can conclude that the linear approximation at $x = 50$ gives us a pretty good estimate of the function's actual value at that point.

Radius of the balloon

Previously, we learned how to use implicit differentiation to take the derivative of an equation in which the x and y were mixed together on the same side of the equation. For instance,

$$x^2y^2 = 2x + 4y$$

is an equation with x and y variables mixed together on both sides of the equation, and no matter how much we try to manipulate it, we can't rewrite the equation in a way that completely separates the variables from each other, putting y on one side and x on the other.

Therefore, to differentiate that equation, we'd have to use implicit differentiation.

Related rates

At this point, we want to turn our attention toward related rates problems, which we can only solve using implicit differentiation.

Related rates problems are usually easy to spot, because they ask us to find how quickly one variable is changing when we know how quickly another variable is changing. For example, these are all related rates questions:

- How fast is the radius of a tire increasing if air is being pumped into the tire at a particular rate?

- How fast is the water level in a swimming pool increasing if water is being fed into the pool at a particular rate?

- How fast is the distance between you and your friend decreasing if you're walking toward him at a particular rate?

These sound tricky, but we'll actually follow a fairly consistent set of steps in order to solve every related rates problem. In general, we want to follow these steps:

1. Build an equation containing all the relevant variables, solving for some of them using other information, if necessary.

2. Implicitly differentiate the equation with respect to time t, before plugging in any of the values we know.

3. Plug in all the values we know, leaving only the one we're trying to solve for.

4. Solve for the unknown variable.

Related rates is one of those concepts where practicing lots and lots of problems really does make a difference, because the more problems you do, the better feel you get for this pattern of problem solving.

Radius of the balloon

There are an infinite number of different kinds of related rates problems we can do, but there are also some related rates problems in particular that are really common. In this section we're going to focus each lesson on one of these most common types, starting here with the inflating/deflating balloon problem.

The scenario is that we have either an inflating balloon, or a deflating balloon. We're usually told to treat the balloon as a perfect sphere, and then we're given information about the volume of the balloon, and/or the radius of the balloon.

Remember from geometry that the formulas for the volume of a sphere and the surface area of a sphere are

$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$

where V is the volume of the sphere and r is the radius of the sphere. We'll usually use these formulas in these kinds of related rates problems.

Let's work through an example where we're given the rate at which volume is increasing, and asked to find the rate at which the length of the radius is increasing.

Example

How fast is the radius of a spherical balloon increasing when the radius is 100 cm, if air is being pumped into it at 400 cm^3/s?

In this example, we're asked to find the rate of change of the radius, given the rate of change of the volume. So right away we know we're looking for a formula that relates the volume of a sphere to the radius of a sphere, and we can get that from the formula for volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Before doing anything else, we use implicit differentiation to differentiate both sides with respect to time t. When we used implicit differentiation in the past, we'd treat x like a normal variable, but we'd treat y as a function of x. That meant that, every time we took the derivative of y, we had to multiply by y'.

For these related rates problems, we treat t like a normal variable, and we treat every other variable as a function of t. Which means that, every time we take the derivative of one of these other variables, we have to multiply by the derivative of that variable.

So when we use implicit differentiation to differentiate both sides of the equation for the volume of a sphere, we'll multiply by V' when we differentiate V, and multiply by r' when we differentiate r.

$$1(V') = \frac{4}{3}\pi(3r^2)(r')$$

$$V' = 4\pi r^2 r'$$

Notice how we took the derivative of V and r like normal, but then we multiplied by V' and r'. Remember that π is a constant, so the $(4/3)\pi$ just acts like a coefficient in front of the r^3, and it stays right where it is.

Because V and r are both functions with respect to time t, we can replace their derivatives V' and r' with dV/dt and dr/dt. Keep in mind that dV/dt

represents the rate at which the volume is changing, dr/dt is the rate at which the radius is changing, and r is the length of the radius at a specific moment in time.

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

Now that we're done with the differentiation, we can substitute for any values we already know. The problem tells us that air is being pumped into the balloon, which means the volume of the balloon will be changing at that rate, so the rate of change of the volume is 400 cm^3/s.

$$400=4\pi r^2\frac{dr}{dt}$$

We've also been told that the length of the radius at the specific moment we're interested in is 100.

$$400=4\pi(100)^2\frac{dr}{dt}$$

$$400=4\pi(10{,}000)\frac{dr}{dt}$$

$$400=40{,}000\pi\frac{dr}{dt}$$

We've been asked to figure out how fast the radius of the balloon is changing. The rate of change of the radius is dr/dt, so we need to solve this equation for dr/dt.

$$\frac{dr}{dt}=\frac{400}{40{,}000\pi}$$

$$\frac{dr}{dt}=\frac{1}{100\pi}$$

This equation says that the rate of change of the radius is $1/100\pi$ cm per second, so we can say that, if air is being pumped into the balloon at 400 cm^3/s, then the length of the radius is increasing at $1/100\pi$ cm/s at the exact moment in time when the length of the radius is 100 cm.

Price of the product

When we looked at the inflating/deflating balloon problem, we weren't given an equation as part of the question. But we knew the formula for the volume of a sphere, so we were able to use that equation to relate the variables for volume and radius.

When we have a "price of the product" problem, we're usually given the equation that relates quantity and price as part of the problem. For instance, we might be told that the quantity and price of an item are related by $q = 20e^{-2p}$.

Along with the equation, we'll be given information about the sale price, the rate of change of the sale price, the quantity (supply) of the item, or the rate of change of the supply.

If we're given the equation that relates price to quantity, then we can differentiate it implicitly, substitute for the values we know, and then solve for the value we're trying to find.

Let's do an example.

Example

The manufacturer is increasing production of an item by 14 units per week. The item currently sells for $\$2$. How fast is the price of the item changing as the manufacturer produces more units?

$$q = 1000e^{-0.5p}$$

The equation for q in terms of p gives us the relationship between the quantity (number of items being manufactured) and price (the retail price at which the item is being sold).

We need to start by differentiating the equation, taking the derivative of both q and p as functions of time t, and therefore multiplying by q' when we differentiate q, and by p' when we differentiate p.

$$1(q') = 1000e^{-0.5p}(-0.5)(1)(p')$$

$$q' = -500e^{-0.5p}p'$$

We know that production (quantity) is increasing by 14 units per week, so $q' = 14$.

$$14 = -500e^{-0.5p}p'$$

And we know that the current price of the item is $\$2$, so $p = \$2$.

$$14 = -500e^{-0.5(2)}p'$$

We were asked to solve for the rate of change of the price, which means we need to solve for p'.

$$-\frac{14}{500} = e^{-1}p'$$

$$-\frac{7}{250} = e^{-1}p'$$

$$p' = -\frac{7e}{250}$$

$$p' \approx -0.08$$

Since we were told in the problem that the manufacturer was increasing production by 14 units per *week*, this result tells us that the retail price of the item is changing by $-\$0.08$ per week (decreasing by 8 cents per week). In other words, the price is going down by 8 cents per week.

Water level in the tank

In the same way that we used the formula for the volume of a sphere to solve the inflating/deflating balloon problems, we can use the formulas for the volume of other geometric figures to solve "water level in the tank" problems.

Formulas and variables

The shape of the tank in the problem will dictate which volume formula we need to use. As a reminder, here are the volume formulas for different geometric figures:

Tank shape	Volume formula
Cube	$V = s^3$
Rectangular prism	$V = lwh$
Triangular prism	$V = (1/2)whl$
Pyramid	$V = (1/3)s^2h$
Cone	$V = (1/3)\pi r^2h$
Sphere	$V = (4/3)\pi r^3$
Cylinder	$V = \pi r^2h$

In these kinds of related rates problems, a tank is either being filled up with water, or some other substance, or the tank is being emptied of that water or substance.

We may be given information about the amount of substance in the tank or how quickly the tank is being filled or emptied, about the height of the substance in the tank or how quickly that height is changing, or (for some tank shapes), the length of the radius of the substance, or how fast the length of the radius is changing.

As with other related rates problems, we'll start with the volume equation, then use implicit differentiation to take the derivative, treating each variable as a function of time t.

Once we've found the derivative, we'll substitute the values we know, and then solve for the value we're trying to find.

Tank vs. substance

There's one thing we need to be really careful of whenever we're doing "water level in the tank" problems, which is that we need to make sure we're distinguishing between the tank itself, and the substance in the tank.

If the tank is completely full, then the substance and the tank will have the same dimensions and the same volume. But if the tank isn't full, then the substance will have a smaller volume and smaller dimensions than the tank.

Usually, we're focused on the substance itself, so when we apply the formula for volume, we're using it as a representation of the volume of the substance, not the volume of the tank.

Let's work through an example in which we're filling up a cone-shaped tank.

Example

An inverted cone-shaped tank with radius 8 meters and height 15 meters is being filled at a rate of 2 cubic meters of water per minute. How fast is the water level rising when the height of the cone of water is 12 meters?

Because the tank is cone-shaped, we'll be using the formula for the volume of a cone.

$$V = \frac{1}{3}\pi r^2 h$$

Eventually, we'll need to solve for dh/dt, which we'll introduce into the equation by differentiating. But first, we need to eliminate r from the volume equation by expressing it in terms of h.

Knowing that the height and radius of the tank are 15 and 8, respectively, we can say

$$\frac{8}{15} = \frac{r}{h}$$

$$r = \frac{8}{15}h$$

This is the relationship between the radius and height of the tank, but the cone of water follows the same relationship, so we can substitute for r into the volume equation,

$$V = \frac{1}{3}\pi\left(\frac{8}{15}h\right)^2 h$$

$$V = \frac{64}{675}\pi h^3$$

then differentiate,

$$\frac{dV}{dt} = \frac{64}{225}\pi h^2\left(\frac{dh}{dt}\right)$$

and finally plug in what we know.

$$2 = \frac{64}{225}\pi(12)^2\left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{25}{512\pi}$$

$$\frac{dh}{dt} \approx 0.0155$$

This answer tells us that the water level is increasing at rate of $h' \approx 0.0155$ meters per minute.

Let’s do one more example, but this time we’ll be emptying a cylindrical tank.

Example

Oil is being pumped out of a cylindrical tank with radius 3 feet at a rate of 10 gallons (0.13 cubic feet) per minute. How fast is the oil level changing?

Because the tank is cylindrical, we’ll use the formula for the volume of a cylinder.

$$V = \pi r^2 h$$

The radius of the oil in the cylindrical tank never changes, so substitute $r = 3$.

$$V = \pi(3)^2 h$$

$$V = 9\pi h$$

Differentiate the equation.

$$1(V') = 9\pi(1)h'$$

$$V' = 9\pi h'$$

The volume of the oil is decreasing as oil is pumped out at 10 gallons (0.13 cubic feet) per minute, so we’ll substitute $V' = -0.13$ to get h' in feet per minute.

$$-0.13 = 9\pi h'$$

We want to know how fast the oil level is falling, so we'll solve for h'.

$$h' = -\frac{0.13}{9\pi}$$

$$h' \approx -0.0046$$

This result tells us that the height of the oil is changing at a rate of $h' \approx -0.0046$ feet per minute (falling by 0.0046 feet per minute).

Observer and the airplane

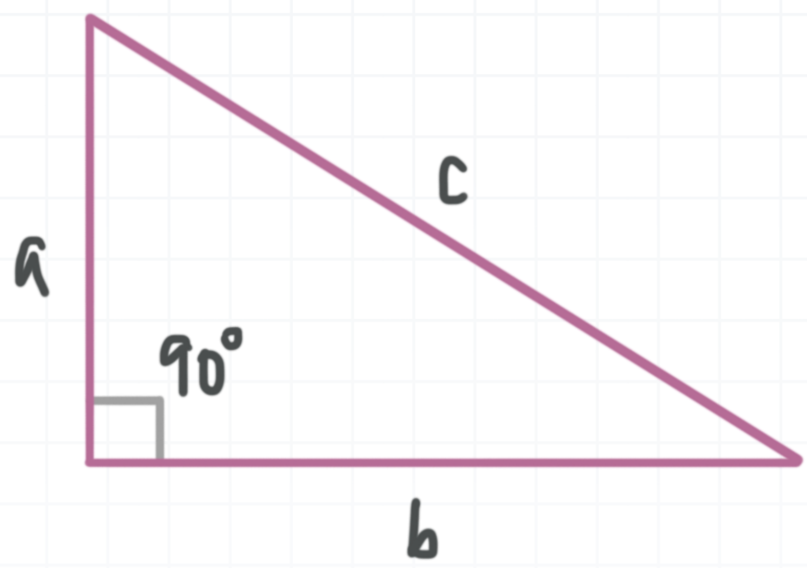
In these kinds of related rates problems, we have two objects, where either one of them is moving toward or away from the other, or where both objects are in motion toward or away from each other.

This could be a stationary observer on the ground, and an airplane flying toward or away from them overhead. It could be two cars in motion, either where both of them are approaching the same intersection, both of them are driving away from the same intersection, or one is driving toward the intersection while the other drives away.

When we tackle these kinds of problems, or really any kind of related rates problem, it's especially helpful to draw a picture of what's happening, and then label the parts of the picture.

Almost always with this type of problem, we'll use the two objects as two vertices of a **right triangle** (a triangle with one 90° angle). Once we have the triangle formed, we'll need the Pythagorean theorem in order to solve for some of the missing values that we need.

Remember that, for a right triangle, the two shorter sides that border the right angle are called a and b and the longest side that's opposite the right angle (the **hypotenuse**) is called c.



Then the **Pythagorean theorem** says that

$$a^2 + b^2 = c^2$$

Let's do a complete example so that we can see what it looks like to apply the Pythagorean theorem as part of a related rates problem.

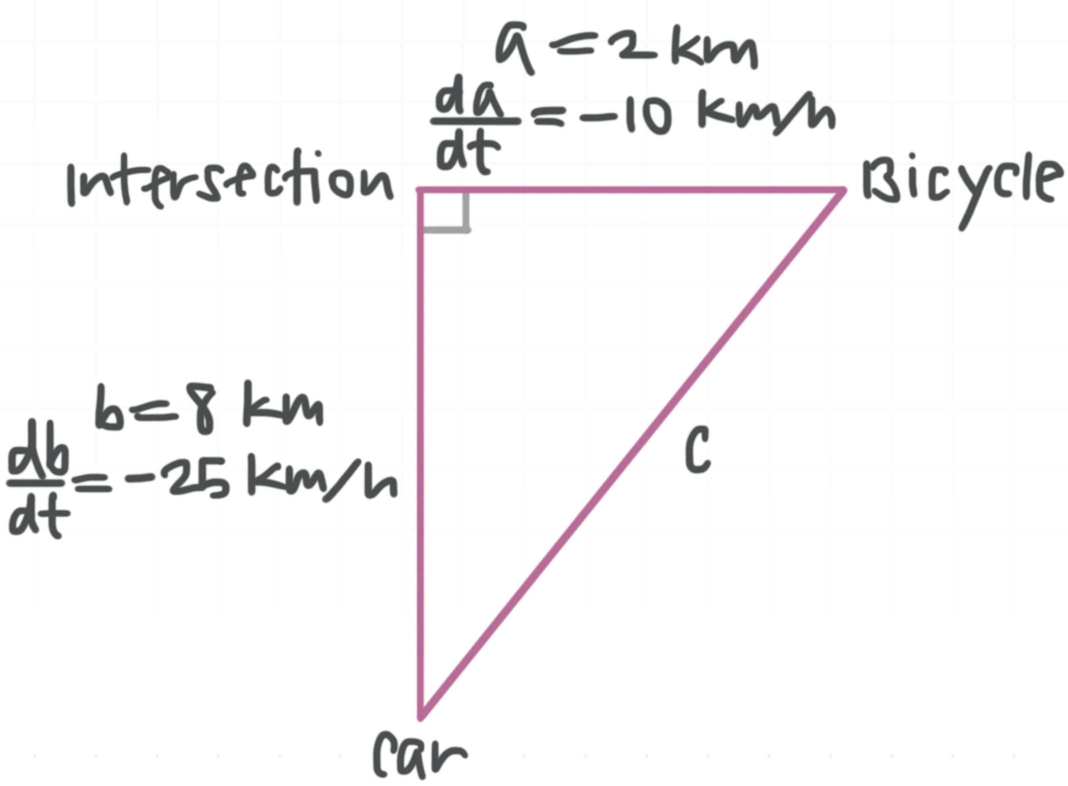
Example

A bicycle is 2 km east of an intersection, traveling toward it at 10 km/h. Meanwhile, a car is 8 km south of the intersection, traveling toward it at 25 km/h. How fast is the distance between them decreasing?

We want to start by sketching the diagram, showing the intersection, the bicycle, and the car.

We can also label the diagram with the values we've been given. The shortest side is the distance between the intersection and the bicycle, so we'll say $a = 2$ km. Because the bicycle is moving toward the intersection, the distance between them is decreasing, which means the length of that side is getting shorter, so we'll give the rate of change of that side length the negative value $da/dt = -10$ km/h.

The other leg is the distance between the intersection and the car, so we'll say $b = 8$ km. Because the car is moving toward the intersection, the distance between them is decreasing, which means the length of that side is getting shorter, so its rate of change is the negative value $db/dt = -25$ km/h.



Ultimately, we're trying to find the rate of change of the distance between the bicycle and the car, which is the rate at which the length of side c is decreasing, or dc/dt. We can use the Pythagorean theorem to find the distance between the bicycle and the car, c, at the time when $a = 2$ and $b = 8$.

$$a^2 + b^2 = c^2$$

$$2^2 + 8^2 = c^2$$

$$4 + 64 = c^2$$

$$68 = c^2$$

$$c = \sqrt{68}$$

$$c = 2\sqrt{17}$$

Then, if we start with the Pythagorean theorem, which relates the three side lengths, $a^2 + b^2 = c^2$, we can use implicit differentiation to take the derivative of both sides.

$$2a\left(\frac{da}{dt}\right) + 2b\left(\frac{db}{dt}\right) = 2c\left(\frac{dc}{dt}\right)$$

Substitute what we already know.

$$2(2)(-10) + 2(8)(-25) = 2(2\sqrt{17})\left(\frac{dc}{dt}\right)$$

$$-40 - 400 = 4\sqrt{17}\left(\frac{dc}{dt}\right)$$

Now solve for dc/dt, which is the rate of change between the bicycle and the car.

$$-440 = 4\sqrt{17}\left(\frac{dc}{dt}\right)$$

$$\frac{dc}{dt} = -\frac{440}{4\sqrt{17}}$$

$$\frac{dc}{dt} = -\frac{110}{\sqrt{17}}$$

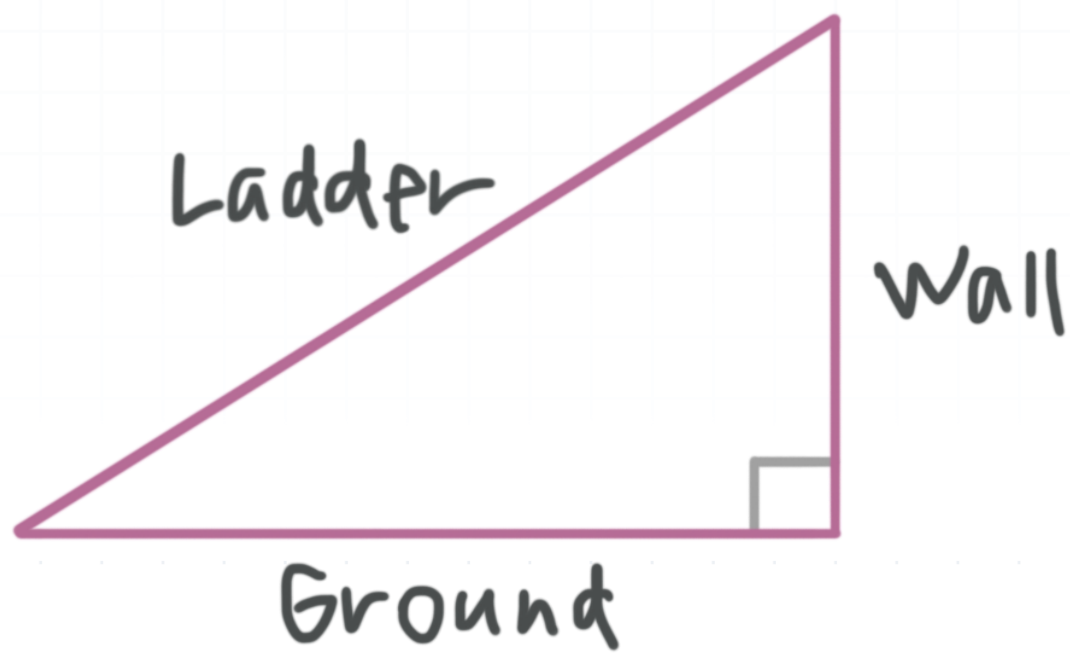
$$\frac{dc}{dt} \approx -26.68$$

This result tells us that the distance between the bicycle and the car is decreasing at a rate of about 26.68 km/h, which means that the car and bicycle are getting closer to each other at that rate.

Ladder sliding down the wall

The last type of related rates problem we want to focus on is the common "ladder sliding down the wall" problem. Sometimes this is a ladder sliding down a wall, sometimes it's a shovel sliding down a garden fence, but the idea is always the same.

Like in the "observer and the airplane" lesson, we'll use the Pythagorean theorem $a^2 + b^2 = c^2$ in these problems, because the ladder, the wall, and the ground will form a right triangle.



One of the things to keep in mind here is that the length of the ladder is constant. Its length won't change as it falls. But as the ladder slides, the height of the wall between the ground and the top of the ladder will decrease. At the same time, the length of the ground between the wall and the bottom base of the ladder will increase.

We also often need to use trigonometric functions to work with the angles inside the triangle formed by the ladder, the wall, and the ground. The trig functions we want to focus on are sine, cosine, and tangent. As a reminder, the sine of an angle inside the triangle is equal to the length of the opposite side, divided by the length of the hypotenuse; the cosine of an angle inside the triangle is equal to the length of the adjacent side, divided by the length of the hypotenuse; and the tangent of an angle inside the triangle is equal to the length of the opposite side, divided by the length of the adjacent side.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

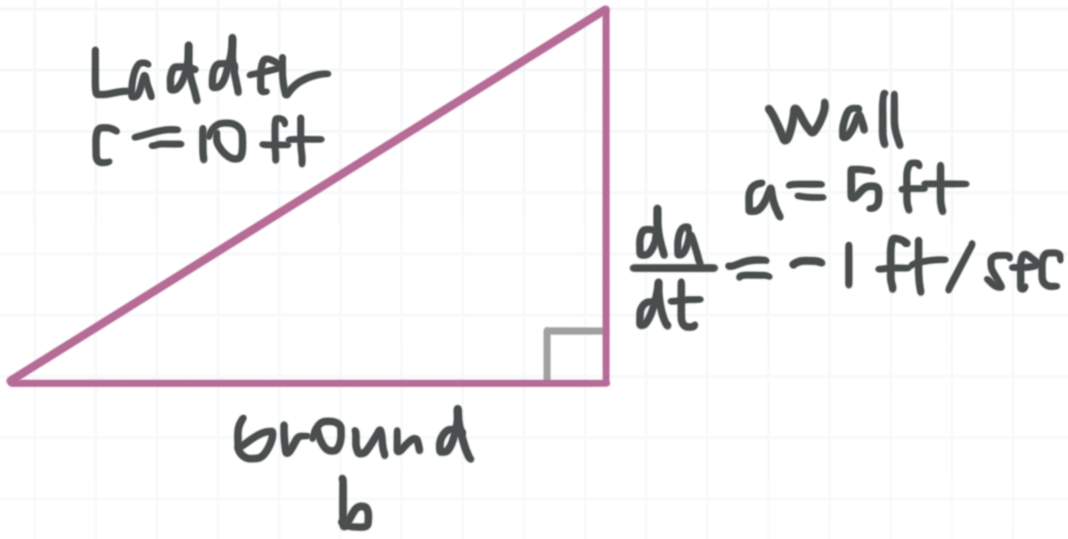
We may also need to utilize the fact that the three interior angles of a triangle always sum to 180°.

With all of this background laid out now, let's work through an example.

Example

A 10 foot ladder is sliding down a wall at a rate of 1 foot per second. How quickly is the base of the ladder sliding along the ground away from the wall when the top of the ladder is 5 feet off the ground?

The first thing we want to do is sketch a diagram, and include everything we know.



Next, we start with the Pythagorean theorem equation, $a^2 + b^2 = c^2$. We don't need to preserve dc/dt in the derivative equation, because we haven't been asked to solve for the rate of change of the length of the ladder, and we know that the length of the ladder stays constant at $c = 10$ ft. So we'll substitute $c = 10$.

$$a^2 + b^2 = 10^2$$

$$a^2 + b^2 = 100$$

Because, at the moment we're interested in, we know the length of sides a and c, we can use the Pythagorean theorem to solve for b.

$$5^2 + b^2 = 10^2$$

$$25 + b^2 = 100$$

$$b^2 = 75$$

$$b = \sqrt{75}$$

$$b = 5\sqrt{3}$$

Use implicit differentiation to find the derivative,

$$2a\left(\frac{da}{dt}\right)+2b\left(\frac{db}{dt}\right)=0$$

$$a\left(\frac{da}{dt}\right)+b\left(\frac{db}{dt}\right)=0$$

then substitute what we know.

$$(5)(-1)+(5\sqrt{3})\left(\frac{db}{dt}\right)=0$$

We've been asked to find the rate of change of the length of side b, which is the db/dt value that remains in the equation. We need to solve for that value.

$$-5+5\sqrt{3}\left(\frac{db}{dt}\right)=0$$

$$5\sqrt{3}\left(\frac{db}{dt}\right)=5$$

$$\frac{db}{dt}=\frac{5}{5\sqrt{3}}$$

$$\frac{db}{dt}=\frac{1}{\sqrt{3}}$$

$$\frac{db}{dt}\approx 0.58$$

The answer tells us that the length of side b, which is the distance between the wall and the base of the ladder, is increasing at a rate of 0.58 feet per second as the base of the ladder slides away from the wall.

Applied optimization

We've learned already how to use optimization to find the extrema of a function. But we can use the optimization process for more than just sketching graphs of functions, or finding the highest and lowest points of the function's graph.

Optimization in the real world

This same optimization process can be used in the real world. When the function we start with models some real-world scenario, then finding the function's highest and lowest values means that we're actually finding the maximum and minimum values in that situation.

For example, these are all things we can find by applying the optimization process to the real world:

- the dimensions of a rectangle that maximize or minimize its area or perimeter
- the maximum product or minimum sum of squares of two real numbers
- the time at which velocity or acceleration is maximized or minimized
- the dimensions that maximize or minimize the surface area or volume of a three-dimensional figure

- the production or sales level that maximizes profit

This is only a tiny fraction of the many ways we can use optimization to find maxima and minima in the real world.

Solving optimization problems

When it comes to actually solving these problems, we'll follow the same kinds of steps we took to solve optimization problems before.

1. Write an equation in one variable that represents the value we're trying to maximize or minimize.

2. Take the derivative, set it equal to 0 to find critical points, and use the first derivative test to determine where the function is increasing and decreasing.

3. Based on the increasing/decreasing behavior of the function, identify the function's maxima and minima.

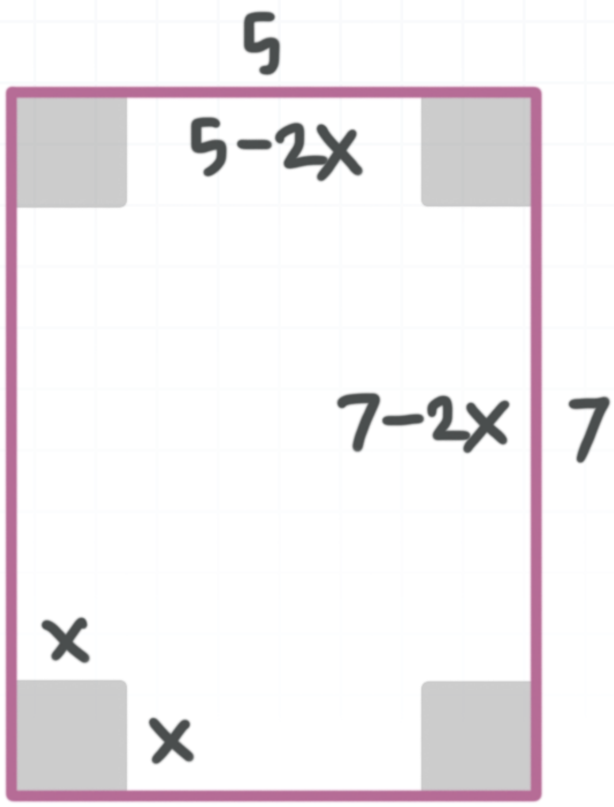
4. Use the extrema to answer the question being asked.

With these steps in mind, let's work through a typical applied optimization example. Keep in mind, there are many different kinds of applied optimization problems, but we solve all of them using this same set of steps.

Example

A 5×7 piece of paper has squares of side-length x cut from each of its corners, such that folding up the sides will create a box with no top. Find the value of x that maximizes the volume of the open-top box.

First, we'll sketch an image of the flat piece of paper.



The diagram shows the 5×7 dimensions of the paper, and the $x \times x$ square that was cut out of each corner. After cutting out the squares from the corners, the width of the open-top box will be $5 - 2x$, and the length will be $7 - 2x$.

We're being asked to maximize the volume of a box, so we'll use the formula for the volume of a box, and substitute in the length, width, and height of the open-top box.

$$V = lwh$$

$$V = (7 - 2x)(5 - 2x)x$$

$$V = (35 - 14x - 10x + 4x^2)x$$

$$V = 35x - 14x^2 - 10x^2 + 4x^3$$

$$V = 4x^3 - 24x^2 + 35x$$

Now that we have the function we want to maximize, find the derivative.

$$\frac{dV}{dx} = 12x^2 - 48x + 35$$

Find critical points by setting the derivative equal to 0,

$$12x^2 - 48x + 35 = 0$$

and then using the quadratic formula to solve for x.

$$x = \frac{48 \pm \sqrt{2{,}304 - 1{,}680}}{24}$$

$$x = \frac{48 \pm \sqrt{624}}{24}$$

$$x = \frac{12 \pm \sqrt{39}}{6}$$

So the critical points of the volume function are $x \approx 3.04$ and $x \approx 0.96$. But before we start testing critical points, we should always consider which of the critical points is even plausible.

Substitute $x = 3.04$ into the equations for the length and width of the open-top box.

$$l = 7 - 2x$$

$$l = 7 - 2(3.04)$$

$$l = 7 - 6.08$$

$$l = 0.92$$

and

$$w = 5 - 2x$$

$$w = 5 - 2(3.04)$$

$$w = 5 - 6.08$$

$$w = -1.08$$

It's nonsensical for the width of the box to be negative, so $x = 3.04$ can't be a solution. If we try $x = 0.96$ in both the length and width equations, we get positive values for both, so $x = 0.96$ can be a potential solution.

So we'll test $x = 0.96$ to make sure there's a local maximum at that point. To test it, we'll pick a value of x less than 0.96, and another greater than 0.96, and plug them into the derivative.

$$\frac{dV}{dx}(0.5) = 12(0.5)^2 - 48(0.5) + 35$$

$$\frac{dV}{dx}(0.5) = 14$$

and

$$\frac{dV}{dx}(2) = 12(2)^2 - 48(2) + 35$$

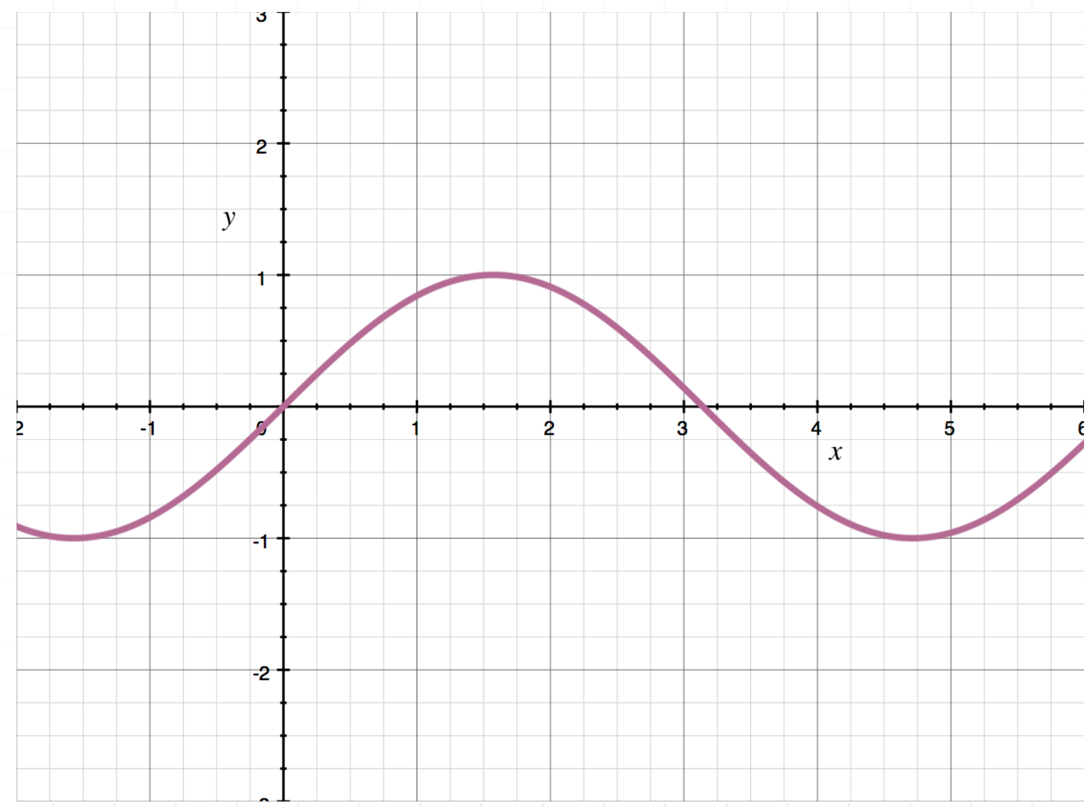
$$\frac{dV}{dx}(2) = -13$$

Because the derivative is increasing $(14 > 0)$ to the left of the critical point, and decreasing $(-13 < 0)$ to the right of it, the function has a maximum at $x = 0.96$, and we can say that the volume of the open-top box is maximized when $x = 0.96$.

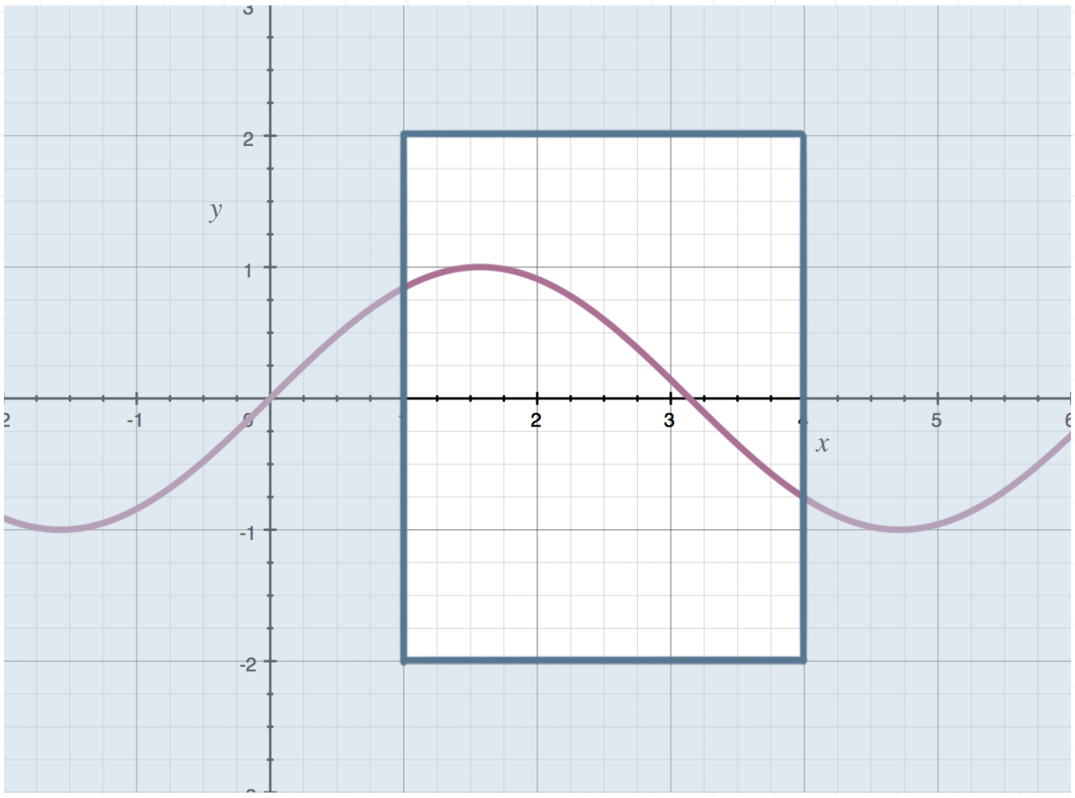
Mean Value Theorem

The Mean Value Theorem, in general, is a theorem that we use to prove that a function has a particular slope in an interval.

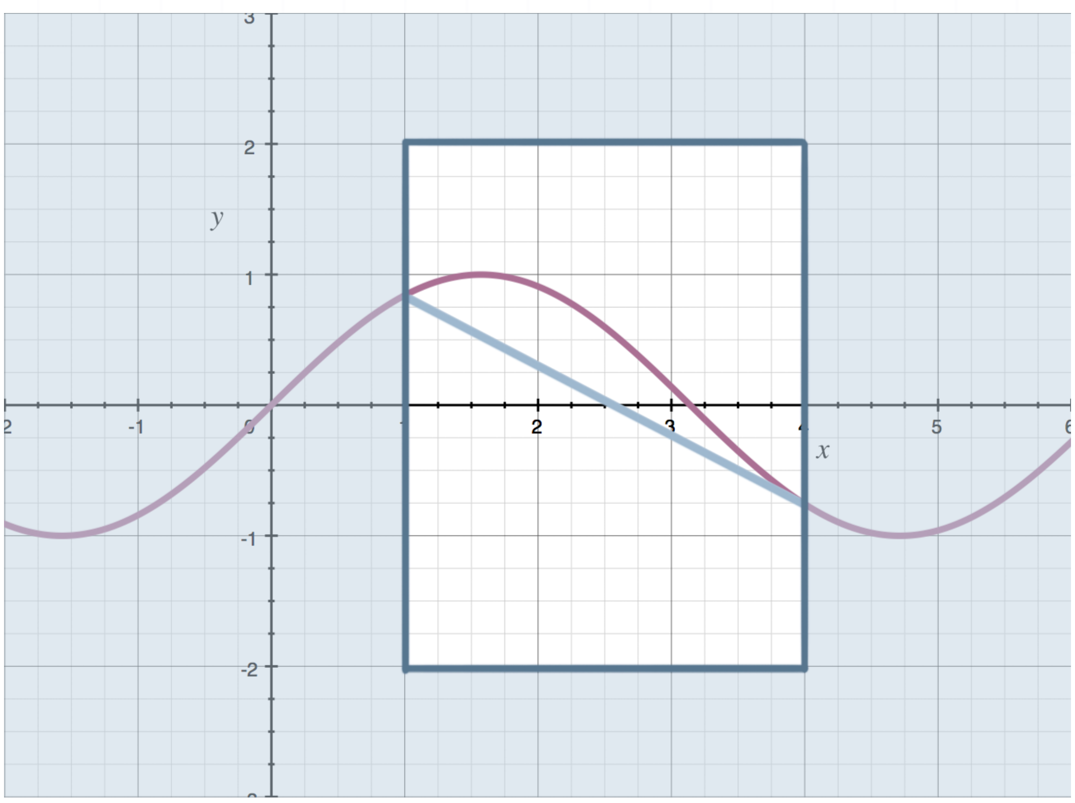
To break this down, let's first think about some generic function. We'll use $y = \sin x$, whose graph looks like this:



We can pick a closed interval, let's say $[1,4]$, and use it to bound the function. All we're saying by picking the interval is that we're only interested in what the function is doing inside that interval. Everything outside of that interval we can ignore.

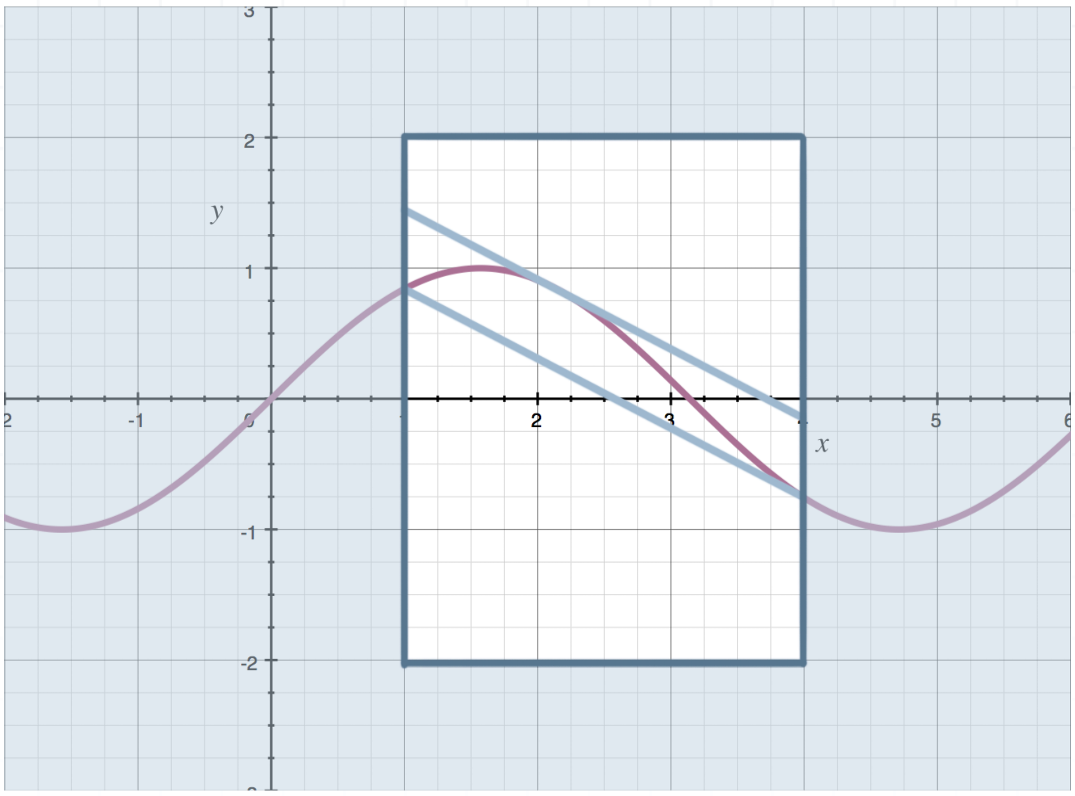


Once we've chosen an interval, look at the endpoints of the function at each edge of the interval, and connect them with a straight line.



The Mean Value Theorem tells us that, as long as the function is continuous (unbroken) and differentiable (smooth) everywhere inside the interval we've chosen, then there must be a line tangent to the curve somewhere in the interval, which is parallel to this line we've just drawn that connects the endpoints.

Here's where that tangent line exists for this particular function we've been using:



Notice how it's parallel to the line that connects the endpoints. This is what the Mean Value Theorem guarantees: a tangent line, somewhere inside the interval, that's parallel to the line that connects the endpoints.

The point at which that tangent line intersects the curve (the point of tangency) will exist at $x = c$. We call the interval $[a, b]$, so c will fall between a and b, $a < c < b$.

Mathematically, the **Mean Value Theorem** says that if $f(x)$ is

1. continuous on the closed interval $[a, b]$

2. differentiable on the open interval (a, b)

then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Practically, the consequence of the Mean Value Theorem is that the instantaneous rate of change at $x = c$ will be equal to the average rate of change over the interval.

If we take that to the real world, it means we can say things like this:

- If our average speed over the course of a long road trip is 80 km/h, then we must have been traveling at exactly 80 km/h at least at one moment during the road trip.
- If the outdoor temperature on average throughout the morning increased by 3° per hour, then the hourly temperature increase must have been exactly 3° at some point during the morning.
- If a child grew on average 0.25 inches per month over the course of a year, then their monthly growth rate must have been exactly 0.25 inches sometime during the year.

Let's work through an example of how to use the Mean Value Theorem to prove that a particular value exists for the function in an interval.

Example

We drive from Florida to California in exactly 48 hours, traveling a distance of $3{,}000$ miles. Along the route we take, the speed limit is 60 mph. Use the

Mean Value Theorem to prove that we were speeding at least at one point during the trip.

Because the trip took 48 hours, we can set the interval at $t=[0,48]$, and we can say that the function that models the trip is $f(t)$. If we plug these values into the Mean Value Theorem, we get

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{f(48)-f(0)}{48-0}$$

$$f'(c)=\frac{3{,}000-0}{48-0}$$

$$f'(c)=\frac{3{,}000}{48}$$

$$f'(c)=\frac{125}{2}$$

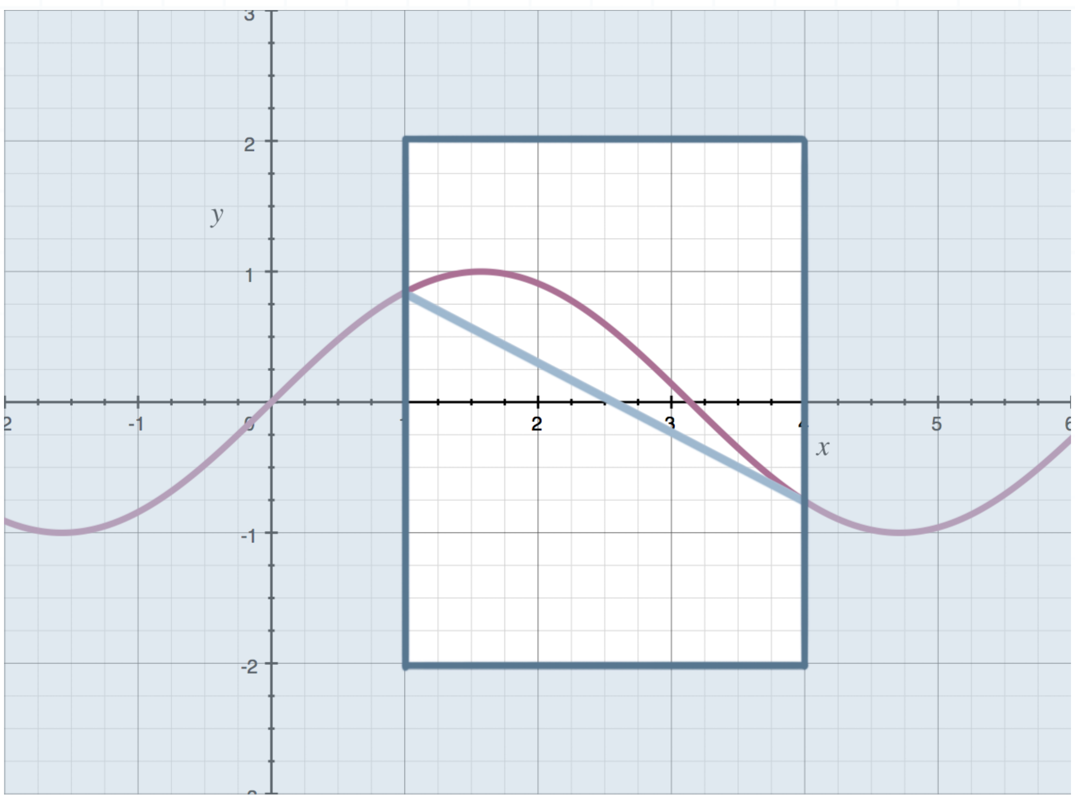
$$f'(c)=62.5$$

This result tells us that, at least at one moment during the trip, we must have been traveling at 62.5 mph, which means we were speeding over the legal limit at some point along the way.

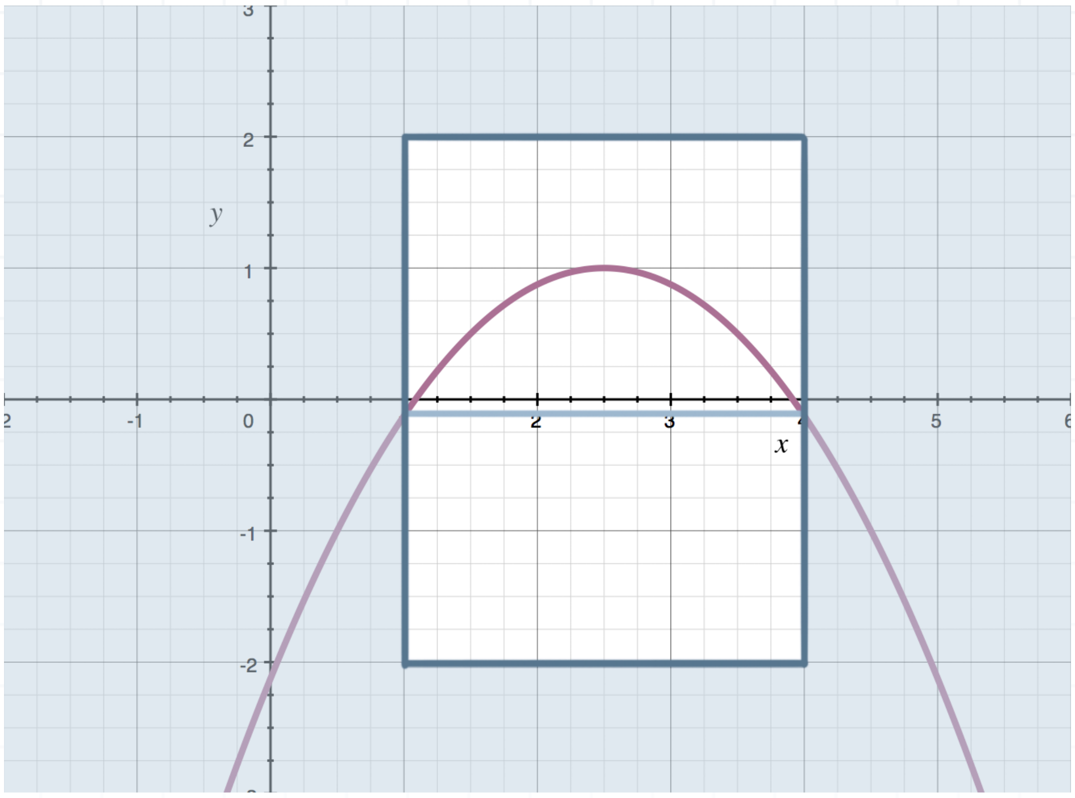
Rolle's Theorem

Rolle's Theorem is a specific instance of the Mean Value Theorem, in which the endpoints of the function at the edges of the interval are equal to one another.

In the Mean Value Theorem lesson, we looked at a function in an interval, and the line that connected the endpoints was slanted.

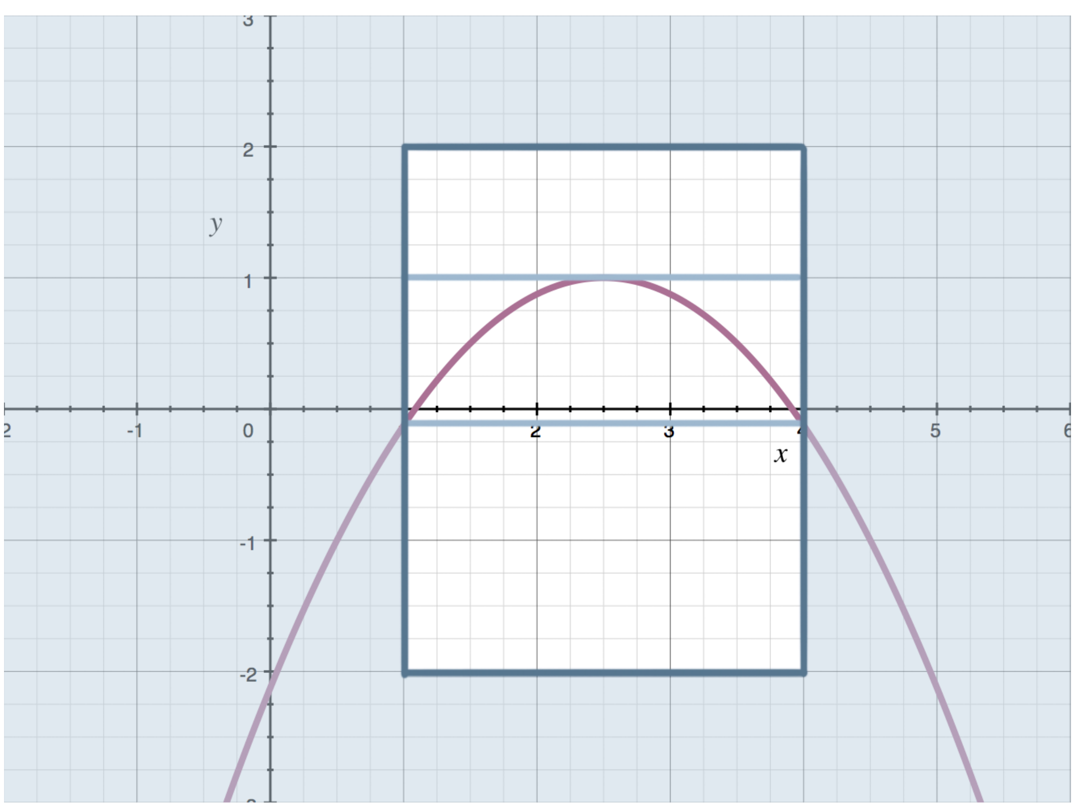


But to apply Rolle's Theorem, the values of the function at the endpoints must be equal, which means the line that connects them will be perfectly horizontal.



Just like the Mean Value Theorem, Rolle's Theorem tells us that, as long as the function is continuous (unbroken) and differentiable (smooth) inside the interval, then there must be a tangent line that's parallel to the horizontal line that connects the endpoints.

Here's where that tangent line exists for this particular function we've been using:



Notice how it's parallel to the line that connects the endpoints. This is what Rolle's Theorem guarantees: a tangent line, somewhere inside the interval, that's parallel to the line that connects the endpoints.

Because the line that connects the endpoints is horizontal, the tangent line will also be horizontal. Remember that horizontal tangent lines exist wherever the derivative is equal to 0, so Rolle's Theorem can prove all of the following:

- The existence of a horizontal tangent line in the interval
- A point at which the derivative is 0 in the interval
- The existence of a critical point in the interval
- A point at which the function changes direction in the interval, either from increasing to decreasing, or from decreasing to increasing

The point at which that tangent line intersects the curve (the point of tangency) will exist at $x = c$. We call the interval $[a, b]$, so c will fall between a and b, $a < c < b$.

We can prove Rolle's Theorem if we start from the Mean Value Theorem. The Mean Value Theorem says

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But if the endpoints are equal, then $f(a) = f(b)$, so we could make a substitution, and get

$$f'(c) = \frac{f(b) - f(b)}{b - a}$$

$$f'(c) = \frac{0}{b - a}$$

$$f'(c) = 0$$

This result tells us that, by assuming the endpoints were equal, we found a value of 0 for the derivative at $x = c$ in the interval, which is exactly what Rolle's Theorem states.

Let's work through an example.

Example

Use Rolle's Theorem to show that the function has a critical point in the interval $[0,2]$.

$$f(x) = -\frac{1}{2}x^4 + 2x^2$$

A polynomial function like this one will be continuous and differentiable everywhere in its domain. Then all we need to do is check to make sure that the function's value is equal at the endpoints of the interval, $x = 0$ and $x = 2$.

$$f(0) = -\frac{1}{2}(0)^4 + 2(0)^2$$

$$f(0) = 0$$

and

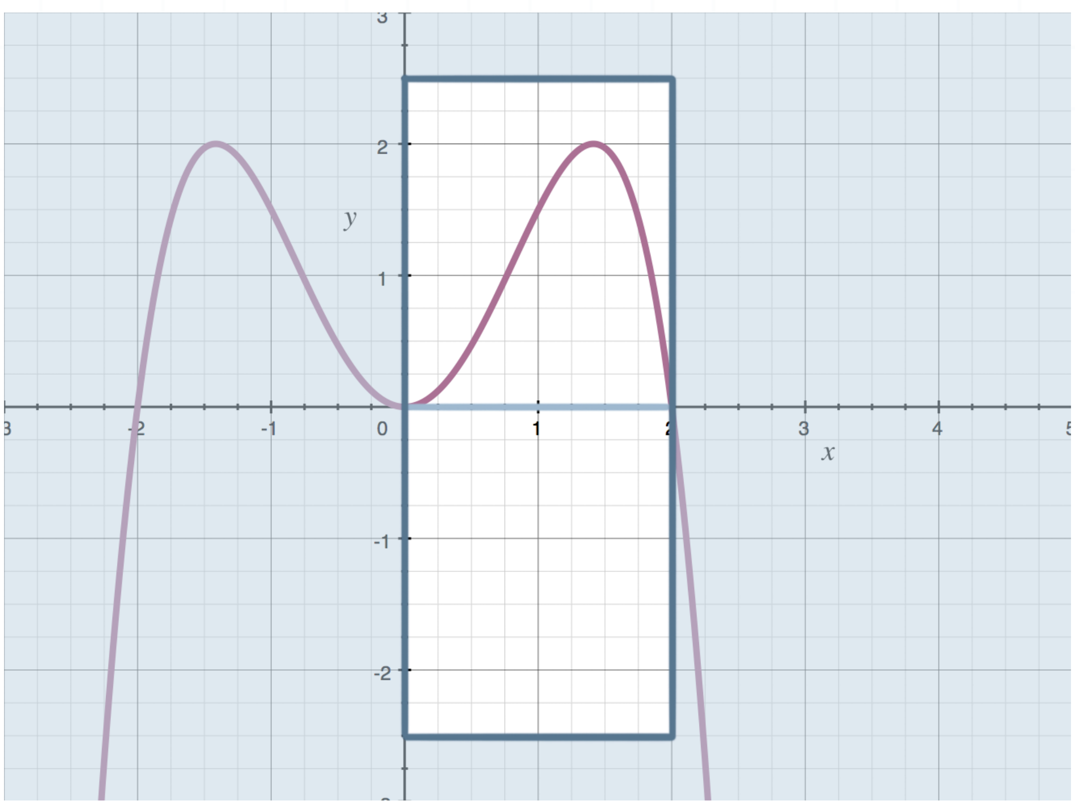
$$f(2) = -\frac{1}{2}(2)^4 + 2(2)^2$$

$$f(2) = -\frac{1}{2}(16) + 2(4)$$

$$f(2) = -8 + 8$$

$$f(2) = 0$$

The value of the function is 0 at both endpoints. It's not necessary, but we could verify this visually if we wanted to double-check ourselves.



Because the endpoints of the interval are equal to one another, Rolle's Theorem tells us that there must be a critical point somewhere in the interval.

Newton's Method

The **root** of a function is a point at which the graph of the function crosses the x-axis. Because $y = 0$ everywhere along the x-axis, a root is any point where the value of the function is 0.

Newton's Method is a tool that allows us to approximate the point at which the root exists. When we use Newton's Method, the function must be in the form $f(x) = 0$. If it isn't, we'll need to put it in that form before we start.

We'll start with one approximating value, and use it to get a better approximation. Then we'll use this new approximation to get an even better approximation. We'll continue that process, over and over, until we get an approximation we're satisfied with.

Usually, we'll choose to find an approximation to a certain number of decimal places, and that's how we'll know when to be "satisfied" with the approximation.

For example, we might choose to find an approximation to three decimal places. If so, then once we get an answer that's stable to three decimal places, meaning that the first three decimal places don't change as we keep taking better and better approximations, then we know we're done.

If we use a starting approximation x_n, then we can say that the next subsequent approximation is the x_{n+1} approximation, and Newton's Method gives us a formula for that x_{n+1} approximation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If we don't know an initial approximation to the solution x_0, we can sketch the graph of the function and use that to get an estimate of the solution, which we can then use as x_0. Or if we know the interval where the function has a solution, then we can use the midpoint of the interval as x_0.

Let's work through an example where we use Newton's Method to find an approximation of the root of a function.

Example

Use Newton's Method to find an approximation of the root of the function to four decimal places, with $x_0 = -1$.

$$f(x) = x^2 - x$$

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^2 - x = 0$$

Take the derivative of the function.

$$f'(x) = 2x - 1$$

We're using $x_0 = -1$ as the starting approximation, which means $n = 0$. We can substitute this value into the Newton's Method formula to get a formula for the next approximation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Substitute the starting approximation $x_0 = -1$.

$$x_1 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{(-1)^2 - (-1)}{2(-1) - 1} = -1 - \frac{1 + 1}{-2 - 1}$$

$$= -1 - \frac{2}{-3} = -1 + \frac{2}{3} = -\frac{1}{3} \approx -0.3333$$

Use this value for x_1 to find the next approximation for x_2.

$$x_2 = -\frac{1}{3} - \frac{f\left(-\frac{1}{3}\right)}{f'\left(-\frac{1}{3}\right)} = -\frac{1}{3} - \frac{\left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right)}{2\left(-\frac{1}{3}\right) - 1} = -\frac{1}{3} - \frac{\frac{1}{9} + \frac{1}{3}}{-\frac{2}{3} - 1}$$

$$= -\frac{1}{3} - \frac{\frac{1}{9} + \frac{3}{9}}{-\frac{2}{3} - \frac{3}{3}} = -\frac{1}{3} - \frac{\frac{4}{9}}{-\frac{5}{3}} = -\frac{1}{3} + \frac{\frac{4}{9}}{\frac{5}{3}} = -\frac{1}{3} + \frac{4}{9}\left(\frac{3}{5}\right)$$

$$= -\frac{1}{3} + \frac{4}{3}\left(\frac{1}{5}\right) = -\frac{1}{3} + \frac{4}{15} = -\frac{5}{15} + \frac{4}{15} = -\frac{1}{15} \approx -0.0667$$

Use this value for x_2 to find the next approximation for x_3.

$$x_3 = -\frac{1}{15} - \frac{f\left(-\frac{1}{15}\right)}{f'\left(-\frac{1}{15}\right)} = -\frac{1}{15} - \frac{\left(-\frac{1}{15}\right)^2 - \left(-\frac{1}{15}\right)}{2\left(-\frac{1}{15}\right) - 1} = -\frac{1}{15} - \frac{\frac{1}{225} + \frac{1}{15}}{-\frac{2}{15} - 1}$$

$$= -\frac{1}{15} - \frac{\frac{1}{225} + \frac{15}{225}}{-\frac{2}{15} - \frac{15}{15}} = -\frac{1}{15} - \frac{\frac{16}{225}}{-\frac{17}{15}} = -\frac{1}{15} + \frac{\frac{16}{225}}{\frac{17}{15}}$$

$$= -\frac{1}{15} + \frac{16}{225}\left(\frac{15}{17}\right) = -\frac{1}{15} + \frac{16}{15}\left(\frac{1}{17}\right) = -\frac{1}{15} + \frac{16}{255}$$

$$= -\frac{17}{255} + \frac{16}{255} = -\frac{1}{255} \approx -0.0039$$

Use this value for x_3 to find the next approximation for x_4.

$$x_4 = -\frac{1}{255} - \frac{f\left(-\frac{1}{255}\right)}{f'\left(-\frac{1}{255}\right)} = -\frac{1}{255} - \frac{\left(-\frac{1}{255}\right)^2 - \left(-\frac{1}{255}\right)}{2\left(-\frac{1}{255}\right) - 1} = -\frac{1}{255} - \frac{\frac{1}{255^2} + \frac{1}{255}}{-\frac{2}{255} - 1}$$

$$= -\frac{1}{255} - \frac{\frac{1}{255^2} + \frac{255}{255^2}}{-\frac{2}{255} - \frac{255}{255}} = -\frac{1}{255} - \frac{\frac{256}{255^2}}{-\frac{257}{255}} = -\frac{1}{255} + \frac{\frac{256}{255^2}}{\frac{257}{255}}$$

$$= -\frac{1}{255} + \frac{256}{255^2}\left(\frac{255}{257}\right) = -\frac{1}{255} + \frac{256}{255}\left(\frac{1}{257}\right) = -\frac{1}{255} + \frac{256}{255 \cdot 257}$$

$$= -\frac{257}{255 \cdot 257} + \frac{256}{255 \cdot 257} = -\frac{1}{255 \cdot 257} = -\frac{1}{65{,}535} \approx -0.0000$$

If we pull together the approximations we've found so far,

$$x_1 = -\frac{1}{3}$$

$$x_2 = -\frac{1}{15}$$

$$x_3 = -\frac{1}{255}$$

$$x_4 = -\frac{1}{65{,}535}$$

we can see that the solution is only getting closer and closer to 0 as the denominator gets larger.

To four decimal places, we've already found $x_4 \approx -0.0000$. Because we're getting closer to 0 with each approximation, the next approximation will also be all zeros to the first four decimal places. Therefore, the approximation of the root of the function to four decimal places is 0.

L'Hospital's Rule

Now that we've covered derivatives pretty extensively, we can introduce L'Hospital's Rule, which ties limits and derivatives together.

If you remember from the lessons on solving limits, we usually want to try solving for the limit using substitution, then factoring if substitution didn't work, then conjugate method if factoring didn't work, and then eventually we might start looking at other limit methods, if necessary.

But when other limit methods fail us, we do have a backup plan, which is L'Hospital's rule. If, no matter what other methods we try, substitution results in an **indeterminate form**, like these,

$$\frac{\pm\infty}{\pm\infty} \qquad \frac{0}{0} \qquad (0)(\pm\infty)$$

$$1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad \infty-\infty$$

then we should try L'Hospital's rule to evaluate the limit.

Applying L'Hospital's Rule

To use L'Hospital's Rule, we need the function we're evaluating to be a fraction. To apply the rule, we replace both the numerator and denominator of the fraction with their own derivatives. In other words, we take the derivative of the numerator and make that the *new* numerator, and we take the derivative of the denominator and make that the *new* denominator.

Once we've replaced both, then we try substitution to evaluate the limit. If we get a real number answer, then L'Hospital's Rule was successful, we've found our answer, and we can stop there.

If, on the other hand, we still get an indeterminate form after using substitution, it's okay. We simply apply L'Hospital's Rule a second time. In fact, we can continue applying the rule over and over again, until eventually substitution gives a real number answer instead of an indeterminate form.

Not quotient rule

Because L'Hospital's Rule has us taking derivatives inside a fraction, people often confuse this process with the Quotient Rule, which was a derivative rule we learned earlier. These two rules are not related!

If we want to find the derivative of a fraction, we use Quotient Rule. But in this case, we're not trying to find the derivative of a fraction. Instead, we're trying to take the limit of a fraction, and that fraction is giving us an indeterminate form when we try to substitute. In this case, we apply L'Hospital's rule, which has us replacing the numerator and denominator with their own derivatives. And that's not at all related to the Quotient Rule.

Let's work through an example where we apply L'Hospital's rule to evaluate a limit as $x \to 0$.

Example

Evaluate the limit.

$$\lim_{x \to 0} \frac{e^x - 1}{\sin(2x)}$$

If we try substitution to evaluate at $x = 0$, we get an indeterminate form.

$$\frac{e^0 - 1}{\sin(2(0))}$$

$$\frac{1 - 1}{\sin(0)}$$

$$\frac{0}{0}$$

Because we get an indeterminate form, let's try applying L'Hospital's Rule. The derivative of the numerator $e^x - 1$ is e^x, and the derivative of the denominator $\sin(2x)$ is $2\cos(2x)$. Rewrite the function by replacing the numerator with its derivative and replacing the denominator with its derivative.

$$\lim_{x \to 0} \frac{e^x}{2\cos(2x)}$$

Try substitution again to evaluate the limit.

$$\frac{e^0}{2\cos(2(0))}$$

$$\frac{1}{2\cos(0)}$$

$$\frac{1}{2(1)}$$

$$\frac{1}{2}$$

In this last example, we were able to get to a real number answer, so we can stop, and this is the value of the limit.

But if evaluating the limit at 0 in this last step had again resulted in an indeterminate form, we would have reapplied L'Hospital's Rule, replacing the numerator e^x with its derivative e^x, and replacing the denominator $2\cos(2x)$ with its derivative $-4\sin(2x)$. Then we would have tried substitution again to evaluate the limit at $x = 0$.

And we'd keep going, applying L'Hospital's Rule and then testing substitution, over and over, until we were able to evaluate the limit and get a real number answer, instead of an indeterminate form. Sometimes we'll have to apply L'Hospital's Rule three or even four times to get to the real number answer.

L'Hospital's Rule works really well on the two indeterminate forms $0/0$ and $\pm\infty/\pm\infty$. With other indeterminate forms, it may work better to write rewrite products as quotients and vice versa, using the rule

$$f(x)g(x) = \frac{g(x)}{\frac{1}{f(x)}}$$

Sometimes we'll need to use L'Hospital's rule on a fraction that's sitting inside some other function, so let's look at an example of a problem like that.

Example

Use L'Hospital's rule to evaluate the limit.

$$\lim_{x \to -3} 2x^{\frac{1}{x+3}}$$

If we try substitution to evaluate at $x = -3$, we get an indeterminate form.

$$2(-3)^{\frac{1}{-3+3}}$$

$$2(-3)^{\frac{1}{\infty}}$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to -3} 2x^{\frac{1}{x+3}}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to -3} \ln(2x^{\frac{1}{x+3}})$$

$$\ln y = \lim_{x \to -3} \frac{1}{x+3} \ln(2x)$$

$$\ln y = \lim_{x \to -3} \frac{\ln(2x)}{x+3}$$

With the limit rewritten, we'll apply L'Hospital's rule to the fraction.

$$\ln y = \lim_{x \to -3} \frac{\frac{1}{2x}(2)}{1}$$

$$\ln y = \lim_{x \to -3} \frac{\frac{1}{x}}{1}$$

$$\ln y = \lim_{x \to -3} \frac{1}{x}$$

Evaluate the limit,

$$\ln y = \frac{1}{-3}$$

$$\ln y = -\frac{1}{3}$$

then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^{-\frac{1}{3}}$$

$$y = e^{-\frac{1}{3}}$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to -3} 2x^{\frac{1}{x+3}}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to -3} 2x^{\frac{1}{x+3}} = e^{-\frac{1}{3}}$$

Position, velocity, and acceleration

The relationship between position, velocity, and acceleration is a common application of derivatives.

That's because velocity is the derivative of position, and acceleration is the derivative of velocity. So if we say that the position function is defined as $x(t)$, velocity by $v(t)$, and acceleration by $a(t)$, then we can describe the relationship between these functions in a table.

Position	$x(t)$
Velocity	$v(t) = x'(t)$
Acceleration	$a(t) = v'(t) = x''(t)$

In these kinds of problems, we're often given the position of some object, like a particle or a car, and then asked to calculate all different kinds of values from that position function.

Some of the most common values we'll compute for these three functions are given in the following table.

Speed (always positive, has no direction)

$$s(t) = |v(t)|$$

Speed is **increasing** when velocity and acceleration have the same sign: $v(t), a(t) > 0$ or $v(t), a(t) < 0$

Speed is **decreasing** when velocity and acceleration have opposite signs: $v(t) > 0$ with $a(t) < 0$, or $v(t) < 0$ with $a(t) > 0$

Velocity (positive or negative, has a direction)

$$v(t) = x'(t)$$

Object is **moving forward** (to the right) when $v(t) > 0$

Object is **moving backward** (to the left) when $v(t) < 0$

Object is **at rest** (not moving) when $v(t) = 0$

Velocity is **increasing** when $a(t) > 0$

Velocity is **decreasing** when $a(t) < 0$

Acceleration (positive or negative)

$$a(t) = v'(t) = x''(t)$$

Notice how we said that speed is always positive. That's because speed is given by the absolute value of velocity. When we take the absolute value of something, it means we'll turn that value positive, which is why speed will always be positive. And speed has no direction. It's simply a positive rate, for example, 40 km/h or 12 inches/second.

Velocity, unlike speed, indicates direction. So if velocity is positive, it means the object is moving forward; but if velocity is negative, it means the object is moving backward.

For these kinds of problems, we also want to remember the difference between "instantaneous" and "average." For instance, instantaneous velocity at $t = a$ is the velocity of the object at the exact moment $t = a$. On the other hand, average velocity on $t = [a, b]$ is the average velocity of the object over the entire interval from $t = a$ to $t = b$.

To find instantaneous velocity, we simply evaluate the velocity function $v(t)$ at $t = a$. But to find average velocity, we'll use

$$\Delta v(a, b) = \frac{x(b) - x(a)}{b - a}$$

Let's work through a full example to see how some of these values get calculated for a particular position function.

Example

From the position function given for a particle, find velocity and acceleration as functions of t. Find the direction in which the particle is moving when $t = 1$, and say whether its velocity and speed are increasing or decreasing at that same point.

$$x(t) = 3t^2 + 8t - 2t^{\frac{5}{2}}$$

To find velocity, take the derivative of the position function, and then to find acceleration, take the derivative of the velocity function.

$$v(t) = x'(t) = 6t + 8 - 5t^{\frac{3}{2}}$$

$$a(t) = v'(t) = x''(t) = 6 - \frac{15}{2}t^{\frac{1}{2}}$$

To find direction when $t = 1$, substitute $t = 1$ into the velocity function.

$$v(1) = 6(1) + 8 - 5(1)^{\frac{3}{2}}$$

$$v(1) = 9$$

Because $v(1)$ is positive, the particle is moving forward.

To determine whether velocity is increasing or decreasing, substitute $t = 1$ into the acceleration function.

$$a(1) = 6 - \frac{15}{2}(1)^{\frac{1}{2}}$$

$$a(1) = -\frac{3}{2}$$

Because acceleration is negative at $t = 1$, velocity is decreasing at that point. And since the velocity is positive and decreasing at $t = 1$, that means that speed is also decreasing at that point.

Ball thrown up from the ground

In this derivative application, we're dealing with the vertical motion pattern of an object that's launched straight up from the ground (or some other height), travels up until it reaches a maximum, and then falls back down to earth.

In these problems, we're always interested in three positions along the flight path of the object:

1. the initial position where the object begins when it's first thrown up,

2. the maximum position where the object stops traveling up and starts traveling back down again, and

3. the position at which the object hits the ground, finishing its flight.

At each of these three positions, we're always interested in three values:

1. the time at which the object reaches the position,

2. the height of the object at that position, and

3. the velocity of the object at that position.

If the ball is thrown from the ground, instead of from some higher spot, then its initial height is $y=0$. And when it hits the ground again after falling back down, its final height will again be $y=0$.

We also can say that the ball flight begins at time $t=0$. And when the ball reaches its maximum point, the velocity is $v=0$, since this is the point where the object changes direction, from increasing on its way up to decreasing on its way down.

Therefore, before we even begin the problem, we can partially fill in the table of values that describes the flight path.

Initial	**Maximum**	**End**
$t=0$		
	$v=0$	
$y=0$		$y=0$

So when we solve these kinds of problems, we're usually working on filling in the missing values in this table. We'll usually start with a position function $y(t)$ that models the flight path of the object, and an initial velocity. If the position function includes g, it's the gravitational constant $g=32$ ft/s^2 or $g=9.8$ m/s^2, and we can substitute for it.

If we differentiate the position function, we get the velocity function, and we can set the velocity function equal to 0, we'll be able to solve for the time t at which the object reaches its maximum height.

Then we can plug that value of t into the position function in order to find the object's maximum height. At that point, we'll have the first two columns from the table completely filled in.

To find the time when the object hits the ground at the end of its path, set the position function equal to 0, and solve for t. We should find $t = 0$ and one other value for t. This other value for t is the time at which the object hits the ground.

Then if we evaluate the velocity function (the derivative) at this second value of t, we'll get the velocity of the object when it hits the ground.

Let's work through a full example.

Example

A ball is thrown straight up from the ground with an initial velocity of $v_0 = 32$ ft/s. Find its height and velocity when it leaves the ground, when it reaches its maximum height, and when it hits the ground again.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

First, substitute $g = 32$, since the gravitational constant is included in the function.

$$y(t) = -\frac{1}{2}(32)t^2 + v_0t + y_0$$

$$y(t) = -16t^2 + v_0t + y_0$$

The initial velocity is given as $v_0 = 32$, so substitute this value as well.

$$y(t) = -16t^2 + 32t + y_0$$

Substitute the initial height of the ball, $y_0 = 0$.

$$y(t) = -16t^2 + 32t + 0$$

$$y(t) = -16t^2 + 32t$$

Fill out the vertical motion table with what's given so far in the question.

Initial	**Maximum**	**End**
$t = 0$		
$v = 32$	$v = 0$	
$y = 0$		$y = 0$

To fill out the rest of the table, start by setting the original function equal to 0, which will give us the time(s) at which the ball is on the ground.

$$-16t^2 + 32t = 0$$

$$-16t(t - 2) = 0$$

$$t = 0, 2$$

This tells us that the ball is on the ground when $t = 0$ and $t = 2$, so we can add $t = 2$ to the vertical motion chart.

Initial	**Maximum**	**End**
$t = 0$		$t = 2$
$v = 32$	$v = 0$	

$y = 0$	$y = 0$

Take the derivative of the original position function to find the velocity function.

$$v(t) = y'(t) = -32t + 32$$

Find the velocity at $t = 2$.

$$v(2) = -32(2) + 32$$

$$v(2) = -64 + 32$$

$$v(2) = -32$$

Add this to the vertical motion chart.

Initial	**Maximum**	**End**
$t = 0$		$t = 2$
$v = 32$	$v = 0$	$v = -32$
$y = 0$		$y = 0$

Set the velocity function equal to 0 to find the time at which the ball reaches its maximum height.

$$-32t + 32 = 0$$

$$32t = 32$$

$$t = 1$$

Add this to the vertical motion chart.

Initial	Maximum	End
$t = 0$	$t = 1$	$t = 2$
$v = 32$	$v = 0$	$v = -32$
$y = 0$		$y = 0$

Substitute $t = 1$ into the original position function to find the maximum height of the ball.

$$y(1) = -16(1)^2 + 32(1)$$

$$y(1) = -16 + 32$$

$$y(1) = 16$$

Add this to the vertical motion chart.

Initial	Maximum	End
$t = 0$	$t = 1$	$t = 2$
$v = 32$	$v = 0$	$v = -32$
$y = 0$	$y = 16$	$y = 0$

If we want to summarize this information, we can say that the ball leaves the ground with an initial velocity of 32 ft/s, reaches its maximum height of 16 feet after $t = 1$ second, at which point it stops increasing in height, starts falling back toward the ground, and eventually hits the ground at a velocity of -32 ft/s after $t = 2$ seconds.

Coin dropped from the roof

In the previous lesson, we were looking at a vertical motion pattern in which we threw an object up from the ground, or from some other height, and the object traveled upward, eventually reached a maximum height, and then fell back down to earth, eventually stopping when it hit the ground.

In this lesson, we're looking at a different vertical motion pattern. This time, we're dropping a coin, or some other object, from some height, letting it fall straight to the ground, eventually stopping when it hits the ground.

Let's work through an example of how to find different values from a position function that models this pattern of vertical motion.

Example

A watermelon is dropped from the top of a building 28 meters high. Find instantaneous velocity at $t=2$, average velocity between $t=0$ and $t=2$, and find the time when the watermelon hits the ground.

Plugging everything we know into the formula for standard projectile motion, we get

$$x(t)=-\frac{1}{2}gt^2+v_0t+y_0$$

$$x(t) = -\frac{1}{2}(9.8)t^2 + 0t + 28$$

$$x(t) = -4.9t^2 + 28$$

Find the velocity function by differentiating the position function.

$$v(t) = x'(t) = -9.8t$$

To find instantaneous velocity at $t = 2$, substitute $t = 2$ into the velocity function.

$$v(2) = -9.8(2)$$

$$v(2) = -19.6$$

This is the instantaneous velocity at $t = 2$. Find average velocity over $t = [0,2]$.

$$\Delta v(a, b) = \frac{x(b) - x(a)}{b - a}$$

$$\Delta v(0,2) = \frac{x(2) - x(0)}{2 - 0}$$

$$\Delta v(0,2) = \frac{-4.9(2)^2 + 28 - (-4.9(0)^2 + 28)}{2}$$

$$\Delta v(0,2) = \frac{-19.6 + 28 - 28}{2}$$

$$\Delta v(0,2) = \frac{-19.6}{2}$$

$$\Delta v(0,2) = -9.8$$

This is the average velocity of the watermelon between $t = 0$ and $t = 2$.

The watermelon will hit the ground when $x(t) = 0$, so we'll set the position function equal to 0 and then solve for t.

$$-4.9t^2 + 28 = 0$$

$$4.9t^2 = 28$$

$$t^2 = \frac{28}{4.9}$$

$$t \approx \sqrt{5.71}$$

$$t \approx 2.39$$

The watermelon hits the ground after $t \approx 2.39$ seconds.

Marginal cost, revenue, and profit

We've been looking at physical applications of derivatives, but there are also economics applications.

In this lesson, we'll look at marginal cost, revenue, and profit. But before we jump into these marginal values, let's look at cost, revenue, and profit in general.

Cost, revenue, and profit

If a business wants to calculate the revenue generated, the cost incurred, and the profit gained by producing x units of a product, it can use the specific formulas.

Revenue	$R(x) = xp$
Cost	$C(x) = F + V(x)$
Profit	$P(x) = R(x) - C(x)$

In these formulas, p is the selling price of an individual unit, so **revenue** is given by the product of selling price and the number of units sold. F is fixed cost and $V(x)$ is variable cost, so **cost** is the sum of the fixed and variable costs. The **profit** is then the difference between the revenue and the cost.

In other words, if a company is making 100 units of their product, the revenue function will tell them how much revenue will be generated by the

100 units, the cost function will tell them how much it'll cost to produce the 100 units, and the profit function will find the total profit gained from producing and then selling the 100 units.

The marginal functions

Of course, every company wants to maximize its profits, but increasing the number of units they produce doesn't always translate to higher profits.

For instance, if an airplane company is making as many planes each month as their current manufacturing space allows, they might need to build a second factory in order to make even one more plane per month. But building a second factory, to make only one more plane, won't necessarily be profitable. On the other hand, if they build a second factory in order to produce 100 more planes each month, that might be a profitable decision.

The marginal revenue, cost, and profit functions are what the company can use to determine whether or not they should increase production. These marginal functions are the derivatives of their associated functions. So the marginal revenue function is the derivative of the revenue function; the marginal cost function is the derivative of the cost function; and the marginal profit function is the derivative of the profit function.

The **marginal revenue** function models the revenue generated by selling one more unit, the **marginal cost** function models the cost of making one more unit, and the **marginal profit** function models the profit made by selling one more unit.

This understanding of what the marginal functions model should make sense to us. Because these marginal functions are derivative functions, they model the slope of the original function, or the change per unit. So if we, for instance, find a marginal cost function as the derivative of the cost function, the marginal cost function should be modeling the change, or slope, of the cost function. And that slope is really just how much the original cost function is increasing or decreasing, per unit.

Let's work through an example where we find all three marginal functions.

Example

Calculate a smartphone manufacturer's marginal cost, marginal revenue, and marginal profit when they're producing 75 smartphones, if the selling price is $p = 6x$ and the cost of making the smartphones is modeled by $C(x)$.

$$C(x) = 6x^2 + 34x + 2{,}500$$

To calculate marginal cost at 75 units, we take the derivative of the cost function and then evaluate the derivative at $x = 75$.

$$C'(x) = 12x + 34$$

$$C'(x) = 12(75) + 34$$

$$C'(x) = 934$$

The marginal cost at $x = 75$ is $\$934$, which means the additional cost associated with producing the 76th unit is $\$934$.

To calculate marginal revenue at 75 units, we need to find a revenue function, take its derivative, and then evaluate the derivative at $x = 75$.

The revenue equation is $R(x) = xp$ where p is the selling price, $p = 6x$.

$$R(x) = x(6x)$$

$$R(x) = 6x^2$$

Taking the derivative of revenue to get marginal revenue, and then evaluating at $x = 75$, we get

$$R'(x) = 12x$$

$$R'(x) = 12(75)$$

$$R'(x) = 900$$

The marginal revenue at $x = 75$ is $\$900$, which means the additional revenue associated with selling the 76th unit is $\$900$.

Finally, to solve for marginal profit we need to find a profit function, take its derivative, and then evaluate the derivative at $x = 75$.

The profit equation is $P(x) = R(x) - C(x)$, where R is the revenue function we found earlier and C is the cost function we were given.

$$P(x) = R(x) - C(x)$$

$$P(x) = (6x^2) - (6x^2 + 34x + 2{,}500)$$

$$P(x) = -34x - 2{,}500$$

Taking the derivative of profit to get marginal profit, and then evaluating at $x = 75$ gives

$$P'(x) = -34$$

$$P'(75) = -34$$

The marginal profit at $x = 75$ is $-\$34$, which means that the smartphone company's profit declines by $\$34$ when they produce and sell the 76th smartphone.

We can also find marginal profit just by subtracting marginal cost from marginal revenue.

$$P'(x) = R'(x) - C'(x)$$

$$P'(75) = 900 - 934$$

$$P'(75) = -34$$

Either way, we find $P'(75) = -34$ and we can say that, if the manufacturer's goal is to maximize profit, they should not increase production.

Half-life

Growth and decay problems are another common application of derivatives.

We actually don't need to use derivatives in order to solve these problems, but derivatives are used to build the basic growth and decay formulas, which is why we study these applications in this part of calculus.

We won't work through how to prove these formulas, because in addition to derivatives, we also use integrals to build them, and we won't learn about integrals until later in calculus.

So, for now, we'll just state that the basic equation for exponential decay is

$$y = Ce^{-kt}$$

where C is the amount of a substance that we're starting with, k is the decay constant, and y is the amount of the substance we have remaining after time t. Since substances decay at different rates, k will vary depending on the substance.

Half-life equation

Every decaying substance has its own half-life, because **half-life** is the amount of time required for exactly half of our original substance to decay, leaving exactly half of what we started with. Because every substance decays at a different rate, each substance will have a different half-life. But regardless of the substance, when we're looking at half-life, we know that

$$y = \frac{C}{2}$$

Because y is the amount of substance that remains as the substance decays, and because C is the amount of substance we started with originally, when the substance has decayed to half of its original amount, y will be equivalent to $C/2$. So we can substitute this value in for y, and then simplify the decay formula.

$$\frac{C}{2} = Ce^{-kt}$$

$$\frac{1}{2} = e^{-kt}$$

So, when we're dealing with half-life specifically, instead of exponential decay in general, we can use this formula we got from substituting $y = C/2$.

Let's do an example problem.

Example

Fermium-253 has a half-life of 3 days. If we start with $1{,}200$ mg of Fermium-253, how much of the substance remains after 10 days?

Before doing anything else, we need to find the value of the decay constant k for Fermium-253. Substitute $C = 1{,}200$ and $t = 3$ into the half-life formula.

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-k(3)}$$

$$\frac{1}{2} = e^{-3k}$$

Apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-3k})$$

$$\ln\frac{1}{2} = -3k$$

$$k = -\frac{1}{3}\ln\frac{1}{2}$$

With a value for k, we can now solve for the amount of substance remaining, y, after $t = 10$ days.

$$y = Ce^{-kt}$$

$$y = 1{,}200e^{-\left(-\frac{1}{3}\ln\frac{1}{2}\right)(10)}$$

$$y = 1{,}200e^{\frac{10}{3}\ln\frac{1}{2}}$$

$$y = 119.06$$

Therefore, after $t = 10$ days, $y = 119.06$ mg of Fermium-253 remain.

Let's try another example.

Example

The half-life of Americium-243 is $7{,}370$ years. How long will it take a mass of Americium-243 to decay to 73% of its original size?

We haven't been told the exact amount in the original mass of Americium-243, but regardless of the size of the mass, we can say that we're starting with 100% of the mass.

To find the decay constant k, we'll substitute $t = 7{,}370$ years into the half-life equation.

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-k(7{,}370)}$$

$$\frac{1}{2} = e^{-7{,}370k}$$

Apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-7{,}370k})$$

$$\ln\frac{1}{2} = -7{,}370k$$

$$k = -\frac{1}{7{,}370}\ln\frac{1}{2}$$

With a value for k, we can now solve for the number of years it'll take for the substance to decay to 73% of its original size.

If we started with 100% of the original substance and 73% of the substance remains, then we can substitute $C = 1$ and $y = 0.73$, along with the value we've just found for the decay constant k.

$$y = Ce^{-kt}$$

$$0.73 = 1e^{-\left(-\frac{1}{7{,}370}\ln\frac{1}{2}\right)t}$$

$$0.73 = e^{\left(\frac{1}{7{,}370}\ln\frac{1}{2}\right)t}$$

Apply the natural logarithm to both sides in order to solve for t.

$$\ln 0.73 = \ln\left(e^{\left(\frac{1}{7{,}370}\ln\frac{1}{2}\right)t}\right)$$

$$\ln 0.73 = \left(\frac{1}{7{,}370}\ln\frac{1}{2}\right)t$$

$$7{,}370\ln 0.73 = \left(\ln\frac{1}{2}\right)t$$

$$t = \frac{7{,}370\ln 0.73}{\ln\frac{1}{2}}$$

$$t \approx 3{,}346.21 \text{ years}$$

This answer tells us that it'll take about $3{,}346.21$ years for the mass of Americium-243 to decay to 73% of its original size.

Newton's Law of Cooling

Newton's Law of Cooling models the way in which a warm object in a cooler environment cools down until it matches the temperature of its environment.

Therefore, this law is similar to the half-life equation we just learned, in the sense that they both model the rate at which something decays. A half-life equation models the rate at which a radioactive substance decays, whereas Newton's Law models the rate at which the temperature is "decaying" from hotter to cooler.

The Law

The law tells us that the rate at which the object cools is proportional to the difference between the object and the environment around it.

In other words, if you put a boiling pot of soup in the freezer, it'll cool down faster than if you simply leave the pot on the counter. That's because the difference between the temperature of the soup and the freezer is greater than the difference between the temperature of the soup and the room-temperature countertop. The greater the temperature difference, the faster the object will cool.

The Newton's Law of Cooling formula is

$$\frac{dT}{dt} = -k(T - T_a) \text{ with } T(0) = T_0$$

where T is the temperature over time t, k is the decay constant, T_a is the temperature of the environment ("ambient temperature"), and T_0 is the initial, or starting, temperature of the hot object.

The point of applying Newton's Law is to generate an equation that models the temperature of the hot, but cooling, object over time. That temperature function, and the solution to the Newton's Law equation, will be

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

The easiest way to understand how Newton's Law applies to a real-world scenario is to work through an example, so let's work through one.

Example

At a local restaurant, a big pot of soup, boiling at 100° C, has just been removed from the stove and set on the countertop, where the ambient temperature is 23° C. After 5 minutes, the soup cools to 98°. If the soup needs to be served to the restaurant's customers at 90° C, how long will it be before the soup is ready to serve?

For Newton's Law problems, it's especially helpful to list out what the question tells us.

$T_0 = 100^\circ$ Initial temperature of the soup

$T_a = 23^\circ$ Ambient temperature on the countertop

$T(5) = 98^\circ$ At time $t = 5$ minutes, the soup has cooled to 98°

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 23 + (100 - 23)e^{-kt}$$

$$T(t) = 23 + 77e^{-kt}$$

Substitute the initial condition $T(5) = 98^\circ$,

$$T(5) = 23 + 77e^{-k(5)}$$

$$98 = 23 + 77e^{-5k}$$

in order to find a value for the decay constant k.

$$75 = 77e^{-5k}$$

$$\frac{75}{77} = e^{-5k}$$

$$\ln\frac{75}{77} = \ln(e^{-5k})$$

$$\ln\frac{75}{77} = -5k$$

$$k = -\frac{1}{5}\ln\frac{75}{77}$$

Substitute this value for k into the equation modeling temperature over time.

$$T(t) = 23 + 77e^{-\left(-\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$T(t) = 23 + 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

We want to find the time t at which the soup reaches 90°, so we'll substitute $T(t) = 90^\circ$.

$$90 = 23 + 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$67 = 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$\frac{67}{77} = e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

Apply the natural logarithm to both sides of the equation.

$$\ln\frac{67}{77} = \ln\left(e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}\right)$$

$$\ln\frac{67}{77} = \left(\frac{1}{5}\ln\frac{75}{77}\right)t$$

$$5\ln\frac{67}{77} = \left(\ln\frac{75}{77}\right)t$$

$$t = \frac{5\ln\frac{67}{77}}{\ln\frac{75}{77}}$$

$$t \approx 26.43$$

The conclusion then is that the pot of soup will cool from 100° C to 90° C in about 26.5 minutes, at which point it'll be ready to serve to the restaurant's customers.

Sales decline

If sales of a product are consistently declining at an exponential rate, we can model that decline with the formula

$$F = Pe^{-rt}$$

where P is the number of items being sold, r is the rate of decline, and F is the number of items being sold after sales have continued to decline for some specified amount of time t.

It's important to remember that the rate r and the time t need to have complementary units. For instance, if r is given as some rate per *month*, then time t needs to be measured in *months*. If r is in years, then t should also be in years.

When the units of rate and time don't match, we'll need to convert one to match the other. For example, if r was given as a rate per year, and t was given in months, we could either divide t by 12 to convert it into years to match the rate, or we could divide r by 12 to convert it into months to match the time.

Let's work through an example where we solve for the rate of decline.

Example

Sales of Isaac Newton bobbleheads have decreased over the last 4 years. Four years ago, $285{,}674$ bobbleheads were sold, but over the last year, sales were only $97{,}546$ bobbleheads. Assuming that sales have declined at a steady exponential rate, what is the rate of decline?

We know that fourth year sales are $F = 97{,}546$, and that first year sales were $P = 285{,}674$. Time is $t = 4$ and, we need to calculate r, so we'll substitute what we know into the sales decline formula.

$$F = Pe^{-rt}$$

$$97{,}546 = 285{,}674e^{-r(4)}$$

Solve for r.

$$\frac{97{,}546}{285{,}674} = e^{-4r}$$

$$0.3415 = e^{-4r}$$

$$\ln 0.3415 = \ln(e^{-4r})$$

$$-1.075 = -4r$$

$$r = 0.269$$

This result tells us that sales of Isaac Newton bobbleheads have declined at an annual rate of about $r = 0.269$, or $r = 26.9\,\%$ per year.

We'll do one more example, and in this one we'll use the exponential model we develop to predict the future sales decline of a product.

Example

One year ago, a company sold $5{,}698$ disposable sandwich bags in a month. But over the last 12 months, sales have decreased at a steady exponential rate of $15\,\%$ per month. How many disposable sandwich bags did the company sell last month, and how many are predicted to sell in 6 months from now?

The rate of decline is $r = 0.15$, the original amount from one year ago (12 months ago) was $P = 5{,}698$. Therefore, we could say that last month is at $t_{\text{last month}} = 12$, and 6 months from now is $t_{\text{in 6 months}} = 18$.

We'll substitute into the sales decline formula to find sales for last month.

$$F = Pe^{-rt}$$

$$F_{\text{last month}} = 5{,}698e^{-0.15(12)}$$

$$F_{\text{last month}} \approx 941.87$$

In the past month, about 942 units of disposable sandwich bags were sold. Now we'll calculate expected sales for 6 months from now.

$$F_{\text{in 6 months}} = 5{,}698e^{-0.15(18)}$$

$$F_{\text{in 6 months}} \approx 382.94$$

Assuming the rate of decline remains the same, in 6 months from now, the company will sell about 383 disposable sandwich bags.

Compounding interest

We've been looking at exponential decay problems, like half-life, Newton's Law of Cooling, and sales decline, but now we want to turn from exponential decay to exponential growth, starting with compound interest problems.

Compound interest

In general, the idea of compound interest is that the interest our money earns, itself earns interest.

For instance, if we deposit $\$100$ into an account that earns 5% interest annually, then we'll earn $\$100 \cdot 5\% = \$5$ in interest the first year.

Then at the beginning of the second year, we'll be starting with $\$105$, instead of just $\$100$. Which means that, at 5%, we'll be earning interest not only on the initial $\$100$ deposit, but also on the $\$5$ of interest we earned in the first year. So during the second year, we'll earn $\$105 \cdot 5\% = \$5.25$.

And this pattern will continue year after year. We'll have only deposited the initial $\$100$, but the interest we earn each year will get compounded.

When interest is **compounded**, it means that the interest gets added to the principal that we initially deposited, and therefore will itself also earn interest. Because of this compounding, we'll earn more and more interest each year, which is what we see in the table.

Year	Interest earned	Running balance
1	\$5.00	\$105.00
2	\$5.25	\$110.25
3	\$5.51	\$115.76
4	\$5.79	\$121.55
5	\$6.08	\$127.63
6	\$6.38	\$134.01
7	\$6.70	\$140.71
8	\$7.04	\$147.75
9	\$7.39	\$155.14
10	\$7.76	\$162.90

Exponential growth

In the table above, we see the exponential growth of the compounding interest. The account is earning more and more interest each year, and the running balance in the account is therefore growing by a larger and larger amount each year.

It's this kind of exponential growth that we model with

$$A = Pe^{rt}$$

where P is the initial investment (the principal), r is annual interest rate, and A is the amount in the account after time t. Sometimes we'll write this formula as

$$FV = PVe^{rt}$$

where $A = FV$ is the "future value" in the account, and $P = PV$ is the "present value" in the account. We only use this formula when interest is compounded continuously. If instead interest is compounded a specific n times per year (monthly, quarterly, semi-annually, annually, etc.), we'd use

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the initial investment, r is the annual interest rate, A is the amount in the account after time t, and n is the number of times the interest is compounded per year.

Let's work through an example where we use the formula for continuous compounding to calculate the amount in the account after a certain period of time.

Example

We invest $\$2{,}000$ at a rate of 3% compounded continuously. How much will the investment be worth after 3 years?

From the question, we know that the principal is $P = \$2{,}000$, the interest rate is $r = 0.03$, and the time is $t = 3$. Substitute these values into the compound interest formula

$$A = Pe^{rt}$$

$$A = 2{,}000e^{0.03(3)}$$

$$A = 2{,}000e^{0.09}$$

$$A \approx 2{,}188.35$$

So after 3 years, at 3% interest, the investment is worth about $\$2{,}188.35$.

Let's try another example. In this one, we'll earn different interest rates over different periods of time, so we'll need to compute each of those time periods separately.

Example

We invest $\$6{,}000$ at 2% for 2 years. After 2 years, the interest rate increases to 3.5% for the next 2 years. Then, after those first 4 years, the interest rate increases to 5%. Find the value of the investment after 3 years, and the value after 6 years.

We'll need to handle each interest rate separately. Let's call the first 2 years the "first term," the second 2 years the "second term," and everything after that the "third term."

We'll use subscripts to denote whether the rate belongs to the first term, second term, or third term.

$$r_1 = 0.02$$

$$r_2 = 0.035$$

$$r_3 = 0.05$$

We know $P_1 = \$6{,}000$ but we don't know P_2 or P_3. Let's find those now since we'll need them to answer both parts of this question. P_2 will equal A_1, the amount of money we have at the end of the first term (after 2 years). But we can find $A_1 = P_1 e^{r_1 t}$ where $t = 2$.

$$A_1 = P_1 e^{r_1 t}$$

$$A_1 = 6{,}000 e^{0.02(2)}$$

$$A_1 = 6{,}244.86$$

$$P_2 = 6{,}244.86$$

Next we remember that P_3 will equal A_2, the amount of money we have at the end of the second term (2 years after the end of the first term, 4 years since the beginning of the first term). So $A_2 = P_2 e^{r_2 t}$ where $t = 2$.

$$A_2 = P_2 e^{r_2 t}$$

$$A_2 = 6{,}244.86 e^{0.035(2)}$$

$$A_2 = 6{,}697.66$$

$$P_3 = 6{,}697.66$$

Armed with r_1, r_2, and r_3, plus P_1, $P_2 = A_1$, and $P_3 = A_2$, we can tackle both parts of this question.

In order to solve for the value of the investment after 3 years, we'll use the data for the second term. But we have to remember that 3 years into the investment is 1 year into the second term, so $t = 1$, $P_2 = 6{,}244.86$ and $r_2 = 0.035$.

$$A_{3\text{ years}} = 6{,}244.86e^{0.035(1)}$$

$$A_{3\text{ years}} = 6{,}467.30$$

The investment would be worth $\$6{,}467.30$ after 3 years.

In order to solve for the value of the investment after 6 years, we'll use the data for the third term. But we have to remember that 6 years into the investment is 2 years into the third term so $t = 2$, $P_3 = 6{,}697.66$ and $r_3 = 0.05$.

$$A_{6\text{ years}} = 6{,}697.66e^{0.05(2)}$$

$$A_{6\text{ years}} = 7{,}402.06$$

The investment would be worth $\$7{,}402.06$ after 6 years.

Now let's do one where interest is compounded quarterly, instead of continuously, so that we can see how to use the other formula.

Example

We deposit $\$6{,}000$ into an account that pays $5.5\,\%$ interest, compounded quarterly. How long before the balance in the account reaches $\$10{,}500$?

Because interest is compounded quarterly, we set $n = 4$, and plug everything we've been given into the formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$10{,}500 = 6{,}000\left(1 + \frac{0.055}{4}\right)^{4t}$$

$$10{,}500 = 6{,}000(1.01375)^{4t}$$

$$1.75 = 1.01375^{4t}$$

Take the natural logarithm of both sides.

$$\ln(1.75) = \ln(1.01375^{4t})$$

$$\ln(1.75) = 4t\ln(1.01375)$$

$$4t = \frac{\ln(1.75)}{\ln(1.01375)}$$

$$t = \frac{\ln(1.75)}{4\ln(1.01375)}$$

$$t \approx 10.2446$$

It'll take a little over 10 years for the balance in the account to grow from $\$6{,}000$ to $\$10{,}500$.

W W W . K R I S T A K I N G M A T H . C O M