

# Intermediate Value Theorem with an interval

The Intermediate Value Theorem is a theorem we use to prove that a function has a root inside a particular interval.

The **root** of a function, graphically, is a point where the graph of the function crosses the  $x$ -axis. Algebraically, the root of a function is the point where the function's value is equal to 0.

Formally, the **Intermediate Value Theorem** states:

Let  $f(x)$  be a function which is continuous on the closed interval  $[a, b]$  and let  $y_0$  be a real number lying between  $f(a)$  and  $f(b)$ . If  $f(a) < y_0 < f(b)$  or  $f(b) < y_0 < f(a)$ , then there will be at least one  $c$  on the interval  $(a, b)$  where  $y_0 = f(c)$ .

The formal theorem is technical, but it's actually saying something simple, which is that, assuming  $f(x)$  is continuous on  $[a, b]$ , if we can show that the function's value is negative at one side of the interval and positive at the other side, then that fact alone proves that the graph of the function must cross the  $x$ -axis at some point inside the interval.

That being said, in order to use the Intermediate Value Theorem, we need to 1) have a closed interval  $[a, b]$  (which means that  $a$  and  $b$  are included as part of the interval), and 2) show that the function is continuous everywhere in the interval.

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## Example

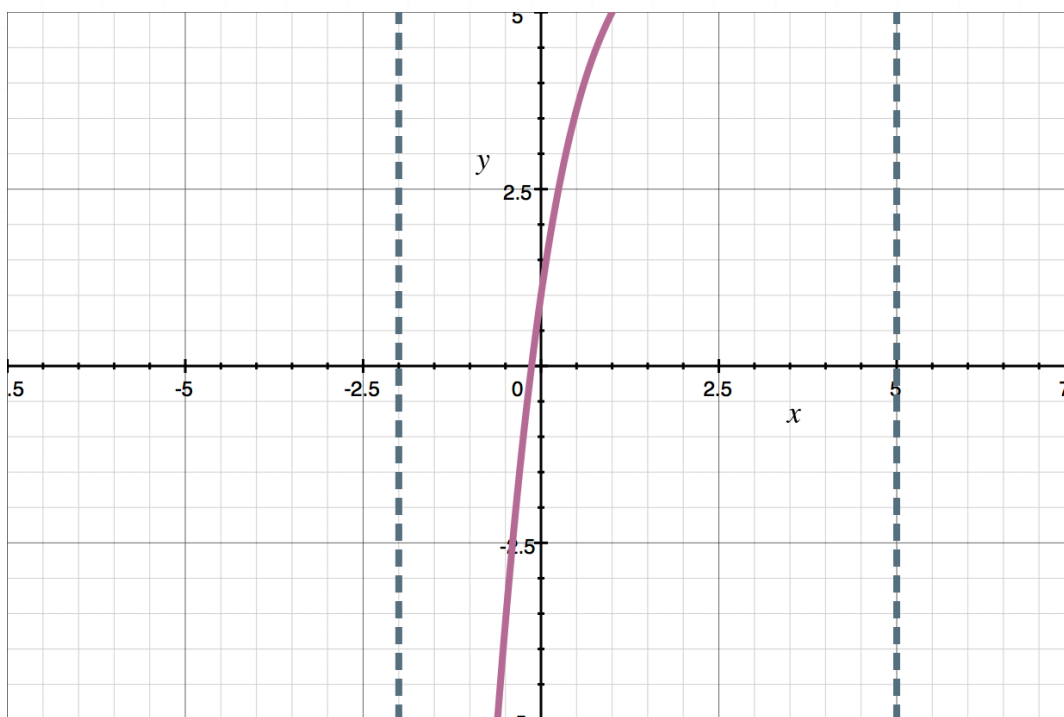
Show that the function has at least one root in the interval  $[-2, 5]$ .



$$f(x) = x^3 - 4x^2 + 7x + 1$$

First, we should confirm that this function is continuous over the given interval.

Continuity breaks typically occur in functions that have variables in the denominator of fractions, trigonometric functions, logarithmic functions, or functions with square roots. This function has none of these elements, so we know that it's continuous. If we want to double-check ourselves, we can graph the function.



Next, we substitute the endpoints into the function. We get

$$f(-2) = (-2)^3 - 4(-2)^2 + 7(-2) + 1$$

$$f(-2) = -8 - 16 - 14 + 1$$

$$f(-2) = -37$$



and

$$f(5) = (5)^3 - 4(5)^2 + 7(5) + 1$$

$$f(5) = 125 - 100 + 35 + 1$$

$$f(5) = 61$$

Since we know that  $f(x) = x^3 - 4x^2 + 7x + 1$  is continuous, and since the function's value at the left edge of the interval is negative (below the  $x$ -axis), and the value of the function at the right side of the interval is positive (above the  $x$ -axis), we know that there's at least one point  $c$  at which the function will cross the  $x$ -axis.

$$-37 < f(c) < 61$$

Therefore, the function has at least one solution (root) in the interval  $[-2, 5]$ .

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