Absolute, relative, and percentage error

Once we've used a linear approximation equation to estimate the function's value at a particular point, we may want to calculate how good or bad the estimate was.

Depending on the point of tangency we chose, and the behavior of the function near that point, the linear approximation we found might be a pretty good estimate of the function's actual value, or it might not be.

To determine whether or not we've found a good estimate, we want to look at error.

Absolute error

The **absolute error** for an estimation at a particular point a is the absolute value of the difference between the function's actual value and the linear approximation at that point.

$$E_A(a) = |f(a) - L(a)|$$

We can use E_A to indicate absolute error. We take the absolute value of the difference to ensure that we always get a positive value for absolute error. That's because absolute error is truly just the distance between the value of the linear approximation line $L(x)$ at $x = a$ and the value of the function $f(x)$ at $x = a$.

If we didn't use absolute value in the absolute error formula, we'd get a positive value for absolute error when the linear approximation line was below the function's curve, but a negative value for absolute error when the linear approximation line was above the function's curve. We're just interested in the distance between the two values, so we take the absolute value.

For instance, if we're trying to find a function's value at $x = 1.9$, the actual value might be $f(1.9) = 2.1081$, while the linear approximation might be $L(1.9) = 2.1$.

So the absolute error at $x = 1.9$ would be

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(1.9) = |f(1.9) - L(1.9)|$$

$$E_A(1.9) = |2.1081 - 2.1|$$

$$E_A(1.9) = |0.0081|$$

$$E_A(1.9) = 0.0081$$

This is the absolute error, which is the distance between the function's actual value at $x = 1.9$ and the linear approximation at $x = 1.9$.

Relative error

Once we've found absolute error, we can use it to find relative error. Think of **relative error** as the amount of error, compared to the function's actual value. As the name suggests, it tells us the amount of error in the approximation, *relative* to the function's actual value.

We find it by dividing absolute error by the function's actual value.

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

Notice how, if we wanted to skip the absolute error step and go straight to relative error, we could have written the relative error formula to include the absolute error calculation.

$$E_R(a) = \frac{|f(a) - L(a)|}{f(a)}$$

In the previous example, we found that the absolute error was $E_A = 0.0081$, and that the function's actual value was $f(1.9) = 2.1081$. So the relative error at $x = 1.9$ would be

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(1.9) = \frac{E_A(1.9)}{f(1.9)}$$

$$E_R(1.9) = \frac{0.0081}{2.1081}$$

$$E_R(1.9) \approx 0.003842$$

The easiest way to conceptualize relative error is to compare different examples.

Let's say that, in one problem, you know that the function's actual value at a point is 100, and the absolute error at that point is 10. Then relative error at that point is $10/100 = 0.10$. Compared to the value of the actual function, 100, the absolute error of 10 doesn't seem too bad, which is reflected in the somewhat small value of the relative error, $E_R = 0.10$.

But if, in another problem, you know that the function's actual value at a point is 12, and the absolute error at that point is still 10 (like in the previous problem), then the relative error at that point is $10/12 \approx 0.83$. Compared to the value of the actual function, 12, the absolute error of 10 is pretty bad, which is reflected in the somewhat large value of the relative error, $E_R \approx 0.83$.

Percentage error

The **percentage error** is simply the relative error expressed as a percentage. So to find percentage error, we just have to multiply the relative error by $100\,\%$. So we could write the formula for percentage error in three ways:

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(a) = 100\% \cdot \frac{E_A(a)}{f(a)}$$

$$E_P(a) = 100\% \cdot \frac{|\,f(a) - L(a)\,|}{f(a)}$$

In the previous example, we found relative error at $x = 1.9$ to be $E_R(1.9) \approx 0.003842$. Converting that to percentage error, we get

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(a) = 100\% \cdot 0.003842$$

$$E_P(a) = 0.3842\,\%$$

Now that we understand how to find each of these error values, let's work through a full example.

Example

Use a linear approximation to estimate the value of $\sqrt{50}$, then find the absolute, relative, and percentage error of the estimate.

We need to realize here that $\sqrt{50}$ is a difficult value to find. But it's very close to $\sqrt{49}$, which we already know is 7. So instead of thinking specifically about $\sqrt{50}$, let's think about $\sqrt{x}$, and therefore use the function $f(x) = \sqrt{x}$.

Differentiate the function,

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f'(x) = \frac{1}{2x^{\frac{1}{2}}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

then evaluate the derivative at $x = 49$.

$$f'(49) = \frac{1}{2\sqrt{49}}$$

$$f'(49) = \frac{1}{2(7)}$$

$$f'(49) = \frac{1}{14}$$

So the linear approximation line intersects $f(x) = \sqrt{x}$ at the point of tangency $(49,7)$, and has a slope of $m = 1/14$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 7 + \frac{1}{14}(x - 49)$$

$$L(x) = 7 + \frac{1}{14}x - \frac{49}{14}$$

$$L(x) = \frac{1}{14}x - \frac{7}{2} + \frac{14}{2}$$

$$L(x) = \frac{1}{14}x + \frac{7}{2}$$

Then the linear approximation at $x = 50$ is

$$L(50) = \frac{1}{14}(50) + \frac{7}{2}$$

$$L(50) = \frac{50}{14} + \frac{7}{2}$$

$$L(50) = \frac{50}{14} + \frac{49}{14}$$

$$L(50) = \frac{99}{14}$$

$$L(50) \approx 7.07142857$$

In comparison, the actual value of $\sqrt{50}$ is

$$f(x) = \sqrt{x}$$

$$f(50) = \sqrt{50}$$

$$f(50) \approx 7.07106781$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(50) = |f(50) - L(50)|$$

$$E_A(50) \approx |7.07106781 - 7.07142857|$$

$$E_A(50) \approx |-0.00036076|$$

$$E_A(50) \approx 0.00036076$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(50) = \frac{E_A(50)}{f(50)}$$

$$E_R(50) \approx \frac{0.00036076}{7.07106781}$$

$$E_R(50) \approx 0.00005102$$

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(50) = 100\% \cdot E_R(50)$$

$$E_P(50) \approx 100\% \cdot 0.00005102$$

$$E_P(50) \approx 0.005102\,\%$$

Given the extremely small percentage error, we can conclude that the linear approximation at $x = 50$ gives us a pretty good estimate of the function's actual value at that point.
