Chain rule with quotient rule

Of course, we'll use chain rule with quotient rule, as well. Sometimes we'll have a quotient nested inside some other power function or product function, and sometimes we'll have power or product functions nested inside quotients.

Let's look at an example in which we differentiate a power function that has a quotient as its "inside function."

Example

Find the derivative of the function.

$$y = \left(\frac{6x^4}{3x}\right)^8$$

First, we'll substitute for the quotient and set

$$u = \frac{6x^4}{3x}$$

We'll need to find u', and since u is a quotient, we'll apply quotient rule to take its derivative.

$$u' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$u' = \frac{(24x^3)(3x) - (6x^4)(3)}{(3x)^2}$$

$$u' = \frac{72x^4 - 18x^4}{9x^2}$$

Going back to the original function, we used substitution to simplify it to

$$y = (u)^8$$

Use power rule to take the derivative, making sure to apply chain rule and multiplying by u'.

$$y = 8(u)^7 u'$$

$$y' = 8\left(\frac{6x^4}{3x}\right)^7\left(\frac{72x^4 - 18x^4}{9x^2}\right)$$

$$y' = 8\left(\frac{6x^4}{3x}\right)^7\left(\frac{54x^4}{9x^2}\right)$$

We could leave the derivative this way, but let's simplify the fractions by factoring and canceling like terms.

$$y' = 8\left(\frac{2x^4}{x}\right)^7\left(\frac{6x^4}{x^2}\right)$$

$$y' = 8(2x^3)^7(6x^2)$$

$$y' = 48x^2(2x^3)^7$$

We can double-check the answer we got in this last example by first simplifying the original function,

$$y = \left(\frac{6x^4}{3x}\right)^8$$

$$y = (2x^3)^8$$

and then differentiating the simplified function.

$$y' = 8(2x^3)^{8-1}(6x^2)$$

$$y' = 48x^2(2x^3)^7$$

Let's look at another example, this time where we have power functions nested inside a quotient.

Example

Find the derivative of the function.

$$y = \frac{(6x^4 - 5)^2}{(7x^2 + 3)^3}$$

Because we have a quotient, we'll need to apply the quotient rule. We know that we'll need the numerator $f(x)$ and denominator $g(x)$, as well as the derivatives of both the numerator and denominator.

$$f(x) = (6x^4 - 5)^2$$

$$f'(x) = 2(6x^4 - 5)^{2-1}(24x^3) = 48x^3(6x^4 - 5)$$

and

$$g(x) = (7x^2 + 3)^3$$

$$g'(x) = 3(7x^2 + 3)^{3-1}(14x) = 42x(7x^2 + 3)^2$$

Plugging these values into the quotient rule gives

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(48x^3(6x^4 - 5))((7x^2 + 3)^3) - ((6x^4 - 5)^2)(42x(7x^2 + 3)^2)}{((7x^2 + 3)^3)^2}$$

$$y' = \frac{48x^3(6x^4 - 5)(7x^2 + 3)^3 - 42x(6x^4 - 5)^2(7x^2 + 3)^2}{(7x^2 + 3)^6}$$

There's a common factor of $(7x^2 + 3)^2$ between the numerator and denominator, so we'll cancel it.

$$y' = \frac{48x^3(6x^4 - 5)(7x^2 + 3) - 42x(6x^4 - 5)^2}{(7x^2 + 3)^4}$$