

# Proving that the limit does not exist

Given what we now know about how the existence of the one-sided limits dictates the existence of the general limit, we can use this relationship as a test for showing whether or not the general limit exists.

Remember we said before that the general limit exists at a point  $x = c$  if

1. the left-hand limit exists at  $x = c$ ,
2. the right-hand limit exists at  $x = c$ , and
3. those left- and right-hand limits are equal to one another.

Therefore, the general limit **does not exist (DNE)** at  $x = c$  if

1. the left-hand limit does not exist at  $x = c$ , and/or
2. the right-hand limit does not exist at  $x = c$ , and/or
3. the left- and right-hand limits both exist, but aren't equal to one another.

Let's do an example where we show algebraically that the limit does not exist.

## Example

Prove that the limit does not exist.

$$\lim_{x \rightarrow 2} \frac{1}{x - 2}$$



If we try substitution, we get an undefined value, because the denominator of the fraction becomes 0.

$$\frac{1}{2-2}$$

$$\frac{1}{0}$$

Because we can't use substitution, we'll instead use values on either side of  $x = 2$ , very close to  $x = 2$ , to determine how the function is behaving as  $x \rightarrow 2$ .

$$f(1.9999) = \frac{1}{1.9999 - 2} = \frac{1}{-0.0001} = -10,000$$

$$f(2.0001) = \frac{1}{2.0001 - 2} = \frac{1}{0.0001} = 10,000$$

From the function's values around  $x = 2$ , we can tell that the function is tending toward  $-\infty$  to the left of  $x = 2$ , and toward  $\infty$  to the right of  $x = 2$ .

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$$

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$$

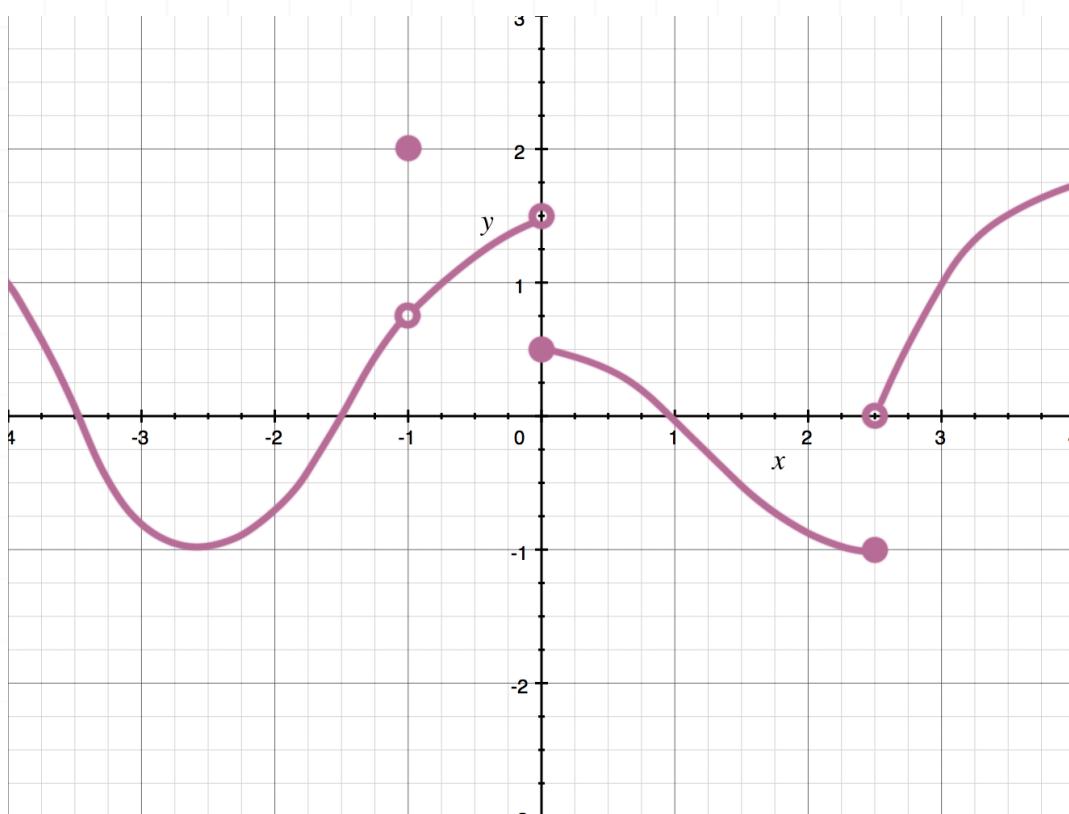
Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at  $x = 2$ .



We can also determine graphically that the limit does not exist.

### Example

Use the graph to determine whether or not the limit exists at  $x = 0$ .



At  $x = 0$ , the function is approaching  $3/2$  from the left side. But from the right side, the function is approaching  $1/2$ . So if we say that the graph represents the function  $f(x)$ , then the one-sided limits are

$$\lim_{x \rightarrow 0^-} f(x) = \frac{3}{2}$$

$$\lim_{x \rightarrow 0^+} f(x) = \frac{1}{2}$$

Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at  $x = 0$ .

## Infinite one-sided limits

We also want to look at what happens to the general limit when both one-sided limits are infinite.

1. If both one-sided limits are  $\infty$ , then the general limit exists and is equal to  $\infty$ .
2. If both one-sided limits are  $-\infty$ , then the general limit exists and is equal to  $-\infty$ .
3. If one of the one-sided limits is  $\infty$  while the other one-sided limit at the same point is  $-\infty$ , then the general limit doesn't exist.

Let's do an example where the limit is infinite.

### Example

Find the limit.

$$\lim_{x \rightarrow 3} \frac{1}{|x - 3|}$$

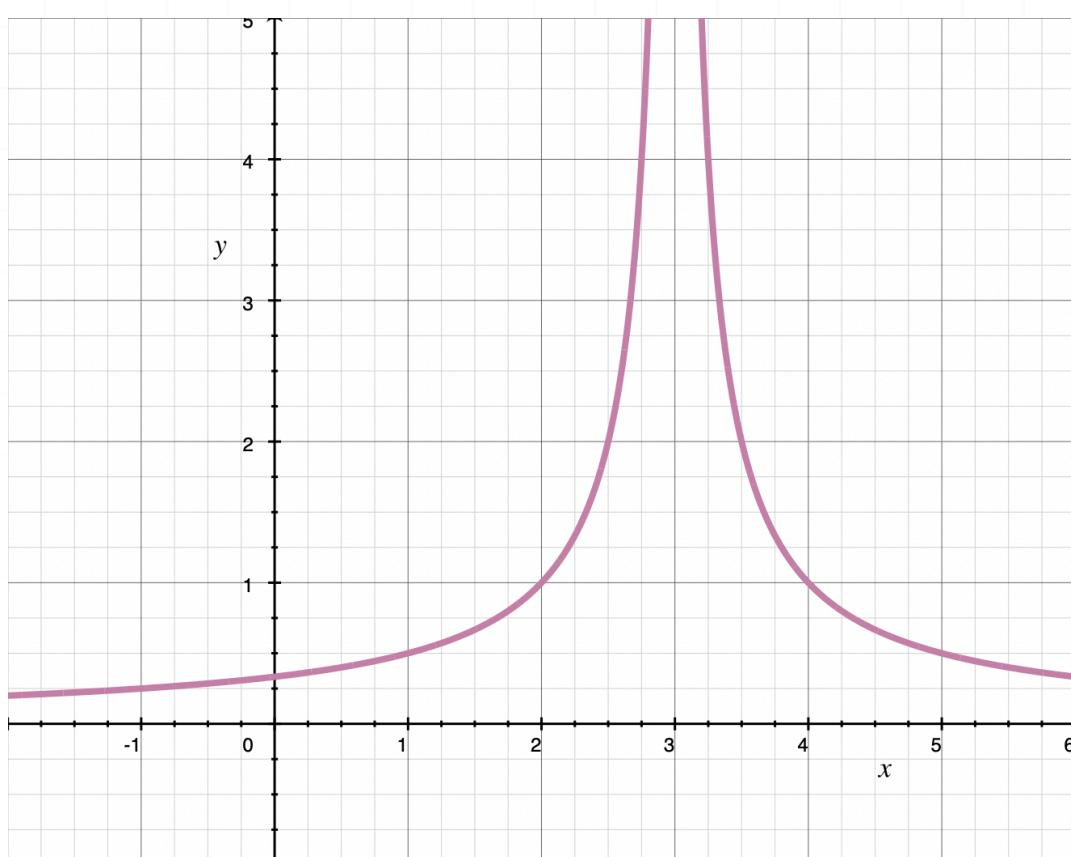
We can get a sense of the one-sided limits if we evaluate the function at values close to  $x = 3$ .



$$\lim_{x \rightarrow 3^-} \frac{1}{|x - 3|} \approx \frac{1}{|2.999999 - 3|} \approx \frac{1}{0.000001} \approx 1,000,000$$

$$\lim_{x \rightarrow 3^+} \frac{1}{|x - 3|} \approx \frac{1}{|3.000001 - 3|} \approx \frac{1}{0.000001} \approx 1,000,000$$

When we evaluate the function at values close to  $x = 3$ , we see that both one-sided limits are headed toward  $\infty$ . If we use the graph to confirm this hunch,



we see that at  $x = 3$ , the function is approaching  $\infty$  from the left side and  $\infty$  from the right side.

$$\lim_{x \rightarrow 3^-} \frac{1}{|x - 3|} = \infty$$

$$\lim_{x \rightarrow 3^+} \frac{1}{|x - 3|} = \infty$$

Because the one-sided limits are equal, the general limit exists and is equal to that same value.

$$\lim_{x \rightarrow 3} \frac{1}{|x - 3|} = \infty$$

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