Big O notation

Sudkamp book – Chap. 14.2

Script chap 2.

$\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{R}$ = reals

Big O notation

Let's take $f, g : \mathbb{N} \to \mathbb{N}$

We say f is of order g if

there is a constan $c > 0 \in \mathbb{N}$ and

$n_0 \in \mathbb{N}$ s.t. $f(n) \leq c \cdot g(n)$

$\forall n \geq n_0$.

threshold.

$O(g) := \{ f \mid f \text{ is of order } g \}$

"Big O of g"

if $f \in O(g)$ we say

g provides an asymptotic upper bound for f.

if $f \in O(g)$ $g \in O(f)$

then we say they are of the same order.

$f = O(g)$ instead $f \in O(g)$

abuse of notation.

correct

$$f(n) = n^2 + O(n)$$

$$f(n) = n^2 + g(n), \quad g(n) \in O(n).$$

$$g \in o(n).$$

Example

$f(n) = n^2$

$g(n) = n^3$

show that $f \in O(g)$

c, n_0 s.t.

$f(n) \leq c \cdot g(n)$

$\forall n \geq n_0$

$c = 1$

$n > 1$ i.e. $n_0 \geq 2$

$f(n) \leq g(n)$

n	f(n)	g(n)
1	1	1
2	4	8
3	9	27

Example

$f(n) = n^2$

$g(n) = n^3$

show that

$g \notin O(f)$

Assume that

$g \in O(f)$

$\exists\, c, n_0$ s.t.

$g(n) \leq c \cdot f(n) \quad \forall n \geq n_0.$

choose $n_1 = 1 + \max\{c, n_0\}$

$n_1^3 = n_1 \cdot n_1^2 > c \cdot n_1^2$ where $n_1 > n_0$.

$\Rightarrow\Leftarrow$.

$n_1 = 1 + \max\{c, n_0\}$

$n_1 = 1 + c$ or $1 + n_0$

$c > n_0$ $n_0 > c$

$1 + c > c$ $1 + n_0 > n_0$

$n_1 > c$

$n_1^3 = \underline{n_1 \cdot n_1^2} > \underline{c \cdot n_1^2}$

we have a contradiction

$\exists\, n \geq n_0$ s.t.

$g(n) \geq c \cdot f(n)$

this is the opposite of our definition.

Thus we may conclude that $g \notin O(f)$.

Proof by Contradiction.

Show P,

$\therefore$ Assume $\neg P$,

if there arises a contradiction in $\neg P$,

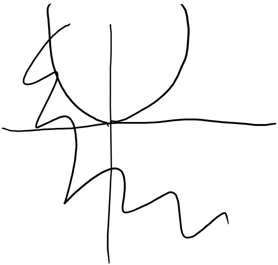
we conclude P,

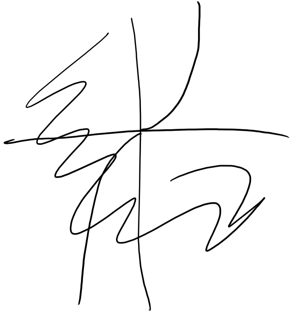
$$\left.\begin{array}{l}\text{Exhaustion.}\\ \text{Induction.}\\ \text{Construction.}\\ \text{Contraposition.}\end{array}\right\}\ \text{Be familiar}$$

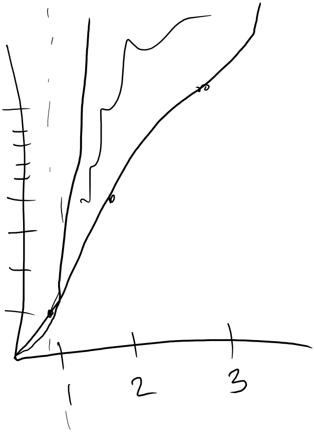
$f = n^2$

$g = n^3$

$\mathbb{N} \to \mathbb{N}$






1
2
3

$f(n) = n^2 + 2n + 5$
$g(n) = n^2$

show $g \in O(f)$

if $c = 1$
$n > 0$

$n^2 \leq n^2 + 2n + 5$

$0 \leq 2n + 5$

Show $f \in O(g)$

$f(n) = n^2 + 2n + 5$ $\qquad$ $g(n) = n^2$

$$n^2 + 2n + 5 \leq n^2 + 2n^2 + 5n^2$$

$$\leq 8n^2$$

$$C = 8$$

$$\forall n > 1.$$