

# 1 Background Materials

This section covers relevant background material to understanding the rest of the course and this script.

## 1.1 Set Theory

Material for this section is drawn from [1, Chapter 1] and [2, Chapters 1.2 and 6].

Intuitively, sets are collections of objects, which are called *elements* of the set. For example, one could consider a class of students to be a set – where each student is an element of that set.

### 1.1 Definition (Common sets and their notation)

- $\mathbb{N}$ : the natural numbers
- $\mathbb{Z}$ : the integers
- $\mathbb{Q}$ : the rational numbers
- $\mathbb{R}$ : the real numbers
- $\mathbb{C}$ : the complex numbers

### 1.2 Remark (Notation)

Sets are denoted using braces:  $\{$  and  $\}$ . There are two common ways of describing sets, as follows.

### 1.3 Definition (Set-Roster Notation)

The elements of a set may be listed explicitly: starting and ending with braces, and with the elements comma separated. Ellipses may be used for infinite sets.

### 1.4 Example (Finite)

$$A = \{a, b, c, d\}$$

$$B = \{1, 2, 3, 4\}$$

$$S = \{a, 1, \%\}$$

### 1.5 Remark (Use of "...")

The elements of a set be defined using clear patterns and ellipses ("..."). This method is generally discouraged in all but the most obvious of use cases.

### 1.6 Example (Infinite)

$$C = \{\dots, -1, 0, 1, \dots\}$$

$$D = \{2, 4, 6, 8, \dots\}$$

### 1.7 Example (Ambiguous Pattern)

$$E = \{2, 4, \dots\}$$

**Exercise 1** Identify at least two patterns that can fit the set-roster notation in the above example.

### 1.8 Definition (The Empty Set)

The set with no elements in it is denoted as  $\emptyset$ .

### 1.9 Remark (Alternate Notation)

The empty set may sometimes be denoted as a pair of braces with nothing in between.

$$\emptyset = \{\}$$

### 1.10 Definition (Set-Builder Notation)

The most powerful – and common – way of describing a set is through set-builder notation. Essentially, it is a way to describe a set via some number of rules that all elements in the set must adhere to.

### 1.11 Example (Representing Even Integers)

Recall that  $\mathbb{Z}$  is the set of all integers.

$$\mathbb{Z}^{\text{even}} = \{2x \mid x \in \mathbb{Z}, x \geq 0\}$$

### 1.12 Example (The Rational Numbers)

We can describe the set of all rational numbers as follows.

$$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}.$$

### 1.13 Remark (Reading Notation)

- The vertical line or “pipe”,  $|$ , is read as “such that”, “when” or “where,”; sometimes a colon is used instead.
- Commas in mathematics are read as “and.”

### 1.14 Definition (Cardinality)

Cardinality of a set  $X$ , denoted as  $|X|$ , is the measure of a set representing the number of unique, distinct elements in that set.

### 1.15 Example (Cardinality)

Let  $X = \{1, 2, a, \{3, 4\}, 9\}$ . Then, the cardinality  $|X| = 5$ .

**Exercise 2** Compute  $|\emptyset|$ .

**Exercise 3** Compute  $|\{\mathcal{R}\}|$ .

### 1.16 Definition (Universal Quantifier)

The universal quantifier  $\forall$  is read as “for all”, indicating that the corresponding condition holds for all cases.

### 1.17 Example (Using $\forall$ )

For example, we may use this qualifier as an indexing operator, similar to a **for** loop in programming where we may want to compute a value for each natural number less than ten. That is,  $\forall n \in \mathbb{N}, \text{ s.t. } n < 10, f(n)$ .

### 1.18 Definition (Existential Quantifier)

The existential quantifier  $\exists$  is read as “exists”, indicating that an arbitrary but particular value exists according to the corresponding condition.

### 1.19 Example (Using $\exists$ )

For example, we may say that there exists a global minima in  $\mathbb{N}$ , as follows.

$$\exists n \text{ s.t. } \forall n' \in \mathbb{N}, n \leq n'$$

**Exercise 4** Prove Example 1.19.

### 1.20 Definition (Improper Subset)

$$A \subseteq B \leftrightarrow \{a \mid a \in A, a \in B\}$$

### 1.21 Example (Improper Subset)

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### 1.22 Definition (Proper Subset)

$A$  is a proper subset of  $B$ , when  $A \subset B$  iff  $A \subseteq B$  and  $\exists b \in B \text{ s.t. } b \notin A$ .

### 1.23 Remark (Superset)

Terminologically, the inverse of a subset is the superset.

### 1.24 Definition (Set Union)

The Set Union  $C$  is denoted as

$$C = A \cup B$$

, where  $A, B, C$  are sets. It is defined as

$$C = \{c \mid c \in A \text{ or } c \in B\}$$

### 1.25 Example (Set Union)

Let  $A = \{\dots, -2, -1, 0\}$  and  $B = \{0, 1, 2, \dots\}$ . Then,  $A \cup B = \mathbb{Z}$

### 1.26 Definition (Set Intersection)

The Set Intersection  $C$  is denoted as

$$C = A \cap B$$

, where  $A, B, C$  are sets. It is defined as

$$C = \{c \mid c \in A \text{ and } c \in B\}$$

### 1.27 Example (Set Intersection)

$$\mathbb{Z} \cap \mathbb{N} = \{0\}$$

**Exercise 5** Compute

$$|\mathbb{Q} \cap \mathbb{Z} \cap \mathbb{N}|$$