Using Limits   2.21 — 2.23 in the script.

Theorem

1. if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \in O(g)$
$g \notin O(f)$

2. if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, $0 < c < \infty$ $f \in \Theta(g)$
$g \in \Theta(f)$

3. if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ then $f \notin O(g)$
$g \in O(f)$

Proof part 1 of theorem 2.21

Assume $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$. for each $\varepsilon > 0$

$\exists \; n_\varepsilon \in \mathbb{N}$

s.t. $\forall n \geq n_\varepsilon$

we have

$\frac{f(n)}{g(n)} < \varepsilon$, thus $f(n) < \varepsilon \cdot g(n)$

So, let $c = \varepsilon = 1$, choose $n_0 = n_\varepsilon$

Then it is the case $f(n) \leq c \cdot g(n)$.

$g \notin O(f)$

Now assume $g \in O(f)$, there must exist a $d > 0$ and $m_0 \in \mathbb{N}$ s.t. $g(n) \leq d \cdot f(n)$ for all $n \geq m_0$,

$$\frac{1}{d} \leq \frac{f(n)}{g(n)}, \quad \forall n \geq m_0.$$

then $\lim\limits_{n \to \infty} \dfrac{f(n)}{g(n)} \geq \dfrac{1}{d} > 0$

which violates our premise. $g \notin O(f)$. $\Rightarrow\Leftarrow$

L'Hospital rule can be helpful

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Recall that prime

f' is the <u>first derivative</u>

Example

$n\log_a(n) \in O(n^2)$ and $n^2 \notin O(n\log_a(n))$

$$\lim_{n\to\infty} \frac{n\log_a(n)}{n^2} = \lim_{n\to\infty} \frac{\log_a(n) + n\log_a(e)/n}{2n}$$

$$\lim_{n\to\infty} \frac{\log_a(n)}{2n} + \lim_{n\to\infty} \frac{\log_a e/n}{2n} = 0 + 0 = 0.$$

Then by theorem 2.21.1 this holds.