2 Big-O Notation

Material for this section is drawn from [6, Chapter 14.2].

2.1 Remark (Common Sets)
$\mathbb{N} = \{0, 1, 2, \dots\}$
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{R}$: the real numbers

2.2 Definition (Big-O Notation)
Let $f, g : \mathbb{N} \to \mathbb{N}$.
We say: f is *of order* g, if there is a constant $c > 0$ and $n_0 \in \mathbb{N}$ s.t. $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.
$O(g) := \{f \mid f \text{ is of order } g\}$ ("big oh of g").
If $f \in O(g)$ we can say that g *provides an asymptotic upper bound on* f.
If $f \in O(g)$ and $g \in O(g)$, then they *have the same rate of growth*, and g *is an asymptotically tight bound on* f (and vice versa).

2.3 Remark (Common Abuse of Notation)
$f = O(g)$ instead of $f \in O(g)$.
$f(n) = n^2 + O(n)$ instead of "$f(n) = n^2 + g(n)$ for some $g \in O(n)$".

2.4 Example
Let $f(n) = n^2$; $g(n) = n^3$.
Show $f \in O(g)$.
[For $c = 1$ and $n > 1$, $n^2 \leq c \cdot n^3$.]

2.5 Example
Let $f(n) = n^2$; $g(n) = n^3$.
Show $g \notin O(f)$.
[Assume $n^3 \in O(n^2)$. Then ex. c, n_0 s.t. $n^3 \leq c \cdot n^2$ for all $n \geq n_0$. Choose $n_1 = 1 + \max\{c, n_0\}$. Then $n_1^3 = n_1 \cdot n_1^2 > c \cdot n_1^2$ and $n_1 > n_0$. $\lightning$]

2.6 Example
$f(n) = n^2 + 2n + 5$; $g(n) = n^2$
$g \in O(f)$ [For $c = 1$ and $n > 0$, $n^2 \leq c \cdot (n^2 + 2n + 5)$.]
$f \in O(g)$
[For $n > 1$ we have $f(n) = n^2 + 2n + 5 \leq n^2 + 2n^2 + 5n^2 = 8n^2 = 8 \cdot g(n)$. Hence, for $c = 8$ and $n > 1$, $f(n) \leq c \cdot g(n)$.]

2.7 Definition
$\Theta(g) := \{f \mid f \in O(g) \text{ and } g \in O(f)\}$

2.8 Example (Big-Theta Notation)
For f, g from Example 2.6, $f \in \Theta(g)$.

2.9 Remark
A *polynomial* (with integer coefficients) *of degree* $r \in \mathbb{N}$ is a function of the form

$$f : \mathbb{N} \to \mathbb{Z} : n \mapsto c_r \cdot n^r + c_{r-1} \cdot n^{r-1} + \cdots + c_1 \cdot n + c_0,$$

with $0 < r \in \mathbb{N}$, *coefficients* $c_i \in \mathbb{Z}$ $(i = 1, \ldots, r)$, $c_r \neq 0$.
For $f, g : \mathbb{N} \to \mathbb{Z}$, we say $f \in O(g)$ if $|f| \in O(|g|)$, where $|f| : \mathbb{N} \to \mathbb{N} : n \mapsto |f(n)|$.

2.10 Example
$f(n) = n^2 + 2n + 5$; $g(n) = -n^2$
$g \in O(f)$
[We have $|g| : n \mapsto n^2$ and $|f| \equiv f$. Thus, from Example 2.6 we know that $|g| \in O(|f|)$.]
$f \in O(g)$ [As before, from Example 2.6.]

2.11 Remark
For $f, g : \mathbb{N} \to \mathbb{R}$, we say $f \in O(g)$ if $\lfloor f \rfloor \in O(\lfloor g \rfloor)$.

2.12 Remark
$\log_a(n) \in O(\log_b(n))$ for all $1 < a, b \in \mathbb{N}$. [$\log_a(n) = \log_a(b) \cdot \log_b(n)$ for all $n \in \mathbb{N}$.]

2.13 Theorem
The following hold.

1. If $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \in O(g)$ and $g \notin O(f)$.
2. If $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c$ with $0 < c < \infty$, then $f \in \Theta(g)$ and $g \in \Theta(f)$.
3. If $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \notin O(g)$ and $g \in O(f)$.

Proof: We show part 1.
Assume $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$, i.e., for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ we have $\frac{f(n)}{g(n)} < \varepsilon$, and hence $f(n) < \varepsilon g(n)$. Now select $c = \varepsilon = 1$ and $n_0 = n_\varepsilon$. Then $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$, which shows $f \in O(g)$.
Now if we also assume $g \in O(f)$, then there must exist $d > 0$ and $m_0 \in \mathbb{N}$ s.t. $g(n) \leq d \cdot f(n)$ for all $n \geq m_0$, i.e., $\frac{1}{d} \leq \frac{f(n)}{g(n)}$ for all $n \geq m_0$. But then $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} \geq \frac{1}{d} > 0$ ↯ . ■

2.14 Remark
The l'Hospital's Rule often comes in handy:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

2.15 Example
$n \log_a(n) \in O(n^2)$ and $n^2 \notin O(n \log_a(n))$

$$\left[\lim_{n\to\infty} \frac{n \log_a(n)}{n^2} = \lim_{n\to\infty} \frac{\log_a(n) + n(\log_a(e)/n)}{2n} = \lim_{n\to\infty} \frac{\log_a(n)}{2n} + \lim_{n\to\infty} \frac{\log_a(e)}{2n} = 0 + 0 = 0\right]$$

2.16 Theorem
Let f be a polynomial of degree r. Then

(1) $f \in \Theta(n^r)$

(2) $f \in O(n^k)$ for all $k > r$

(3) $f \notin O(n^k)$ for all $k < r$

2.17 Theorem
The following hold.

(1) $\log_a(n) \in O(n)$ and $n \notin O(\log_a(n))$

(2) $n^r \in O(b^n)$ and $b^n \notin O(n^r)$

(3) $b^n \in O(n!)$ and $n! \notin O(b^n)$

2.18 Remark (The Big-O Hierarchy)
The following is the hierarchy of complexities, according to their Big-O Notation, as well as the natural language terms frequently used to describe them.

$O(1)$	constant	"sublinear"	"subpolynomial"
$O(\log_a(n))$	logarithmic	"sublinear"	"subpolynomial"
$O(n)$	linear		"subpolynomial"
$O(n \log_a(n))$	$n \log n$		"subpolynomial"
$O(n^2)$	quadratic		"polynomial"
$O(n^3)$	cubic		"polynomial"
$O(n^r)$	polynomial ($r \geq 0$)		
$O(b^n)$	exponential ($b > 1$)		"exponential"
$O(n!)$	factorial		"exponential"