Appendix A. Lagrangian Mechanics

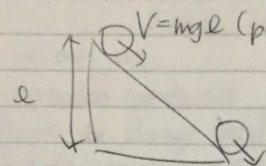
Newton mechanics
- Explicitly describing force in ‘vector’ form.
- Proper in cartesian coordinates

Lagrangian Mechanics
- Same physical principle
- Energy-based, scalar form
- Independent from coordinates -> can easily transforming coordinates

Intuition

In conservative system

such as



$V = mgl$ (potential energy)

$V = T$ (energy conservation)

$T = \frac{1}{2}mv^2$. (kinetic energy)

Easy derivation

in any spatial coordinates, q

$T = \frac{1}{2} m \dot{q}^2$.

$\rightarrow \frac{\partial T}{\partial \dot{q}} = m\dot{q}$

$\rightarrow \frac{\partial T}{\partial q} = 0$.

$V = V(q)$

$\rightarrow \frac{\partial V}{\partial \dot{q}} = 0$

$\rightarrow \frac{\partial V}{\partial q} = -F_c$

Newton's law

$$\frac{d}{dt}(m\dot{q}) = F_c.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = -\frac{\partial V}{\partial q}.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}} - \frac{\partial V}{\partial \dot{q}}\right) = \frac{\partial T}{\partial q} - \frac{\partial V}{\partial q}.$$

if $L = T - V$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

In non-conservative system.

$$\frac{d}{dt}(m\dot{q}) = F_c + \underbrace{F_{NC}}_{=Q}$$

$$\rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q.}$$

Standard Derivation

- Path integral of L(=T-V, difference bet. kinetic and potential energy) from time t1 to t2

$$\min J = \int_{t_1}^{t_2} L\,dt \quad \rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

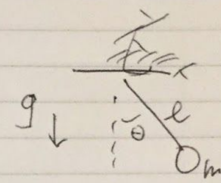
- Calculus of variation, Hamilton's principle

예제: simple pendulum

Simple pendulum

Kinematics

$$p = \begin{bmatrix} \ell \sin\theta \\ -\ell\cos\theta \end{bmatrix} \rightarrow \dot{p} = \begin{bmatrix} \ell\dot{\theta}\cos\theta \\ \ell\dot{\theta}\sin\theta \end{bmatrix}$$



kinetic energy

$$T = \frac{1}{2}\dot{p}^T m \dot{p} = \frac{1}{2} m\ell^2\dot{\theta}^2.$$

potential energy

$$U = -mg\ell\cos\theta$$

Lagrangian

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta. \quad -①$$

Dynamics

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \underline{Q} \quad -②$$

generalized force.

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}. \quad -③$$

$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta \quad -④$$

$$①,④ \rightarrow ② : \quad \underline{m\ell^2\ddot{\theta} + (mg\ell\sin\theta)\dot{\theta} = Q}$$

Reference

[1] http://www.physicsinsights.org/lagrange_1.html
[2] http://www.nyu.edu/classes/tuckerman/stat.mech/lectures/lecture_1/node3.html
[3] http://www.astro.uwo.ca/~houde/courses/PDF%20files/physics350/Lagrange.pdf
[4] https://en.wikipedia.org/wiki/Lagrangian_mechanics