Math 228 **Exam 2** Spring 2020

Instructions: Write all answers in the space provided, showing work and providing explanations. Open notes, open book, but you must WORK ALONE and may NOT search the internet for answers. You have two hours to complete the exam.

By signing below, I certify that the work on this take-home exam is solely my own, that I did not receive assistance from anyone other than my instructor, and did not use resources other than my own notes and the course textbook.

Signature: ____________ **Date:** 4/20/20

Start time: 10:15 **Stop time:** 11:51

(4pts) 1. Select the best answer to each counting problem below.

(a) You are at a fancy soup tasting party. There are 12 soups available and you can select 5 of them to try. Your order will be delivered all together (in separate bowls of course). How many ways can you order different soups?

A. $12! = 4791001600$ B. $P(12,5) = 95040$ C. $\binom{12}{5} = 792$ D. $5! = 120$

(b) Before you are able to order, the server informs you that the kitchen is running behind and so the soups will be brought out one by one instead of all at once. How many ways can you order 5 out of the 12 soups now?

A. $12! = 4791001600$ B. $P(12,5) = 95040$ C. $\binom{12}{5} = 792$ D. $5! = 120$

(8pts) 2. Explain the relationship between the two counting problems above both numerically and in terms of fancy soups. Be specific: don't just say which is larger, say how many times larger it is, and why this makes sense.

Part b is 5! times larger than part a. This makes sense because 5! would be the amount of times a soup is repeated. Part a only counts that specific order whereas part b counts all orders. And it'll count 5! more orders, which are the ones where soups can be repeated.

(12pts) 3. Phil's Fresh Fish Friends store sells 18 different kids of fish. For each part below, include the answer and a very brief explanation of why your answer is correct.

(Hint: no two answers on this page will be the same.)

(a) How many different purchases can you make if you want 6 unique fish to put in one big tank?

$\binom{18}{6}$ this is correct because you want each combination without any repeats.

18,564

(b) You decide you don't care if all of the fish are different. How many ways can you buy 6 (not necessarily unique) fish to put in one big tank?

$\binom{23}{5}$ or $\binom{23}{18}$

33,649 Stars and bars 18 possibilities for each fish

(c) You remember you already have 6 fish bowls (all different sizes and shapes) at home already and don't need to buy a new tank. How many ways can you buy 6 unique fish and arrange them in the different tanks?

$P(18, 6)$ this is correct because we need unique combos and we get that from permutations

13,366,080

(d) You remember that you don't care if the fish are different anymore. How many ways can you purchase 6 fish, not necessarily unique, to put in the 6 very unique fish bowls?

18^6 For each of 6 fish, there are 18 options

(12pts) 4. Your brewery has 13 identical kegs of its famous *starfish ale* to distribute to the 5 bars in town.

(a) How many ways can you distribute the kegs? Explain your answer, including an example of one outcome (way to distribute the kegs) and what sort of diagram you use to represent that outcome.

Stars and bars

$\binom{17}{4}$

2,380

You can distribute kegs so that not all bars get one. 1 bar could get all the kegs. This is why we start with 17.

1 2 3 4 5

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(b) How many ways could you distribute the 13 kegs to the 5 bars so that no bar gets more than 3 kegs? Explain what the entries in your use of the Principle of Inclusion and Exclusion represent.

$$\binom{17}{4} - \left[\binom{5}{1}\binom{13}{4} - \binom{5}{2}\binom{9}{4} + \binom{5}{3}\binom{5}{4}\right]$$

2,380 5(715) 10(126) 10(5)

2350 $-[3575 - 1260 + 50]$

$= 15$

Here we are excluding all the possibilities where a bar gets 4 kegs or more. And where 2 or 3 bars get 4 or more kegs

5. Your young cousin is doing school online too. Their math program gives them 4 problems the first day. Each day after that it gives them 3 more problems than the day before (so the second day there are 7 problems, and 10 problems the third day, etc.).

(6pts) (a) Let $(a_n)_{n\geq 1}$ be the sequence where a_n is the number of problems that have to be done on the nth day. Give a recursive definition and a closed formula for a_n (clearly mark which is which).

non recursive

$a_n = 3n+1$

recursive

$a_n = a_{n-1} + 3$

(8pts) (b) Let $(b_n)_{n\geq 1}$ be the sequence where b_n is the *total* number of problems solved after the nth day's work was completed (so the sequence starts $4, 11, 21, 34, \ldots$). Give a recursive definition and a closed formula for b_n.
Show your work (especially for the closed formula).

Closed

0 1 2

0, 4, 11, 21, 34, 50

4 7 10 13 16

3 3 3 3 power of 2

$4 = a + b$ $a = 4 - b$ $an^2 + bn + c$

$11 = 4a + 2b$ $an^2 + bn + 0$

$11 = 4(4-b) + 2b$

$= 16 - 4b + 2b$

$11 = 16 - 2b$

$2b = 5$

$b = 2.5$

$b_n = 1.5n^2 + 2.5n + 0$

recursive

$b_n = b_{n-1} + 3n + 1$

(16pts) 6. For each sequence described below, circle the form of the closed formula and briefly explain. Then write down the system of equations you would need to solve to find the constants a, b, etc. (You do not need to find the constants.)

(a) The sequence $(a_n)_{n\geq 0}$ with recursive definition $a_n = a_{n-1} + (2n^2 + 1)$; $a_0 = 15$. (The sequence starts $15, 18, 27, 46, 79, \ldots$.)

A. $an^2 + bn + c$ B. $an^3 + bn^2 + cn + d$ C. $a(-3)^n + b4^n$ D. $a1^n + b12^n$

Briefly explain:

15, 18, 27, 46, 79

1st power 3 9, 19, 33

2nd power 6, 10, 14

3rd power 4 4

the differences are constant at the third power so B is correct.

System of equations:

d=15

18 = a+b+c+15

27 = 8a+4b+2c+15

46 = 27a+9b+3c+15

(b) The sequence $(b_n)_{n\geq 0}$ with recursive definition $b_n = b_{n-1} + 12b_{n-2}$; $b_0 = 1$, $b_1 = 2$. (The sequence starts $1, 2, 14, 38, 206, \ldots$.) $b_2 = b_1 + b_0(12)$

2 + 12
= 14

A. $an^2 + bn + c$ B. $an^3 + bn^2 + cn + d$ C. $a(-3)^n + b4^n$ D. $a1^n + b12^n$

Briefly explain:

1, 2, 14, 38, 206

1, 12, 24, 168

11 12 144

1 132

cannot be A or B

System of equations:

(18pts) 7. To keep yourself busy during social isolation, you start **mathmemes.com**, a site to collect awesome math memes. You start (day 0) with 7 memes. As your friends and internet weirdos start to post things, you find that each day the number of memes triples, but then after that, 6 get taken down due to copyright claims.

Let $(a_n)_{n\geq 0}$ be the sequence giving the number of memes on the site after the nth day (after the tripling and the removal of 6). Note $a_0 = 7$.

(a) Write down a recurrence relation to describe a_n and briefly explain. 7, 15

$a_n = (a_{n-1})3 - 6$

each day it will triple from the previous value then 6 will get taken down

(b) Prove that the number of memes at the end of each day will always be odd.

Any number multiplied by 3 will be odd

$\rightarrow$ $7 * 3 = 21$
$15 * 3 = 45$

Any odd number minus an even number will remain odd

$\rightarrow$ $21 - 6 = 15$
$45 - 6 = 39$

Since the first number is odd, when it is multiplied by 3 then 6 is subtracted, the result will be odd. Subsequently this means that the next number is odd

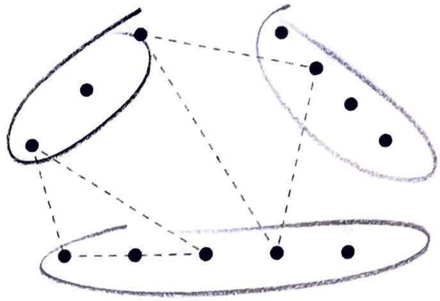
(c) Give a careful proof by mathematical induction that $a_n = 4 \cdot 3^n + 3$.

base case $(n=0) \rightarrow a_0 = 4 \times 3^0 + 3 = 7$ which is right

induction $\rightarrow a_1 = 4 \times 3^1 + 3 = 15$ which is right

(16pts) 8. Give two different expressions that count the number of triangles you can form using the 12 dots below as vertices. (Two such triangles are drawn for your reference.)

Both answers should involve binomial coefficients, but should contain a different number of them. Explain your answers.



(a) First answer and explanation:

$$\binom{12}{3} - \left[\binom{5}{2} + \binom{3}{2} + \binom{4}{2}\right]$$

$$220 - \quad 10 + 3 + 6 \quad = 201$$

12 dots to begin with and need to choose 3. but nomore then 2 dots can be from the same group of dots which is why I have the exclusion bracket being subtracted.

(b) Second answer and explanation:

(10bn-pts) **9. BONUS!! Prove the combinatorial identity:**

$$\binom{x+y+z}{3} - \binom{x}{3} - \binom{y}{3} - \binom{z}{3} = (y+z)\binom{x}{2} + (x+z)\binom{y}{2} + (x+y)\binom{z}{2} + xyz.$$