Linearly Constrained Separable Optimization using PiecewiseQuadratics.jl and LCSO.jl

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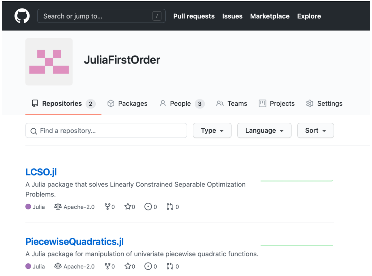
Outline

PiecewiseQuadratics.jl

LCSO.jl

JuliaFirstOrder

Portfolio Optimization



Portfolio Construction as
Linearly Constrained Separable Optimization

Nicholas Moehle   Jack Gindi   Stephen Boyd
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March 10, 2021

[2103.05455]

Outline

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PiecewiseQuadratics.jl

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  - derivative
  - convex envelope
  - proximal operator
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- Implements several methods useful for optimization
  - sum
  - derivative
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  - proximal operator
  - ...
- Applications to cost functions
  - Higher fidelity XXX(What does this mean? -Nick)
  - Computationally tractable

Example

$$f(x) = \begin{cases} x^2 - 3x - 3 & \text{if } x \in [-\infty, 3] \\ -x + 3 & \text{if } x \in [3, 4] \\ 2x^2 - 20x + 47 & \text{if } x \in [4, 6] \\ x - 7 & \text{if } x \in [6, 7.5] \\ 4x - 29 & \text{if } x \in [7.5, \infty] \end{cases}$$

XXX As a formatting issue, can terms of the same order be aligned?
-Nick

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```
using PiecewiseQuadratics
f = PiecewiseQuadratic([
  # BoundedQuadratic(   l,   u,   p,      q,      r),
    BoundedQuadratic(-Inf, 3.0, 1.0,  -3.0,   3.0),
    BoundedQuadratic( 3.0, 4.0, 0.0,  -1.0,   3.0),
    BoundedQuadratic( 4.0, 6.0, 2.0, -20.0,  47.0),
    BoundedQuadratic( 6.0, 7.5, 0.0,   1.0,  -7.0),
    BoundedQuadratic( 7.5, Inf, 0.0,   4.0, -29.0)
]);
```

Plot

```
using Plots
plot(get_plot(f); ...)
plot!(get_plot(simplify(envelope(f))); ...)
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XXX We would make the font size larger to match the font size of the rest of the slide. (Basically, print out a smaller image.) -Nick XXX Change 'piece-wise' to 'piecewise'

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Linearly Constrained Separable Optimiaziton

A *linearly constrained separable optimization* (LCSO) problem:

$$\begin{array}{ll} \text{minimize} & f(x) = \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & Ax = b, \end{array} \tag{1}$$

where

- the decision variable is $x \in \mathbf{R}^n$
- the parameters are $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, and the functions f_i.

[1] See: arXiv:2103.05455

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LCSO problems can be solved using ADMM.[1]

- Exactly, if the f_i are convex
- Approximately, if they aren't

[1]See: arXiv:2103.05455

Extended-Form LCSO Problems

An extended-form LCSO problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & Ax = b, \end{array} \qquad (2)$$

where

- $q \in \mathbf{R^n}$, $P \in \mathbf{S^n_+}$ is a symmetric positive semidefinite matrix

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$$
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\text{subject to} & Ax = b,
\end{array}
\tag{2}
$$

where

- $q \in \mathbf{R^n}$, $P \in \mathbf{S^n_+}$ is a symmetric positive semidefinite matrix
- Extended-form LCSO problems can be reduced to standard LCSO problem (using an eigendecomposition)[2]

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- The separable functions f_i must be piecewise quadratic (see `PiecewiseQuadratics.jl`)
- (In theory, could be extended to any f_i that support a `prox` method)

Example

```julia
using LCSO
using PiecewiseQuadratics

n = 4  # num features
m = 2  # num constraints

# construct problem data
x0 = rand(n)
X = rand(n, n)

# ensure P is PSD
P = X'X
q = rand(n)
A = rand(m, n)
b = A * x0
```

```julia
# x_1 has to be ∈ [−1, 2] ∪ [2.5, 3.5]
#   with quadratic penalty ∈ [−1, 2]
#   and linear penalty ∈ [2.5, 3.5]
g1 = PiecewiseQuadratic([
    BoundedQuadratic(-1, 2, 1, 0, 0),
    BoundedQuadratic(2.5, 3.5, 0, 1, 0)
])
# x_2 has to be ∈ [−20, 10]
g2 = indicator(-20, 10)

# x_3 has to be ∈ [−5, 10]
g3 = indicator(-5, 10)

# x_4 has to be exactly 1.2318
g4 = indicator(1.2318, 1.2318)
```

Example

```
g = [g1, g2, g3, g4]

# solve
params = AdmmParams(P, q, A, b, g)
vars, stats = optimize(params)

print(vars.x)  # optimal x
# [-0.04933, 1.2180, -1.9325, 1.2318]
```

Applications

- Portfolio optimization
- Radiation treatment planning
- Dynamic energy management
- ...

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JuliaFirstOrder

- GitHub organization for first-order methods in Julia
  - https://github.com/JuliaFirstOrder
- Plan to migrate several related packages
  - ProximalOperators.jl
  - ProximalAlgorithms.jl
  - StructuredOptimization.jl
  - ...
- Thank you to Miles Lubin for the introductions!

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Portfolio Optimization

- The original mean–variance portfolio optimization problem of Markowitz, *i.e.*, "Risk Minimization"

$$\begin{array}{ll} \text{minimize} & \gamma x^T \Sigma x - \mu^T x \\ \text{subject to} & \mathbf{1}^T x = 1. \end{array} \tag{3}$$

- The decision variable $x \in \mathbf{R}^n$ is the fraction of the portfolio value in each of n assets.
- The vector $\mu \in \mathbf{R}^n$ is the expected return forecast for the n assets, meaning $\mu^T x$ is the expected portfolio return.
- The matrix $\Sigma \in \mathbf{S}^n_{++}$ is the asset return covariance matrix, meaning $x^T \Sigma x$ is the variance of the portfolio return.
- $\gamma \in \mathbf{R}_{\geq 0}$ is the risk aversion parameter

...With Separable Costs!

- The portfolio optimization problem with separable costs:

$$\begin{array}{ll} \text{minimize} & x^T \Sigma x - \gamma_{\mathsf{risk}} \mu^T x + \sum_{i=1}^n f_i(x_i) \\ \text{subject to} & \mathbf{1}^T x = 1 \end{array} \tag{4}$$

- f_i are asset-level penalties, like taxes and trading costs, and are piecewise quadratic (more later)

Separable cost examples

- Trading costs:

$$f_i^{\mathrm{trd}}(x_i) = s_i|x_i - x_{\mathrm{init},i}| + \begin{cases} 0 & x_i = x_{\mathrm{init},i} \\ c_i^{\mathrm{trd}} & \text{otherwise} \end{cases}$$

- Holding cost:

$$f_i^{\mathrm{hold}}(x_i) = \begin{cases} 0 & x_i = 0 \\ c_i^{\mathrm{hold}} & \text{otherwise} \end{cases}$$

- Position limits:

$$f_i^{\mathrm{lim}}(x_i) = \begin{cases} 0 & h_{\mathrm{lb},i} \le x_i \le h_{\mathrm{ub},i} \\ \infty & \text{otherwise} \end{cases}$$

- Combinations of these:

$$f_i(x_i) = f_i^{\mathrm{trd}}(x_i) + f_i^{\mathrm{hold}}(x_i) + f_i^{\mathrm{lim}}(x_i)$$

Relaxation

- Problems are often nonconvex
- One way to handle this: **Relax** the problem by replacing f_i with its **convex envelope**

$$f_i^{**}(x) = \sup\{g(x) \mid g \text{ is convex and } g(x) \leq f_i(x),\ x \in \mathbf{dom}(f_i)\},$$

  then use the result to recover a solution to the original problem
- Easy to compute when f_i are piecewise quadratic

Questions?