In the final section, section 2.5, we discuss some of the special features and implications of the intertemporally separable utility function with constant rate of time preference used in the chapter, and examine alternative formulations.

2.1 The Ramsey Problem

Frank Ramsey[2] posed the question of how much a nation should save and solved it using a model that is now the prototype for studying the optimal intertemporal allocation of resources. The model presented in this section is essentially that of Ramsey.

The population, N_t, grows at rate n; it can be thought of as a family, or many identical families, growing over time. The labor force is equal to the population, with labor supplied inelastically. Output is produced using capital, K, and labor. There is no productivity growth.

The output is either consumed or invested, that is, added to the capital stock. Formally,

$$Y_t = F(K_t, N_t) = C_t + \frac{dK_t}{dt}. \tag{1}$$

For simplicity, we assume that there is no physical depreciation of capital, or that Y_t is net rather than gross output.[3] The production function is homogeneous of degree one: that is, there are constant returns to scale.

In per capita terms

$$f(k_t) = c_t + \frac{dk_t}{dt} + nk_t, \tag{2}$$

where lowercase letters denote per capita (equal to per worker) values of variables so that k is the capital-labor ratio and $f(k_t) \equiv F(K_t/N_t, 1)$; we assume $f(\cdot)$ to be strictly concave and to satisfy the following conditions, known as Inada conditions:

$$f(0) = 0, \qquad f'(0) = \infty, \qquad f'(\infty) = 0.$$

We also assume that the economy starts with some capital so that it can get production off the ground:

$$k_0 > 0.$$

The preferences of the family for consumption over time are represented by the utility integral:

$$U_s = \int_s^{\infty} u(c_t) \exp[-\theta(t - s)]\, dt. \tag{3}$$

The family's welfare at time s, U_s, is the discounted sum of instantaneous *utilities* $u(c_t)$. *The function* $u(\ )$ *is known as the instantaneous utility function*, or as "felicity"; $u(\ )$ is nonnegative and a concave increasing function of the per capita consumption of family members. The parameter θ is the rate of time preference, or the subjective discount rate, which is assumed to be strictly positive.[4]

The Command Optimum

Suppose that a central planner wants at time $t = 0$ to maximize family welfare. The only choice that has to be made at each moment of time is how much the representative family should consume and how much it should add to the capital stock to provide consumption in the future. The planner has to find the solution to the following problem:

$$\max U_0 = \int_0^{\infty} u(c_t) \exp(-\theta t)\, dt \tag{4}$$

subject to (2) and the constraints

k_0 given; $\qquad k_t, c_t \geqslant 0$ for all t.

We characterize the solution using the maximum principle.[5] The optimal solution is obtained by setting up the present value Hamiltonian function:

$$H_t = u(c_t) \exp(-\theta t) + \mu_t[f(k_t) - nk_t - c_t]. \tag{5}$$

The variable μ is called the *costate* variable associated with the *state* variable k; equivalently it is the multiplier on the constraint (2). The value of μ_t is the marginal value as of time zero of an additional unit of capital at time t.

It is often more convenient to work, instead, with the marginal value, as of time t, of an additional unit of capital at time t, $\lambda_t \equiv \mu_t \exp(\theta t)$; we shall do so here. Replacing μ_t by λ_t in (5) gives

$$H_t = [u(c_t) + \lambda_t(f(k_t) - nk_t - c_t)] \exp(-\theta t). \tag{5'}$$

We do not explicitly impose the nonnegativity constraints on k and c.

Necessary and sufficient conditions for a path to be optimal under the assumptions on the utility and production functions made here are that[6]

$$H_c = 0,$$

$$\frac{d\mu_t}{dt} = -H_k,$$

$$\lim_{t\to\infty} k_t\mu_t = 0.$$

Using the definition of $H(\cdot)$ and replacing μ by λ, we get

$$u'(c_t) = \lambda_t, \tag{6}$$

$$\frac{d\lambda_t}{dt} = \lambda_t[\theta + n - f'(k_t)], \tag{7}$$

$$\lim_{t\to\infty} k_t u'(c_t)\exp(-\theta t) = 0. \tag{8}$$

Equations (6) and (7) can be consolidated to remove the costate variable λ, yielding

$$\frac{du'(c_t)/dt}{u'(c_t)} = \theta + n - f'(k_t), \tag{7'}$$

or equivalently

$$\left[\frac{c_t u''(c_t)}{u'(c_t)}\right]\left(\frac{dc_t/dt}{c_t}\right) = \theta + n - f'(k_t).$$

The expression $cu''(c)/u'(c)$ will recur often in this book. It reflects the curvature of the utility function. More precisely, it is equal to the elasticity of marginal utility with respect to consumption. If utility is nearly linear and if marginal utility is nearly constant, then the elasticity is close to zero. This elasticity is itself closely related to the *instantaneous elasticity of substitution*. The elasticity of substitution between consumption at two points in time, t and s, is given by

$$\sigma(c_t) \equiv -\frac{u'(c_s)/u'(c_t)}{c_s/c_t}\frac{d(c_s/c_t)}{d[u'(c_s)/u'(c_t)]}.$$

Taking the limit of that expression as s converges to t gives $\sigma = -u'(c_t)/u''(c_t)c_t$ so that $\sigma(c_t)$ is the inverse of the negative of the elasticity of marginal utility. When utility is nearly linear, the elasticity of substitution is very large. Using the definition of σ, (7′) can be rewritten as

$$\frac{dc_t/dt}{c_t} = \sigma(c_t)[f'(k_t) - \theta - n]. \tag{7''}$$

The key conditions are (7) [or (7′) or (7″)] and (8). Equation (7) is the Euler equation, the differential equation describing a necessary condition that has to be satisfied on any optimal path. It is the continuous time analogue of the standard efficiency condition that the marginal rate of substitution be equal to the marginal rate of transformation, as we shall show shortly. The condition is also known as the Keynes-Ramsey rule. It was derived by Ramsey in his classic article, which includes a verbal explanation attributed to Keynes. We now develop an intuitive explanation of this repeatedly used condition.

The Keynes-Ramsey Rule

The easiest way to understand the Keynes-Ramsey rule is to think of time as being discrete and to consider the choice of the central planner in allocating consumption between time t and $t + 1$. If he decreases consumption at time t by dc_t, the loss in utility at time t is equal to $u'(c_t)\, dc_t$. This decrease in consumption at time t, however, allows for more accumulation and thus more consumption at time $t + 1$: consumption per capita can be increased by $(1 + n)^{-1}[1 + f'(k_t)]\, dc_t$, leading to an increase in utility at $t + 1$ of $(1 + n)^{-1}[1 + f'(k_t)]u'(c_{t+1})\, dc_t$. Along the optimal path small reallocations in consumption must leave welfare unchanged so that the loss in utility at time t must be equal to the discounted increase in utility at time $t + 1$. Thus

$$u'(c_t) = (1 + \theta)^{-1}(1 + n)^{-1}[1 + f'(k_t)]u'(c_{t+1}).$$

This condition can be rewritten as

$$\frac{(1 + \theta)^{-1}u'(c_{t+1})}{u'(c_t)} = \frac{1 + n}{1 + f'(k_t)}, \tag{9}$$

which states that the marginal rate of substitution (MRS) between consumption at times t and $t + 1$ is equal to the marginal rate of transformation (MRT), from production, between consumption at times t and $t + 1$. If the period is short enough, this condition reduces to equation (7′).

A more rigorous argument runs as follows: Consider two points in time, t and s, $s > t$. We now imagine reallocating consumption from a small interval following t to an interval of the same length following s. Decrease c_t by amount Δc_t at time t for a period of length Δt, thus increasing capital accumulation by $\Delta c_t \Delta t$. That capital is allowed to accumulate between $t + \Delta t$ and s, with consumption over that interval unchanged from its

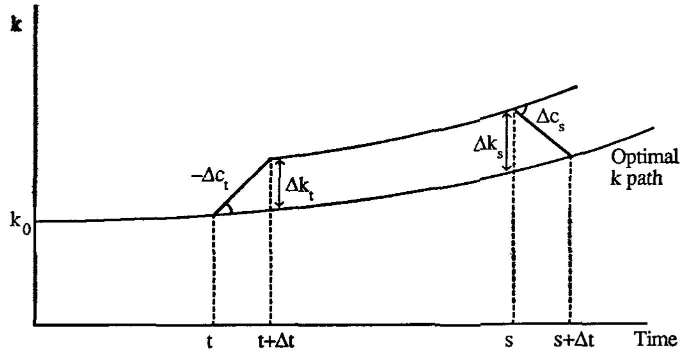

Figure 2.1
The Keynes-Ramsey rule

original value. All the increased capital is consumed during an interval of length Δt starting at s, with consumption thereafter being unchanged from the level on the original path. This variation from the optimal path is illustrated in figure 2.1.

For sufficiently small Δc and Δt, such a reallocation should have no effect on welfare, provided the path is optimal. Thus

$$u'(c_t)\Delta c_t \Delta t + u'(c_s)\exp[-\theta(s-t)]\Delta c_s \Delta t = 0.$$

The relation between Δc_t and Δc_s is implied by

$$\Delta c_t \Delta t = \Delta k_t, \qquad \Delta c_s \Delta t = \Delta k_s,$$

and

$$\Delta k_s = -\Delta k_t \exp\left\{\int_{t+\Delta t}^{s} [f'(k_v) - n]\, dv\right\}.$$

Capital accumulated in the first interval Δt grows at the rate $f'(k) - n$ between $t + \Delta t$ and s.

Eliminating Δc's and Δk's from the preceding relations gives

$$\frac{u'(c_t)}{u'(c_s)\exp[-\theta(s-t)]} = \exp\left\{\int_{t+\Delta t}^{s} [f'(k_v) - n]\, dv\right\}. \tag{10}$$

Equation (10) has the same interpretation as equation (9), namely, that marginal rates of substitution and transformation are equal.

As this equality must hold for all t and s, it follows that

$$\lim_{s \to t} \frac{d\text{MRS}(t, s)}{ds} = \lim_{s \to t} \frac{d\text{MRT}(t, s)}{ds}.$$

Applying this to (10) gives equation (7′).

The Keynes-Ramsey rule, in discrete or continuous time, implies that consumption increases, remains constant, or decreases depending on whether the marginal product of capital (net of population growth) exceeds, is equal to, or is less than the rate of time preference. This rule is quite fundamental and quite intuitive: the higher the marginal product of capital relative to the rate of time preference, the more it pays to depress the current level of consumption in order to enjoy higher consumption later. Thus, if initially the marginal product of capital is high, consumption will be increasing over time on the optimal path. Equation (7″) shows the specific role of the elasticity of substitution in this condition: the larger this elasticity, the easier it is, in terms of utility, to forgo current consumption in order to increase consumption later, and thus the larger the rate of change of consumption for a given value of the excess of the marginal product over the subjective discount rate.

The Transversality Condition

Equation (8), the transversality condition, is best understood by considering the same maximization problem with the infinite horizon replaced by a finite horizon T. In this case, if $u'(c_T)\exp(-\theta T)$ were positive (i.e., if the present value of the marginal utility of terminal consumption were positive), it would not be optimal to end up at time T with a positive capital stock because it could, instead, be consumed.[7] The condition would be

$$k_T u'(c_T)\exp(-\theta T) = 0.$$

The infinite horizon transversality condition (TVC) can be thought of as the limit of this condition as T becomes large.[8]

Two Useful Special Cases

CRRA

Two instantaneous utility functions are frequently used in intertemporal optimizing models. The first is the constant elasticity of substitution, or isoelastic, function:[9]

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \qquad \text{for } \gamma > 0,\ \gamma \neq 1,$$

$$= \ln c, \qquad \text{for } \gamma = 1.$$

The basic economic property of this function is implied by its name. The elasticity of substitution between consumption at any two points in time, t and s, is constant and equal to $(1/\gamma)$. Thus, in equation (7″), σ is no longer a function of consumption. The elasticity of marginal utility is equal to $-\gamma$.

When this instantaneous utility function is used to describe attitudes toward risk, something we shall do later in the book when we allow for uncertainty, γ has an alternative interpretation. It is then also the coefficient of relative risk aversion, defined as $-u''(c)c/u'(c)$. Thus this function is also called the constant relative risk aversion (CRRA) utility function.[10]

Substantial empirical work has been devoted to estimating σ under the assumption that it is indeed constant, by looking at how willing consumers are to shift consumption across time in response to changes in interest rates. Estimates of σ vary substantially but usually lie around or below unity: the bulk of the empirical evidence suggests a relatively low value of the elasticity of substitution.

CARA

The second often used class of utility functions is the exponential, or constant absolute risk aversion (CARA), of the form

$$u(c) = -\left(\frac{1}{\alpha}\right)\exp(-\alpha c), \qquad \alpha > 0.$$

Under this specification the elasticity of marginal utility is equal to $-\alpha c$, and the instantaneous elasticity of substitution is equal to $(\alpha c)^{-1}$; thus σ is decreasing in the level of consumption.

When interpreted as describing attitudes toward risk, this function implies constant absolute risk aversion, with α being the coefficient of absolute risk aversion, $-u''(c)/u'(c)$. Constant absolute risk aversion is usually thought of as a less plausible description of risk aversion than constant relative risk aversion; the CARA specification is, however, sometimes analytically more convenient than the CRRA specification and thus also belongs to the standard tool kit.

For the CARA utility function, the Euler equation becomes

$$\frac{dc}{dt} = \alpha^{-1}[f'(k) - n - \theta]. \tag{7‴}$$

In this case the change in consumption is proportional to the excess of the marginal product of capital (net of population growth) over the discount rate.

Steady State and Dynamics

The optimal path is characterized by equations (7′), (8), and the constraint (2). We start with the steady state. In steady state both the per capita capital stock, k, and the level of consumption per capita, c, are constant. We denote the steady state values of these variables by k^* and c^*, respectively.

The Modified Golden Rule

From (7), with dc/dt equal to zero, we have the modified golden rule relationship:

$$f'(k^*) = \theta + n. \tag{11}$$

The marginal product of capital in steady state is equal to the sum of the rate of time preference and the growth rate of population. Corresponding to the optimal capital stock k^* is the steady state level of consumption, implied by (2):

$$c^* = f(k^*) - nk^*. \tag{12}$$

The *golden rule* itself is the condition $f'(k) = n$: this is the condition on the capital stock that maximizes *steady state* consumption per capita.[11] The modification in (11) is that the capital stock is reduced below the golden rule level by an amount that depends on the rate of time preference. Even though society or the family could consume more in a steady state with the golden rule capital stock, the impatience reflected in the rate of time preference means that it is not optimal to reduce current consumption in order to reach the higher golden rule consumption level.

The *modified golden rule* condition is a very powerful one: it implies that ultimately the productivity of capital, and thus the real interest rate,[12] is determined by the rate of time preference and n. Tastes and population growth determine the real interest rate ($\theta + n$), and technology then determines the capital stock and level of consumption consistent with that interest rate.[13] Later in the chapter we will explore the sensitivity of the modified golden rule result to the formulation of the utility function $u(\cdot)$ in (1).

Dynamics

To study dynamics, we use the phase diagram in figure 2.2, drawn in (k, c) space.[14] All points in the positive orthant are feasible, except for points on

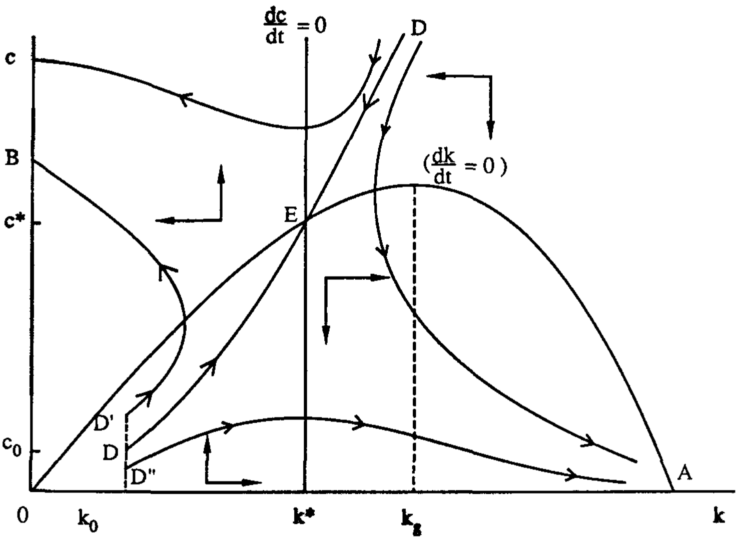

Figure 2.2
The dynamics of capital and consumption

the vertical axis above the origin: without capital (i.e., if $k = 0$), output is zero, and thus positive c is not feasible.

The locus $dk/dt = 0$ starts from the origin, reaches a maximum at the golden rule capital stock k_g at which $f'(k_g) = n$, and crosses the horizontal axis at point A where $f(k) = nk$. The $dc/dt = 0$ locus is, from (7′), vertical at the modified golden rule capital stock, k^*.

Anywhere above the $dk/dt = 0$ locus, the capital–labor ratio k is decreasing: consumption is above the level that would just maintain k constant (i.e., the level of c on the $dk/dt = 0$ curve.) Similarly, k is increasing at points below the $dk/dt = 0$ locus. In the case of the $dc/dt = 0$ locus, consumption is increasing to the left of the locus, where $f'(k) > \theta + n$, and decreasing to the right of the locus. The vertical arrows demonstrate these directions of motion.[15]

There are three equilibria, the origin, if $\sigma^{-1}(0)$ is different from zero (see note 15), point E, and point A. In appendix A we show that only the trajectory DD, the *saddle point path,* that converges to E satisfies the necessary conditions (2), (7′), and (8). On all other paths, either the Keynes-Ramsey condition eventually fails or the transversality condition is not satisfied.[16]

The central planner's solution to the optimizing problem (1) is fully summarized by the path DD. For each initial capital stock, this implies a unique initial level of consumption. For instance, with initial capital stock k_0, the optimal initial level of consumption is c_0. Convergence of c and k to c^* and k^* is monotonic. Note that in this certainty model the central planner knows at time 0 what the level of consumption and the capital stock will be at every moment in the future.

Local Behavior around the Steady State

Linearization of the dynamic system (2) and (7′) yields further insights into the dynamic behavior of the economy. Linearizing both equations in the neighborhood of the steady state gives

$$\frac{dc}{dt} = -\beta(k - k^*), \qquad \beta \equiv [-f''(k^*)c^*]\sigma(c^*) > 0, \tag{13}$$

and

$$\begin{aligned}\frac{dk}{dt} &= [f'(k^*) - n](k - k^*) - (c - c^*) \\ &= \theta(k - k^*) - (c - c^*).\end{aligned} \tag{14}$$

The solution to this system of linear differential equations is most easily found by reducing it to a single second-order equation in k. Differentiating (14) with respect to time, and using (13) to substitute for dc/dt, gives

$$\frac{d^2k}{dt^2} - \theta\left(\frac{dk}{dt}\right) - \beta k = -\beta k^*. \tag{15}$$

The roots of the characteristic equation associated with the second-order differential equation are $\theta \pm (\sqrt{\theta^2 + 4\beta})/2$. One root is positive and the other negative, implying the saddle point property: the presence of a positive root implies that for arbitrary initial conditions, the system explodes; for any given value of k_0, there is a unique value of dk/dt such that the system converges to the steady state (see appendix B).

Let λ be the negative, stable root. The solution for k_t such that, starting from k_0, the system converges to k^* is

$$k_t = k^* + (k_0 - k^*)\exp(\lambda t).$$

The speed of convergence is thus given by $|\lambda|$. In turn $|\lambda|$ is an increasing function of f'' and of σ, and a decreasing function of θ. The higher the elasticity of substitution, the more willing people are to accept low consumption early on in exchange for higher consumption later and the faster capital accumulates and the economy converges to the steady state.[17]