Problems and Solution for the GMM

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Outline

A simple example of GMM

OLS estimator is a MM estimator which both of them are the one step GMM

IV estimator is a MM estimator which both of them are the one step GMM

MLE estimator is a MM estimator which both of them are the one step GMM

Two step GMM for a Rational Expectations model

Question 1

Example

- Observations in sample **A** are assumed to have been drawn from a distribution with mean equal to μ.
- Observations in sample **B** are assumed to have been drawn from a distribution with a mean equal to $\mu + 5$.
- All observations are uncorrelated. The sample means are $\bar{y}_A = 7$ and $\bar{y}_A = 9$, and their variances are estimated to be $Var(\bar{y}_A) = 1$ and $Var(\bar{y}_B) = 0.2$.

Example (Cont.)

1. Write a quadratic form that is a function of μ and that is minimized at the asymptotically effient GMM estimate of μ.

2. Compute the numerical value of this GMM estimate.

3. Compute the value of the GMM test statistic for testing the assumption about the relationship between the means of samples **A** and **B**. State the asymptotic null distribution of this test statistic, including degrees of freedom.

Question 1: Solution

1. Write a quadratic form that is a function of μ and that is minimized at the asymptotically effient GMM estimate of μ.

$$J(\mu, y_A, y_B) = \begin{bmatrix} \bar{y}_A - \mu \\ \bar{y}_B - (\mu + 5) \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{y}_A - \mu \\ \bar{y}_B - (\mu + 5) \end{bmatrix}$$

$$J(\mu, y_A, y_B) = (7 - \mu)^2 + 5(4 - \mu)^2$$

   - Note: Variance of $(\bar{y} + \alpha)$ is $var(\bar{y}) + 0$, $\bar{m} = \bar{y}$ and $m = \mu$.

2. Compute the numerical value of this GMM estimate. Minimize $J(\mu, y_A, y_B)$ with respect to μ, where $J(.)$ is the formula from (1). Then

$$\frac{\partial(\mu, y_A, y_B)}{\partial \mu} = -2(7 - \mu) - 2(5)(4 - \mu) = 0$$

$$\hat{\mu} = 4.5$$

Question 1: Solution

1. Compute the value of the GMM test statistic for testing the assumption about relationship between the means of samples **A** and **B**. State the asymptotic null distribution of this test statistic, including degrees of freedom.

This statistic is obtained by substituting the GMM estimate of $\hat{\mu}$ into the quadratic form $J(\mu, y_A, y_B)$ that it minimizes,

$$J(\mu, y_A, y_B) = (2.5)^2 + 5(-0.5) = 7.5$$

Now one can pustulate the hypothesis that:

$$H_0 : E(y|B) = E(y|A) + 5$$

this hypothesis can be tested by the $J(\mu, y_A, y_B)$ statistic which is aymptotically distrebuted as χ_2 with 1 degree of freedom.

Question 2: Solution

Example (OLS estimator is the MM estimator which both of them are the one step GMM)

- Consider the simple regression model with no intercept term:

  $$y_t = \beta x_t + u_t$$

  show that the its GMM estimator is the same as OLS(or MM)
  Solution:
  - The quadratice form is
    $J(\beta, y, x) = [\sum x_t(y_t - x_t\beta)]' W [\sum x_t(y_t - x_t\beta)]$

    $$= W[\sum x_t(y_t - x_t\beta)]^2$$

    $$\frac{\partial J(.)}{\partial \beta} = -2W \sum x_t^2 \left[\sum x_t y_t - \sum x_t^2 \beta\right] = 0$$

- The same is true for $\boldsymbol{Y} = \boldsymbol{X}\beta + \boldsymbol{U}$, How?

Question 2: Solution

Example (OLS estimator is a MM estimator which both of them are the one step GMM)

- The same is true for $\boldsymbol{Y} = \boldsymbol{X}\beta\boldsymbol{)} + \boldsymbol{U}$, How?
  - The objective function of GMM for moment equations of the multivariate linear regression is:

$$J(\beta, y, x) = \left[\boldsymbol{X}'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right]' \boldsymbol{W} \left[\boldsymbol{X}'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right]$$

then the first order conditions for the minimization proiblem in vector form is:

$$\frac{\partial J(.)}{\partial \boldsymbol{\beta}} = -2\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}' \left[\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{X}'\boldsymbol{X}\beta\right] = \mathbf{0}$$

- The similar results are also proved for IV estimators as th GMM.

Question 3

Example (IV estimator is the MM estimator which both of them are the one step GMM)

- The similar results are also proved for IV estimators of $\boldsymbol{Y} = \boldsymbol{X}\beta) + \boldsymbol{U}$, How?
  - The objective function of GMM for moment equations of the multivariate linear regression with endogenous covariates is:

$$J(\beta, y, x) = \left[\boldsymbol{Z}'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right]' \boldsymbol{W} \left[\boldsymbol{Z}'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right]$$

    then the first order conditions for the minimization problem in vector form is:

$$\frac{\partial J(.)}{\partial \beta} = -2\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{W}' \left[\boldsymbol{Z}'\boldsymbol{Y} - \boldsymbol{Z}'\boldsymbol{X}\beta\right] = \mathbf{0}$$

    Where, the $\boldsymbol{W}$ is a $q \times q$ and symmetric matrix, why?
- The similar results are also proved for MLE estimators as th GMM.

Question 4

Example (MLE estimator is a MM estimator which both of them are the one step GMM)

- See the FOC for a log of likelihood function when our regression is of the form, $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}$

$$\partial lnL(\boldsymbol{\beta}, \sigma^2|\boldsymbol{w})/\partial\boldsymbol{\beta} = -2/\sigma^2(-\boldsymbol{X}'\boldsymbol{y} + \boldsymbol{X}'\boldsymbol{X}\beta)$$

- Write $J(\beta, y, x)$ function nad find the GMM estimator
  - FOC are the moment conditions and one can write them as:

$$\bar{\boldsymbol{m}}(\boldsymbol{X}, \boldsymbol{Y}, \beta) = 2/\sigma^2\boldsymbol{X}'(\boldsymbol{y} - \boldsymbol{X}\beta)$$

and

$$\boldsymbol{m}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\beta}) = \boldsymbol{0}$$

Question 4

Example (MLE estimator is a MM estimator which both of them are the one step GMM)

- Then,
  - the corresponding objective function for GMM is defined as:

$$J(\beta, y, x) = \left[2/\sigma^2 \boldsymbol{X}'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})\right]' \boldsymbol{W} \left[2/\sigma^2 \boldsymbol{X}'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})\right]$$

  - Now find the FOC for $J(\beta, y, x)$ and show that GMM gives the same estimator as MLE, provided that the disturbance term of regression holds CLR assumptions.

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

Long live GMM

Question 5

Example (Two step GMM for a Rational Expectations model)

- Imagin you are given the following model with 4 Instrumental variables, namely you are faced with 4 moment conditions:

$$E\left[\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+r_{t+1})-1\right)\boldsymbol{z_{1t}}\right]=0$$

$$\ldots$$

$$E\left[\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+r_{t+1})-1\right)\boldsymbol{z_{4t}}\right]=0$$

Question 5

Example (Two step GMM for a Rational Expectations model)

- Then their sample counterparts are:

$$\frac{1}{T}\sum_t \left[\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+r_{t+1})-1\right)\boldsymbol{z_{1t}}\right]$$

$$\ldots$$

$$\frac{1}{T}\sum_t \left[\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+r_{t+1})-1\right)\boldsymbol{z_{4t}}\right]$$

- Now find $\boldsymbol{W}$, $J(\beta, y, x)$, and the FOC for objective function of GMM estimators.
- Algebraic operations are long but as easy as eating a piece of cake.
- Rest of the way is starteforward, go ahead and do it.

Long live GMM