Differential Forms

Gauss law in differential form

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Consider an infinitely long cylinder with charge density given by:

$$\rho(r) = \begin{cases} Cr^2 & r \leq R \quad \text{inside} \\ 0 & r > R \quad \text{outside} \end{cases}$$

What is $\vec{E}$ at all points in space?

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

• Keep in mind that this is purly radian ($\hat{r}$)

$$\frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \cancel{\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}} = \frac{Cr}{\varepsilon_0}$$

we do not need this because it only depends on $\hat{r}$

• We want to solve for electric field

$$\frac{1}{r}\frac{\partial(rE_r)}{\partial r} = \frac{Cr^2}{\varepsilon_0} \Rightarrow \frac{\partial(rE_r)}{\partial r} = \frac{Cr^3}{\varepsilon_0} \Rightarrow \int \frac{\partial(rE_r)}{\partial r} = \int \frac{Cr^3}{\varepsilon_0}$$

$$rE_r = \frac{Cr^4}{4\varepsilon_0} \Rightarrow rE_r = \frac{Cr^4}{4\varepsilon_0} + A$$

A is just a constant like +C we use A because we already have C in equation

We will set the constant to zero because the electric field at the center of a uniformly charged cylinder is zero

$$\boxed{E_r = \frac{Cr^3}{4\varepsilon_0}}$$

Now we Find the eletric Field For $r > R$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r} \frac{\partial (r E_r)}{\partial r} = \frac{0}{\varepsilon_0} \implies \frac{\partial (r E_r)}{\partial r} = 0$$

• we move r to other side

but we need to integrate zero we will get a constant also we are integrating with respect with dr

Constant

$$\int \frac{\partial (r E_r)}{\partial r} = \int 0 \implies E_r = \frac{B}{r}$$

Notice the integration constant does not vanish; instead we need the electric Field to continous at the boundary $r = R$ when we say the electric Field is continous we mean the Field inside and Just outside that Point are equal $r = R$ usally there is No Surface Charge on that boundary

$$E_{inside} = E_{outside}$$

$$\frac{C r^3}{4 \varepsilon_0} = \frac{B}{R}$$

• we can replace r with R

$$\frac{C R^4}{4 \varepsilon_0} = B$$

$$\boxed{E = \frac{C R^4}{4 r \varepsilon_0}}$$

a infinitely long cylindrically symmetric charge distribution of radius R has the following relationship between density ρ and the radial coordinate r:

$$\rho(r) = \begin{cases} -\beta r & 0 \le r \le R \\ 0 & r > R \end{cases}$$

What are the units of β?

$$\rho(r) = -\beta r$$

$$\frac{C}{m^3} = -\beta\,(m)$$

$$\boxed{-\beta = \frac{C}{m^4}}$$

What is the charge Q contained in a region of the cylinder L?

Keep in mind ρ is not a constant you must solve the integral

$$Q = \frac{\rho}{\varepsilon_0}$$

$$Q = \int \frac{\rho\, dV}{\varepsilon_0} \Rightarrow Q = \int \frac{\rho\, 2\pi r L\, dr}{\varepsilon_0} \Rightarrow -\int_0^R \frac{\beta\, 2\pi r^2 L}{\varepsilon_0}\, dr$$

$$-\frac{2\pi L \beta}{\varepsilon_0} = \int_0^R r^2\, dr \Rightarrow -\frac{2\pi L \beta}{\varepsilon_0}\left[\frac{r^3}{3}\right]_0^R$$

$$\boxed{Q = \frac{-2\pi \beta L R^3}{3\varepsilon_0}}$$

the charge Q contained in the region L but we will integrate over radius R

Made with Goodnotes

Using gauss law in differential Form Find $\vec{E}$ in the region inside the cylinder

$r \leq R$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r}\frac{\partial(rE_r)}{\partial r} = \frac{-\beta r}{\varepsilon_0}$$

$$\frac{\partial(rE_r)}{\partial r} = \frac{-\beta r^2}{\varepsilon_0}$$

$$\int \frac{\partial(rE_r)}{\partial r} = -\frac{\beta}{\varepsilon_0}\int r^2\,dr$$

$$rE_r = \frac{-\beta r^3}{3\varepsilon_0}$$

$$E_r = \frac{-\beta r^2}{3\varepsilon_0} + C \qquad C = 0$$

$$\boxed{E_r = \frac{-\beta r^2}{3\varepsilon_0}}$$

So on the inside $r=0$ which means $E=0$ the reason why is Symmetry So $C=0$ on inside if we wanted to relate inside and outside Fields we can say $r=R$ because of Continuity what does Continuous or continuity mean? image Standing right on the Surface of a Charged Cylinder or Sphere if there is No Surface charge, the Field lines Pass Smootly through the boundary (E is continous) but if there is charge right on that Surface, Field lines get denser (E is discontinous)

Use gauss law in integral form to confirm you get the answer for the previous answer

$$\oint E\,dA = \frac{Q}{\varepsilon_0} \qquad Q = \rho\,dV \Rightarrow -\beta\, 2\pi \frac{x}{2} L\,dx$$

$$\oint E\,dA = \frac{1}{\varepsilon_0}\int -B\,2\pi x^2\,dr$$

✱ x is our dummy variable its only job is to be integrated so we can then place r into it

$$EA = \frac{-2\pi L B}{\varepsilon_0}\int_0^r x^2\,dx$$

$$E(2\pi r L) = \frac{-2\pi L B}{\varepsilon_0}\left[\frac{x^3}{3}\right]_0^r$$

$$E(\cancel{2\pi}\cancel{r}\cancel{L}) = \frac{-\cancel{2\pi}\cancel{L}\beta r^{\cancel{3}2}}{3\varepsilon_0}$$

$$E = \frac{-\beta r^2}{3\varepsilon_0}$$

Using gauss law in differential form find the electric field outside the cylinder also you need to solve for this using the electric field is continous at the boundry $r=R$

$r > R$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r}\frac{\partial(rE_r)}{\partial r} = \frac{0}{\varepsilon_0} \Rightarrow \frac{\partial(rE_r)}{\partial r} = 0$$

$$\int \frac{\partial(rE_r)}{\partial r} = \int 0$$

$$rE_r = C \Rightarrow E_r = \frac{C}{r}$$

$$E_{inside} = E_{outside}$$ when $r = R$

$$\frac{-Br^2}{3\varepsilon_0} = \frac{C}{r}$$ replace r with R

$$\frac{-BR^2}{3\varepsilon_0} = \frac{C}{R}$$

$$C = \frac{-BR^3}{3\varepsilon_0}$$

✱ Plug this back into C in the equation you got

$$E_r = \frac{C}{r} \Rightarrow \boxed{E = \frac{-BR^3}{3r\varepsilon_0}}$$

What is the electric Potential in the region inside the cylinder? Please Pick a guage in which V=0 on the outside the cylinder r=R

$$\boxed{V = \frac{\beta r^3}{9\varepsilon_0} - \frac{\beta R^3}{9\varepsilon_0}}$$

$$E = -\frac{dV}{dr}$$

$$\int \frac{\beta r^2}{3\varepsilon_0} dr = \int \frac{dV}{dr}$$

$$\frac{\beta}{3\varepsilon_0} \int r^2 dr = V$$

$$\frac{\beta r^3}{9\varepsilon_0} + C = V$$

• we were told to set V=0

$$0 = \frac{\beta r^2}{9\varepsilon_0} + C \Rightarrow C = \frac{-\beta R^2}{9\varepsilon_0}$$ (we said at V=0 r=R so replace r with R)

What is the electric Potential V in the region outside the cylinder, in the same gauage as in the last Problem

$$\boxed{\frac{\beta R^3}{3\varepsilon_0} \cdot \ln(r) - \frac{\beta R^3}{3\varepsilon_0} \ln(R) = V}$$

$$E = -\frac{dV}{dr}$$

$$\frac{-\beta R^3}{3r\varepsilon_0} = -\frac{dV}{dr}$$

$$\int \frac{\beta R^3}{3r\varepsilon_0} dr = \int \frac{dV}{dr}$$

☆ do Not Forget to Plug the C back into equation

$$\frac{\beta R^3}{3\varepsilon_0} \int \frac{1}{r} dr = V$$

$$\frac{\beta R^3}{3\varepsilon_0} \cdot \ln(r) + C = V$$

$$\frac{\beta R^3}{3\varepsilon_0} \ln(r) + C = 0 \Rightarrow C = -\frac{\beta R^3}{3\varepsilon_0} \ln(R)$$

(we said at V=0 r=R So replace r with R) a cylindricaly symmetric charge distribution where all charge is confined to be within an infinitiy long cylinder of radius R centerd around the origin

$$V(r) = \frac{\alpha^2 r^4}{40\,\varepsilon_0} \quad \text{For} \quad r \le R$$

What is the electric field inside the cylinder radius r?

$$E = -\frac{dv}{dr}$$

$$E = -\frac{d\left(\frac{\alpha^2 r^4}{40\varepsilon_0}\right)}{dr} \implies E = \frac{-\alpha^2}{40\,\varepsilon_0}\,\frac{d(r^4)}{dr}$$

$$E = \frac{-4r^3\alpha^2}{40\,\varepsilon_0} \implies \boxed{E = \frac{-r^3\alpha^3}{10\,\varepsilon_0}}$$

a spherically charged distribution has a charge density:

$$P(r) = \begin{cases} A\left(r + \frac{1}{R}\right) & r \le R \quad \text{inside} \\ 0 & r > R \quad \text{outside} \end{cases}$$

Find $\vec{E}$ and V assuming $V \to 0$ and $r \to \infty$

$$\nabla \cdot E = \frac{P}{\varepsilon_0}$$

$$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} = \frac{1}{\varepsilon_0} \cdot A\left(1 - \frac{r}{R}\right)$$

$$\frac{\partial(r^2 E_r)}{\partial r} = \frac{Ar^2}{\varepsilon_0} - \frac{Ar^3}{\varepsilon_0 R}$$

$$\int \frac{\partial(r^2 E_r)}{\partial r} = \int \frac{Ar^2}{\varepsilon_0} - \int \frac{Ar^3}{\varepsilon_0 R}$$

$$\int \frac{\partial (r^2 E_r)}{\partial r} = \int \frac{Ar^2}{\varepsilon_0} - \int \frac{Ar^3}{\varepsilon_0 R}$$

$$r^2 E_r = \frac{A}{\varepsilon_0} \int r^2 \, dr - \frac{A}{\varepsilon_0 R} \int r^3 \, dr$$

$$r^2 E_r = \frac{Ar^3}{3\varepsilon_0} - \frac{Ar^4}{4\varepsilon_0 R} + C$$

$$E = \frac{Ar}{3\varepsilon_0} - \frac{Ar^2}{4\varepsilon_0 R} + \frac{C}{r^2} \Rightarrow E_r = \frac{Ar}{3\varepsilon_0} - \frac{Ar^2}{4\varepsilon_0 R}$$

Now we observe that, by symmetry (remember the electric field will point directly inward or outward because of symmetry) the electric field must vanish at $r=0$ as there equal amounts of charges at equal distances from this point so we observe $C=0$

Now solve solve for the outside $r > R$

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r^2} \frac{\partial (r^2 E_r)}{\partial r} = \frac{0}{\varepsilon_0} \Rightarrow \frac{1}{r^2} \frac{\partial (r^2 E_r)}{\partial r} = 0$$

We will integrate the integral of 0 is just a constant move the r to the other side after taking the integral of the partial derivative (which has become a derivative due to it having only one variable) you should pull out a r^2

$$\frac{\partial (r^2 E_r)}{\partial r} = 0$$

$$\int \frac{\partial (r^2 E_r)}{\partial r} = \int 0$$

$$r^2 E_r = c$$

$$E_r = \frac{c}{r^2}$$

Notice since $r=0$ is outside our domain, the divergence at $r=0$ for nonzero c is Not a Problem we can solve for C by observing the absence of a surface charge so the electric field must be continous at all points in space so they must match at the boundary $r=R$

• we will set $r=R$

$$\frac{Ar}{3\varepsilon_0} - \frac{Ar^2}{4\varepsilon_0 R} = \frac{c}{r^2}$$

$$\frac{AR}{3\varepsilon_0} - \frac{AR^{\cancel{2}}}{4\varepsilon_0 \cancel{R}} = \frac{c}{R^2}$$

$$\frac{AR^3}{3\varepsilon_0} - \frac{AR^2}{4\varepsilon_0} = c$$

$$\boxed{E_r = \frac{AR^3}{3\varepsilon_0 r^2} - \frac{AR^2}{4\varepsilon_0 r^2}}$$

$$\vec{E} = \begin{cases} 0 & \text{if } 0 \le r < a \\ \frac{x}{r}\hat{r} & \text{if } a \le r \le 2a \\ \frac{2xa}{r^2}\hat{r} & \text{if } 2a < \end{cases}$$

units of x

$E = \frac{x}{r}$

$\frac{n}{c} = \frac{x}{m} \Rightarrow \boxed{x = \frac{n \cdot m}{c}}$

is there surface charge on $r=a$ if so what is the total surface charge Q_a and σ_a

$$\sigma_a = \frac{Q}{A} = \frac{\cancel{4\pi} x \cancel{a} \varepsilon_0}{\cancel{4\pi} a^{\cancel{2}}} = \frac{x\varepsilon_0}{a}$$

$$\boxed{\sigma_b = \frac{x\varepsilon_0}{a}}$$

$$\oint E \, dA = \frac{Q}{\varepsilon_0}$$

$$E(4\pi a^2) = \frac{Q}{\varepsilon_0}$$

• here σ is calculating in terms of area

$$\left(\frac{x}{\cancel{a}}\right)(4\pi a^{\cancel{2}}) = \frac{Q}{\varepsilon_0}$$

$$\boxed{Q = 4\pi x a \varepsilon_0}$$

what is the charge density ρ in the region $a < r < 2a$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

• here the ρ is calculating in terms of volume

$$\frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{\rho}{\varepsilon_0}$$

$$\boxed{\frac{x}{a^2} \cdot \varepsilon_0 = \rho}$$

$$\frac{1}{r^2}\frac{\partial(r^{\cancel{2}} \frac{x}{\cancel{r}})}{\partial r} = \frac{\rho}{\varepsilon_0}$$

$$\frac{x}{r^2}\frac{\partial(r)}{\partial r} = \frac{\rho}{\varepsilon_0}$$

• Sphere

$$V(r) = \begin{cases} Ae^{-r/R} & \text{if } r \leq R \quad \text{inside} \\ \dfrac{AR}{er} & \text{if } r > R \quad \text{outside} \end{cases}$$

What is the electric Field?

$$E = -\frac{dV}{dr} \Rightarrow E = -\frac{d(Ae^{-r/R})}{dr} \Rightarrow E = -A\frac{d(e^{-r/R})}{dr}$$

$$\boxed{E = \frac{Ae^{-r/R}}{R}}$$

• inside

$$E = -\frac{dV}{dr} \Rightarrow E = -\frac{d\left(\frac{AR}{er}\right)}{dr} \Rightarrow E = -\frac{AR}{e}\frac{d(1/r)}{dr}$$

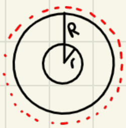
• outside

$$\boxed{E = \frac{AR}{er^2}}$$

What is the total charge contained by the sphere of radius R centerd at the origin

$$\oint E\,dA = \frac{Q}{\varepsilon_0} \Rightarrow EA = \frac{Q}{\varepsilon_0} \Rightarrow \frac{AR}{e\cancel{R^2}} \cdot (4\pi \cancel{R^2}) = \frac{Q}{\varepsilon_0}$$

$$\boxed{Q = AR4\pi e^{-1}\varepsilon_0}$$



• this is going to be our gaussian surface we want it to be @ $r = R$ because we get the surface and and Volume

What is the charge density $\rho(r)$ in the region for $r<R$

$$\vec{\nabla}\cdot E = \frac{\rho}{\varepsilon_0} \implies \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r^2}\frac{\partial\left(r^2\cdot\frac{Ae^{-r/R}}{R}\right)}{\partial r} = \frac{\rho}{\varepsilon_0} \implies \frac{A}{r^2 R}\frac{\partial(r^2\cdot e^{-r/R})}{\partial r} = \frac{\rho}{\varepsilon_0}$$

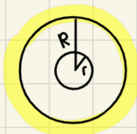
$$\frac{df}{dr}\cdot g + f\cdot\frac{dg}{dr} \qquad f=r^2 \quad \frac{df}{dr}=2r \qquad g=e^{-r/R} \quad \frac{dg}{dr}=\frac{e^{-r/R}}{R}$$

$$\boxed{\rho = \frac{A}{r^2 R}\left(2re^{-r/R} + \frac{r^2 e^{-r/R}}{R}\right)}$$

$$(2r)(e^{-r/R}) + (r^2)\left(\frac{e^{-r/R}}{R}\right)$$

What is the uniform surface density σ on the surface of radius R?

$$\sigma = \frac{Q}{A} = \frac{A\cancel{R}\,\cancel{4\pi}\,e^{-1}\varepsilon_0}{\cancel{4\pi}\,R^{\cancel{2}}} = \boxed{\frac{A\varepsilon_0}{Re}}$$



• thats the surface charge they are looking for is there charge on the surface $r = 2a$ if so what is Q_a and σ_a

Notice the only place r is equal to $2a$ is $a \leq r \leq 2a$ but now we must replace r with $2a$

$$\oint E\, dA = \frac{Q}{\varepsilon_0}$$

$$E\, dA = \frac{Q}{\varepsilon_0}$$

$$\frac{x}{\cancel{2}\cancel{a}}\left(\overset{2}{\cancel{4}}\pi a^{\cancel{2}}\right) = \frac{Q}{\varepsilon_0}$$

$$\boxed{\frac{2xa\pi}{\varepsilon_0} = Q}$$

Made with Goodnotes

$E(r) = \beta r^3 \hat{r}$ For $r \leq R$

Find the electric Potential inside the Cylinder

$$E = -\frac{dv}{dr}$$

$$\int \beta r^3 \hat{r} = \int -\frac{dv}{dr}$$

$$-\frac{\beta r^4}{4} + C = V$$

$$-\frac{\beta r^4}{4} = V$$

Problem 5 (40 points total): An insulating sphere of radius R is made from a dielectric material of dielectric constant K. Inside the sphere (i.e., for the region $r < R$), the electric potential $V(r)$ is given by

$$V(r) = -Ar^3 \tag{2}$$

where A is a positive constant. Outside the sphere is vacuum.

a). (10 points) What is the electric field $\vec{E}$ as a function of r inside the sphere of radius R?

• insulating

$$E = -\frac{dV}{dr} \Rightarrow E = -\frac{d(-Ar^3)}{dr}$$

$$\vec{E} = A\frac{d(r^3)}{dr} \Rightarrow \boxed{E = 3r^2 A}$$

What is the Free charge density ρ_f inside the sphere of radius r?

$$\vec{D} = K\varepsilon_0 \vec{E} \Rightarrow D = K\varepsilon_0 (3Ar^2)$$

$$\vec{\nabla} \cdot D = \rho_f \Rightarrow \frac{1}{r^2}\frac{d(r^2 D_r)}{\partial r} = \rho_f$$

$$\frac{1}{r^2}\frac{d(r^2 K\varepsilon_0 3Ar^2)}{dr} = \rho_f \Rightarrow \frac{K\varepsilon_0 3A}{r^2}\frac{d(r^4)}{dr} = \rho_f$$

$$\rho_f = \frac{12K\varepsilon_0 A r^{\cancel{3}}}{\cancel{r^2}} \Rightarrow \boxed{\rho_f = 12K\varepsilon_0 Ar}$$

Find the total bound charge Q_b contained within the sphere

$$-\rho_f\left(1-\frac{1}{K}\right)=\rho_b$$

$$-12K\varepsilon_0 A r\left(1-\frac{1}{K}\right)=\rho_b$$

$$Q_b=\int_0^R \rho_b \, dV \implies \int_0^R -12K\varepsilon_0 r A\left(1-\frac{1}{K}\right)\cdot 4\pi r^2 dr$$

$$-48\pi K\varepsilon_0 A\left(1-\frac{1}{K}\right)\int_0^R r^3 dr \implies -48\pi K\varepsilon_0 A\left(1-\frac{1}{K}\right)\left[\frac{r^4}{4}\right]_0^R$$

$$\boxed{Q_b = -12\pi K\varepsilon_0 A R^4\left(1-\frac{1}{K}\right)}$$

What is the bound surface charge density on the surface of the dielectric sphere?

$$Q_b^{volume} = -Q_b^{surface}$$

$$-12\pi K\varepsilon_0 \cancel{A} R^4\left(1-\frac{1}{K}\right)=\sigma_b \cancel{A}$$

$$\boxed{\sigma_b = 12\pi K\varepsilon_0 R^4\left(1-\frac{1}{K}\right)}$$

an insulating sphere of charge has radius R and a charge density ρ as a function of position given by:

$$\rho(r) = \frac{\alpha}{r^2}$$

in Si, what are the units of α?

$$\frac{C}{m^2} = \frac{1}{m^2} \qquad \boxed{\alpha = \frac{C}{m}}$$

What is the electric field in the region inside the sphere?

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \implies \frac{1}{\cancel{r^2}} \frac{d(r^2 E_r)}{dr} = \frac{\alpha}{\varepsilon_0 \cancel{r^2}}$$

the full sphere has radius R but the gaussian surface we are drawing is inside the sphere has radius

$$\int \frac{\partial (r^2 E_r)}{\partial r} = \int_0^r \frac{\alpha}{\varepsilon_0} \, dr'$$

$$r^2 E_r = \frac{\alpha r}{\varepsilon_0}$$

$$\boxed{E = \frac{\alpha}{\varepsilon_0 r}}$$

What is the electric potential V in the region inside the sphere, assuming a gauage where V=0 on the surface of the sphere?

$$E = -\frac{dV}{dr} \implies \int \frac{\alpha}{\varepsilon_0 r} \, dr = \int -\frac{dV}{dr}$$

• V=0 at r=R

$$-\frac{\alpha}{\varepsilon_0} LN(r) + C = V$$

$$+\frac{\alpha}{\varepsilon_0} LN(R) = C$$

$$V = \frac{\alpha}{\varepsilon_0} LN(R) - \frac{\alpha}{\varepsilon_0} LN(r) = \boxed{\frac{\alpha}{\varepsilon_0} LN\left(\frac{r}{R}\right) = V}$$

What is the self energy of charge

$$U = \frac{1}{2}\int D \cdot E \, dr \implies D = \varepsilon_0 K \vec{E} = \frac{\alpha}{r} \qquad E = \frac{\alpha}{\varepsilon_0 r}$$

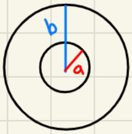
$$\frac{1}{2}\int_0^R \frac{\alpha}{r} \cdot \frac{\alpha}{\varepsilon_0 r} 4\pi r^2 dr \implies \frac{2\alpha^2 \pi}{\varepsilon_0}\int_0^R \frac{1}{\cancel{r^2}} \cancel{r^2} dr$$

$$\boxed{U = \frac{2\alpha^2 \pi R}{\varepsilon_0}}$$

☆ We got rid of the dot product because the $\vec{D}$ and $\vec{E}$ are going in the radial direction when you take the dot product of two unit vectors we get zero an insulating hollow sphere of inner radius a and outer radius b is made from a dielectric of dielectric constant k that has a free charge ρ_f as a function of the radial coordinate r given by

$$\rho(r) = \frac{A(r-a)}{b^2}$$

What is the electric flux density $\vec{D}$ in the region $a<r<b$



$$\vec{\nabla}\cdot D = \rho_f$$

$$\frac{1}{r^2}\frac{\partial(r^2 D_r)}{dr} = \frac{A(r-a)}{b^2}$$

$$\int \frac{\partial(r^2 D_r)}{\partial r} = \int \frac{A(r^3 - ar^2)}{b^2}$$

$$r^2 D_r = \frac{A}{b^2}\int r^3 - ar^2$$

$$D_r = \frac{A}{b^2 r^2}\left[\frac{r^4}{4} - \frac{ar^3}{3}\right]$$

$$D_r = \frac{A}{b^2}\left[\frac{r^2}{4} - \frac{ar}{3}\right] + \frac{C}{r^2}$$

$$0 = \frac{A}{b^2}\left[\frac{a^2}{4} - \frac{a^2}{3}\right] + \frac{C}{a^2}$$

$$0 = \frac{Aa^2}{b^2}\left[\frac{1}{4} - \frac{1}{3}\right] + \frac{C}{a^2}$$

$$-\frac{Aa^2}{b^2}\left[\frac{1}{4} - \frac{1}{3}\right] = \frac{C}{a^2}$$

$$C = \frac{Aa^4}{12 b^2}$$

$$\boxed{D_r = \frac{A}{b^2}\left[\frac{r^2}{4} - \frac{ar}{3} + \frac{a^4}{12 r^2}\right]}$$

We need to solve for C because we integrated from a to b while the inside of this sphere is hollow the constant C represents the "effect" of the hollow region or more precisley, it represents how the field behaves at the inner boundary @ r=a we also know the inside of the hollow sphere most be D=0 at r=a if we replaced that hollow space with a conducting sphere we would still need to see how the boundary if affected because a hollow sphere is just like a conducting one they both have $\vec{E}=0$ and $\vec{D}=0$ just the conducting spheres charges rise to the surface

• Dielectrics

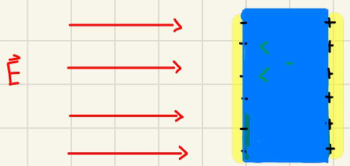
recall that the electric dipole Potential energy was minimized when dipole aligned with a Field (the dipole moment is Pointing in the Same direction as the electric Field)

by Placing non-Polar Molecules (a non-Polar molecule is one where the electrons are Shared or distributed evenly) in a electric Field you can incude a dipole moment and cause the Molecules to align with the electric Field with a dipole moment P Pointing in the direction of electric Field

Pointing in the most Favorable way the Positive charges are going with the electric Field While Negitive are going away So the $\vec{P}$ Points From negitive to Positive

if we were to Place this dipole in a electric Field the Positive Charge on the nucleus is going to Feel a Force that Pushes it in the direction of the electric Field and negitivly Charged cloud is going to experience a Force that Pushes it oppisite the direction of the electric Field and that will stretch it out

• notice if we have bulk material made of dipole Placed in a electric Field, the material will Polarize



this is the electric Field that the blue block is Placed in when this eletric Field goes through the blue material the Positive charges Follow this Field and the Negitive Charges go in oppisite

• these charges are just stuck here they only exist be we Put them in this electrical Field they are called bound Charges they are Not Free

even when the Positives and negitives are in different sides there still will be a electric Field that is Produced from the Positive charge to the negitive charges this Field will limit this electric Field its not canceling it Fully but Partially canceling it this is called a induced electric Field which means it was created by this electric Field our model of a dielectric is a model of a insulator its a because these charges do not move so that means the affective charge is only lying on the surface

• Polarization

- New Vector Field $\underline{\underline{\vec{P}}}$ (capital P) For Polarization

this tells us the net total dipole we will define it as:

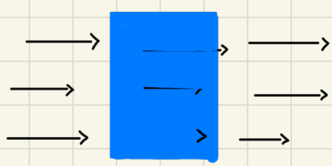
$$\underline{\underline{\vec{P}}} = n\,\vec{p}$$

Polarization

dipole moment

number density
number of dipoles per unit volume

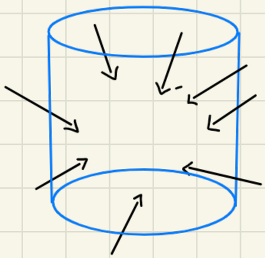
• consider a localized set of identical oriented dipoles



• the net bound charge that crosses a given surface of area dA (blue) is clearly the number of dipoles that strike the surface which by analogy to Flux is:

$$dQ = n\vec{p} \cdot dA \Rightarrow \underline{\underline{\vec{P}}} \cdot dA$$

the amount of charge striking this surface is given by this dot product



if we have this $\underline{\underline{\vec{P}}}$ Field Flowing inward that would mean we have Positive charge inside the surface and if my Flux was outward that would mean we would have negitive charge inside the surface

thus because the surface normal points outward, the bound charge left inside a volume with closed surface A is given by the negitive integral:

$$Q_{b,\text{enclosed}} = -\oint_A \underline{\underline{\vec{P}}} \cdot d\vec{A}$$

because the Flux being positive actually means that its Negitive charges inside the closed surface and the Flux being negitive is actually telling you that there's positive charges inside

bound

bound means not moving and staying in place

by definition, the bound charge density ρ_b is such that:

$$\int_V \rho_b \, dV = Q_{b,\text{enclosed}}$$

now applying the divergence therom:

$$\int_V \rho_b \, dV = Q_{b,\text{enclosed}} = -\oint_A \underline{\underline{\vec{P}}} \cdot d\vec{A} = -\int_V \nabla \cdot \underline{\underline{\vec{P}}} \, dV$$

Since the volume is arbitrary (the surface can be any surface, dosen't matter what surface i pick this has to be true everywhere), we may equate the integrands:

$$\rho_b = -\nabla \cdot \underline{\underline{\vec{P}}}$$

gauss law still applies in dielectrics

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

but now notice ρ includes both bound and free charges

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho_b + \rho_f}{\varepsilon_0}}$$

• any other charge would be a free charge for example: point charge, line of charge, or conducting net surface

We rearange it so we just get the free charge:

$$\nabla \cdot \varepsilon_0 \vec{E} = \rho_b + \rho_F$$

$$\rho_F = \nabla \cdot \varepsilon_0 \vec{E} - \rho_b$$

do keep in mind that we already took account for the ρ_b (bound charges) in the deravation for $\vec{D}$ thats why $\vec{D}$ only cares about the ρ_F (free charges)

We know that $\rho_b = -\nabla \cdot \underline{\underline{\vec{P}}}$

$$\rho_F = \nabla \cdot \varepsilon_0 \vec{E} + \nabla \cdot \underline{\underline{\vec{P}}}$$

Factor out ∇

$$\rho_F = \nabla \cdot (\varepsilon_0 \vec{E} + \underline{\underline{\vec{P}}})$$

this makes sense to define a new field:

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \underline{\underline{\vec{P}}}}$$

• always true vaild for any material (linear or non-linear)

this is called the electric flux density, and such that:

$$\boxed{\nabla \cdot \vec{D} = \rho_F}$$

• the dielectric constant

- as we noted previously, in most material the polarization is in the direction of the electric field and proportional in strength of the field

$$\underline{\vec{P}} = \varepsilon_0 (K-1)\vec{E}$$

• only for linear materials

for some proportionality constant K

K is unitless

K must be $K \geq 1$

K is a property of whatever the material is in other words it depends on the material

We know that:

$$\vec{D} = \varepsilon_0 \vec{E} + \underline{\vec{P}}$$

• we know that $\underline{\vec{P}}$ is $\varepsilon_0 (K-1)\vec{E}$ we can plug it in for $\underline{\vec{P}}$

$$\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 (K-1)\vec{E} \Rightarrow \vec{D} = \varepsilon_0 K \vec{E}$$

$$\vec{D} = \varepsilon_0 K \vec{E}$$

this equation only applies to linear

☆ this has K in it which means it only relates to the free charge because of bound charges is already built into K

• Permittivity

$$\varepsilon_0 K = \varepsilon$$

this is Permittivity

• observe that since vacuum corresponds to no Polarization, $K_{vac} = 1$ and

$$\varepsilon_0 = \varepsilon$$

hence the ε_0 is called Vacuum Permittivity

if $K=1$ we get $\vec{\underline{P}} = 0$

$$\vec{\underline{P}} = \varepsilon_0 (K-1)\vec{E} \implies \vec{\underline{P}} = \varepsilon_0 (1-1)\vec{E} = 0$$

which would mean there are no dipole to Polarize

• notice by our usual argument, gauss law in dielectrics can also be written as an integral

$$\boxed{\oint_A \vec{D} \cdot d\vec{A} = Q_{F,\text{enclosed}}}$$

• always true even when:
- non-linear
- linear

★ only counts free charges

$$\oint_A \vec{E} \cdot d\vec{A} = \frac{Q_{f,\text{enclosed}}}{K\varepsilon_0}$$

• only on linear
• needs symmetry

$$\vec{D} = K\varepsilon_0\vec{E} \implies \vec{E} = \frac{D}{K\varepsilon_0}$$

• we can take this and Plug it in

$$\vec{D} = \varepsilon_0\vec{E} + \vec{\underline{\underline{P}}}$$

$$\vec{D} = \cancel{\varepsilon_0}\left(\frac{D}{\cancel{\varepsilon_0} K}\right) + \vec{\underline{\underline{P}}}$$

$$\vec{D} - \frac{\vec{D}}{K} = \vec{\underline{\underline{P}}}$$

$$\vec{D}\left(1 - \frac{1}{K}\right) = \vec{\underline{\underline{P}}}$$

• we multiply both sides with ∇

$$\nabla\vec{D}\left(1 - \frac{1}{K}\right) = \nabla\cdot\vec{\underline{\underline{P}}}$$

• and we use this relationship $\nabla\cdot\vec{D} = \rho_f$ and $-\nabla\cdot\vec{\underline{\underline{P}}} = \rho_b$

$$\rho_f\left(1 - \frac{1}{K}\right) = -\rho_b$$

$$\boxed{-\rho_f\left(1 - \frac{1}{K}\right) = \rho_b}$$

• this equation tells us if we know either ρ_f or ρ_b we can get the other equation

☆ if $K=1$ then $\rho_b = 0$

explation:

at $K=1$ $\quad P = (K-1)\varepsilon_0\vec{E} = (1-1)\varepsilon_0\vec{E} = 0$

So if there is No Polarization then there is No bound charges but on the other hand you can have Free charges even if there's No Polarization

• energy

$U = \frac{1}{2} \vec{D} \cdot \vec{E}$ and $\frac{1}{2} K \varepsilon_0 \vec{E} \cdot \vec{E} \Rightarrow \frac{1}{2} K \varepsilon_0 E^2$

non-linear

linear

or you can do

$$U = \frac{1}{2} \int \rho(r) V(r) \, dr$$

$\rho(r) =$ roh or $\rho = \frac{Q}{V}$

$V(r) =$ electric Potential

$$
\begin{tikzcd}
 & Q \arrow[r, leftrightarrow, "Q = \int \rho\, dv"] \arrow[d, leftrightarrow, "Q = Q_f + Q_b"'] & \rho \arrow[r, leftrightarrow, "\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}"] \arrow[d, leftrightarrow, "\rho = \rho_b + \rho_f"] & E \arrow[r, leftrightarrow, "\vec{E} = -\nabla V"] \arrow[d, leftrightarrow, "\vec{D} = K\varepsilon_0 \vec{E}"] & V \\
 & Q_F \arrow[r, leftrightarrow, "Q_F = \int \rho_F\, dV"] & \rho_F \arrow[r, leftrightarrow, "\nabla \cdot \vec{D} = \rho_f"] \arrow[d, leftrightarrow, "\rho_b = -\rho_F\left(1 - \frac{1}{K}\right)"] & D \arrow[d, dash, "\vec{D} = \varepsilon_0 \vec{E} + \vec{P}"] & \\
Q_{surface} \arrow[r, leftrightarrow, "Q_{surface} = -Q_{bulk}"] \arrow[d, leftrightarrow] & Q_{bulk} \arrow[r, leftrightarrow, "Q_{bulk} = \int \rho_b\, dV"] & \rho_b \arrow[r, leftrightarrow, "\rho_b = -\nabla \cdot \vec{P}"'] & \underline{\underline{\vec{P}}} & \\
\sigma_b = \dfrac{Q_b \text{ surface}}{\text{Area}} & & & &
\end{tikzcd}
$$

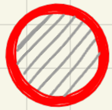
- $E = -\frac{dV}{dr}$
  - when you are given electric potential and told to find $\vec{E}$ you will use this equation and plug the equation into V in the dervative and then take dervative in terms of r
  - when you are given E and asked to find the potential energy you will plug the equation for E into the equation and integrate

if it has a dot you will need vector calc examples:

$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$, $\nabla \cdot \vec{D} = \rho_F$, $\rho_b = -\nabla \cdot \underline{\underline{\vec{P}}}$

• extra stuff you might need

• What Jacobian Factor to use



area $= 4\pi r^2$

thickness $= dr$

- if you have a sphere

  $dV = 4\pi r^2 dr$

- if you have a ring

  $d\lambda = 2\pi x \, dx$

- if you have a cylinder

  $dV = 2\pi r h \, dr$

You might need to change h to L

$$Q = \int_{\text{Volume}} \rho(r)\, dV \quad , \quad Q = \int_{\text{Surface}} \sigma(r)\, dA \quad , \quad Q = \int_{\text{length}} \lambda(r)\, dx$$

$\rho = C/m^3$ $\qquad$ $\sigma = C/m^3$ $\qquad$ $\lambda = C/m$