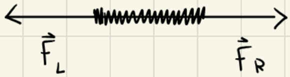
Waves and the wave equation

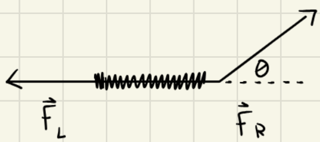
- Deriving the wave equation

Consider a differential segment of string of mass dm and length dx under tension



$$|\vec{F}| = T = |\vec{F}_R|$$

now consider what happens when one end of the segment is displaced in the vertical direction



$$|\vec{F}| = T = |\vec{F}_R|$$

Since θ is small we can use leading order taylor approximations in the x and y direction

- $F_x \cos\theta$
- $F_y \sin\theta$

Since the displacment is small, $\theta < 1$ we can use small angel approximations

$$\cos\theta = 1 \quad \text{and} \quad \sin\theta = \theta$$

the horizontal (x) components of forces are:

$$F_{net\,x} = F_R \cos\theta - F_L \cos(0)$$

- on the left side its not being lifted so its just zero

- **left Side**

So we can say if $\cos(0)=1$ then $F_L\cos(0)=T$ the Force is Just tension

- **right Side**

We can say $F_r\cos\theta \Rightarrow T\cos(\theta)$ because the Force is Just tension

applying small angel approximations

$$\cos\theta = 1$$

$F_{net_x} = T\cos\theta - T$ but we said $\cos\theta$ is equal to one so:

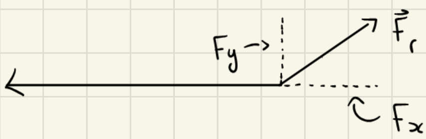
$F_{net_x} = T(1) - T = 0$

- why this makes sense
  - horizontally, the string doesn't accelerate so tension balances out

the vertical component is:

$$F_y = T\sin\theta$$

the vertical force is contributed by the right end



the net vertical force is

$$F_{net_y} = T\sin\theta_R - T\sin\theta_L$$

applying small-angel approximation

$$\sin\theta \approx \tan\theta$$

- this happens because $\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$ when $\cos\theta = 1$ and this occurs when applying small angel approximations

$$F_{net\,y} = T\tan\theta - T\cancel{\sin(0)}^{\to 0}$$

✫ on the left side its not being lifted so the degree is zero

$$F_{net\,y} = T\tan\theta$$

newtons second law says

$$F_{net\,y} = ma_y$$

For our segment $\mu\,dx$ where μ is the linear mass density

$$\mu = \frac{\text{mass of string}}{\text{length of string}} \quad \frac{kg}{m}$$

Vertical acceleration: $a_y = \frac{\partial^2 y}{\partial t^2}$

So we can write:

✫ acceleration is equal to $a = \frac{F}{m}$ or $\frac{F_{net}}{dm}$

$$|\vec{a}| = a_y = \frac{\partial^2 y}{\partial t^2} \approx \frac{F_{net\,y}}{\partial m} = \frac{T\tan\theta}{\mu\,dx}$$

✫ $\approx$ means approximation

geometrically $\tan\theta$ is the slope of the string

$$\tan\theta = \frac{\text{rise}}{\text{run}} = d\left(\frac{dy}{dx}\right)$$

- the d is for differential

So replace $\tan\theta$ with $d\left(\frac{dy}{dx}\right)$

$$\frac{d^2y}{dt^2} = \frac{T\tan\theta}{\mu\, dx} \Rightarrow \frac{T}{\mu} \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

So now we have:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \cdot \frac{\partial^2 y}{\partial x^2}$$

but if we analyze the units of T/μ we will realize that its the same as velocity but squared

$$\frac{T}{\mu} = \frac{\cancel{kg} \cdot m \cdot s^{-2}}{\cancel{kg} \cdot m^{-1}} = \frac{m^2}{s^2} = v^2$$

$$v^2 = \frac{T}{\mu} \Rightarrow v = \sqrt{\frac{T}{\mu}}$$

this is the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

- but the wave equation itself is not the solution but it is just a diffrential equation that describes how the wave behaves

- think of the wave equation like Newtons second law ($F=ma$) its not going to answer motion by itself

The Solution to the wave equation is:

$$y(x,t) = F(kx \pm \omega t)$$

Where f is any arbirtary Function and k and ω are constants

applying Partial dervatives

- Partial dervatives treat everything like a constant except the wanted variable

$$\frac{dy}{dt} = \omega f'(kx \pm \omega t) \Rightarrow \frac{d^2y}{dt^2} = \omega^2 F''(kx \pm \omega t)$$

$$\frac{dy}{dx} = kf'(kx \pm \omega t) \Rightarrow \frac{d^2y}{dt^2} = k^2 F''(kx \pm \omega t)$$

Plug these into the wave equation

$$V^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$$

$$V^2(k^2 \cancel{F''(kx \pm} \omega t) = \omega^2(\cancel{F''(kx \pm} \omega t))$$

$$\sqrt{V^2 k^2} = \sqrt{\omega^2}$$

$$Vk = \omega$$

We now have:

$$V = \frac{\omega}{k}$$

this equation is the Phase Velocity of a wave

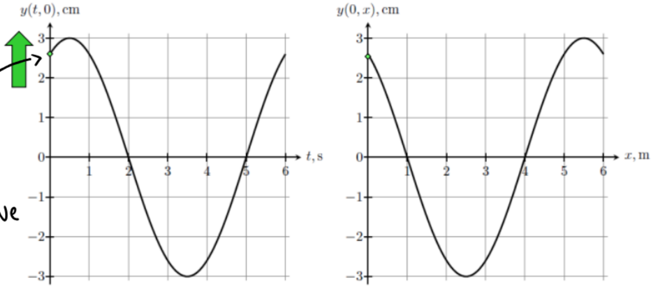
☆ You might want this: a waves velocity is its wave divided by its Period

left Moving Vs right Moving

left moving wave = $Kx + wt$

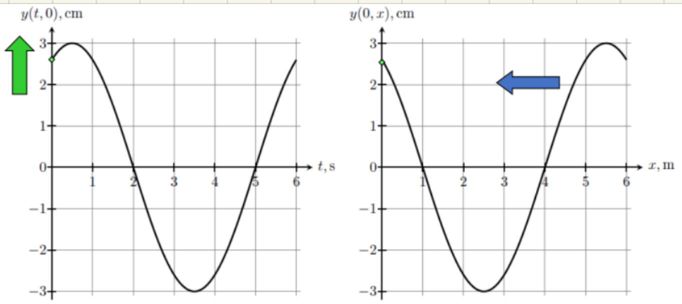
right Moving wave = $Kx - wt$

this is the Point both graphs have in common

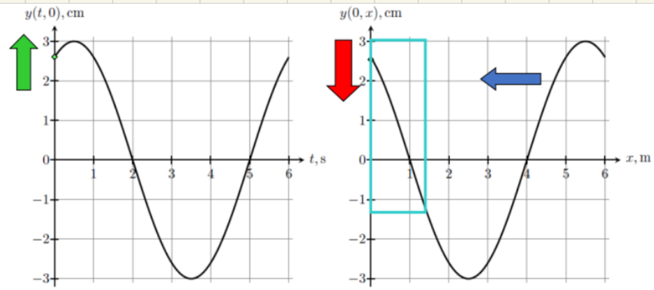


Now we look at the y vs. t graph, since time always increases as the wave travels. We identify if the point in common increases or decreases with time. Here, the point in common is **increasing.**

after we Found the Point in common we look at the y(t) graph and we notice that Point will be increasing ☆ For the y(t) graph we will imagine the Point is moving while For the y(x) graph the Point will stay Still but you will imagine the wave Moving either left or right So here we see on the y(t) graph if we move the Point it will be increasing and if we take a look at y(x) graph imagine the Point being Still and move the wave left but doing that will cause the Point to go downward if the wave traveld left which does Not Match our y(t) but if we move the wave to the right the Point in common would Match our y(t) graph



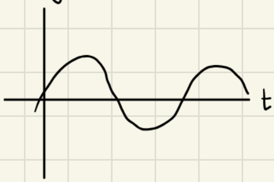
We now look at the y vs. x graph and find the way the function must shift to match the y vs. t behavior. For now, we will do this by guess and test. Assume the wave moves to the left, as indicated.



Notice if the wave moves left, the future of the point in common is what is boxed in **blue**. This would result in the point traveling **downward** if the wave traveled to the left, which does not match our *y* vs. *t* graph.

think of these graphs as screenshots

$y(t)$ at $x = -5$cm



- this is the wave at $x = -5$cm this is kind of a screenshot showing you how time is changing at $x = -5$cm

$y(x)$, at $t = 0$ sec

- this is the wave at $t = 0$ this is also kind of a screenshot showing you how position is changing at $t = 0$ sec

the Sinusoidal Solution

in this class we will assume a wave equation in Form of a Sinusoid

$$y(x,t) = A\sin(Kx \pm \omega t + \phi_0) + y_0$$

So everytime we want to analyze a wave we will use the Solution with a Sine Function in it

- A is termed the amplitude it Parameterizes the Size of a wave
- K is termed the wave number it relates to the wavelength(λ) by:

$$K = \frac{2\pi}{\lambda}$$

- ω is termed angular Frequency it relates to the Period (T) by:

$$\omega = \frac{2\pi}{T}$$

- ϕ_0 is termed the Phase constant it Shifts the entire Function left and right
- y_0 is termed the equilibrium Position it Shifts the entire Function up and down

recall that to obey the wave equation

$$V = \frac{\omega}{K} \Rightarrow V = \frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} = \frac{\lambda}{T}$$

So $\boxed{V = \frac{\omega}{K}}$ is equal $\boxed{V = \frac{\lambda}{T}}$

Furthermore, since $-1 \leq \sin\theta \leq 1$

$\sin\theta$ can only intake a number between -1 and 1

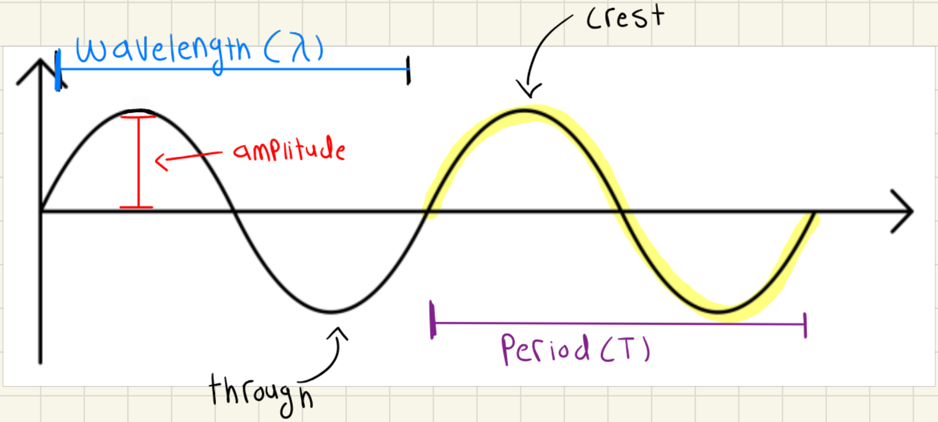
So that means:

$$y_{min} = y_0 - A$$

$$y_{max} = y_0 + A$$

When you given two graphs $y(x)$ and $y(t)$ you can find amplitude from either one

☆ the period can only be read from the $y(t)$, period is talking about time so it would make sense to get it from $y(t)$ graph while the wavelength comes from the $y(x)$ graph the wavelength is talking about the length of the wave so getting it from $y(x)$ makes sense



one Full cycle

- the Period Can only be gotten From y(t) graph
- the wavelength Can only be read From the y(x) graph

Summary of equations

$\mu = \dfrac{\text{mass of string}}{\text{length of string}}$ $\dfrac{kg}{m}$

$y_{min} = y_0 - A$

$y_{max} = y_0 + A$

$V = \sqrt{\dfrac{T}{\mu}}$ $\dfrac{m}{s}$ • wave speed on a stretched string

$V = \dfrac{\omega}{k}$ $\dfrac{m}{s}$ or $V = \dfrac{\lambda}{T}$ $\dfrac{m}{s}$

$y(x,t) = A\sin(kx \pm \omega t + \phi_0) + y_0$

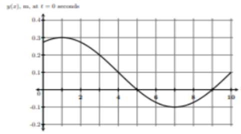
$k = \dfrac{2\pi}{\lambda}$ m^{-1}

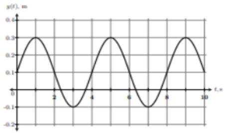
$\omega = \dfrac{2\pi}{T}$ s^{-1}

1.1 Class 1: Waves and the Wave Equation

1.1.1 Finding the Parameters

The picture below to the left gives a snapshot of an example one dimensional sinusoidal wave on a 10 m long string at an instant in time $t = 0$ seconds. Below to the right is a graph of how the wave oscillates in time at position $x = 4$ meters.





In this activity, we will practice using graphs to determine the parameters for a 1D sinusoidal wave of the form

$$Y(x,t) = A \sin(kx \pm \omega t + \phi_0) + y_0 \tag{1.1}$$

for ω and k such that

$$v = \frac{\omega}{k} \tag{1.2}$$

1. What are A, the amplitude, and y_0, the equilibrium position, of this 1D wave?

   These can be read off either graph as $A = 0.2$ m and $y_0 = 0.1$ m, respectively.

2. What is λ for the wave pictured above? What, therefore, is k? Hint: make sure you include units for both λ and k!

   Notice that (from the graph on the left) $\frac{\lambda}{2}$ is 7 m -1 m $= 6$ m. Hence $\lambda = 12$ m and

   $$k = \frac{2\pi}{12\text{ m}} = \frac{\pi}{6}\text{ m}^{-1} \approx 0.522\text{ m}^{-1}$$

3. For this one dimension wave, what is T? What is ω? Again, don't forget to include units.

   T can be read off the graph on the right as 4 seconds. Hence

   $$\omega = \frac{2\pi}{4\text{ s}} = \frac{\pi}{2}\text{ s}^{-1} \approx 1.57\text{ s}^{-1}$$

4. What is the speed of this one-dimensional wave?

   Notice by modifying equation 1.2, we have:

   $$v = \frac{\omega}{k} = \frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} = \frac{\lambda}{T} = \frac{12\text{ m}}{4\text{ s}} = 3\text{ m/s}$$

5. To find the direction the wave is traveling, we first must identify the point in common between the $y(x)$ and $y(t)$ graphs, as this is the only point in space and time we have multiple pieces of information about. What is the only point (x,t) in space and time that appears on both graphs pictured above?

   This is the point $x = 4$ m and $t = 0$ seconds, since all points on the left graph are at $t = 0$ s, and all points on the right graph are at $x = 4$ m.

6. Once we have identified the point in common, we look at the $y(t)$ graph. As time moves forward, will the point in common move upward or downward? Now look at the point in common on the $y(x)$ graph. In order to move the correct direction in time, should this wave be traveling to the right or to the left?

   At time $t = 0$, the graph $y(t)$ is moving upward. In order for the graph of $y(x)$ at $x = 4$ to move upward, we need the portion of the wave lying to the left of $x = 4$ to be the future of this point. Hence the wave must move to the right.

7. Based on your answer to 6, do you want to use a $+$ or a $-$ on the ωt term in the sinusoidal wave equation solution?

   We established that right moving waves correspond to the negative sign on the ωt term, so we want that negative sign present.

8. You will now notice that we have found every parameter in equation 1.1 except ϕ_0, the *fixed phase constant*. Solve for ϕ_0 now by picking a known maximum and setting the total phase at this position and time equal to $\frac{\pi}{2}$.

   Let's pick the maximum at $t = 0$ s, $x = 1$ m on the $y(x)$ as an example. Then

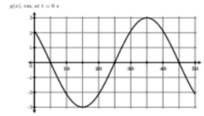
   $$\left(\frac{\pi}{6}\text{ m}^{-1}\right)(1\text{ m}) - \left(\frac{\pi}{2}\text{ s}^{-1}\right)(0\text{ s}) + \phi_0 = \frac{\pi}{2} \Rightarrow \frac{\pi}{6} + \phi_0 = \frac{\pi}{2} \Rightarrow \phi_0 = \frac{\pi}{3}$$

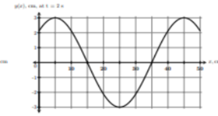
9. Is the value for ϕ_0 you found in question 8 unique? That is to say, are there other values of ϕ_0 that also reproduce the same graphs of our 1D wave?

   Adding an integer multiple of 2π to ϕ_0 returns the exact same graph - e.g., $\phi_0 = \frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$. Indeed, if you picked a different maximum in question 8, you might very easily have returned one of these equivalent values of ϕ_0.

1.1.2 Two $y(x)$ Graphs

Now consider a different 1D wave on a string. The two pictures below show two snapshots this wave, the first at time $t = 0$ and the second at time $t = 2$ seconds.





1. Find A, y_0, λ, and k for this wave.

   The first three can be read off of either graph as $A = 3$ cm, $y_0 = 0$ cm, and $\lambda = 40$ cm. This means

   $$k = \frac{2\pi}{40\text{ cm}} = \frac{\pi}{20}\text{ cm}^{-1} \approx 0.157\text{ cm}^{-1}$$

2. From just the information given, is it possible to determine ω? Why or why not?

   No it is not possible, since we don't know where the maximum at position $x = 35$ cm at $t = 0$ seconds appears on the $t = 2$ second graph, if at all.

3. Assume that the crest at $x = 35$ m at time $t = 0$ is at position $x = 5$ m at time $t = 2$ s. If this is the case, find the velocity and direction in which this wave is traveling. Then find ω and ϕ_0, and use this information to write down an equation $y(x,t)$ for this wave.

   In this case, the wave is moving to the left at speed

   $$v = \frac{35\text{ cm} - 5\text{ cm}}{2\text{ s}} = 15\text{ cm/s}$$

   Hence

   $$\omega = vk = \left(\frac{\pi}{20}\text{ cm}^{-1}\right)(15\text{ cm/s}) = \frac{3\pi}{4}\text{ s}^{-1} \approx 2.36\text{ s}^{-1}$$

   To find ϕ_0, then, we pick the maximum at $t = 2$ seconds, $x = 5$ cm, being sure to use the $+$ sign for a left moving wave;

   $$\frac{\pi}{2} = \left(\frac{\pi}{20}\text{ cm}^{-1}\right)(5\text{ cm}) + \left(\frac{3\pi}{4}\text{ s}^{-1}\right)(2\text{ s}) + \phi_0 = \frac{\pi}{4} + \frac{6\pi}{4} + \phi_0 \Rightarrow \phi_0 = \frac{2\pi}{4} - \frac{7\pi}{4} = -\frac{5\pi}{4}$$

   Hence, in total

   $$Y(x,t) = (3\text{ cm})\sin\left(\left(\frac{\pi}{20}\text{ cm}^{-1}\right)x + \left(\frac{3\pi}{4}\text{ s}^{-1}\right)t - \frac{5\pi}{4}\right)$$

   Of course, this answer is not unique, since ϕ_0 is not unique.

4. Assume that the crest at $x = 35$ m at time $t = 0$ is at position $x = 45$ m at time $t = 2$ s. For this case, find the velocity and direction in which this wave is traveling. Then find ω and π_0, and use this information to write down an equation $y(x,t)$ for this wave.

   This time, the wave is moving to the right at speed

   $$v = \frac{45\text{ cm} - 35\text{ cm}}{2\text{ s}} = 5\text{ cm/s}$$

   Hence

   $$\omega = vk = \left(\frac{\pi}{20}\text{ cm}^{-1}\right)(5\text{ cm/s}) = \frac{\pi}{4}\text{ s}^{-1} \approx 0.785\text{ s}^{-1}$$

   To find ϕ_0, we can still pick the maximum at $t = 2$ seconds, $x = 5$ cm, but now we must use the $-$ sign for the right moving wave;

   $$\frac{\pi}{2} = \left(\frac{\pi}{20}\text{ cm}^{-1}\right)(5\text{ cm}) - \left(\frac{\pi}{4}\text{ s}^{-1}\right)(2\text{ s}) + \phi_0 = \frac{\pi}{4} - \frac{2\pi}{4} + \phi_0 \Rightarrow \phi_0 = \frac{2\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{4}$$

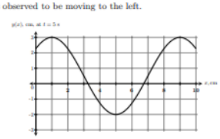
   which is (for good mathematical reason) actually equivalent to our previous ϕ_0 in part c! Hence, in total

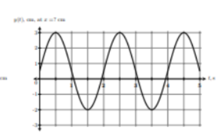
   $$Y(x,t) = (3\text{ cm})\sin\left(\left(\frac{\pi}{20}\text{ cm}^{-1}\right)x - \left(\frac{\pi}{4}\text{ s}^{-1}\right)t + \frac{3\pi}{4}\right)$$

   Of course, this answer is also not unique.

1.1.3 An Unknown Point in Common

The graphs below show the same 1D wave on a 10 cm long string. The graph at the left shows a snapshot of the wave at time $t = 5$ seconds, while the graph on the right shows the behavior of the wave in time at an unknown position x. The wave is observed to be moving to the left.





1. What is the speed of the wave on this string?

   The wave has a wavelength of 8 cm (evident from the graph on the left) and a period of 2 seconds (evident from the graph on the right). Hence

   $$v = \frac{\lambda}{T} = \frac{8\text{ cm}}{2\text{ s}} = 4\text{ cm/s}$$

2. What equation $Y(x,t)$ describes this one dimensional wave?
   Notice the wave oscillates between 3 cm and -2 cm. Hence

   $$A = \frac{3-(-2)}{2} = 2.5\text{ cm}$$

   and

   $$y_0 = \frac{3-2}{2} = 0.5\text{ cm}$$

   From our previous work, we know

   $$T = 2\text{ s} \Rightarrow \omega = \frac{2\pi}{2} = \pi\text{ s}^{-1}$$

   and

   $$\lambda = 8\text{ cm} \Rightarrow k = \frac{2\pi}{8} = \frac{\pi}{4}\text{ cm}^{-1}$$

   Since the wave travels left, we want the plus sign on the ωt term. This leaves only the ϕ_0 unknown. Notice we need to use a maximum on the $y(x)$ graph, since the position for the $y(t)$ graphs is unknown (unless you wanted to do part 3 first!). I pick the maximum at $x = 1$ cm, $t = 5$ s, which implies, based on our usual procedure

   $$\frac{\pi}{4\text{ cm}}(1cm) + (\pi\text{ s}^{-1})(5\text{ s}) + \phi_0 = \frac{\pi}{2} \Rightarrow \phi_0 = \frac{\pi}{2} - \frac{\pi}{4} - 5\pi = -\frac{19}{4}\pi$$

   which is equivalent to any number of correct answers (e.g., $\frac{5\pi}{4}$). Hence

   $$Y(x,t) = (2.5\text{ cm})\sin\left(\frac{\pi}{4\text{ cm}}x + (\pi\text{ s}^{-1})t + \frac{5\pi}{4}\right) + 0.5\text{ cm}$$

3. At what position x is the graph of $y(t)$ located at? How do you know?

   Notice on the $y(t)$ graph, the wave is at y-position $y = 0.5$ cm at $t = 5$ seconds. Hence the two possible positions (based on the $y(x)$ graph) are $x = 3$ cm and $x = 7$ cm, as these are the only two points on the wave at $y = 0.5$ cm. Notice as time goes on, the position of the $y(t)$ graph must be going down. since the wave moves left, this happens at $x = 3$ cm (at $x = 7$ cm, the left moving wave would result in the point moving upward originally). Hence the graph on the right is at position $x = 3$ cm.