Moments of inertia, rotational energy and Physical Pendula

Momentum of interia is like mass but for rotations the further the mass is from the axis the harder it is to spin the closer it is the easier it is to spin but the moment of interia can change depending on the shape, and where the axis of rotation is

- general equation for moment of inertia

$$I = \sum_{i=1}^{n} mr^2 \quad \text{or} \quad I = mr^2$$

- this equation was generalized for density

$$\int \rho r^2 \, dV$$

- Moment of interia of uniform sphere

$$\boxed{I = \frac{2}{5} mR^2}$$

- rotation is center of mass

m = mass
R = radius

Moment of interia of a cylinder

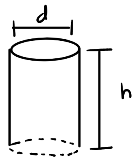
$$\boxed{I = \frac{1}{2} \cdot mR^2}$$

- rotation is center of mass

- m = mass
- R = radius

expiation and work of moment of interia of cylinder

- axis of rotation is in the middle



$dV = 2\pi r h$ (lateral surface area)

$\rho = \frac{m}{V} = \frac{m}{\pi R^2 h} \leftarrow$ (volume)

$$\int_0^R \left(\frac{m}{\pi \cancel{h} R^2}\right) r^2 \cdot 2\pi r \cancel{h}$$

all of that is ρ

this is all constant

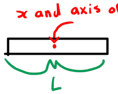
$$\frac{2m}{R^2} \int_0^R r^3 \, dr \Rightarrow \frac{2m}{R^2}\left[\frac{R^4}{4}\right] \approx \boxed{\frac{mR^2}{2}}$$

the limit of integration is going from zero to R. zero represents the axis of rotation and r "radius" represents distance

- For 2d objects like a disk we use dA because we are dealing with area
- For 3d objects like a cylinder we use dV because we are dealing with volume

- what if we wanted to derive a equation for a rod length L and we let x be the distance between the center of mass of the rod and the axis of rotation

x and axis of rotation



$$\int \rho r^2 \, dV$$

$\rho = \frac{m}{V} = \frac{m}{L}$ (1d has no volume)

$r^2 = x^2$

constant

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{m}{L} \cdot x^2 \, dx$$

- the rod has no area or volume all we are doing is integrating along the x

$$\frac{m}{L}\int_{-L/2}^{L/2} x^2 \, dx \approx \frac{m}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2}$$

$$\frac{m}{L}\left[\frac{(L/2)^3}{3} - \frac{(-L/2)^3}{3}\right] \Rightarrow \left[\frac{L^3}{8}\cdot\frac{1}{3} + \frac{L^3}{8}\cdot\frac{1}{3}\right]$$

$$\Rightarrow \frac{m}{L}\left[\frac{2L^3}{24}\right] \Rightarrow \boxed{\frac{mL^2}{12}}$$

Parallel axis theorm

- lets say you have a rod thats spining on the center of its axis we know the equation but what if it said the axis of rotation was shifted 3m deriving a would be challeging but this is where the parallel axis comes in

equation: $I = I_{com} + Md^2$

com
center of mass

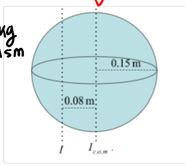
m = mass of object
d = new distance

- this means that it has to shift from the center of mass
- if it is not shifting from the com you need to derive it if its shifting from the end you need to find that equation and use it for "com"

example

C.O.M

mass = 0.46kg
radius = 0.15m



- $I_{sphere} = \frac{2}{5} mR^2$

$I = I_{com} + md^2$

$I = \frac{2}{5} mR^2 + md^2$

$I = \frac{2}{5}(0.46)(0.15)^2 + (0.46)(0.08)^2$

$\boxed{I = 7.084 \times 10^{-4}\ kg \cdot m^2}$

- notice that it shifted from the center of mass the equation $\frac{2}{5} mR^2$ only account when the the rotation is from the center of mass

rotation kinetic energy

$K_{rot} = \frac{1}{2} I \omega^2$

this equation will give you the kinetic energy of a rotating object

- you can plug in the moment of interia for different shapes using this formula

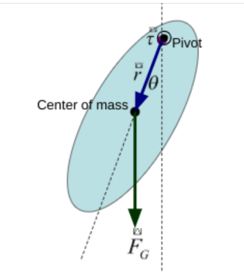
example:
what is the kinetic energy of a rotating sphere

$I = \frac{2}{5} mR^2$ $\quad K_{rot} = \frac{1}{2} I \omega^2$

$K_{rot} = \frac{1}{2}(\frac{2}{5} mR^2)\omega$

$K_{rot} = \frac{1}{5} mR^2 \omega$ that gives you a Krot specifically for sphere

The Physical Pendulum



$$\tau = |\vec{r}|\cdot|\vec{F_g}|\sin\theta$$

$$I\cdot\alpha = L\cdot mg\cdot\theta$$

$$-I\cdot\frac{d^2\theta}{dt} = L\cdot mg\cdot\theta$$

$$\frac{d^2\theta}{dt} = -\frac{L\cdot mg\theta}{I}$$

$$\cancel{-}\omega^2\cancel{\theta} = \cancel{-}\frac{L\cdot mg\cancel{\theta}}{I}$$

$$\omega^2 = \frac{L\cdot mg}{I}$$

$$\boxed{\omega = \sqrt{\frac{L\cdot mg}{I}}}$$

$r = L$ • r (radius) can be represented as the length from the pivot to the center of mass or "L"

$\tau = I\alpha$ • torque can be represented as this equation

$|\vec{F_g}| = mg$ • force here is mass • gravity

α • angular acceleration is $\frac{d^2\theta}{dt^2}$ or the second derivative of θ

$\sin\theta \approx \theta$ • this is the small angle approximation it says when a angle θ is small (in radians) this works because the higher power of θ in taylor series become tiny and can be ignored

$$\frac{d^2x}{dt} = -\omega^2 x \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\omega^2\theta$$

• this is almost the same form as the standard equation for simple harmonic motion so you can just replace it with $-\omega^2\theta$

- $\omega = \frac{2\pi}{T}$
- $T = \frac{2\pi}{\omega}$

$$T = \frac{2\pi}{\omega} = T = \frac{2\pi}{\sqrt{\frac{L\cdot mg}{I}}} \Rightarrow \boxed{T = 2\pi\cdot\sqrt{\frac{I}{L\cdot mg}}}$$

- $T = 2\pi\cdot\sqrt{\frac{I}{L\cdot mg}}$

Physical pendulum is just a regular pendulum but instead of a point mass at the end of the string you have a real object with mass and shape and you must use moment of interia depending on the shape and the axis of rotation

Summary of equation

- the units for I are $kg \cdot m^2$ for all shapes / axis of rotation

- $I_{sphere} = \frac{2}{5} m R^2$ • rotating at the center of mass

- $I_{cylinder} = \frac{1}{2} m R^2$ • rotating at the center of mass

- $I_{rod} = \frac{1}{12} m L^2$ • rotating at the center of mass

- $I_{general} = MR^2$

- $I = \int \rho r^2 \cdot dV$ • generlized for density

- $I = I_{c.o.m} + md^2$

- $K_{rot} = \frac{1}{2} I \omega^2$ • (J)

- $\omega = \sqrt{\frac{L \cdot mg}{I}}$ • (rad/s)

- $\omega = \frac{2\pi}{T}$ and $T = \frac{2\pi}{\omega}$

- $T = 2\pi \cdot \sqrt{\frac{I}{L \cdot mg}}$ • (seconds)