Final Material

Circular Motion

① Main idea ((Characteristics of all uniform Circular Motion)):
- Centripetal acceleration ⇒ $\boxed{a_c = \frac{v^2}{r}}$
- Centripetal force ⇒ $\boxed{F_{net} = F_c = m \cdot a_c}$

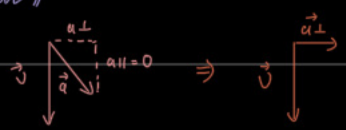
Directed toward the center of circle...

② How did we derive centripetal acceleration ?!
- [A] In a uniform circular motion ⇒ ((Velocity Constant)), assuming $(a_{\parallel} = 0)$ & dt small
- [B] Similar triangles ⇒ $v\,\theta\,v$ (Δv) & $r\,\theta\,r$ ($v\Delta t$), we are assuming $d\theta$ is small...

[A] As an object moving in a Circular motion, it experience 2 acceleration:
- $a_{(\perp \text{ to } v)}$ ⇒ a_c ((Does no work - only provides Circular rotation))
- $a_{(\parallel \text{ to } v)}$ ⇒ $a_T = 0$ ((Tangential acceleration))

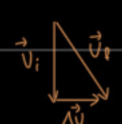
((Uniform Circular Motion))

$$\vec{v}_{(after)} = \vec{v}_{(initial)} + \vec{\Delta v} = \vec{v}_{(initial)} + \vec{a} \cdot dt = -v\hat{j} + a_{\perp} \cdot dt\,\hat{i}$$



$$\vec{v}_{(after)} = \sqrt{(-v)^2 + (a_{\perp} \cdot dt)^2} = \sqrt{v^2 + a_{\perp}^2 \cdot dt^2}$$

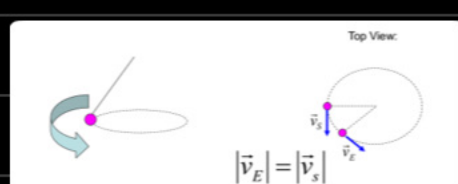
equal 0

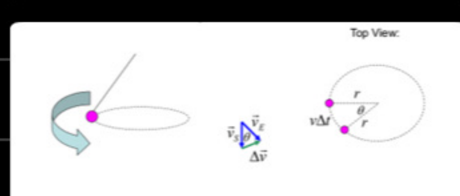


$\vec{v}_{(after)} = v$ ⇒ $v_{(after)} = v_{(initial)}$ ((Magnitude only))

[B]

$v \cdot \Delta t$ = arc length





$$\frac{\Delta v}{v} = \frac{v \cdot \Delta t}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r} \Rightarrow \boxed{a_c = \frac{v^2}{r}}$$

- It is the net force of the system: $F_{net} = m \cdot a_c$
- Does no work since $a \perp$ to velocity: $\Delta KE = 0$ (constant v)
- provides a uniform rotation since $a_{\parallel} = 0$

Moments of Inertia

① Main idea:
1. Where is the axis of rotation located in an object ?!
2. How to calculate moments of inertia ?!
3. Understanding physical pendulum along with moments of inertia...

② Where is the axis of rotation located in an object ?!

The total momentum of each molecules add up to equal the momentum of the whole system...

(# of molecules)

[1] $P_{total} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m\,\vec{v}_i$ ⇒ [2] $\boxed{X_{c.o.m} = \sum_{i=1}^{n} \frac{m_i\,\vec{x}_i}{M}}$

Rotational Motion

① Main idea:
- ⓐ What is the difference between Rotational & Circular Motion ?!
- ⓑ What kind of relationship it has with transitional Motion (linear vector) ?!
- ⓒ What affects Rotational Motion ?!
- ⓓ Understanding the whole story !!

② Difference between Rotational & circular motion
- Rotational Motion
  - is the motion of an object around an internal axis (axis inside the object itself)
  - as a ball spinning or revolving around itself...
  - has 2 forces acting upon it:
    - Centripetal (F_c) $(F_{\parallel})$
      - Maintains the object's shape (due to intermolecular forces)...
      - points towards the axis of rotation ((usually Center of mass))...
    - Tangential (F_T) $(F_{\perp})$
      - it's the reason behind the change of Rotation speed...
      - always tangent to the object ((perpendicular to radius))
- Circular Motion
  - is the motion of an object around an external axis (axis separate than the object)
  - as a car turning around a circular track...
  - has 2 forces acting upon it:
    - Centripetal (F_c) $(F_{\parallel})$
      - Maintains a uniform circular Motion ((due to gravity, friction, or tension))
      - points toward the center of the circle..
    - Tangential (F_T) $(F_{\perp})$
      - it's the reason behind the change of circular speed...
      - always tangent to the circle ((perpendicular to radius))

③ What kind of relationship does a Rotational motion have with transitional Motion ?!

(linear) transitional $\xrightarrow{\boxed{r}}$ Rotational (polar)

$S = r \cdot \theta$

$V_T = r \cdot \omega$

$a_T = r \cdot \alpha$

$a_c = r \cdot \omega^2$

(a_T, a_c: a_{net}; θ, ω, α: angular)

$\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Rotational Kinematics

$\omega_f = \alpha \cdot \Delta t + \omega_i$

$\Delta\theta = \frac{1}{2} \cdot \alpha \cdot \Delta t + \omega_i \cdot \Delta t$

$2 \cdot \alpha \cdot \Delta\theta = \omega_f^2 - \omega_i^2$

$\Delta\theta = \frac{\omega_f + \omega_i}{2} \cdot \Delta t$

$\Delta\theta = \#\,\text{revolution} \cdot 2\pi$

$\boxed{T = \frac{\Delta t}{\#\,\text{revolution}}}$ $\boxed{T = \frac{2\pi r}{V_T}}$

Direction of (θ, ω, α)

curl right fingers to to the direction of rotation... and the thumb points to the direction of (θ, ω, α)

④ What affects Rotational Motion ?!

⇒ Torque: how much a force causes an object to rotate about an axis... it makes things spin...

$\boxed{\tau = r \cdot F_T}$

$\boxed{\tau = I \cdot \alpha}$

$\boxed{F_T = F \cdot \sin\theta}$

$\boxed{I = \sum_{i=1}^{n} m_i \cdot r_i^2}$

⇒ What affects Spinning speed (ω) ?!
- External force, depends on:
  - How much force exerted → (F)
  - How far from axis of rotation → (r)
  - How much (F) is perpendicular to (r) → $(\sin\theta)$
- Moments of inertia (I)
  - the closer the mass is to the axis of rotation, the easier it is to spin the object, and requiring lesser tangential force to produce the same amount of spinning...

$\boxed{KE_{Rotational} = \frac{1}{2} \cdot I \cdot \omega^2}$ ⇒ $\boxed{E_{total} = K + K_{rot} + U}$ $\boxed{\Delta E = \Delta K + \Delta K_{rot} + \Delta U = 0}$

If an object is rolling down a hill, then the stored potential energy goes to both linear & Rotational kinetic energy...

$$M_{total} \cdot V_{total} = \sum_{i=1}^{n} m_i \cdot \frac{d\vec{x_i}}{dt}$$

$$V_{total} = \frac{d\left(\sum_{i=1}^{n} \frac{m_i \cdot \vec{x_i}}{M}\right)}{dt}$$

when we added up the masses of each molecule based on how far it is from a selected coordinated point and devide it by total mass it gives us the exact location of where the center of mass located... it is like when we calculate our GPA... each class we take has its own weight on the GPA based on the score we got and the # of credit hours and when we added up all the classes and devide it by total credit hours taken... it represents GPA

$$GPA = \sum_{i=1}^{n} \frac{hour \cdot Grade}{Total\ hours}$$

It represent our average grade based on hours. However, ($x_{c.o.m}$) is the average distance of all molecules based on the molecule mass...

[3] In the reference of frame where the c.o.m is stationary:

$$\frac{d\left(\sum_{i=1}^{n} \frac{m_i \cdot \vec{x_i}}{M}\right)}{dt} = 0$$

$\boxed{\text{which means that all molecules are rotating around the c.o.m ... and the axis of rotation is always at the c.o.m ... unless the axis of rotation was set to be other than that ...}}$

(5) Understanding the whole story ?!

- In both cases, as the tangential force increases, the rotation or circulation speed increases... therefore, increasing the centripetal force required to maintain the object's shape or uniform circular motion... meaning there is a maximum speed at which the object maintains its Rotational or circular Motion... for simplicity, we either maintain a constant speed (no tangential force) or if the speed is changing, we calculate at a certain period of time...
- In Rotational Motion, if we zoom inside the object, the molecules move in a circular path... and its movement is restricted by its neighboring molecule... However, the farther away the molecule from center of rotation, the more tangential force is required for all molecules to have the same period, angular acceleration, & angular speed... as a consequence, enabling the object to Rotate as one... as a result, when an external tangential force is applied to the edge of an object, the force is not distributed evenly between molecules... and that is why the edge of an object always break first when the object is rotating at really high speeds...

(r) increase → (F_T) increases → (a_T) increases → (V_T) increases → (a_c) increases → (F_c) increases

(I) increases → (I) increases → $\boxed{\text{For every point on an object to maintain the same } ((T, \alpha, \omega))...}$

Therefore, if we say a person is rotating with the merry-go-round, then that person must have the same ((T, α, ω)) since its one with the system...

(($d \propto r$)) $90°$

Distance the molecules travels $= r \cdot \theta$

Inner molecule ________

middle molecule ________

edge molecule ________

For the molecules to finish their own distances at the same time...

Inner		middle		edge
V_T	<	V_T	<	V_T
a_T	<	a_T	<	a_T
F_T	<	F_T	<	F_T

(3) How to calculate moments of inertia (I) ?!

[1] When axis of rotation is at c.o.m

$$\boxed{I_{c.o.m} = \int r^2 \cdot dm}$$

dm: infinitesimal mass element... Which means how much does the molecule's mass added up to the total mass as radius increases

((how much will the next taken class affect our GPA))

[2] When axis of rotation is out of c.o.m

$$\boxed{I_{o.c.o.m} = I_{c.o.m} + \underbrace{M}_{\text{total mass}} \underbrace{d^2}_{\text{distance from c.o.m}}}$$

[3] ⇒ dm = Density × infinitesimal (volume, area, length)

- for 1D object (like a thin rod):
  - Linear mass density $(\lambda) = \frac{M}{L}$
  - $\boxed{dm = \lambda \cdot dx}$
- for 2D object (like a disk or a plate)
  - surface mass density $(\sigma) = \frac{M}{A}$
  - $\boxed{dm = \sigma \cdot dA}$
- for 3D object (like a solid sphere or cylinder):
  - Volume mass density $= (\rho) = \frac{M}{V}$
  - $\boxed{dm = \rho \cdot dV}$

$I_{c.o.m} = \int \rho\, r^2 \cdot dV$ ((like our slide ☺))

[4] Memorizing $I_{c.o.m}$ values:

- Sphere $= \frac{2}{5} MR^2$
- Cylinder or disk $= \frac{1}{2} MR^2$
- Hollow disk (ring) $= MR^2 = M\frac{(a^2 + b^2)}{2}$ (a: outer radius, b: Inner radius)
- bar $= \frac{1}{12} ML^2$

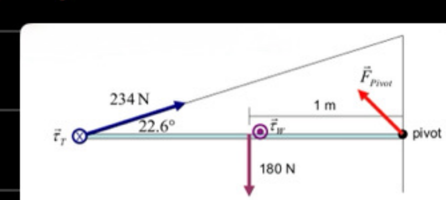
[5] Note: moments of inertia is the angular analogue of inertial mass... As inertial mass resists changes in transitional motion, moments of inertia resists changes in Rotational motion...

Statics

(1) Static Equilibrium: is a phrase used to describe a stationary object ((at rest))... meaning it is in a perfectly balanced state...

(2) $a = 0$ & $\alpha = 0$

$\sum F = 0$ & $\sum \tau = 0$

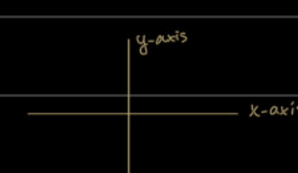


(3) We always set our axis of rotation at the unknown force that acts on our object

(4) When solving for $\sum F = 0$

- $\sum F_y = 0$
- $\sum F_x = 0$

⇒



(4) Understanding physical pendulum along with moments of inertia ?!

[1]

$$\vec{\tau} = \vec{r} \cdot \vec{F_g} \cdot \sin\theta$$

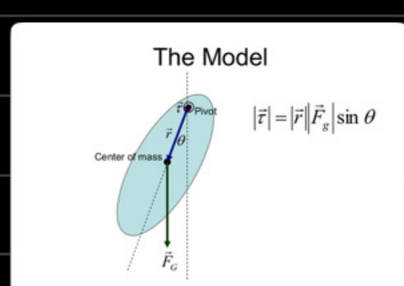
$$-I \cdot \alpha = \ell \cdot mg \cdot \sin\theta$$

$$\frac{d^2\theta}{dt} = -\frac{\ell\, mg \cdot \sin\theta}{I}$$

$$\frac{d^2\theta}{dt} = -\frac{\ell mg}{I} \cdot \theta$$

$\vec{\tau}$ negative since body is rotating in the direction of decreasing θ...

$\sin\theta \approx \theta$



$\boxed{\omega = \sqrt{\frac{\ell mg}{I}}}$, where ℓ equal d (distance between axis of rotation and c.o.m)...

[2] when $I = I_{o.c.o.m}$ (ring) ⇒ $\omega = \sqrt{\frac{\ell mg}{mr^2 + md^2}} = \sqrt{\frac{\ell \cdot g}{r^2 + \ell^2}}$

Angular Momentum

$$\Rightarrow \lim_{\boxed{l \gg r}} \sqrt{\frac{l \cdot g}{l^2}} \Rightarrow \omega = \sqrt{\frac{g}{l}}$$

this means when l is exponentially larger than r, then r^2 is neglible compared to l^2 $\Rightarrow r^2 + l^2 \approx l^2$ ... it returns back to being a simple pendulum... since l is way larger than r it means that the axis of rotation exists farther away from the object itself (axis outside of the object)... it is no longer experiencing rotational motion... instead it is circular motion ☺...

3 $I \propto m \Rightarrow$ However, ω is independent of mass since we are able to cancel the numerator mass with the denominator mass...

$\boxed{\omega \propto l \;\&\; r}$ it depends on how the mass is spread out !!

Universal Gravitation & Kepler's Third Law

① $\boxed{F_g = G \frac{m \cdot M}{r^2}}$

r: distance between 2 c.o.m

② $\boxed{g = \frac{GM}{r^2}}$

- when above the surface of (M): $\boxed{r = r_{(planet)} + h}$
- when on the surface of (M): $\boxed{r = r_{(planet)}}$

③ $F_g = F_c$, assuming planets move in a circular orbit

$$G \frac{m \cdot M}{r^2} = m \cdot \frac{v^2}{r} \Rightarrow \boxed{v = \sqrt{\frac{GM}{r}}}$$

$$T = \frac{2\pi r}{v} \Rightarrow \boxed{T^2 = \frac{4\pi^2}{GM} \cdot r^2}$$

④ Kepler's Third Law

$$\boxed{T^2 \propto r^3 \Rightarrow \frac{T^2}{r^3} = \frac{T^2}{r^3}}$$

$\Downarrow$

The farther the planet from star, the longer the time period...

⑤ Universal potential

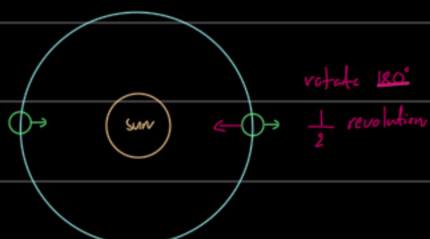
$$F_g = \frac{-du}{dr} \Rightarrow \int du = \int G \frac{m \cdot M}{r^2} \cdot dr$$

$\Rightarrow \boxed{U = -G \frac{m \cdot M}{r}}$ $\Rightarrow$ this is a more accurate way to find potential energy since it accounts for the change of radius over time...

⑥ # of earth's revolution around itself = $T_{(circular)}$ + 1 day = 366.24 / year

as the earth orbits around the sun, its position is in constant change according to its orbiting speed...

Therefore, requiring the earth to make additional rotations to always have the same sun's position throughout the day.. these additional rotations add up to equal 1/year...



rotate 180°

$\frac{1}{2}$ revolution

$\boxed{T_{(orbit)} \neq T_{(spin)}}$

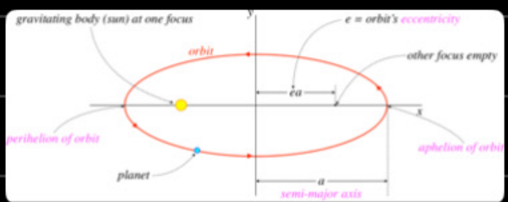
Kepler's First Law

① Main Idea: 1 What is Keplers first law ?!

2 What solutions are used to reach to Kepler's first law ?! ...

② What is Kepler's first Law ?!

All planets revolve around the sun in an ((elliptical)) orbit with sun at one of the focci...



He explained the types of orbit that a planet takes ((Circular, ellipse, parabolic, & hyperbolic)) which are called ((Conic Sections))... since these shapes are sections cut from a cone at different angle...

① Angular Momentum: tells us how hard it would be to stop a rotating object

② Categorized into 2 types based on what type of object:

1 Rigid whole body (System)

$\boxed{L_{(system)} = I \cdot \omega}$

$\Downarrow$

ⓐ $\tau = \frac{dL}{dt}$

- If only (ω) was changing over time: $\boxed{\tau = I\alpha}$
- If (I) & (ω) were changing over time: $\boxed{\tau = I\alpha + \frac{dI}{dt} \cdot \omega}$

ⓑ $\tau_{net} = 0 \Rightarrow \frac{dL}{dt} = 0 \Rightarrow$ (Angular momentum Conserved)

By newton's third Law $\Rightarrow \boxed{\Delta L_1 = -\Delta L_2}$

2 Single point mass (one molecule)

$\boxed{L_{(point)} = r \cdot p \cdot \sin\theta}$

or

$\boxed{L_{(point)} = m \cdot r^2 \cdot \omega}$ $\Rightarrow$ only for objects in a circular motion

ⓐ applies to all point mass... even if the motion is ((not circular))... meaning from my frame of reference, every object that is moving past me not towards or away from me has an angular momentum...

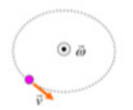
((object moving in a straight line))

((no angular momentum)) $\sin 0 = 0$

* It depends on the frame of reference we choose $\Rightarrow$ in Rotational examples ((axis of rotation))

ⓑ Proof of the equation:

- Consider a point mass rotating around a circular path of radius r with angular velocity ω



$\boxed{\hat{r} = 1}$

$$r \cdot \vec{\omega} = \vec{v} \Rightarrow \vec{\omega} = 1 \times \frac{v}{r}$$

$\boxed{\hat{r} = \frac{\vec{r}}{r}}$

$$\Rightarrow \vec{\omega} = \hat{r} \cdot \frac{\vec{v}}{r} \Rightarrow \vec{\omega} = \frac{\vec{r}}{r} \cdot \frac{\vec{v}}{r}$$

$$\Rightarrow \boxed{\vec{\omega} = \vec{r} \cdot \frac{v}{r^2}} \Rightarrow L = I \cdot \omega = m r^2 \cdot \vec{r} \cdot \frac{v}{r^2}$$

$$\Rightarrow L = \vec{r} \cdot m\vec{v} \Rightarrow \boxed{L = \vec{r} \cdot \vec{p}}$$

Kepler's Problem & Second Law

① Main Idea: 1 What is the Kepler's problem ?!

2 How is the Kepler's problem solved ?!

3 What is Kepler's second law ?!

② What is the Kepler's problem ?!

We assumed previously that a planet as earth orbits the stationary sun... However, the gravitational force affects the two body masses regardless of how large is one of the masses... meaning the sun is <u>not stationary</u> and it is in fact wobbling due to the earth's gravitational force... This complicates equation, such as when solving for the orbital period...

③ How is the Kepler's problem solved ?!

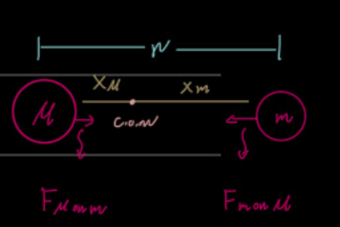
The solution to such problem... is to pretend the larger body is stationary in the center of mass of these two bodies, and the smaller body orbits with a ((Reduced Mass)) at the same separation distance from the larger body as in the unmodified problem...

Therefore, the two body problem is reduced to a ((One body problem)), focused solely on the smaller's body orbit...

Only applied for two bodies, no external force & momentum is conserved, meaning c.o.m experience no acceleration:

$$\frac{d^2 x_{c.o.m}}{dt^2} = 0 \Rightarrow V_{c.o.m} = \text{constant or } \underline{\text{stationary}}$$

our reference frame



$$\vec{r} = X_M - X_m \Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{d^2 X_m}{dt^2} - \frac{d^2 X_M}{dt^2}$$

③ What Solutions are used to reach to Kepler's first Law ?!

↳ Qualitative :

$E_{total} = K + U \Rightarrow E_{total} = K_{trans} + K_{rot} + U$

$\Rightarrow E_{total} = \underbrace{\frac{1}{2} \cdot \mu \left(\frac{dr}{dt}\right)^2}_{\text{radial velocity}} + \underbrace{\frac{1}{2} \cdot I \cdot \left(\frac{d\theta}{dt}\right)^2}_{\text{angular velocity}} - G \frac{mM}{r}$

- $\frac{d\theta}{dt} = \omega$
- $L = \mu \cdot r^2 \cdot \omega$
- $I = \mu r^2$

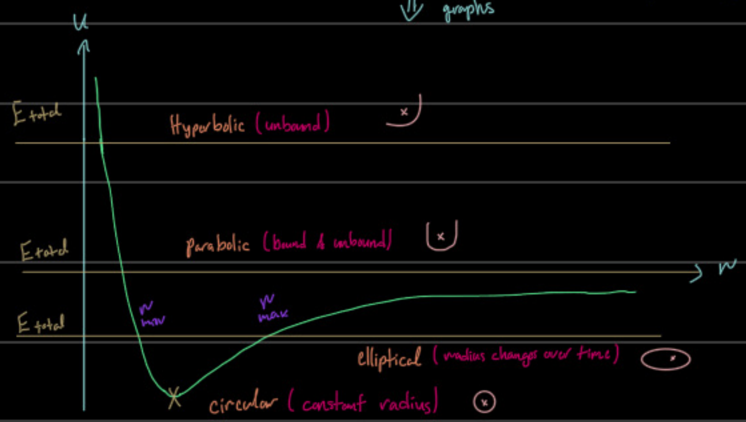
radial velocity: how fast is the radius between two bodies changing

angular velocity: how fast an object sweeps out an angle as it moves along a circular or curved path...

$E_{total} = \frac{1}{2} \cdot \mu \cdot \left(\frac{dr}{dt}\right)^2 + \boxed{\frac{L^2}{2\mu r^2} - G \frac{mM}{r}} \Rightarrow U$ (fictitious)

since it depends only on radius

⇓ graphs



↳ Quantitative : ((high calculus))

$$\boxed{\frac{1}{r} = C\,(1 + e\cos\theta)}$$

$\theta = 180° \Rightarrow r_{max}$, $\theta = 0° \Rightarrow r_{min}$

$$\boxed{e = \sqrt{1 + \frac{2 \cdot E_{total} \cdot L^2}{G^2 \cdot m^2 \cdot M^2 \cdot \mu}}}$$

$$\boxed{C = \frac{G \cdot m \cdot M \cdot \mu}{L^2}}$$

$$\boxed{e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}}$$

$$\boxed{R_{(major\ axis)} = \frac{R_{(perihelion)} + R_{(aphelion)}}{2}}$$

perihelion: closest to sun; aphelion: farthest from sun

↳ $e = 0$ ((circle))

↳ $0 < e < 1$ ((elliptical))

↳ $e = 1$ ((parabolic))

↳ $e > 1$ ((hyperbolic))

⇓

$$\boxed{\begin{aligned} r_{max} &= r_{(major\ axis)} \cdot (1 + e) \\ r_{min} &= r_{(major\ axis)} \cdot (1 - e) \end{aligned}}$$

$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{F_{M\ on\ m}}{m} - \frac{F_{m\ on\ M}}{M} \Rightarrow \frac{F_{M\ on\ m}}{\mu} = \frac{F_{M\ on\ m}}{m} + \frac{F_{M\ on\ m}}{M}$

$\Rightarrow \frac{1}{\mu} = \frac{m + M}{mM} \Rightarrow \boxed{\mu = \frac{mM}{m + M}}$

④ What is Kepler's second Law ?!

A line joining the planet and the sun sweeps equal areas in equal time intervals...

Which means ((Conservation of angular Momentum)):

$$\boxed{\begin{aligned} r_1^2 \cdot \omega_1 &= r_2^2 \cdot \omega_2 \\ r_1 \cdot v_1 &= r_2 \cdot v_2 \end{aligned}}$$

$$\boxed{L = \mu \cdot r^2 \cdot \omega}$$