Simple Harmonic Oscillator

Simple Harmonic Oscillator (SHO) ?!

↳ What ?! ⇒ A system that experience a simple Harmonic motion (SHM), in which the object moves back & forth in a repeated cyclical motion...
As long as energy is conserved, the system oscillates ((indefinitely))...

↳ Examples ⇒ mass attached to a ((Spring or pendulum))...

↳ How ?!

↳ As an external force acts upon a mass attached to a spring... It streches the Spring storing potential energy in the Spring/mass system

↳ Once the external force is removed, the spring releases the stored potential energy in the form of restoring force ((F_{Hooke}))... Here, U_{Hooke}, F_{Hooke}, & acceleration is ((Maximized))...

↳ As the spring is being restored back to equilibrium... the attached mass increases in both velocity & kinetic energy reaching its ((Maximum)) at equilibrium length...
At that moment, U_{Hooke}, F_{Hooke}, & acceleration equal zero ((Minimized))...

↳ Since the attached mass gained kinetic energy, it will not stop at equilibrium length... instead it will continue it motion compressing the spring... During which the springs acts as a barrier storing potential energy for the system... and minimizing the mass's kinetic energy & velocity...

↳ Once the attached mass - kinetic energy reaches zero ((Minimized))... the system releases its stored potential energy propelling the mass back to equilibrium...

↳ The process is repeated forever only if no external force other than the first one acts on the mass attached to the Spring...

↳ Equation :

↳ Spring :

↳ The only force acting on the attached mass is ($F_{Hooke} = -kx$), $x_0 = 0$

$$F_{net} = F_{Hooke} = -kx$$

$$* \ F_{net} = ma = m\frac{d^2x}{dt^2}$$

$$m \cdot \frac{d^2x}{dt^2} = -kx$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \quad (1)$$

SHO equation is a differential equation & physicists solve equation through trial and error :

$x(t) = A.\cos(\omega t + \phi) + x_0$ or $x(t) = A.\sin(\omega t + \phi) + x_0$

$\Downarrow$

$V(t) = \frac{dx}{dt} = -A\omega.\sin(\omega t + \phi) + x_0 = -W x(t)$

$\boxed{V(t) = -W x(t)}$ * (($V_m = x_m = KE_m$)) m: maximized

$a(t) = \frac{d^2x}{dt} = -A\omega^2.\cos(\omega t + \phi) = -\omega^2 x(t)$

$\boxed{a(t) = -\omega^2 x(t)}$ 2 * (($a_m = x_m = U_m = F_m$))

$(1 = 2) \Rightarrow -\frac{k}{m}.x = -\omega^2.x \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$ 3

Angular frequency (W) :

Definition : is the number of oscilation in terms of radians per second...

we can derive from it :

period : is the time required to complete a whole cycle ((whole osscilation))

$\boxed{T = \frac{2\pi}{\omega}}$

frequency : is the number of osscilation per second...

$\boxed{f = \frac{\omega}{2\pi}}$ $\boxed{f = \frac{1}{T}}$ (Hz)

Amplitude (A) :

peak displacement $(x_m - x_0)$:

the difference between maximized length & equilibrium length...

$E_{Total} = KE\,max = U\,max = \frac{1}{2}.k.(x_m - x_0)^2 = \frac{1}{2}.k.A^2$

$E_{total} = \frac{1}{2}.k.A^2 \Rightarrow \boxed{U + KE = \frac{1}{2}.k.A^2}$

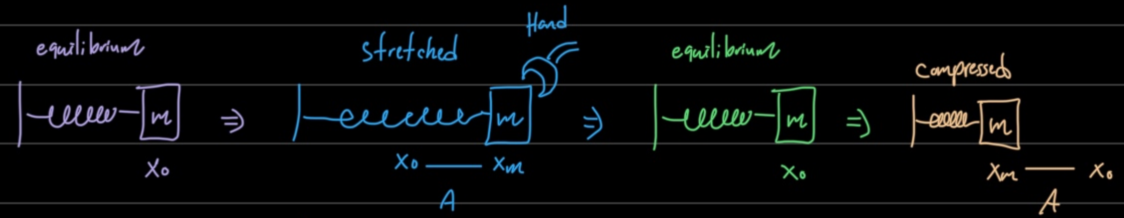
↳ phase Constant (ϕ): Represents starting position of osscilation in Radians

↳ Amplitude & phase Constant:

Represent the integration constant (C)...

They are determined by the external force ((initiator)) that set the mass/spring system into motion... not the system itself...

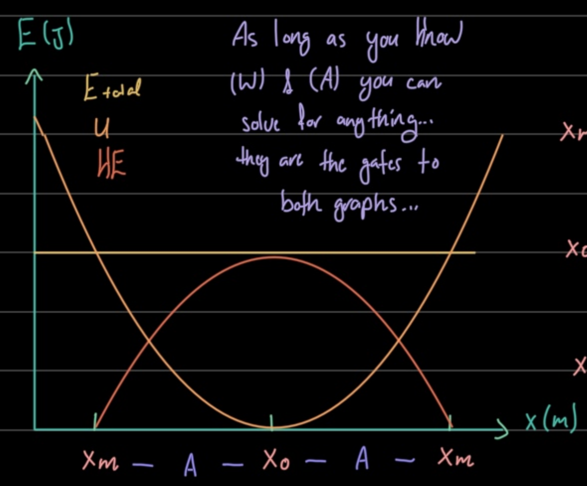
Remember, before setting the mass/spring system to oscillate, it was stationary, which means in ((equilibrium state))



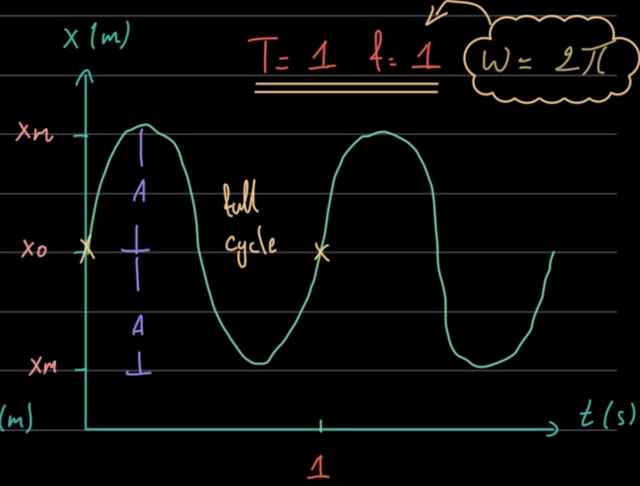
nothing happening ((Stationary))	$F_{extend} = F_{Hooke}$		
	U_m & a_m	KE_m & V_m	U_m, a_m, $F_{Hooke\ m}$
	KE & $V = 0$	U, a, & $F_{Hooke} = 0$	KE & $V = 0$

In a Conserved Energy system, once the hand is Released, the Only force is acting upon the mass/spring system is ((F_{Hooke}))...

The springs keeps osscillating until the end of the world !!



As long as you know (ω) & (A) you can solve for anything... they are the gates to both graphs...



$T = 1 \quad f = 1$

$\omega = 2\pi$

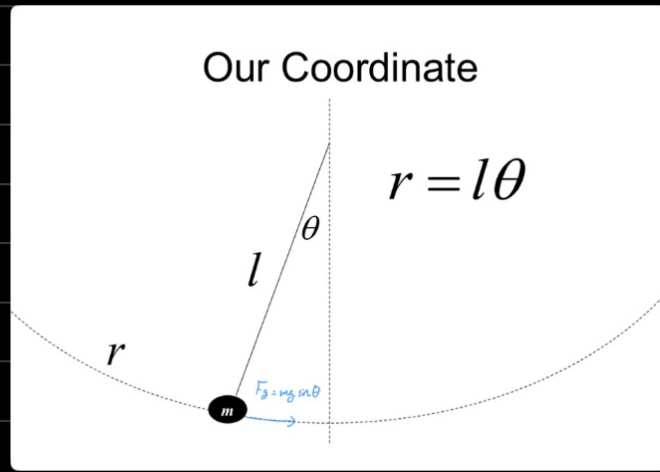
pendulum (SP)

(distance)

$x = r = l\,\theta$

$V = \dfrac{dr}{dt} = l\,\dfrac{d\theta}{dt}$

$$\boxed{a = \frac{d^2 r}{dt^2} = l\,\frac{d^2\theta}{dt^2}}$$



$F_g = \ominus\, mg \cdot \sin\theta$

$m\,a = -mg\sin\theta$

$l\,\dfrac{d^2\theta}{dt^2} = -g \cdot \sin\theta$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l} \cdot \sin\theta}$$

negative implies that the restoring gravitational force is opposite to the displaced object...
All restoring forces in physics or chemistry are negative signaling that an object will always prefer lower potential energy reaching equilibrium...

Remember, the smaller $\theta \rightarrow$ the closer it is to lower potential... which means if we assume θ to be very small, then we can say that it behaves like a spring...
Therefore, we can steal all the formula of osscillation to use it for pendulum... By ((Taylors expansion)):

$$\boxed{\sin\theta \approx \theta}$$

$$\boxed{\frac{d^2\theta}{dt^2} = \frac{-g}{l} \cdot \theta}$$

$$\boxed{W = \sqrt{\frac{g}{l}}}$$

SP equations:

$$\boxed{\begin{aligned} \theta(t) &= A \cdot \cos(Wt \cdot \emptyset) + \theta_0 \\ r(t) &= A \cdot l \cdot \cos(Wt \cdot \emptyset) + r_0 \end{aligned}}$$

$V(t) = -A \cdot l \cdot w\,\sin(Wt \cdot \emptyset) = -l \cdot w \cdot \theta$ or $-w \cdot r$

$$\boxed{V(t) = -l \cdot w \cdot \theta \text{ or } -w \cdot r}$$

$$a(t) = -A.\,l.\,\omega^2 \cos(\omega t.\,\phi) = -l.\,\omega^2.\,\theta \text{ or } -\omega^2 r$$

$$\boxed{a(t) = -l.\,\omega^2.\,\theta \text{ or } -\omega^2 r}$$

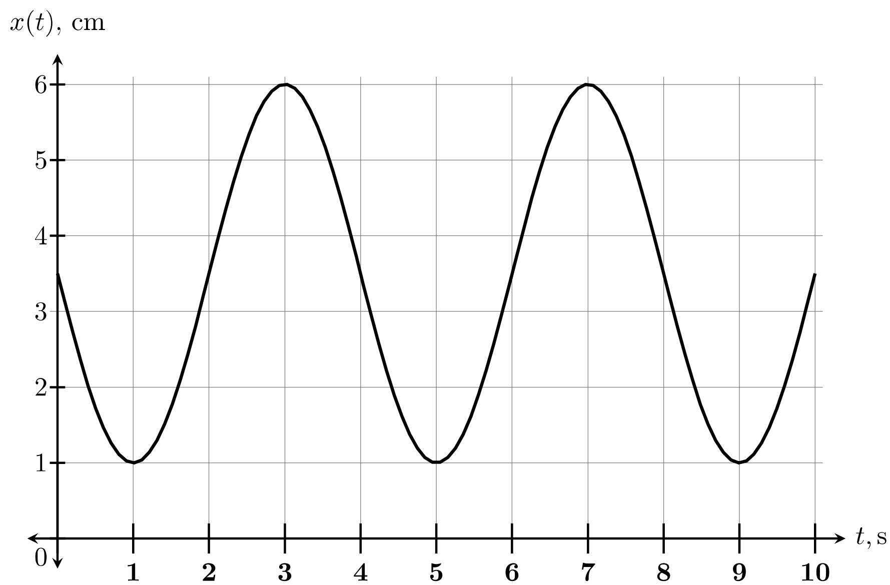
2.3 The Simple Harmonic Oscillator

2.3.1 Analyzing the $x(t)$ Graph of a Simple Harmonic Oscillator

The picture below shows the graph of the position x of a 4 kg mass attached to a spring of unknown spring constant k as a function of time t. Our initial goal in this activity will be to come up with an equation in the form:

$$x(t) = A\cos(\omega t + \phi_0) + x_0 \tag{2.5}$$

that matches the graph pictured. Notice this equation differs from the equation discussed in the lecture slides by the inclusion of an x_0 term, which merely allows for the situation in which the equilibrium position of the spring is not at zero.



1. What are the maximum and minimum positions x_{max} and x_{min} of this oscillator's motion?

2. Notice that the amplitude A and equilibrium position x_0 of the graph, are such that $x_{max} = x_0 + A$ and $x_{min} = x_0 - A$. Using your answer to question 1 and algebra, find A and x_0. Note: don't forget to include units!

3. What is the period, T, of the oscillator? Remember that T is the time between two successive minima or two successive maxima of the graph. What then is the frequency, f, of the oscillator? What about the angular frequency, ω?

4. To find the phase constant, notice that $\cos\theta$ has a maximum when $\theta = 0$ (and also when $\theta = 2n\pi$ for any integer n). Thus, if we identify a time t_{max} where the function has a maximum and then set the entire argument of the cosine function equal to 0, like this:
$$\omega t_{max} + \phi_0 = 0 \tag{2.6}$$
we can solve algebraically for ϕ_0. Find a value for ϕ_0 for the graph pictured on the previous page. Is this value of ϕ_0 unique? What are some other values of ϕ_0 that would also give the same $x(t)$ graph.

5. Using all your previous results, write down an equation for $x(t)$ for the oscillator pictured above.

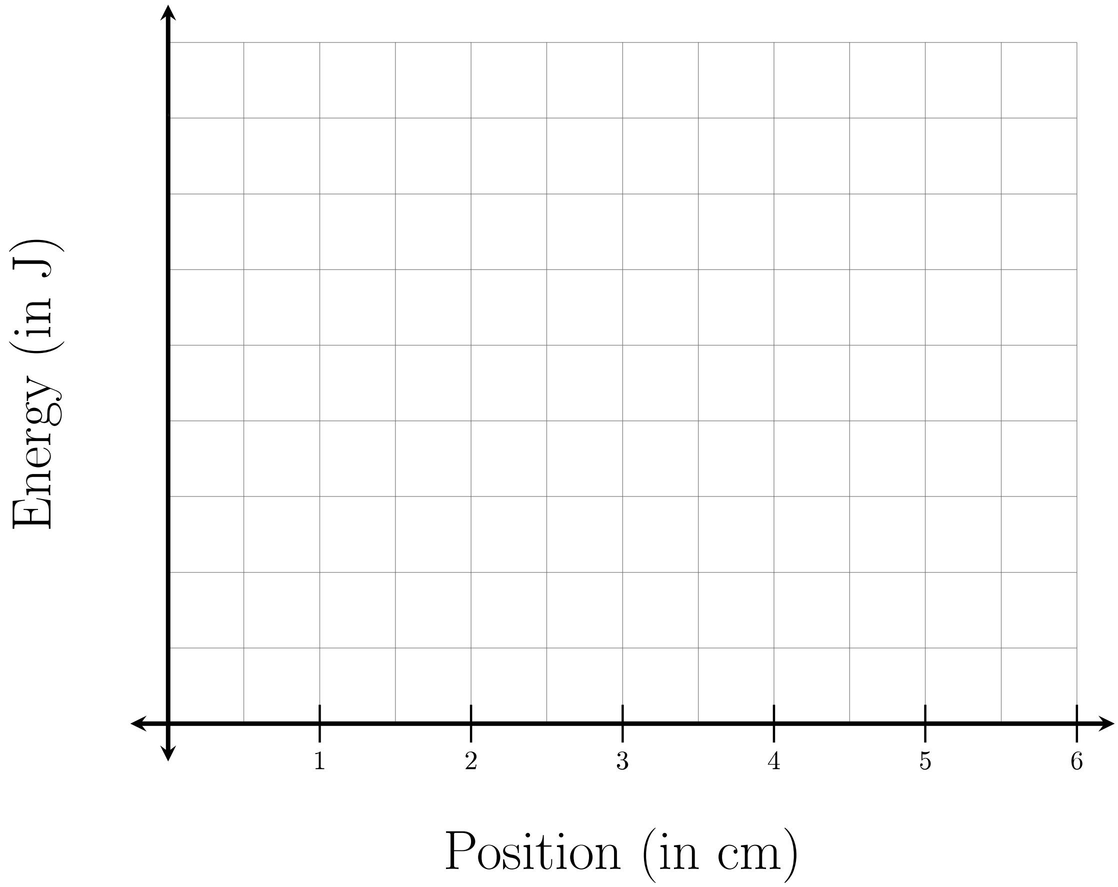
6. What is the spring constant, k, of the unknown spring? Leaving your answer in terms of π is okay, but please be sure to include units!

7. What is E_{tot}, the total energy of the oscillator? Please give your answer in joules. (Hint: remember that $E_{tot} = U(x_{max})$. Also, be careful about units!)

8. What is $v(t)$, the velocity of the oscillator as a function of time? What is v_{max}, the maximum speed the mass attains? Based on this, what is K_{max} the maximum kinetic energy of the oscillator. Does this agree with your answer in question 2?

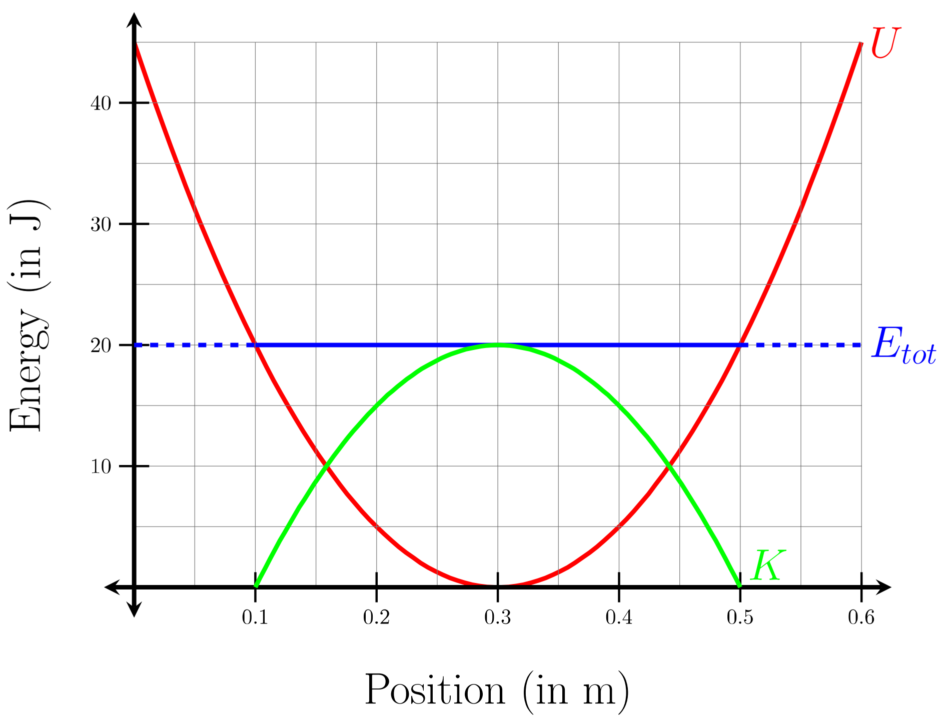
9. What is $a(t)$, the acceleration as a function of time for this oscillator? At what time t does $a(t)$ have its first maximum? What is the position x at this time t?

10. On the graph below, sketch an energy vs. position curve for this harmonic oscillator. Be sure to include and label lines for $E_{tot}(x)$, $U(x)$, and $K(x)$. I have labeled the position axis for you, but you will need to label the energy axis.

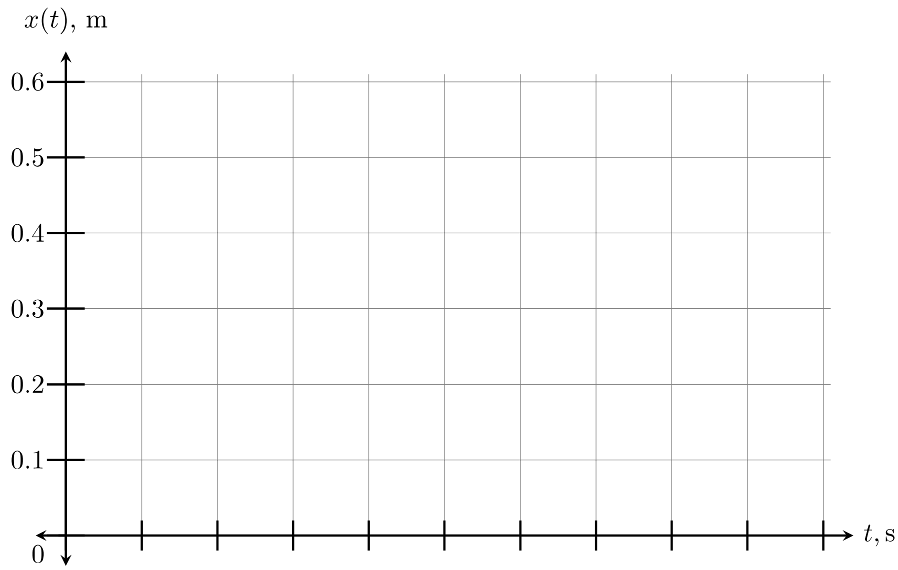


2.3.2 From $U(x)$ to $x(t)$

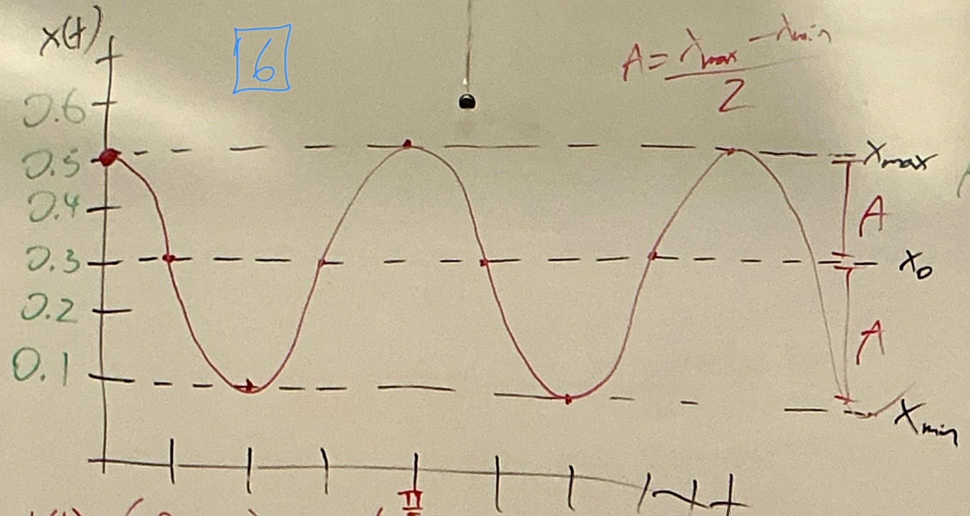
The picture below shows the energy versus position curves of a particular harmonic oscillator, composed of a spring of unknown spring constant k and a mass $m = 10$ kg.



1. What are x_0 and A, the equilibrium position and amplitudes, respectively, of this oscillator?
2. What is the unknown spring constant, k?
3. What is angular frequency ω?
4. Assuming at $t = 0$, the oscillator is at position $x = 0.5$ m, what is the phase constant ϕ_0?
5. Write a function for $x(t)$, the position of the mass on the spring as a function of time.
6. In the space below, draw a graph of $x(t)$ that you found in question 4. I have labeled the positions for you, but you will need to label the times.



$\boxed{6}$



$A = \frac{x_{max} - x_{min}}{2}$

$\boxed{1}$

$x_0 = 0.3$

$A = 0.2$

$\boxed{4}$

$0 = t_{max}$

$wt_{max} + \phi_0 = 0$

$0 + \phi_0 = 0$

$\Rightarrow \phi_0 = 0$

$\boxed{2}$ $U = \frac{1}{2}k(x - x_0)^2$

$5 = \frac{1}{2}k(0.2 - 0.3)^2$

$k = 1000\ kg/s^2$

$m = 10\ kg$

$\boxed{3}$ $w = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}}\ s^{-1}$

$= 10\ s^{-1}$

$\boxed{5}$ $x(t) = (0.2m)\cos(10s^{-1}t) + 0.3m$

$T = \frac{2\pi}{w} = \frac{2\pi}{10} = \frac{\pi}{5}\ s$

$\boxed{2.3.1}$ $\quad * \; m = 4\,kg \qquad * \; x(t) = A\cos(\omega t + \phi_0) + x_0$

$\boxed{1}$ $\quad x_{max} = 6, \quad x_{min} = 1$

$\boxed{2}$ $\quad x_{max} = x_0 + A \Rightarrow 6 = x_0 + A$

$x_{min} = x_0 - A \Rightarrow 1 = x_0 - A$

$\Rightarrow 7 = 2x_0$

$\boxed{x_0 = 3.5\,m}$

$\boxed{A = 2.5\,m}$

$\boxed{3}$ $\quad \boxed{T = 4\,sec}, \quad \boxed{f = \frac{1}{4}\,Hz}, \quad \omega = 2\pi \cdot f = 2\pi \times \frac{1}{4} = \boxed{\frac{\pi}{2}\ \text{radian/s}}$

$\boxed{4}$ $\quad \omega t_{max} + \phi_0 = 0 \Rightarrow \frac{\pi}{2} \cdot 3 + \phi_0 = 0 \Rightarrow \phi = \boxed{-\frac{3\pi}{2}}$

$\Downarrow$

For $x(t)$ to be maximized, we would need $\cos(0) = 1$:

$x(t) = A \cdot 1 + x_0 = 2.5 + 3.5 = \boxed{6m} \Rightarrow$ which is X_{max}

$\boxed{5}$ $\quad \boxed{x(t) = 2.5 \cdot \cos\left(\frac{\pi}{2}t + \frac{-3\pi}{2}\right) + 3.5}$

$\boxed{6}$ $\quad k = \omega^2 \cdot m = \left(\frac{\pi}{2}\right)^2 \cdot 4 = \boxed{\pi^2\ N/m}$

$\boxed{7}$ $\quad U_{Hooke} = \frac{1}{2} k (x - x_0)^2 = \frac{1}{2} \cdot \pi^2 \frac{(6 - 3.5)^2}{(10^2)^2} = \boxed{3.125 \times 10^{-4} \cdot \pi^2\ J}$

$\boxed{8}$ $V_{max} \to K_{max} \to U_{min} = 0$

$$\Rightarrow K + U = E_{total} \Rightarrow \frac{1}{2} m V^2 + 0 = 3.125 \times 10^{-4} \cdot \pi^2$$

$$4 \times V^2 = 6.25 \times 10^{-4} \pi^2$$

$$\boxed{V_{max} = 1.25 \times 10^{-2} \pi \ m/s}$$

$\boxed{9}$

$$x(t) = 2.5 \cdot \cos\left(\frac{\pi}{2} t + \frac{-3\pi}{2}\right) + 3.5$$

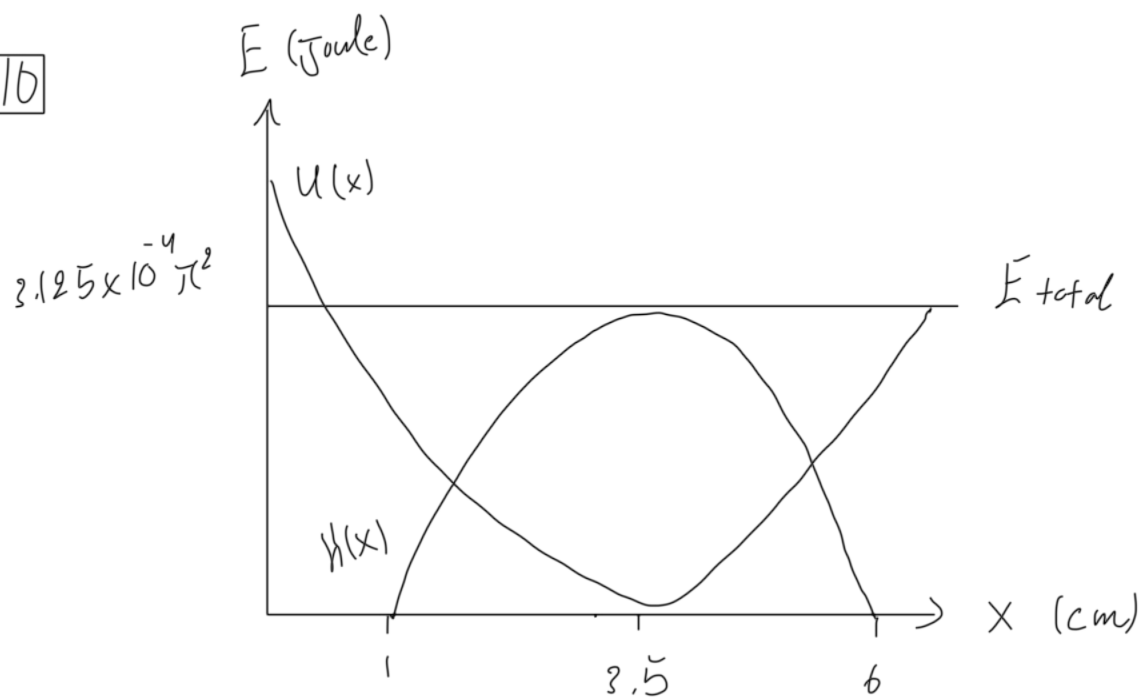
$$V(t) = -2.5 \sin\left(\frac{\pi}{2} t + \frac{-3\pi}{2}\right) \cdot \frac{\pi}{2}$$

$$\boxed{a(t) = -2.5 \cdot \frac{\pi^2}{4} \cos\left(\frac{\pi}{2} t + \frac{-3\pi}{2}\right)}$$

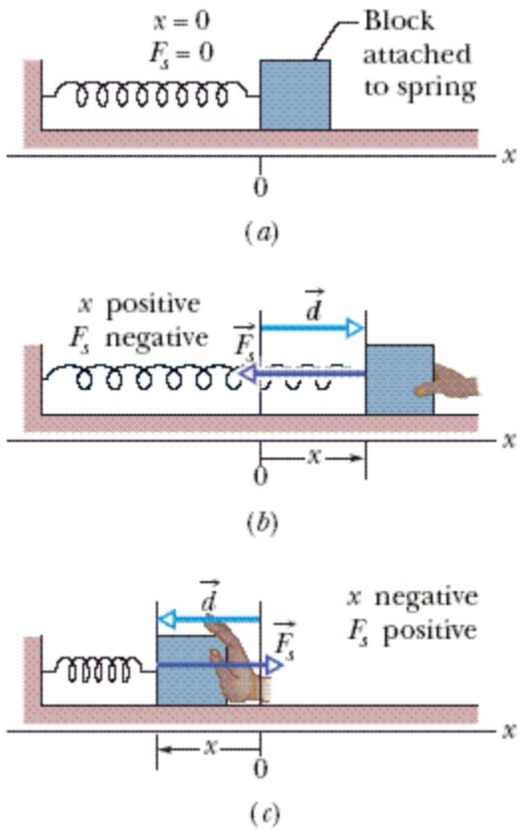
first maximum for $a(t)$ is when $t = 1$

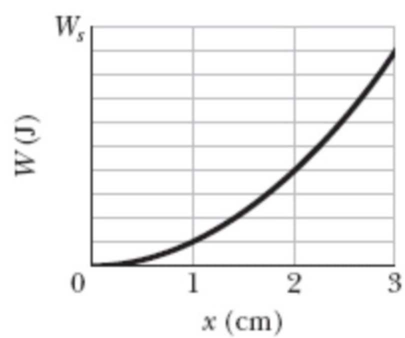
" " for $V(t)$ is when $t = 0$

$\boxed{10}$



In the arrangement of the first figure, we gradually pull the block from $x = 0$ to $x = +3.0$ cm, where it is stationary. The second figure gives the work that our force does on the block. The scale of the figure's vertical axis is set by $W_s = 1.0$ J. We then pull the block out to $x = +7.0$ cm and release it from rest. How much work does the spring do on the block when the block moves from $x_i = +7.0$ cm to **(a)** $x = +4.0$ cm, **(b)** $x = -1.0$ cm, and **(c)** $x = -7.0$ cm?





Solution

Assistance Used

The work done by the spring force is given by equation: $W_s = \frac{1}{2}k\,(x_i^2 - x_f^2)$. The spring constant k can be deduced from the figure which shows the amount of work done to pull the block from 0 to $x = 3.0\ \text{cm}$. The parabola $W_a = kx^2/2$ contains $(0, 0)$, $(2.0\ \text{cm}, 0.40\ \text{J})$ and $(3.0\ \text{cm}, 0.90\ \text{J})$. Thus, we may infer from the data that $k = 2.0 \times 10^3\ \text{N/m}$.

(a) When the block moves from $x_i = +7.0\ \text{cm}$ to $x = +4.0\ \text{cm}$, we have

$$W_s = \frac{1}{2}(2.0 \times 10^3\ \text{N/m})\,[(0.07\ \text{m})^2 - (0.04\ \text{m})^2] = 3.30\ \text{J}.$$

(b) Moving from $x_i = +7.0\ \text{cm}$ to $x = -1.0\ \text{cm}$, we have

$$W_s = \frac{1}{2}(2.0 \times 10^3\ \text{N/m})\,[(0.07\ \text{m})^2 - (-0.01\ \text{m})^2] = 4.80\ \text{J}.$$

(c) Moving from $x_i = +7.0\ \text{cm}$ to $x = -7.0\ \text{cm}$, we have

$$W_s = \frac{1}{2}(2.0 \times 10^3\ \text{N/m})\,[(0.07\ \text{m})^2 - (-0.07\ \text{m})^2] = 0\ \text{J}.$$

A 6.42 g marble is fired vertically upward using a spring gun. The spring must be compressed 4.08 cm if the marble is to just reach a target 23.6 m above the marble's position on the compressed spring. **(a)** What is the change ΔU_g in the gravitational potential energy of the marble-Earth system during the 23.6 m ascent? **(b)** What is the change ΔU_s in the elastic potential energy of the spring during its launch of the marble? **(c)** What is the spring constant of the spring?

(a) Number 1.48 Units J

(b) Number -1.48 Units J

(c) Number 1783.95 Units N/m

Solution

THINK As the marble moves vertically upward, its gravitational potential energy increases. This energy comes from the release of elastic potential energy stored in the spring.

EXPRESS We take the reference point for gravitational potential energy to be at the position of the marble when the spring is compressed. The gravitational potential energy when the marble is at the top of its motion is $U_g = mgh$. On the other had, the energy stored in the spring is $U_s = kx^2/2$. Applying mechanical energy conservation principle allows us to solve the problem.

ANALYZE

(a) The height of the highest point is $h = 23.6\ \text{m}$. With initial gravitational potential energy set to zero, we find

$$\Delta U_g = mgh = (6.42 \times 10^{-3}\ \text{kg})(9.8\ \text{m/s}^2)(23.6\ \text{m}) = 1.48\ \text{J}.$$

(b) Since the kinetic energy is zero at the release point and at the highest point, then conservation of mechanical energy implies $\Delta U_g + \Delta U_s = 0$, where ΔU_s is the change in the spring's elastic potential energy. Therefore, $\Delta U_s = -\Delta U_g = -1.48\ \text{J}$.

(c) We take the spring potential energy to be zero when the spring is relaxed. Then, our result in the previous part implies that its initial potential energy is $U_s = 1.48\ \text{J}$. This must be $\frac{1}{2}kx^2$, where k is the spring constant and x is the initial compression. Consequently,

$$k = \frac{2U_s}{x^2} = \frac{2(1.48\ \text{J})}{(0.0408\ \text{m})^2} = 1780\ \text{N/m}.$$

LEARN In general, the marble has both kinetic and potential energies:

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + mgy$$

At the maximum height $y_{\text{max}} = h, \quad v = 0 \quad \text{and} \quad mgh = kx^2/2, \quad \text{or} \quad h = \frac{kx^2}{2mg}.$

An oscillator consists of a block attached to a spring (k = 306 N/m). At some time t, the position (measured from the system's equilibrium location), velocity, and acceleration of the block are x = 0.147 m, v = -17.7 m/s, and a = -130 m/s^2. Calculate **(a)** the frequency of oscillation, **(b)** the mass of the block, and **(c)** the amplitude of the motion.

(a) Number 4.73 Units Hz

(b) Number 0.346 Units kg

(c) Number 0.613 Units m

eTextbook and Media

Hint Assistance Used

Solution Assistance Used

Please Note: the variables in this *video solution* correspond to the problem as it is presented in your *textbook*.

Video Solution

(a)$a(t) = -\omega^2 x(t)$ leads to

$$a = -\omega^2 x \Rightarrow \omega = \sqrt{\frac{-a}{x}} = \sqrt{\frac{130\ \text{m/s}^2}{0.147\ \text{m}}} = 29.738\ \text{rad/s}.$$

Therefore, $f = \omega/2\pi = 4.73\ \text{Hz}.$

(b)$\omega = \sqrt{k/m}$ provides a relation between ω (found in the previous part) and the mass:

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow m = \frac{306\ \text{N/m}}{(29.738\ \text{rad/s})^2} = 0.346\ \text{kg}.$$

(c) By energy conservation, $\frac{1}{2}kx_m^2$ (the energy of the system at a turning point) is equal to the sum of kinetic and potential energies at the time t described in the problem.

$$\frac{1}{2}kx_m^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \Rightarrow x_m = \sqrt{\frac{m}{k}v^2 + x^2} = \sqrt{\left(\frac{v}{\omega}\right)^2 + x^2}.$$

Consequently, $x_m = \sqrt{\left(\frac{17.7\ \text{m/s}}{29.738\ \text{rad/s}}\right)^2 + (0.147\ \text{m})^2} = 0.613\ \text{m}.$

A 24 g particle undergoes SHM with an amplitude of 5.8 mm, a maximum acceleration of magnitude 5.5×10^3 m/s^2, and an unknown phase constant ϕ. What are **(a)** the period of the motion, **(b)** the maximum speed of the particle, and **(c)** the total mechanical energy of the oscillator? What is the magnitude of the force on the particle when the particle is at **(d)** its maximum displacement and **(e)** half its maximum displacement?

(a)	Number	0.0064	Units	s
(b)	Number	5.648	Units	m/s
(c)	Number	0.3828	Units	J
(d)	Number	132	Units	N
(e)	Number	66	Units	N

Solution

Assistance Used

The textbook notes that the acceleration amplitude is $a_m = \omega^2 x_m$, where ω is the angular frequency and $x_m = 5.8 \times 10^{-3}$ m is the amplitude. Thus, $a_m = 5.5 \times 10^3\ \mathrm{m/s^2}$ leads to $\omega = 973.79\ \mathrm{rad/s}$. Using Newton's second law with $m = 24 \times 10^{-3}$ kg, we have

$$F = ma = m(-a_m \cos(\omega t + \varphi)) = -(132.0\ \mathrm{N})\cos(973.79t + \varphi)$$

where t is understood to be in seconds.

(a) Equation $\omega = \dfrac{2\pi}{T}$ gives $T = 2\pi/\omega = 0.0065$ s.

(b) The relation $v_m = \omega x_m$ can be used to solve for v_m, or we can pursue the alternate (though related) approach of energy conservation. Here we choose the latter. By $\omega = \sqrt{\dfrac{k}{m}}$, the spring constant is $k = \omega^2 m = 22758\ \mathrm{N/m}$. Then, energy conservation leads to

$$\tfrac{1}{2}kx_m^2 = \tfrac{1}{2}mv_m^2 \quad \Rightarrow v_m = x_m\sqrt{\tfrac{k}{m}} = 5.6\ \mathrm{m/s}.$$

(c) The total energy is $\dfrac{1}{2}kx_m^2 = \dfrac{1}{2}mv_m^2 = 0.38$ J.

(d) At the maximum displacement, the force acting on the particle is

$$F = kx = (22758\ \mathrm{N/m})(5.8 \times 10^{-3}\mathrm{m}) = 130\ \mathrm{N}.$$

(e) At half of the maximum displacement, $x = 2.9$ mm, and the force is

$$F = kx = (22758\ \mathrm{N/m})(2.9 \times 10^{-3}\mathrm{m}) = 66\ \mathrm{N}.$$

Suppose that a simple pendulum consists of a small 54 g bob at the end of a cord of negligible mass. If the angle θ between the cord and the vertical is given by
$\theta = (0.045 \text{ rad}) \cos[(3.6 \text{ rad/s})\, t + \phi]$,
what are **(a)** the pendulum's length and **(b)** its maximum kinetic energy?

(a) Number 0.756 Units m

(b) Number 0.00041 Units J

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Hint — Assistance Used

Solution — Assistance Used

(a) Comparing the given expression to $x(t) = x_m \cos(\omega t + \varphi)$ (after changing notation $x \to \theta$), we see that $\omega = 3.6 \text{ rad/s}$. Since $\omega = \sqrt{g/L}$ then we can solve for the length: $L = 0.7562 \text{ m} \approx 0.76 \text{ m}$.

(b) Since $v_m = \omega x_m = \omega L \theta_m = (3.6 \text{ rad/s})(0.7562 \text{ m})(0.045 \text{ rad})$ and $m = 54 \times 10^{-3} \text{ kg}$, then we can find the maximum kinetic energy: $\frac{1}{2} m {v_m}^2 = 4.1 \times 10^{-4} \text{ J}$.