Damped & Driven Harmonic Oscillator

2 Types of Harmonic Oscillator:

- Undriven Harmonic Oscillator (Initial external force is not repeated)
  - Simple Harmonic Oscillator (SHO)

$$\boxed{\frac{d^2x}{dt^2} + w^2 x = 0}$$

$\Rightarrow$ SHO oscillation is repeated forever if the only external force applied to the system is the initial force

$F_{(external)} = -F_{(Hooke)}$ $\Rightarrow$ negative sign implies that $F_{(Hooke)}$ is in the opposite direction of $F_{(external)}$ [displaced Object]

  - Damped Harmonic Oscillator (DHO)

$$\boxed{\frac{d^2x}{dt^2} + \left(\beta \frac{dx}{dt} + w_0^2 x = 0\right)}$$

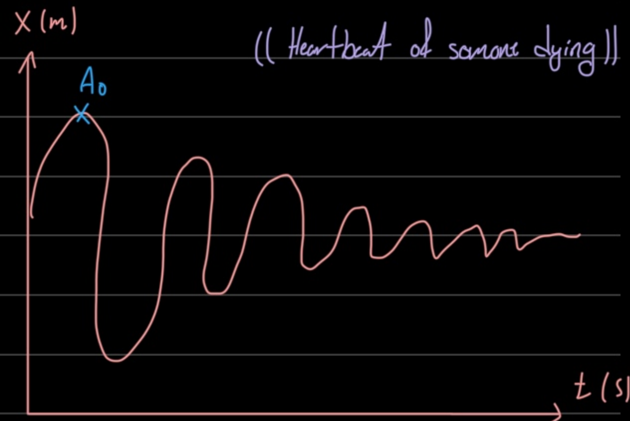
$\Rightarrow$ DHO oscillation is not repeated forever and will eventually stop... returning back to its equilibrium state... This is due to the involvement of non-conservative forces... eating away the system total energy until it reaches zero... which means amplitude should be decreasing with time...

- Equations: (equation for exponential decay)

$$\boxed{a + \beta v + w_0^2 x = 0} \quad (1)$$

To find the relationship between W, W_0, & β we found the derivates of the below equation (2), (3), & (4) and plugged it back to (1) to get (5)...

((Heartbeat of someone dying))



(2)

$$\boxed{x(t) = A_0 e^{-\frac{\beta}{2}t} \cos(Wt + \phi_0) + x_0}$$

$$\boxed{A(t) = A_0 e^{-\frac{\beta}{2}t}}$$

The things we can interpret from graph:

- A_0 & A (Initial A & Damped A)
- T (only Damped T not T_0)

→ using those information we can find:

- β (damped factor)

(3)

$$\boxed{V(t) = -\left(\frac{\beta}{2} + W\right) \cdot x(t)}$$

$\rightarrow$ W (damped angular frequency)

$\rightarrow$ Now we can find (W_0) which is essential to know (k) constant Hooke

$$\boxed{a(t) = \left(\frac{\beta^2}{4} - w^2 + \beta w\right) \cdot x(t)} \quad (4)$$

$$\boxed{W = \sqrt{W_0^2 - \frac{\beta^2}{4}}} \quad (5)$$

$\rightarrow$ 3 types of DHO :

- $\rightarrow$ underdamped $\Rightarrow$ $W_0 > \frac{\beta}{2}$ $\Rightarrow$ Oscillates + gradually returns to equilibrium...
- $\rightarrow$ critically damped $\Rightarrow$ $W_0 = \frac{\beta}{2}$ $\Rightarrow$ no oscillation + quickly returns to equilibrium...
- $\rightarrow$ Overdamped $\Rightarrow$ $W_0 < \frac{\beta}{2}$ $\Rightarrow$ no Oscillation + slowly returns to equilibrium... Imaginary slowest

$\rightarrow$ Damped Ratio (δ) : measures how damped is the system...

$$\boxed{\delta = \frac{\beta}{2W_0}}$$

$\Rightarrow$ $\delta = 0$ (undamped) $\quad$ $0 < \delta < 1$ (underdamped)

$\delta = 1$ (critically damped) $\quad$ $\delta > 1$ (Overdamped)

$\rightarrow$ Quality factor (Q) : measures how fast is the energy being depleted from the system...

$$\boxed{Q = \frac{1}{2\delta} = \frac{W_0}{\beta}}$$

$\Rightarrow$ The larger the value of (δ), the lower the value of (Q), and the faster the energy is being depleted...
This is why (Overdamped systems) is the slowest at reaching equilibrium due to its lower levels of Total energy...

$\rightarrow$ Driven Harmonic Oscillator (Initial external force repeated over time)

$$\boxed{\frac{d^2x}{dt^2} + \beta \cdot \frac{dx}{dt} + W_0^2 x = \sin(W_D)}$$

$\rightarrow$ This equation implies that for a damped system to be able to contiously oscillate without reaching equilibrium... we would need to repeatedly applie the initial external force over time... replenishing the energy that was lost to non-conservative forces...

In DHO, we've learned that the Amplitude & phase constant change over time until it reaches equilibrium... This change is decreasing at a constant rate of ($e^{-\frac{\beta}{2}t}$) which means it follows a predictable pattern...

However, with driven harmonic oscillator, the change of amplitude & phase constant depends on several factors:

- The repeated initial external force (Driven force)
  - How well our driven force's angular frequency (W_D) matches the system's natural frequency (W_0)... when ($W_D = W_0$), we call it ((Resonance))..
    - The energy supplied to the system by driven force is being fully absorbed ((not wasted))... maximizing our increased Amplitude...
    - It is the moment the driven force is applied to the System when it reaches its maximum amplitude... as when you push someone riding a swing that is about to swing back... It is all about ((timing))...
  - How strong is our Driven Force...
  - How damped is our system (β)...

Therefore, to account for such factors we use the following equation:

$$\boxed{x(t) = A\sin(W_D t) + B\cos(W_D t)}$$

$$\boxed{\underbrace{\sqrt{A^2 + B^2}}_{\text{(Total Amplitude)}} = \frac{1}{\sqrt{(W_0^2 - W_D^2)^2 + \beta^2 W_D^2}}}$$

	SHO	underdamped	critically damped	overdamped	Driven harmonic oscillator
Initial force	(All 4 cars have driven over one speed bump)				multiple speed bumps
car's suspension system	Bounces forever	Bounces in a certain time frame	only compresses & quickly returns to its initial state	only compresses & slowly returns to its initial state	Oscillation never stop as SHO... However, how it oscillates depends on how damped the system...

2.4 Damped and Driven Oscillators

2.4.1 Exploring Damping in Desmos

For this activity, please turn on the computers that have been given to each group and open Desmos, a link to which can be found on the course Moodle page.

1. Start by plotting the function

$$y(t) = Ae^{-\frac{\beta}{2}t}\cos\left(\left(\sqrt{\omega_0^2 - \frac{\beta^2}{4}}\right)t + \phi_0\right) \tag{2.7}$$

in Desmos, adding a slider for A, β, ω_0 and ϕ_0. I would set the scale on your sliders so that all four parameters are strictly positive. (Note: feel free to use other names for the variables, like w instead of ω_0; I couldn't get Desmos to recognize ω!)

2. Now try varying the parameters by adjusting the settings on your slider. First vary A and ϕ_0. What changes in the graph? Does this match what you expect, based on what you know about the undamped simple harmonic oscillator?

3. Now try varying ω_0 while keeping β fixed. What happens to the graph? What happens to the rate of decay of the amplitude as ω_0 is varied?

4. Now keep ω_0 fixed and vary β. What happens as β is varied?

5. Now set β and ω_0 to the critical damping condition. What does the graph look like now? What functional form would you say this resembles?

6. Now set β and ω_0 so that the system is overdamped. On a scale of 1 to 10, how lame is Desmos for not being able to take the cosine of an imaginary number?

7. To graph the overdamped region, we have to trick Desmos by using the definition of the hyperbolic cosine

$$\cosh(x) = \cos(ix) \tag{2.8}$$

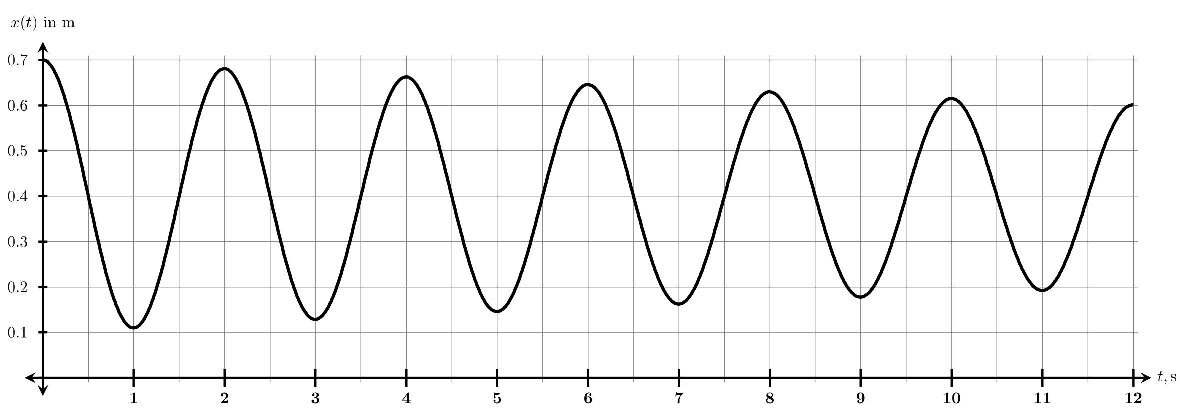
Hence, if we modify our input to

$$y(t) = Ae^{-\frac{\beta}{2}t}\cosh\left(\left(\sqrt{\frac{\beta^2}{4} - \omega_0^2}\right)t + \phi_0\right) \tag{2.9}$$

we can explore the overdamped region. Keeping ω_0 fixed, gradually increase β. What do you notice? As β - i.e., the damping - gets larger, does it take a longer or shorter time for the oscillator to decay? Does this answer surprise you?

2.4.2 A Damping Problem

The graph below shows the position as a function of time for a 0.200 kg mass attached to a spring of unknown spring constant k.



1. Notice that our damped harmonic oscillator acts like a simple harmonic oscillator whose amplitude A is changing in time according to the equation
$$A(t) = A_0 e^{-\frac{\beta}{2}t} \tag{2.10}$$
By estimating $A(t)$ at two different points in time, use the graph above to find β. (Hint: you will have to use logarithms!)

2. What is the approximate quality factor Q of this oscillator? Hint: remember the period you read from the graph is T, the effective period, not T_0, the natural period!

3. What is the unknown spring constant k?

2.4.3 Exploring Resonance in Desmos

In lecture we saw that the amplitude response of the driven harmonic oscillator was given by

$$A = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_D^2\right)^2 + \beta^2\omega_D^2}} \tag{2.11}$$

1. Using Desmos, graph A as a function of ω_D, the driving frequency (that is to say, let $y = A$ and $x = \omega_D$) on the region $0 \leq \omega_D \leq 5$. You will want to add a slider for both β and ω_0; for the moment, we will set the slider for $\omega_0 = 1$. (This is equivalent to setting up the scale of your graph x axis so that it is in units of the natural angular frequency.) Set the β slider so that it goes between 0 and 3, with increments of 0.05. Start with $\beta = 0.2$. What do you notice about the curve, for the amplitude of the induced oscillations as a function of the input angular frequency? Is there a maximum - that is, a particular frequency for which the response of the oscillator has the largest amplitude? If so, about where does this maximum occur?

2. Now press the "play" button next to the β slider, which will cause Desmos to cycle through the various different curves with different βs. As the strength of the damping increases, what do you notice happens to the peak of the resonance curve? See if you can say something about both the size of the peak and the peak location on the ω_D axis.

3. Describe some general features of these resonance curves. Regardless of the setting of β, what happens to the amplitude as ω_D gets very small relative to ω_0? What about if ω_D gets very large relative to ω_0?

4. Does the peak eventually disappear as β is varied? If so, at approximately what ζ does this peak go away? Remember, ζ is the *damping ratio*, defined such that
$$\zeta = \frac{\beta}{2\omega_0} \tag{2.12}$$
(Hint: you might have to zoom in on the curve to see exactly where the maximum disappears!)

5. To find the exact value of where the resonance peak has a zero, we can find the critical points of $A(\omega_D)$. To find such critical points, observe that, since inverse square roots are monotonically decreasing functions, $A(\omega_D)$ has a maximum wherever the argument of the square root
$$f(\omega_D) = (\omega_0^2 - \omega_D^2)^2 + \beta^2\omega_D^2 \tag{2.13}$$
has a minimum. Find $\frac{df}{d\omega_D}$, and then set this result equal to zero to find the critical points of $f(\omega_D)$ and therefore the value for ω_D (in terms of ω_0 and β) that gives the maximum amplitude response.

6. For what value of ζ does this critical point disappear? How close is this exact answer to the approximate answer you found previously?

7. Now set the β slider to a constant value and try varying the value for ω_0. What changes about the graph as ω_0 is varied?

$\boxed{2.4.1}$

$\boxed{2}$ A: by changing Amplitude, it increases peak displacement

ϕ: by changing phase constant, we are shifting the function changing the starting point of function

$\boxed{3}$ W_0: Angular frequency increases # of oscillation per second having the same rate of decay...

$\boxed{4}$ β: the rate of decay increases as β increase

$\boxed{5}$ looks like exponential function ...

$\boxed{6}$ pretty lame...

$\boxed{7}$ As we increase β... it takes longer for the function to reach zero...

$\boxed{2.4.2}$ (m= 0.200kg)

$\boxed{1}$

$$A(t) = A_0 e^{-\frac{\beta}{2}t} \Rightarrow \frac{2}{3} = e^{-6\beta}$$

$$A(t) = 0.3\, e^{-\frac{\beta}{2}t} \qquad \ln\frac{2}{3} = -6\beta \Rightarrow \boxed{\beta = 0.0675\ s^{-1}}$$

$$0.2 = 0.3\, e^{-\frac{\beta}{2}\cdot 12} \qquad \beta = \frac{\ln(2/3)}{-6}$$

$\boxed{2}$

$$Q = \frac{1}{2\delta} = \frac{1}{2} \times \frac{2W_0}{\beta} = \frac{W_0}{\beta} \to \boxed{??}$$

$$\Downarrow$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{W_0^2 - \frac{\beta^2}{4}}} \Rightarrow 2 = \frac{2\pi}{\sqrt{W_0^2 - \frac{\beta^2}{4}}} \Rightarrow W_0^2 - \frac{\beta^2}{4} = \pi^2$$

$$W_0 = \sqrt{\pi^2 + \frac{\beta^2}{4}} = \sqrt{\pi^2 + \frac{0.0676}{4}} = 3.144 \ s^{-1}$$

$$Q = \frac{3.144}{0.0675} = \boxed{46.52}$$

How many oscillation until it loses most of its energy...

$\boxed{3}$ $W_0 = \sqrt{\frac{K}{m}} \Rightarrow K = W_0^2 \cdot m = 3.144^2 \cdot 0.2$

$$\boxed{K = 1.977 \ N/m}$$

$\boxed{2.4.3}$

$\boxed{1}$ there is a maximum A occuring at 5...

$\boxed{2}$ The A decreases & W_0 decreases

$\boxed{3}$

$\boxed{5}$

$$\frac{df(\omega_D)}{d\omega_D} = (\omega_0^2-\omega_D^2)^2 + \beta^2\omega_D^2$$

$$0 = 2(\omega_0^2-\omega_D^2)(-2\omega_D) + 2\beta^2\omega_D$$

$$0 = 2\cancel{\omega_D}\left(-2\omega_0^2+2\omega_D^2+\beta^2\right)$$

$$0 = -\omega_0^2+\omega_D^2+\frac{\beta^2}{2}$$

$$\omega_D = \sqrt{\omega_0^2-\frac{\beta^2}{2}}$$

Peak disappears!

$$\omega_0^2 = \frac{\beta^2}{2}$$

$$\omega_0 = \frac{\beta}{\sqrt{2}}$$

$\boxed{6}$

$$\zeta = \frac{\beta}{2\omega_0}$$

$$\zeta = \frac{\cancel{\beta}}{2\frac{\cancel{\beta}}{\sqrt{2}}}$$

$$\zeta = \frac{\sqrt{2}}{2}$$