Moments of Inertia

① Main idea : [1] Where is the axis of rotation located in an object ?!
[2] How to calculate moments of inertia ?!
[3] Understanding physical pendulum along with moments of inertia ..

② Where is the axis of rotation located in an object ?!

The total momentum of each molecules add up to equal the momentum of the whole system...

\# of molecules

[1]

$$P_{total} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m \vec{v}_i$$

$$M_{total} \cdot V_{total} = \sum_{i=1}^{n} m \cdot \frac{d\vec{x}_i}{dt}$$

$$V_{total} = \frac{d\left(\sum_{i=1}^{n} \frac{m \cdot \vec{x}_i}{M}\right)}{dt}$$

[2]

$$\Rightarrow \boxed{X_{c.o.m} = \sum_{i=1}^{n} \frac{m \cdot \vec{x}_i}{M}}$$

when we added up the masses of each molecule based on how far it is from a selected coordinated system and devide it by total mass it gives us the exact location of where the center of mass located... it is like when we calculate our GPA... each class we take has its own weight on the GPA based on the score we got and the # of credit hours and when we added up all the classes and devide it by total Credit hours taken... it represents GPA

$$GPA = \sum_{i=1}^{n} \frac{hour \cdot Grade}{Total\ hours}$$

It represent our average grade based on hours However, ($X_{c.o.m}$) is the average distance of all molecules based on the molecule mass...

[3] In the reference of frame where the c.o.m is stationary :

$$\frac{d\left(\sum_{i=1}^{n} \frac{m \cdot \vec{x}_i}{M}\right)}{dt} = 0$$

which means that all molecules are rotating around the c.o.m ... and the axis of rotation is always at the c.o.m ... unless the axis of rotation was set to be other than that ..

③ How to calculate moments of inertia (I) ?!

[1] When axis of rotation is at c.o.m

[2] When axis of rotation is out of c.o.m

$$\boxed{I_{c.o.m} = \int r^2 \cdot dm}$$

$$\boxed{I_{o.c.o.m} = I_{c.o.m} + \underbrace{M}_{\text{total mass}}\underbrace{d^2}_{\text{distance from c.o.m}}}$$

dm : infinitesimal mass element... which means how much does the molecule's mass added up to the total mass as radius increases ((how much will the next fallen class affect our GPA))

3

⇒ dm = Density x infinitesimal (volume, area, length)

- for 1D object (like a thin rod):
  - linear mass density $(\lambda) = \frac{M}{L}$
  - $\boxed{dm = \lambda \cdot dx}$
- for 2D object (like a disk or a plate)
  - surface mass density $(\sigma) = \frac{M}{A}$
  - $\boxed{dm = \sigma \cdot dA}$
- for 3D object (like a solid sphere or cylinder):
  - Volume mass density $= (\rho) = \frac{M}{V}$
  - $\boxed{dm = \rho \cdot dV}$

⇓

$I_{c.o.m} = \int \rho r^2 \cdot dV$ ((like our slide ☺))

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Memorizing $I_{c.o.m}$ values:

- Sphere $= \frac{2}{5} MR^2$
- Cylinder or disk $= \frac{1}{2} MR^2$
- Hallow disk (ring) $= MR^2 = M \frac{(a^2 + b^2)}{2}$ (a: outer radius, b: Inner radius)
- bar $= \frac{1}{12} ML^2$

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Note: moments of inertia is the angular analogue of inertial mass... As inertial mass resists changes in transitional motion, moments of inertia resists changes in Rotational motion...

④ Understanding physical pendulum along with moments of inertia ?!

1

$\vec{\tau} = \vec{r} \cdot \vec{F_g} \cdot \sin\theta$

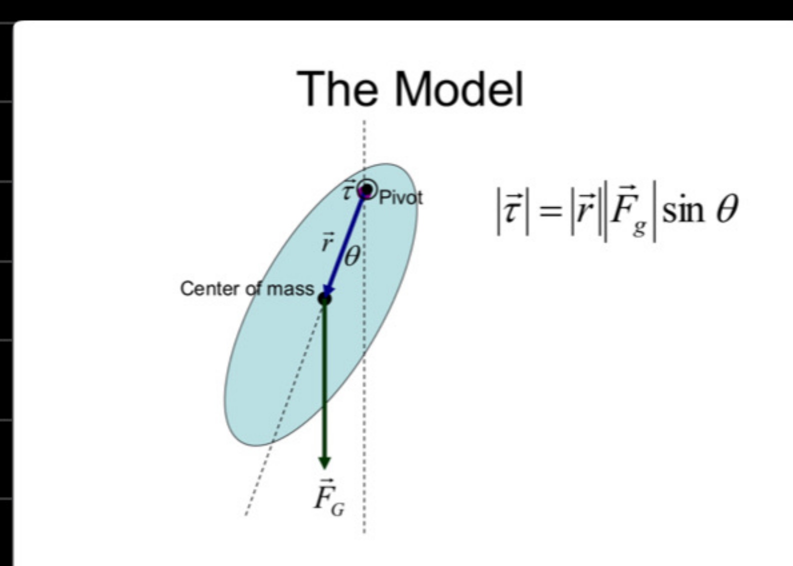
$-I \cdot \alpha = l \cdot mg \cdot \sin\theta$

$\frac{d^2\theta}{dt} = \frac{-l\, mg \cdot \sin\theta}{I}$

$\frac{d^2\theta}{dt} = -\frac{l mg}{I} \cdot \theta$

$\vec{\tau}$ negative since body is rotating in the direction of decreasing θ ...

$\sin\theta \approx \theta$



$$\boxed{W = \sqrt{\frac{l\, mg}{I}}}$$

, where l equal d (distanse between axis of rotation and C.o.m)...

2 when $I = I_{o.c.o.m}$ (ring) $\Rightarrow W = \sqrt{\dfrac{l\,m\,g}{mr^2+md^2}} = \sqrt{\dfrac{l.g}{r^2+l^2}}$

$$\Rightarrow \lim_{\boxed{l>>r}} \sqrt{\dfrac{l.g}{l^2}} \Rightarrow W = \sqrt{\dfrac{g}{l}}$$

this means when l is exponentially larger than r, then r^2 is neglible compared to l^2

$\Rightarrow r^2 + l^2 \approx l^2$ ... it returns back to being a simple pendulum... since l is way larger than r it means that the axis of rotation exists farther away from the object itself (axis outside of the object)... it is no longer experiencing rotational motion... instead it is circular motion ☺...

3 $I \propto m \Rightarrow$ However, W is independent of mass since we are able to cancel the numerator mass with the denominator mass...

$\boxed{W \propto l \;\&\; r}$ it depends on how the mass is spread out !!