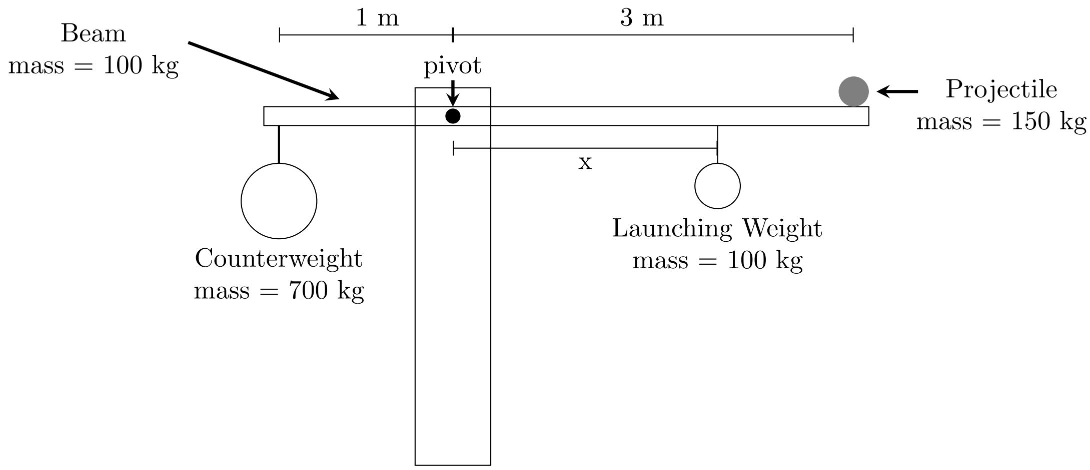
Problem 1: (25 points) A type of ancient military weapon called a trebuchet is shown in the figure below.



(a) (15 points) If the trebuchet is observed to be in static equilibrium, what is the distance x between the launching weight and the pivot point?

Notice there are several ways to solve this problem, such as by finding x such that the center of mass of the entire system is at the pivot point, but I shall show a solution that uses torque balance. Notice we have four torques: that due to the weight force of counterweight, which points out of the screen, and those due to the weight forces of the beam, projectile, and launching weight, all which point inward. Notice for all four cases, $\vec{r}$ is perpendicular to $\vec{F}$, and for the beam, which is assumed to be uniform, the center of mass is in the middle, one meter to the right of the pivot. Hence for the torques to balance

$$\begin{aligned}(1\text{ m})(700\text{ kg})g &= (1\text{ m})(100\text{ kg}) + (3\text{ m})(150\text{ kg})g + x(100\text{ kg})g \\ \Rightarrow 700\text{ kg}\cdot\text{m} - 100\text{ kg}\cdot\text{m} - 450\text{ kg}\cdot\text{m} &= (100\text{ kg})x \Rightarrow x = 1.5\text{ m}\end{aligned} \tag{1}$$

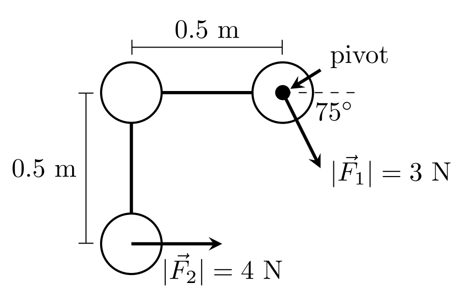
(b) (10 points) What is the magnitude and direction of the force exerted at the pivot point?

For the entire system to be in static equilibrium, there must be no net force on the beam. Hence, the four weight forces plus the force on the pivot must add up to give zero. Hence the force on the pivot must be equal and opposite the entire weight force of the beam and the various weights. That is:

$$\left|\vec{F}_{pivot}\right| = M_{total}g = (700\text{ kg} + 100\text{ kg} + 100\text{ kg} + 150\text{ kg})(9.80\text{ m/s}^2) = 10290\text{ N} \tag{2}$$

with a direction that must be upward, opposite the direction of the weight forces.

Problem 2: (25 points) Three 0.2 kg balls are connected by massless rods, as shown in the figure below. The system is constrained to rotate about the labeled pivot point. Two forces act on the system at the locations shown, and you may ignore all other forces acting on the system (such as the force of gravity).



(a) (15 points) What is the angular acceleration $\vec{\alpha}$ of this system about the pivot point? Be sure to give both magnitude and direction.

First we find the net torque around the given pivot. Notice that the torque due to $\vec{F}_1$ is zero, since $\vec{F}_1$ is applied at the pivot point, and therefore $\vec{r}_1 = 0$. For the torque due to $\vec{F}_2$, observe that the lever arm - i.e., the part of $\vec{r}_2$ perpendicular to $\vec{F}_2$ - is 0.5 m. Hence

$$\tau_2 = \tau_{net} = l\left|\vec{F}_2\right| = (0.5 \text{ m})(4 \text{ N}) = 2 \text{ N}\cdot\text{m} \tag{3}$$

Notice also that, by our right hand rule, this net torque is directed out of the plane of the screen. Now we find the moment of inertia around the axis of rotation.

$$I = \sum_i m_i r_i^2 = (0.2 \text{ kg})(0 \text{ m})^2 + (0.2 \text{ kg})(0.5 \text{ m})^2 + (0.2 \text{ kg})\left(\sqrt{(0.5 \text{ m})^2 + (0.5 \text{ m})^2}\right)^2 = 0.15 \text{ kg}\cdot\text{m}^2 \tag{4}$$

So, via our Newton's second law for rotational motion

$$\tau_{net} = I\alpha \Rightarrow \alpha = \frac{\tau}{I} = \frac{2 \text{ N}\cdot\text{m}}{0.15 \text{ kg}\cdot\text{m}^2} \approx 13.3 \text{ rad/s}^2 \tag{5}$$

Note that the direction of α is the same as τ_{net}, so out of the plane of the screen.

(b) (10 points) If the angular acceleration you found in part in part (a) were constant and the system started from rest, how many complete revolutions would the system pass through in 3 seconds?

Notice we know $\Delta t = 3$ s, $\alpha = 13.3$ rad/s^2, and $\omega_i = 0$, while we seek $\Delta\theta$. Hence we use the constant α rotational kinematics equation

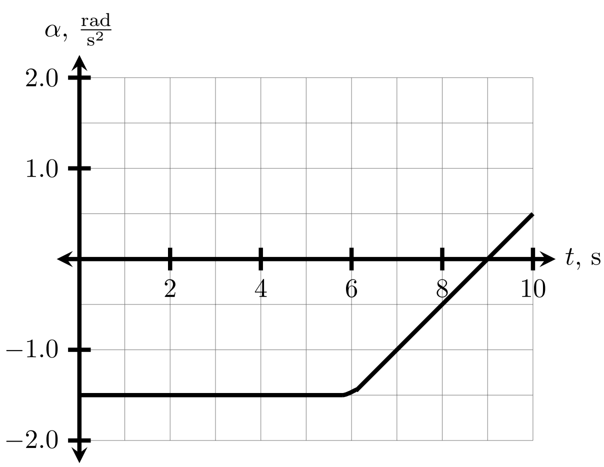
$$\Delta\theta = \frac{1}{2}\alpha\left(\Delta t\right)^2 + \omega_i \Delta t = \frac{1}{2}(13.3 \text{ rad/s}^2)(3 \text{ s})^2 + 0 = 60 \text{ rad} \tag{6}$$

To get the number of revolutions, we then divide by 2π, since there are 2π radians in one revolution.

$$\frac{60 \text{ rad}}{2\pi} \approx 9.55 \text{ revolutions} \tag{7}$$

Hence the system passes through nine complete revolutions in those 3 seconds!

Problem 3: (25 points) The plot below shows the angular acceleration α of a uniform sphere of mass 1.25 kg and radius 3.56 cm about an axis through its center as a function of the time t. At time $t = 2$ seconds, the uniform sphere is observed to be rotating at an angular speed of 3.0 rad/s in the direction of positive θ.



a). (15 points) What is the rotational kinetic energy of the uniform sphere at $t = 9$ seconds?

Notice we first have to find the change in ω between $t = 2$ (where ω is known) and $t = 9$ (where it is sought). In direct analogue to the linear variables, the integral of α gives $\Delta\omega$. This integral is approximately a rectangle plus a triangle. Hence

$$\Delta\omega = (-1.5 \text{ rad/s}^2)(6 \text{ s} - 2 \text{ s}) + \frac{1}{2}(9 \text{ s} - 6 \text{ s})(-1.5 \text{ rad/s}^2) \approx -8.25 \text{ s}^{-1} \tag{8}$$

Hence at $t = 9$ seconds

$$\omega_f = \omega_i + \Delta\omega = 3.0 \text{ rad/s} - 8.25 \text{ rad/s} = -5.25 \text{ rad/s} \tag{9}$$

and thus

$$I = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}(1.25\text{ kg})(0.0356\text{ m})^2\right)(-5.25\text{ rad/s})^2 \approx 0.00873\text{ J} \tag{10}$$

b). (10 points) What is the total change of angle $\Delta\theta$ of the sphere between $t = 0$ and $t = 5$ seconds?

Notice the angular acceleration α is a constant $\alpha = -1.5$ rad/s^2 in this region; we need only find the initial ω to use rotational kinematics! Notice between $t = 0$ and $t = 2$, our integral is a rectangle; hence

$$\Delta\omega = (-1.5\text{ rad/s}^2)(2\text{ s}) = -3\text{ rad/s} \Rightarrow \omega_i = \omega_f - \Delta\omega = 3.0\text{ rad/s} - (-3\text{ rad/s}) = 6.0\text{ rad/s} \tag{11}$$

Notice then by rotational kinematics

$$\Delta\theta = \frac{1}{2}\alpha(\Delta t)^2 + \omega_0\Delta t = \frac{1}{2}(-1.5\text{ rad/s}^2)(5\text{ s})^2 + (6.0\text{ rad/s})(5\text{ s}) \approx 11.25\text{ rad} \tag{12}$$

Problem 4: (55 points total): The rocky planet Lilliania has a rotational period of 14.5 hours, a total mass of 2.23×10^{24} kg, and a radius of 4440 km. Lilliania orbits the star Babababa (which has a mass of 3.45×10^{30} kg) with an orbital period of 205.6 earth days and an eccentricity of 0.056.

(a) (20 points) From the information given, estimate the perihelion and aphelion distances for the planet Lilliania.

Notice the semi-major axis of Lilliania's orbit can be estimated by

$$T^2 = \frac{4\pi^2}{GM}r^3 \Rightarrow r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} \tag{13}$$

where M is the mass of the star and

$$T = 205.6\text{ days} \cdot \frac{24\text{ hr}}{1\text{ day}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{60\text{ s}}{1\text{ m}} \approx 1.776 \times 10^7\text{ s} \tag{14}$$

Hence

$$r = \frac{r_a + r_p}{2} \approx \sqrt[3]{\frac{(6.67 \times 10^{-11}\text{ m}^3/\text{kg}\cdot\text{s}^2)(3.45 \times 10^{30}\text{ kg})(1.776 \times 10^7\text{ s})^2}{4\pi^2}} \approx 1.225 \times 10^{11}\text{ m} \tag{15}$$

Hence

$$r_a = 2(1.225 \times 10^{11}\text{ m}) - r_p \tag{16}$$

But of course the eccentricity is such that

$$\begin{aligned} e = 0.056 = \frac{r_a - r_p}{r_a + r_p} &\Rightarrow 0.056 = \frac{2(1.225 \times 10^{11}\text{ m}) - 2r_p}{2(1.225 \times 10^{11}\text{ m})} \\ \Rightarrow 0.056 = 1 - \frac{r_p}{1.225 \times 10^{11}\text{ m}} &\Rightarrow r_p = (1 - 0.056)(1.225 \times 10^{11}) \approx 1.157 \times 10^{11}\text{ m} \end{aligned} \tag{17}$$

and thus

$$r_a = 2(1.225 \times 10^{11}\text{ m}) - 1.157 \times 10^{11}\text{ m} \approx 1.294 \times 10^{11}\text{ m} \tag{18}$$

(b) (10 points) How far from the surface of Lilliania must a satellite in a circular orbit be placed in order to be in the equivalent of geosynchronous orbit around Lilliania?

Notice we apply the same equation (equation 1 above), but this time

$$T = 14.5\text{ hr} \cdot \frac{3600\text{ s}}{1\text{ hr}} = 52200\text{ s} \tag{19}$$

and the mass is the mass of Lilliania. Thus

$$r \approx \sqrt[3]{\frac{(6.67 \times 10^{-11}\text{ m}^3/\text{kg}\cdot\text{s}^2)(2.23 \times 10^{24}\text{ kg})(52200\text{ s})^2}{4\pi^2}} \approx 2.17 \times 10^7\text{ m} \tag{20}$$

This is measured from the center of Lilliania. In order to answer the question, we must subtract the radius of Lilliania (4.40×10^6 m) to get

$$2.17 \times 10^7\text{ m} - 4.40 \times 10^6\text{ m} = 1.73 \times 10^7\text{ m} \tag{21}$$

(c) (10 points) An alien on the surface of Lilliania builds a physical pendulum by attaching a 2.5 kg point mass to one end of a 1.5 kg uniform bar of length 36 cm, and then fixing the bar to rotate about the other end. How far from this axis of rotation is the center of mass of this physical pendulum?

Using our formula for the center of mass, and using that the center of mass of the uniform bar is at its center, we have

$$x_{c.o.m} = \sum_i \frac{m_i x_i}{M} = \frac{(1.5\text{ kg})(0.18\text{ m})}{1.5\text{ kg} + 2.5\text{ kg}} + \frac{(2.5\text{ kg})(0.36\text{ m})}{1.5\text{ kg} + 2.5\text{ kg}} \approx 0.2925\text{ m} \tag{22}$$

so the center of mass is 0.2925 m away from the axis of rotation.

(d) (15 points) If this alien were to set his physical pendulum into motion, what would be the period of the observed small oscillations?

Notice g on Lilliania can be found by (for M the mass of Lilliania and r Lilliania's radius)

$$g = \frac{GM}{r^2} \approx \frac{(6.67 \times 10^{-11}\text{ m}^3/\text{kg}\cdot\text{s}^2)(2.23 \times 10^{24}\text{ kg})}{(4.4 \times 10^6\text{ m})^2} \approx 7.68\text{ m/s}^2 \tag{23}$$

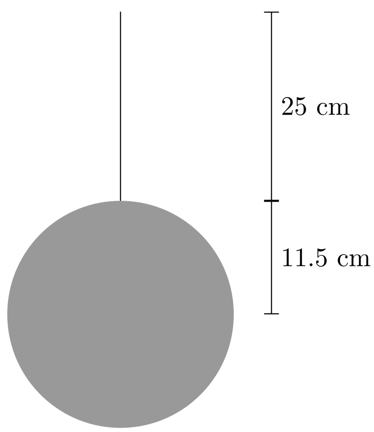
Notice the moment of inertia I of the pendulum about the axis about the top is

$$I = (2.5\text{ kg})(0.36\text{ m})^2 + \frac{1}{3}(1.5\text{ kg})(0.36\text{ m})^2 \approx 0.3888\text{ kg}\cdot\text{m}^2 \tag{24}$$

so the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{lmg}} = 2\pi\sqrt{\frac{(0.3888\text{ kg}\cdot\text{m}^2)}{(0.2925\text{ m})(4\text{ kg})(7.68\text{ m/s}^2)}} \approx 1.31\text{ s} \tag{25}$$

Problem 5: (20 points) On a distant moon with a known radius of 3400 km, a physical pendulum is made by hanging a uniform sphere of total mass $M = 12.5$ kg and radius $r = 11.5$ cm from a massless 25 cm long string attached to the top of the sphere, as shown in the picture below.



If the period of this physical pendulum is observed to be 1.54 seconds, then what is the total mass of the distant moon?

Notice the period is such that

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{lmg}} \tag{26}$$

Here

$$I = I_{c.o.m} + md^2 = \frac{2}{5}(12.5\text{ kg})(0.115\text{ m})^2 + (12.5\text{ kg})(0.115 + 0.250\text{ m})^2 \approx 1.73\text{ kg}\cdot\text{m}^2 \tag{27}$$

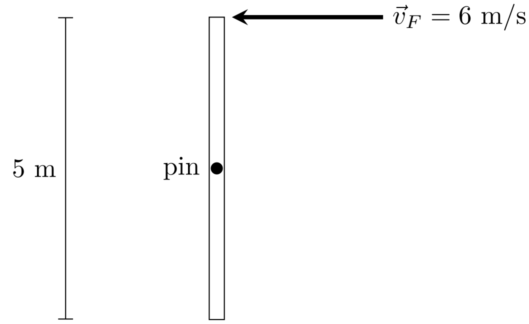
while $l = 0.115 + 0.250 = 0.365$ m, $m = 12.5$ kg, and

$$g = \frac{GM}{R^2} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{IR^2}{lmGM}} \Rightarrow \frac{T^2}{4\pi^2} = \frac{IR^2}{lmGM} \Rightarrow M = \frac{4\pi^2 IR^2}{lmGT^2} \tag{28}$$

Plugging in everything (since $R = 3.4 \times 10^6$ m and $G \approx 6.67 \times 10^{-11}$ m^3/kg·s^2) we have

$$M \approx \frac{4\pi^2(1.73 \text{ kg} \cdot \text{m}^2)(3.4 \times 10^6 \text{ m})^2}{(0.365 \text{ m})(12.5 \text{ kg})(6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2})(1.54 \text{ s})^2} \approx 1.10 \times 10^{24} \text{ kg} \tag{29}$$

Problem 6: (25 points) A 5 m long uniform density bar with a total mass of 50 kg is constrained to rotate about its center via a pin, as shown in the picture below, which illustrates the bar as seen from above. Felicity (mass 16 kg) running at an initial speed of 6 m/s in the direction indicated below, jumps and lands on the very end of the bar, which is initially stationary.



(a) (10 points) It can be shown that the moment of inertia of a uniform bar of mass M and length L **about an axis through one end** is given by $I = \frac{1}{3}ML^2$. Use this fact and the parallel axis theorem to find the total moment of inertia of the bar and Felicity after Felicity lands on the end of the bar.

the total moment of inertia of the system is the moment of inertia of Felicity plus the bar. By the parallel axis theorem, the moment of inertia through the center of the bar, which is its center of mass, must be such that

$$I_{end} = I_{c.o.m} + Md^2 \Rightarrow \frac{1}{3}ML^2 = I_{c.o.m} + M\left(\frac{L}{2}\right)^2 \Rightarrow I_{c.o.m} = \frac{1}{3}ML^2 - \frac{1}{4}ML^2 = \frac{1}{12}ML^2 \tag{30}$$

Hence the total moment of inertia of the bar/Felicity system is

$$I_{tot} = \frac{1}{12}(50 \text{ kg})(5 \text{ m})^2 + (16 \text{ kg})(2.5 \text{ m})^2 \approx 204 \text{ kg} \cdot \text{m}^2 \tag{31}$$

(b) (15 points) Assuming no net external torques, what is the angular velocity ω of the bar after Felicity lands on the end? Please specify both the magnitude and direction of ω.

Since there is no net external torque, the angular momentum of the system is conserved. Before Felicity lands on the bar, the total angular momentum is just the angular momentum that Felicity has about the axis of rotation

$$\vec{L}_i = \vec{L}_{F,i} = \vec{r} \times \vec{p} \Rightarrow L = (2.5 \text{ m})(6 \text{ m/s})(16 \text{ kg}) = 240 \text{ kg} \cdot \text{m}^2/\text{s} \tag{32}$$

which we observe is directed out of the plane of the page. After Felicity lands, we can use that $L_f = I_{tot}\omega$ for I_{tot} found in part a, which must equal $\vec{L}_i$ since angular momentum is conserved. Hence:

$$240 \text{ kg} \cdot \text{m}^2/\text{s} = (204 \text{ kg} \cdot \text{m}^2)\omega \Rightarrow \omega = \frac{240 \text{ kg} \cdot \text{m}^2/\text{s}}{204 \text{ kg} \cdot \text{m}^2} \approx 1.18 \text{ rad/s} \tag{33}$$

with the direction being the same as the direction of Felicity's initial $\vec{L}$ - i.e., out of the plane of the page, as pictured.

Problem 7: (20 points) A particular comet orbits the sun with an eccentricity of 0.983. At perihelion, the comet is 6.75×10^{10} meters from the sun, while at aphelion, the comet is traveling at a speed of 750 m/s. (Recall that the mass of the sun is about 1.99×10^{30} kg.)

(a) (10 points) What is the approximate distance between the comet and the sun at aphelion?

Notice that

$$e = 0.983 = \frac{r_a - r_p}{r_a + r_p} \Rightarrow er_a + er_p = r_a - r_p \Rightarrow r_a(1-e) = r_p(1+e) \Rightarrow r_a = \frac{1+e}{1-e} r_p \tag{34}$$

Hence

$$r_a = \frac{1+0.983}{1-0.983}(6.75 \times 10^{10} \text{ m}) \approx 7.87 \times 10^{12} \text{ m} \tag{35}$$

(b) (10 points) What is the approximate speed of the comet at perihelion?

Using that angular momentum is conserved, we have

$$l_a = l_p \Rightarrow \mu r_a^2 \omega_a = \mu r_p^2 \omega_p \Rightarrow r_a v_a = r_p v_p \Rightarrow v_p = \frac{r_a}{r_p} v_a \tag{36}$$

Hence

$$v_p \approx \frac{7.87 \times 10^{12} \text{ m}}{6.75 \times 10^{10} \text{ m}}(750 \text{ m/s}) \approx 87500 \text{ m/s} = 87.5 \text{ km/s} \tag{37}$$

Problem 8: (25 points) Felicity (with a mass of 16 kg) stands on the rim of a merry-go-around that is rotating with a constant angular velocity of 2.5 rad/s, directed in the upward direction. The merry go around has a mass of 24 kg, a radius of 0.85 m, and can be modeled as a solid cylinder.

(a) (10 points) If the centripetal force Felicity experiences is due to the static frictional force between her feet and the merry-go-around, then what is the minimum coefficient of static friction between her shoes and the merry-go-around required to keep Felicity from flying off?

Notice since Felicity's normal force is equal and opposite her weight force, her maximum static frictional force has magnitude $f_{s,max} = \mu_s mg$. Meanwhile, the centripetal force has magnitude

$$\left|\vec{F}_C\right| = mr\omega^2 \Rightarrow \mu_s mg = mr\omega^2 \Rightarrow \mu_s = \frac{r\omega^2}{g} = \frac{(0.85 \text{ m})(2.5 \text{ rad/s})^2}{9.8 \text{ m/s}^2} = 0.542 \tag{38}$$

(b) (15 points) Felicity now walks to the center of the merry-go-around, so she is standing right on the axis of rotation. Assuming there are no external torques, what is the final angular velocity of the merry-go-around?

Notice the initial moment of inertia of the entire system is notice the total moment of inertia is

$$I_i = I_m + I_{Fi} = \frac{1}{2}(24 \text{ kg})(0.85 \text{ m})^2 + (16 \text{ kg})(0.85 \text{ m})^2 = 20.23 \text{ kg} \cdot \text{m}^2 \tag{39}$$

Meanwhile the final moment of inertia is

$$I_f = \frac{1}{2}(24 \text{ kg})(0.85 \text{ m})^2 + (16 \text{ kg})(0)^2 = 8.67 \text{ kg} \cdot \text{m}^2 \tag{40}$$

Notice that since the net external torque is zero, then angular momentum is conserved.

$$L_i = L_f \Rightarrow I_i\omega_i = I_f\omega_f \Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i = \frac{(20.23 \text{ kg} \cdot \text{m}^2)}{(8.67 \text{ kg} \cdot \text{m}^2)}(2.5 \text{ rad/s}) \approx 5.83 \text{ rad/s} \tag{41}$$

Problem 9: (35 points total) The famous Halley's comet has a highly elliptical orbit around the sun. At closest approach, the comet is approximate 0.586 AU from the sun. (One AU or *astronomical unit* is defined to be the average distance from the earth to the sun, or approximately 1.50×10^{11} m.) At aphelion, Halley's comet is approximately 35.1 AU from the sun.

(a) (10 points) What is the eccentricity of Halley's comet's orbit?

To find the eccentricity, we apply our results from class:

$$e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{35.1 - 0.586}{35.1 + 0.586} \approx 0.967 \tag{42}$$

(b) (15 points) Using the semi-major axis as the mean separation distance between Halley's comet and the sun, estimate how many years it takes Halley's comet to complete one orbit. If Halley's comet was last at perihelion in February of 1986, approximately when will its next perihelion be?

First we find the semi-major axis, which is just the average of the aphelion and perihelion distances

$$r = \frac{r_a + r_p}{2} = \frac{35.1 + 0.586}{2} \approx 17.8 \text{ AU} \tag{43}$$

Now we apply Kepler's Third Law

$$T^2 \propto r^3 \Rightarrow \frac{T_H^2}{r_H^3} = \frac{T_E^2}{r_E^3} \Rightarrow T_H = \sqrt{\frac{r_H^3}{r_E^3}} T_E = \sqrt{\frac{(17.8 \text{ AU})^3}{(1\text{AU})^3}} (1 \text{ year}) \approx 75.4 \text{ year} \tag{44}$$

Since 0.4 years is about five months, if Halley's comet last came in February 1986, we would expect a return in the year 1986+75 = 2061, in the month of July (which is five months after February). This matches very closely with the expected next perihelion of Halley's comet, which is July 28th, 2061. (Mark your calendars now!)

(c) (10 points) At perihelion, Halley's comet orbits at a speed of approximately 54.6 km/s. What is the approximate speed of Halley's comet when it is at aphelion?

Here we use conservation of angular momentum

$$l_a = l_p \Rightarrow \mu r_a v_a = \mu r_p v_p \Rightarrow v_a = \frac{r_p}{r_a} v_p = \frac{0.586 \text{ AU}}{35.1 \text{ AU}} (54.6 \text{ km/s}) \approx 0.911 \text{ km/s} = 911 \text{ m/s} \tag{45}$$

Problem 10: (25 points) Talladega Speedway has a particular banked turn of radius 300 meters banked at an angle of 33°.

(a) (10 points) At what speed can a stock car drive through this turn without requiring any friction between the tires and the road?

Notice that, if no friction is required, then the only two forces acting on the car are gravity and the normal force, with the net force necessary in the horizontal direction to provide the centripetal force. Thus the y component of the normal force must be equal and opposite the weight force, meaning

$$\left|\vec{N}\right| \cos\theta = \left|\vec{F}_G\right| \Rightarrow \left|\vec{N}\right| = \frac{mg}{\cos\theta} \tag{46}$$

The net force is then the x component of the normal force, hence

$$\left|\vec{F}_{net}\right| = \left|\vec{N}\right| \sin\theta = \frac{mg\sin\theta}{\cos\theta} = mg\tan\theta \tag{47}$$

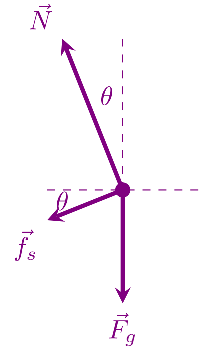
So, since this is the centripetal force

$$m\frac{v^2}{r} = mg\tan\theta \Rightarrow v = \sqrt{rg\tan\theta} = \sqrt{(300 \text{ m})(9.8 \text{ m/s}^2)\tan(33°)} \approx 43.7 \text{ m/s} \tag{48}$$

Note that since this problem was done in the lecture slides, you did not need to go through this entire derivation, You could simply have deployed the final equation (equation 48).

(b) (15 points) If the coefficient of static friction between the tires and the road is 0.70, then what is the maximum speed a stock car can safely travel through the turn?

Notice in this case, our free body diagram looks something like this, since the frictional force opposes the motion of the car sliding up the inclined plane:



Once again, the net force is in the x direction (really, negative x as drawn), so in the y direction the forces must add up to give zero. Hence, assuming $\vec{f_s} = \vec{f}_{s,max}$ for the extremal case:

$$\left|\vec{F_g}\right| + \left|\vec{f_s}\right| \sin\theta = \left|\vec{N}\right| \cos\theta \Rightarrow mg + \mu_s \left|\vec{N}\right| \sin\theta = \left|\vec{N}\right| \cos\theta \Rightarrow \left|\vec{N}\right| = \frac{mg}{\cos\theta - \mu_s \sin\theta} \tag{49}$$

Hence the net force, in the x direction, has magnitude

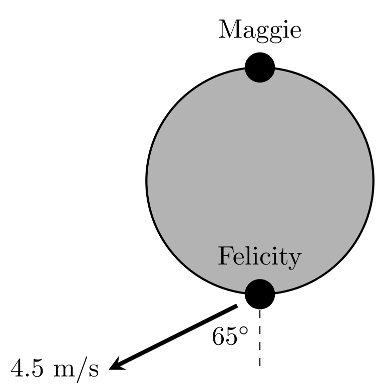
$$\left|\vec{F}_{net}\right| = \left|\vec{f_s}\right| \cos\theta + \left|\vec{N}\right| \sin\theta = \mu_s \left|\vec{N}\right| \cos\theta + \left|\vec{N}\right| \sin\theta = \frac{mg(\mu_s \cos\theta + \sin\theta)}{\cos\theta - \mu_s \sin\theta} \tag{50}$$

But this of course is the centripetal force, so

$$m\frac{v^2}{r} = \frac{mg(\mu_s \cos\theta + \sin\theta)}{\cos\theta - \mu_s \sin\theta} \Rightarrow v = \sqrt{rg\frac{\mu_s \cos\theta + \sin\theta}{\cos\theta - \mu_s \sin\theta}} \approx 89.0 \text{ m/s} \tag{51}$$

Notice, just as in part (a), since this work was done in an in-class activity, you need not have derived the equation used in 51. All you needed to do was deploy it correctly!

Problem 11: (45 points total) Felicity, with a mass of 23 kg, and Maggie, with a mass of 13 kg, stand on opposite ends of an initially stationary merry-go-round, as shown in the picture below (from a top view). The merry-go-round is a uniform disk of radius 1.10 m and mass 45 kg, constrained to rotate about its central axis. Felicity then jumps off the merry-go-round in the direction pictured at a speed of 4.5 m/s.



(a) (10 points) After Felicity jumps off of the merry-go-round, what is the total moment of inertial of the merry-go-round and Maggie system?

Notice I here would be the sum of I for a uniform cylinder plus that of a point mass (specifically Maggie) located on the rim. Hence

$$I = \frac{1}{2}M_{merry}R^2 + M_{Mag}R^2 = \frac{1}{2}(45 \text{ kg})(1.10 \text{ m})^2 + (13 \text{ kg})(1.10 \text{ m})^2 \approx 42.96 \text{ kg}\cdot\text{m}^2 \tag{52}$$

(b) (10 points) Assuming no net external torques while Felicity jumps off the merry-go-round, what is the final angular velocity ω of the merry-go-round after Felicity jumps off. **Please specify both the magnitude and direction of this angular velocity.**

Notice if we assume no net external torques, $L_f = L_i = 0$ for the full system of Felicity plus Maggie plus the merry-go-round. After jumping off, Felicity has a angular momentum

$$L_{F,f} = \vec{r} \times \vec{p} \Rightarrow |L_{F,f}| = rmv\sin(\theta) = (1.10\text{ m})(23\text{ kg})(4.5\text{ m/s})\sin(65^\circ) \approx 103.2\text{ kg}\cdot\text{m}^2/\text{s} \tag{53}$$

directed into the page by our right-hand-rule. Hence the angular momentum of Maggie and the merry-go-round together must be 103.2 kg$\cdot$m^2/s, directed out of the page. Using $L = I\omega$ for I found above, this means

$$\omega = \frac{L}{I} = \frac{103.2\text{ kg}\cdot\text{m}^2/\text{s}}{42.96\text{ kg}\cdot\text{m}^2} \approx 2.402\text{ rad/s} \tag{54}$$

with a direction the same as L for Maggie and the merry-go-round - i.e., out of the page as viewed from above.

(c) (10 points) After Felicity jumps off the merry-go-round, what is the minimum coefficient of static friction necessary to prevent Maggie from sliding off of the now spinning merry-go-round?

Notice the centripetal force has magnitude

$$F_c = mr\omega^2 \tag{55}$$

This force must be being provided by the force of static friction, so using that $\vec{F}_N$ is equal and opposite $\vec{F}_g$ for Maggie standing on the horizontal merry-go-round and $f_{s,max} = \mu_s \left|\vec{F}_N\right|$, we have

$$mr\omega^2 = \mu_{s,min} mg \Rightarrow \mu_{s,min} \approx \frac{r\omega^2}{g} = \frac{(1.10\text{ m})(2.402\text{ rad/s})}{(9.80\text{ m/s}^2)} \approx 0.648 \tag{56}$$

(d) (15 points) After Felicity jumps off the merry-go-round, the merry-go-round is observed to go through 4.5 full rotations before coming to a complete stop. Assuming the net torque due to frictional forces is constant as the merry-go-round spins, what is the magnitude and direction of this constant τ?

Notice $\omega_i = 2.402$ rad/s, $\Delta\theta = 2\pi(4.5) = 9\pi$, $\omega_f = 0$, and α is unknown. Solving for α, we get

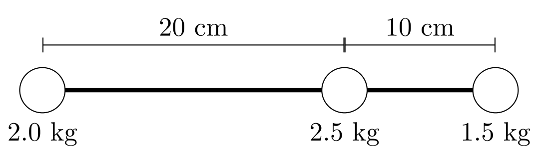
$$\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta \Rightarrow \alpha = \frac{0^2 - (2.402\text{ rad/s})^2}{2(9\pi)} \approx -0.102\text{ rad/s}^2 \tag{57}$$

where the negative sign means α points opposite the direction of ω - i.e., into the plane of the page when viewed from above. This means

$$\tau = I\alpha = (42.96\text{ kg}\cdot\text{m}^2)(0.102\text{ rad/s}^2) \approx 4.38\text{ N}\cdot\text{m} \tag{58}$$

where we note τ points the same direction as α - i.e., into the plane of the page.

Problem 12: (25 points) Three point masses are connected in a line by massless rods, as shown in the picture below.



(a) (15 points) What is the moment of inertia of this system about the center of mass?

First we need to find the center of mass. Using our regular formula and setting $x = 0$ to be where the 2.0 kg mass is located:

$$x_{c.o.m} = \frac{(2.0\text{ kg})(0\text{ m}) + (2.5\text{ kg})(0.2\text{ m}) + (1.5\text{ kg})(0.3\text{ m})}{2.0\text{ kg} + 2.5\text{ kg} + 1.5\text{ kg}} \approx 0.1583\text{ m} \tag{59}$$

Now we can find the moment of inertia using the definition for point masses

$$I = (2.0\text{kg})(0.1583\text{ m})^2 + (2.5\text{kg})((0.2 - 0.1583)\text{ m})^2 + (1.5\text{kg})((0.3 - 0.1583)\text{ m})^2 \approx 0.0846\text{ kg}\cdot\text{m}^2 \tag{60}$$

(b) (10 points) The system experiences a constant angular acceleration about the center of mass due to a constant net torque. If the system starts from rest and reached an angular speed of 8.2 rad/s after going through 3 full rotations, then what is the magnitude of this constant net torque?

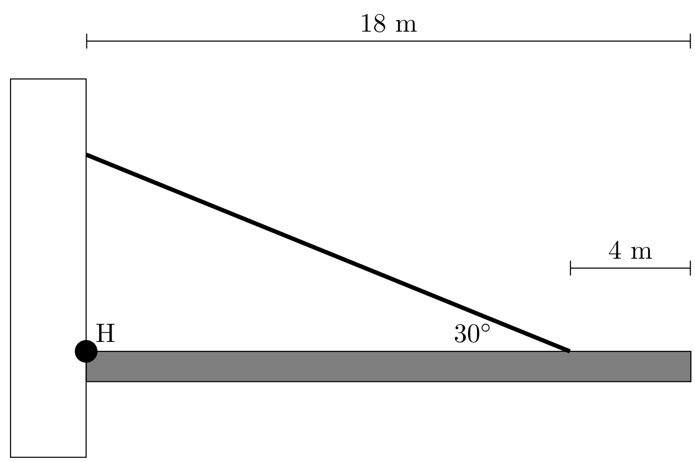
Here we deploy rotational kinematics. Notice $\omega_i = 0$, $\omega_f = 8.2$ rad/s, $\Delta\theta = 3(2\pi) = 6\pi$, and α is the unknown we are seeking. Then:

$$\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta \Rightarrow \alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta} \approx \frac{(8.2 \text{ rad/s})^2 - 0^2}{2(6\pi \text{ rad})} \approx 1.784 \text{ rad/s}^2 \tag{61}$$

We can then find the net torque by deploying Newton's second law for rotations and using the I from part a.

$$\tau = I\alpha \approx (0.0846 \text{ kg} \cdot \text{m}^2)(1.784 \text{ rad/s}^2) \approx 0.151 \text{ N} \cdot \text{m} \tag{62}$$

Problem 13: (25 points) A drawbridge is held up by a cable and attached to the support column by a hinge, located at the point labeled H in the picture below. The tension in the cable is 3200 N, and the bridge deck is of uniform density.



(a) (10 points) What is the mass of the deck of the bridge?

Here we use torque balance around point H. Observe that the torque due to the cable points out of the screen, while the torque due to the weight of the bridge deck points into the screen, hence if the bridge is static these two torques must be equal in magnitude. Thus:

$$m(9.8 \text{ m/s}^2)(9 \text{ m})\sin(90°) = (14 \text{ m})(3200 \text{ N})\sin(30°) \Rightarrow m = \frac{(14 \text{ m})(3200 \text{ N})\sin(30°)}{(9.8 \text{ m/s}^2)(9 \text{ m})} \approx 254 \text{ kg} \tag{63}$$

(b) (15 points) What is the magnitude of the force on the hinge? In what direction does it point?

We know that, as vectors,

$$\vec{F}_H + \vec{F}_G + \vec{T} = 0 \tag{64}$$

since the system is static. Hence, in the x direction

$$F_{Hx} = -F_{Gx} - T_x = 0 - (-3200\cos(30°) \text{ N} \approx 2771 \text{ N} \tag{65}$$

where I have used that the x component of the tension force is negative, since it points to the left. Meanwhile, in the y direction

$$F_{Hy} = -F_{Gy} - T_y = -(-254 \text{ kg})(9.80 \text{ m/s}^2) - 3200\sin(30°) \approx 889 \text{ N} \tag{66}$$

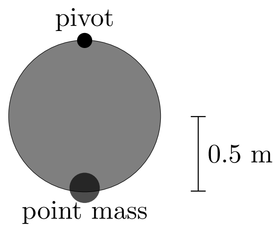
Hence the magnitude of the force on the hinge is

$$\left|\vec{F}_H\right| = \sqrt{F_{Hx}^2 + F_{Hy}^2} \approx \sqrt{(2771 \text{ N})^2 + (889 \text{ N})^2} \approx 2910 \text{ N} \tag{67}$$

while the vector lies in the first quadrant, making an angle of

$$\theta = \tan^{-1}\left(\frac{889 \text{ N}}{2771 \text{ N}}\right) \approx 17.8° \tag{68}$$

Problem 14: (25 points) A physical pendulum is made by attaching a 2 kg point mass to the rim of 3 kg solid, uniform density disk of radius 0.5 m. The entire system is then pinned so that it rotates around a pivot point on the rim of the disk, opposite the position of the point mass, as shown below.



(a) (10 points) What is the moment of inertia of this pendulum about the fixed pivot point?

The moment of inertia is the sum of the moment of inertia of the disk plus the moment of inertia of the point mass. Notice to find the moment of inertia of the disk, we can employ the parallel axis theorem

$$I = I_{c.o.m} + Md^2 = \frac{1}{2}(3 \text{ kg})(0.5 \text{ m})^2 + (3 \text{ kg})(0.5 \text{ m})^2 = 1.125 \text{ k} \cdot \text{m}^2 \tag{69}$$

So the total moment of inertia of the full system is just

$$I_{tot} = 1.125 \text{ kg} \cdot \text{m}^2 + (2 \text{ kg})(1 \text{ m})^2 = 3.125 \text{ kg} \cdot \text{m}^2 \tag{70}$$

(b) (15 points) What is the period, T, of small oscillations of this physical pendulum?

Notice the distance from the pivot to the center of mass of this system can be found using our normal formula for the center of mass, in which we treat the solid disk as a point mass with a center of mass at the center. Hence

$$l = \frac{(3 \text{ kg})(0.5 \text{ m}) + (2 \text{ kg})(1 \text{ m})}{2 \text{ kg} + 3 \text{ kg}} = 0.7 \text{ m} \tag{71}$$

So hence

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{3.125 \text{ kg} \cdot \text{m}^2}{(5 \text{ kg})(9.8 \text{ m/s}^2)(0.7 \text{ m})}} \approx 1.90 \text{ s} \tag{72}$$

Problem 15: (30 points) The planet Uranus has an orbital period of approximately 84 years, a rotational period of approximately 17.4 hours, a mass of 8.68×10^{25} kg, a minimum orbital speed of 6.49 km/s, and a maximum orbital speed of 7.11 km/s.

(a) (10 points) Using the fact that the earth's orbit has a semimajor axis of approximately 1.50×10^{11} m, estimate the semi-major axis of Uranus's orbit.

Here we can apply the proportionality version of Kepler's third law

$$T^2 \propto r^3 \Rightarrow \frac{T_U^2}{r_U^3} = \frac{T_E^2}{r_E^3} \Rightarrow r_U^3 = \frac{T_U^2}{T_E^2} r_E^3 \Rightarrow r_U = \sqrt[3]{\left(\frac{84 \text{ year}}{1 \text{ year}}\right)^2} (1.50 \times 10^{11} \text{ m}) \approx 2.88 \times 10^{12} \text{ m} \tag{73}$$

(b) (15 points) What are the approximate perihelion and aphelion distances for Uranus's orbit?

Notice the semimajor axis is the average of r_a and r_p. Hence

$$\frac{r_a + r_p}{2} = 2.88 \times 10^{12} \text{ m} \tag{74}$$

Meanwhile, conservation of angular momentum gives us that

$$l_a = l_p \Rightarrow \mu r_a^2 \omega_a = \mu r_p^2 \omega_p \Rightarrow r_a^2 \left(\frac{v_a}{r_a}\right) = r_p^2 \left(\frac{v_p}{r_p}\right) \Rightarrow r_a v_a = r_p v_p \tag{75}$$

Notice that, by Kepler's qualitative second law, $v_a = v_{min} = 6.49$ km/s while $v_p = v_{max} = 7.11$ km/s. Hence

$$(6.49 \text{ km/s})r_a = (7.11 \text{ km/s})r_p \Rightarrow r_a = \frac{7.11}{6.49} r_p \approx 1.096 r_p \tag{76}$$

This can now be substituted back into equation (2) above to solve for r_p

$$2.88 \times 10^{12} \text{ m} = \frac{1.096 r_p + r_p}{2} = \frac{2.096}{2} r_p \Rightarrow r_p = (2.88 \times 10^{12} \text{ m}) \left(\frac{2}{2.096} \right) \approx 2.75 \times 10^{12} \text{ m} \tag{77}$$

This is the perihelion distance. To find the aphelion, we plug back into equation 4

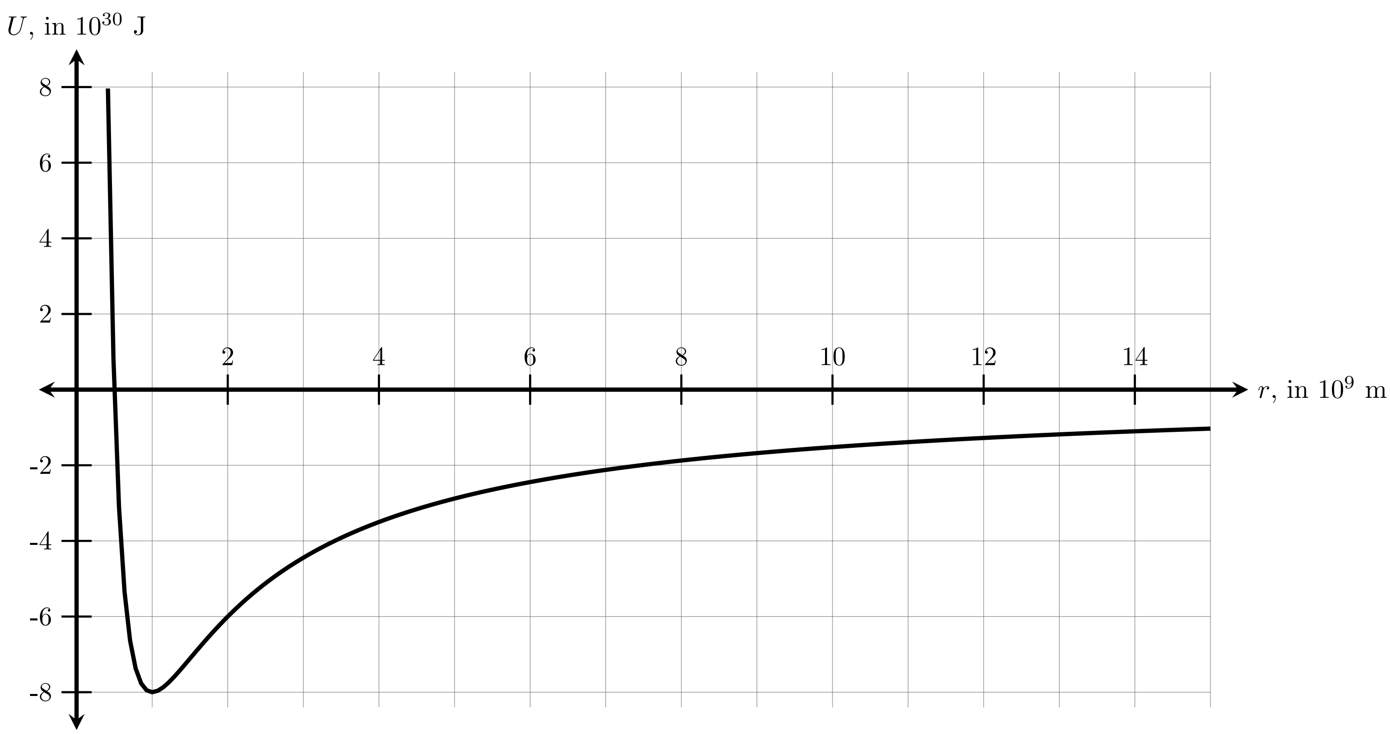
$$r_a = 1.096 r_p \approx 1.096(2.75 \times 10^{12} \text{ m}) \approx 3.01 \times 10^{12} \text{ m} \tag{78}$$

(c) (5 points) What is the eccentricity of Uranus's orbit?

To find the eccentricity, we can deploy our favorite equation from practice final question 1

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{(3.01 \times 10^{12} \text{ m} - 2.75 \times 10^{12} \text{ m})}{3.01 \times 10^{12} \text{ m} + 2.75 \times 10^{12} \text{ m}} \approx 0.046 \tag{79}$$

Problem 16: (15 points) The graph below shows the fictitious potential energy U_{fict} as a function of r for orbits of a particular moon around a gas giant planet.



(a) (5 points) If the orbit of this moon was circular, what would the orbital radius of this moon be?

Notice that the circular orbit is at the bottom of the well, so $r = 1 \times 10^9$ m.

(b) (10 points) If this moon/planet system has a a total energy of -2×10^{30} J, then what is the approximate eccentricity of the moon's orbit?

Notice at this total energy, $r_{min} \approx 0.6 \times 10^9$ m while $r_{max} \approx 7.3 \times 10^9$ m. Hence

$$e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} \approx \frac{7.3 - 0.6}{0.6 + 7.3} \approx 0.848 \tag{80}$$

Problem 17: (30 points) Maggie (with a mass of 8.5 kg) stands on the rim of a 0.75 m radius merry-go-round. The merry-go-round has a mass of 20 kg and can be modeled as a solid cylinder. At time $t = 0$, the merry-go-round has an angular velocity $\omega = 1.2$ rad/s directed downward.

(a) (10 points) What constant net torque must be applied to the merry-go-round to give it a constant angular acceleration of 0.8 rad/s^2, directed in the upward direction? Please give both the magnitude and direction of this torque.

Notice the moment of inertia for the Maggie and merry-go-round system is

$$I = \frac{1}{2}(20 \text{ kg})(0.75 \text{ m})^2 + (8.5 \text{ kg})(0.75 \text{ m})^2 \approx 10.41 \text{ kg} \cdot \text{m}^2 \tag{81}$$

Newton's second law for rotations then implies

$$\vec{\tau} = I\vec{\alpha} \Rightarrow \tau = (10.41 \text{ kg} \cdot \text{m}^2)(0.8 \text{ rad/s}^2) \approx 8.33 \text{ N} \cdot \text{m} \tag{82}$$

with a direction that is the same as α - i.e., upward.

(b) (20 points) The coefficient of static friction between Maggie's shoe and the merry-go-around surface is 0.45. For how long must the constant net torque found in part a be applied in order to send Maggie flying off of the merry-go-around, assuming the only force keeping her on the merry-go-around is static friction?

Notice the constant net torque leads to a constant acceleration of 0.8 rad/s^2, as per the requirements in part a. Meanwhile, we know the maximum net force that can be applied by static friction, since Maggie's normal force is equal and opposite her weight force, must have magnitude

$$\left|\vec{f}_{s,max}\right| = \mu_s mg \tag{83}$$

Notice that this force due to static friction must provide both the centripetal force (with magnitude $mr\omega^2$) and the force in the tangential direction causing Maggie's tangential acceleration (with magnitude $ma_t = mr\alpha$). Since these are perpendicular to one another, the total magnitude must be the square root of their squares. Hence, the maximum ω Maggie can have is such that

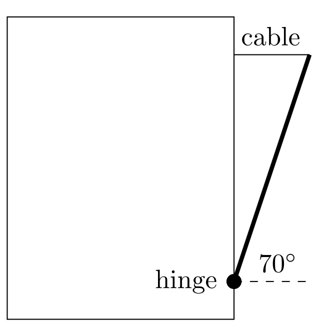
$$\begin{aligned} \mu_s mg &= \sqrt{(mr\omega^2)^2 + (mr\alpha)^2} = mr\sqrt{\omega_{max}^4 + \alpha^2} \\ \Rightarrow \omega_{max}^4 + (0.8 \text{ rad/s}^2)^2 &= \left(\frac{\mu_s g}{r}\right)^2 = \left(\frac{0.45(9.80 \text{ m/s}^2)}{0.75 \text{ m}}\right)^2 \\ \Rightarrow \omega_{max} &= \sqrt[4]{\left(\frac{0.45(9.80 \text{ m/s}^2)}{0.75 \text{ m}}\right)^2 - \left(0.8 \text{ rad/s}^2\right)^2} \approx 2.41 \text{ rad/s} \end{aligned} \tag{84}$$

Hence we now have a rotational kinematics problem in which $\omega_0 = -1.2$ rad/s (note that this initial angular velocity is negative because the angular acceleration is opposite the direction of the initial angular velocity), $\omega_f = 2.41$ rad/s, and $\alpha = 0.8$ rad/s^2. Hence:

$$\omega_f = \alpha\Delta t + \omega_i \Rightarrow 2.41 \text{ rad/s} = (0.8 \text{ rad/s}^2)\Delta t - 1.2 \text{ rad/s} \Rightarrow \Delta t \approx \frac{2.41 + 1.2 \text{ rad/s}}{0.8 \text{ rad/s}^2} \approx 4.52 \text{ s} \tag{85}$$

Note that, since α is small relative to ω_{max}^2, you actually get a pretty good approximation to ω_{max} ignoring the tangential component of the force on Maggie. But technically, it is not correct to do so!

Problem 18: (25 points) The drawbridge at Maggie's castle is held up at an angle of 70° by a horizontal cable attached to its end, as shown in the picture below. The drawbridge is of uniform density and has a total mass of 240 kg, and is attached to the castle tower by a hinge. The whole system is in static equilibrium.



(a) (15 points) What is the magnitude of the tension force in the cable?

Here we use torque balance with the hinge as the pivot point. Observe the force of gravity points downward, and is applied at the midpoint of the bridge (i.e., at point $\frac{L}{2}$ if the bridge is length L), giving a torque into the page with a magnitude

$$\vec{\tau} = \vec{r} \times \vec{F}_G \Rightarrow |\vec{r}| \left|\vec{F}_G\right| \sin\theta = \frac{L}{2}(240\text{ kg})(9.80\text{ m/s})\sin(20^\circ) \tag{86}$$

where we have used that the angle between $\vec{r}$ and $\vec{F}_G$ is actually the compliment of the angle given. This torque must be equal and opposite the torque due to the tension in the cable in order for the torques to balance and the system to be in static equilibrium. Hence

$$\begin{aligned} |\vec{r}| \left|\vec{T}\right| \sin(70^\circ) &= \frac{L}{2}(240\text{ kg})(9.80\text{ m/s})\sin(20^\circ) \\ \Rightarrow \quad \left|\vec{T}\right| &= \frac{L(240\text{ kg})(9.80\text{ m/s})\sin(20^\circ)}{2L\sin(70^\circ)} \approx 428\text{ N} \end{aligned} \tag{87}$$

Notice that L doesn't matter, which is why it wasn't given in the problem!

(b) (10 points) What is the magnitude of the net force applied on the drawbridge at the hinge? In what direction does this force point?

Notice we need the sum of weight force, the tension force, and the force on the hinge to equal zero for the net force to be zero and the system to be in static equilibrium Hence

$$\begin{aligned} \vec{F}_H + \vec{T} + \vec{F}_G = 0 &\Rightarrow \vec{F}_H = -\vec{T} - \vec{F}_G \\ &\Rightarrow \vec{F}_H = -(-428\text{ N})\hat{\imath} - (-240\text{ kg})(9.8\text{ m/s}^2)\hat{\jmath} = 428\hat{\imath} + 2352\hat{\jmath}\text{ N} \end{aligned} \tag{88}$$

The magnitude is then given by the Pythagorean theorem:

$$\left|\vec{F}_H\right| = \sqrt{(428\text{ N})^2 + (2352\text{ N})^2} \approx 2391\text{ N} \tag{89}$$

while the direction is given by trigonometry; the force lies in the first quadrant, at an angle

$$\theta = \tan^{-1}\left(\frac{2352\text{ N}}{428\text{ N}}\right) \approx 79.8^\circ \tag{90}$$

above the x axis.

Problem 19: (35 points) Miranda is a moon of Uranus that has a mass of 6.4×10^{19} kg and a radius of 2.36×10^5 m. Miranda orbits Uranus in an almost perfectly circular orbit of radius 1.29×10^8 meters. Recall from question 1 that Uranus has a rotational period of approximately 17.4 hours and a mass of 8.68×10^{25} kg.

(a) (10 points) How many Uranian days does it take Miranda to complete one full revolution around Uranus? (Hint: 1 hour is 3600 seconds.)

To solve this question, we deploy Kepler's Third Law in explicit form

$$T^2 = \frac{4\pi^2}{GM}r^3 \Rightarrow T = \sqrt{\frac{4\pi^2}{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(8.68 \times 10^{25}\text{ m})}(1.29 \times 10^8\text{ m})^3} \approx 120990\text{ s} \tag{91}$$

Hence, since a Uranian day is approximately $17.4(3600) = 62640$ seconds, it takes

$$\frac{120990\text{ s}}{62640\text{ s per Uranian Day}} \approx 1.93\text{ Uranian Days} \tag{92}$$

for Miranda to complete one orbit around Uranus.

(b) (15 points) An astronaut on Miranda builds a physical pendulum from an object of mass $M = 1.45$ kg that has a moment of inertia of 0.055 kg$\cdot$m^2 about its center of mass. If the axis of rotation of this pendulum is 0.60 m from the center of mass, then what is the natural angular frequency ω_0 of this pendulum on Miranda?

First let us find the acceleration due to gravity, g, on Miranda. Recall that

$$g = \frac{GM}{r^2} = \frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(6.4 \times 10^{19}\ \mathrm{kg})}{(2.36 \times 10^5\ \mathrm{m})^2} \approx 0.0766\ \mathrm{m/s^2} \tag{93}$$

Now we apply our rules for a physical pendulum. Notice we want I about the pivot point, which we can get from I about the center of mass by applying the parallel axis theorem.

$$I_p = I_{c.o.m.} + Md^2 = 0.055\ \mathrm{kg \cdot m^2} + (1.45\ \mathrm{kg})(0.60\ \mathrm{m})^2 \approx 0.577\ \mathrm{kg \cdot m^2} \tag{94}$$

The physical pendulum formula now gives

$$\omega_0 = \sqrt{\frac{lmg}{I}} \approx \sqrt{\frac{(0.6\ \mathrm{m})(1.45\ \mathrm{kg})(0.0766\mathrm{m/s^2})}{0.577\ \mathrm{kg \cdot m^2}}} \approx 0.34\ \mathrm{rad/s} \tag{95}$$