# Marginal Effects and Coefficient Interpretation*

Empirical Methods - Fall 2019

In this course, we focus on linear regressions of the form $Y = X\beta + \epsilon$. For example, we might be interested in the determinants of hourly wages $W$ and write the following model

$$W = \beta_0 + \beta_1 Ed + \beta_2 Age + \epsilon. \tag{1}$$

Why do we estimate such models in general?

- Descriptive analysis
- Causal analysis
- Prediction

This course is mostly about **causal inference**. What assumptions do we need to interpret the results from a regression based on model (1) causally?

A1. The model (1) is **structural/causal**. It describes a causal relationship between our predictors and the dependent variable, stating what would be an individual's wage if we could **manipulate** (change) her level of education or her age while holding everything else constant. In other words, wage is a function of education, age and unobservables $\epsilon$, and this function is given by our model

$$W = f(Ed, Age, \epsilon) = \beta_0 + \beta_1 Ed + \beta_2 Age + \epsilon.$$

A2. **Linear conditional expectation function (CEF)**:

$$E(W|Ed, Age) = \beta_0 + \beta_1 Ed + \beta_2 Age \tag{2}$$

This is equivalent to jointly assuming linearity and zero conditional mean error $E(\epsilon|Ed, Age) = 0$. In words, we are assuming that the CEF of wage given education and age (a statistical object that summarises the association between wage, education and age in the population) is linear and corresponds to our causal model. This is a **very strong** assumption!

*Written by Alexandre Jenni. These notes are intended as a prelude for Matteo Greco's notes on coefficient interpretation in linear models.

Taken jointly, these assumptions can be stated as the **linear structural CEF** assumption.

Using assumption A1, $\beta_1$ is the **marginal effect** of increasing education on wage while holding age constant:

$$\frac{\partial W}{\partial Ed} = \frac{\partial f(Ed, Age, \epsilon)}{\partial Ed} = \beta_1$$

This is a definition of a **causal impact**: how does a variable $y$ change if we were to manipulate $x$ only (change it to a different value)? What is even better, we know that under assumption A2, the OLS estimate $\hat{\beta}_1$ is an unbiased estimate for $\beta_1$. Hence under assumptions 1 and 2, $\hat{\beta}_1$ will be an **unbiased estimator of the marginal effect** of education on wage.

What if there is **heterogeneity** in the marginal effect of education, that is $\beta_{1i}$ changes with each individual $i$? It seems reasonable to assume that some individuals benefit more from getting additional education than others. This can be expressed using the following model specification

$$W_i = f_i(Ed_i, Age_i, \epsilon_i) = \beta_0 + \beta_{1i} Ed_i + \beta_2 Age_i + \epsilon_i$$

We won't be able to identify $\beta_{1i}$ since we only observe every individual with one given level of education. But defining $\beta_1 = \mathbb{E}(\beta_{1i}) = \mathbb{E}(\frac{\partial W_i}{\partial Ed_i})$, and assuming that the conditional expectation of $\beta_{1i}$ is constant $\mathbb{E}(\beta_{1i}|Ed_i, Age_i) = \beta_1$. Then the linear CEF assumption (2) still holds and $\hat{\beta}_1$ will be an **unbiased estimator of the average marginal effect** of education on wage $\mathbb{E}(\frac{\partial W_i}{\partial Ed_i})$. Note that this is also a **strong** assumption. It will be for example violated if workers with a high level of education benefit more from additional years of educations than workers with a low level of education, which might be the reason why they chose to stay longer in school in the first place. To conclude, under the two assumptions **A1** and **A2** we can think of our **regression coefficients as estimates of average marginal effects**.

What if our assumptions 1 and 2 do not hold? We can still run a regression of $W$ on $Ed$, $Age$ and a constant. By definition of a population regression, we will obtain

$$W = \gamma_0 + \gamma_1 Ed + \gamma_2 Age + \epsilon, \qquad \mathbb{E}(\epsilon) = \mathbb{E}(Ed \cdot \epsilon) = \mathbb{E}(Age \cdot \epsilon) = 0.$$

Note that the parameters have changed, since they might not correspond to the parameters of the causal model ($\gamma_k \neq \beta_k$). How do we interpret $\gamma_1$?

- If the conditional expectation function is truly linear

$$\mathbb{E}(W|Ed, Age) = \gamma_0 + \gamma_1 Ed + \gamma_2 Age,$$

  then $\gamma_1$ is the average change in wage that is associated with a change in education in the population holding age constant. Note that this association is purely **descriptive** (or **statistic**), this is the reason why I write $\gamma_1$ and not $\beta_1$ (the structural parameter).

- If the CEF is not linear, $\gamma_1$ gives us the best linear approximation to this relationship.

One last remark before we dive into the interpretation of coefficients in linear models. The term linear models describes models that are **linear in the parameters**. Which of the following models are linear given this definition?

$$W_i = \beta_0 + \beta_1 Ed_i + \beta_2 Age_i + \epsilon_i \tag{3}$$
$$W_i = \beta_0 + \beta_1 \log Ed_i + \beta_2 Age_i + \beta_3 Age_i^2 + \beta_4 \log Ed_i \times Age_i + \epsilon_i \tag{4}$$
$$W_i = \beta_0 + Ed_i^{\beta_1} + \beta_2 Age_i + \epsilon_i \tag{5}$$

Models (3) and (4) are linear in the parameters. By carefully specifying the vector $x_i$, each of them can be rewritten as $W_i = x_i'\beta + \epsilon_i$. This is true even though model (4) is not linear in $Age_i$. Although models (3) and (4) can be treated identically for estimation purposes, the computation of causal or marginal effects in these two models differ. Indeed, we have seen that given our assumptions A1 and A2, $\beta_2$ in model (4) can be interpreted as the effect of changing $Age_i$ while holding $logED_i$, $Age_i^2$ and $\log Ed_i \times Age_i$ constant. But even to conduct counterfactual analysis, it seems to hold $Age_i^2$ constant when changing $Age_i$.

On the other hand, equation (5) is nonlinear in the parameters and cannot be estimated by OLS. It also requires a different set of assumptions to identify average marginal effects. The appendix provides a very short introduction to the identification of average marginal effects in nonlinear models for the interested readers.

For specific examples on how to interpret models which are linear in the parameters but may not be linear in the independent variables, have a look at Matteo's notes on coefficient interpretation.

## References

[1] Bruce Hansen. *Econometrics*. Online book, 2019, https://www.ssc.wisc.edu/ bhansen/econometrics/.

## Appendix: Marginal effects in nonlinear models[1]

What happens if we relax the linearity assumption? A fully general representation for a structural model is

$$y = h(X, U) = h(x_1, \ldots, x_n, u_1, \ldots, u_n),$$

where $X$ is a vector of observables and $U$ a vector of unobservables. Note that the structural function $h$ is left unspecified at this stage. Going back to our previous example, this could be

$$W = h(Ed, Age, U)$$

where U=(ability, soft skills, ...).

We are still interested in estimating the marginal effect of exogenously changing the level of education. With a nonlinear model, this effect will depend on the level of education and the age (which was not the case in our linear example). Hence the parameter of interest is the **average conditional marginal effect**

$$\mathbb{E}\left(\frac{\partial W}{\partial Ed}|Ed, Age\right) = \mathbb{E}\left(\frac{\partial h(Ed, Age, U)}{\partial Ed}|Ed, Age\right),$$

where we are taking the expectation over all the unobservables in $U$.

In general, identification of the average conditional marginal effect of education requires $Ed$ and $U$ to be independent given $Age$ (**conditional independence assumption**). If the model is additively separable (with the error term), such that

$$W = g(Ed, Age) + f(U),$$

we only need the conditional mean independence assumption

$$E(f(U|Ed, Age)) = E(f(U|Age)).$$

See Hansen (2019, Sections 2.14 and 2.30 for more details).

[1] This section was not covered in class and this material is not relevant for the exam.