Empirical Methods

Topic 2c:

Instrumental Variables Estimation

Instrumental Variables (IV)

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Instrumental Variables Overview I

- We turn finally to Instrumental Variables estimation
  - This is a critically important topic
  - As it is the first and most important tool in handling the violation of the most important CLRM assumption

$$E(\epsilon_i | x_i) = 0$$

IV Basics

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Instrumental Variables I

- Let's formalize how IV "works" using our familiar notation

$$y_i = x_i'\beta + \epsilon_i$$

- But assume $E(\epsilon_i|x_i) \neq 0$, in particular
  - Suppose some elements of x_i are *endogenous* while others are *exogenous*, i.e. $x_i' = [x_{1i}' \quad x_{2i}']$ with
    - x_{1i} a $K_1 \times 1$ vector of included endogenous variables
    - x_{2i} a $K_2 \times 1$ vector of included exogenous variables
    - And $K_1 + K_2 = K$
  - Note difference in notation here compared to what we had earlier
    - Where $\tilde{M}$ and $\tilde{K}$ were the numbers of included endogenous and exogenous variables

Instrumental Variables II

- To estimate our model when $E(\epsilon_i|x_i) \neq 0$, we need extra information
  - In particular, we need some variables to "move x_i" in ways that are uncorrelated with ϵ_i
- These variables are called Instrumental Variables
  - Let z_i be a $L \times 1$ vector of instruments with $z_i' = [z_{1i}' \quad x_{2i}']$ with
    - ★ z_{1i} a $L_1 \times 1$ vector of excluded (from this equation) exogenous variables
    - ★ (Note in our earlier notation, $L_1 = K^*$)
    - ★ x_{2i} (still) is a $K_2 \times 1$ vector of included exogenous variables
    - ★ And $L_1 + K_2 = L$

Instrumental Variables III

Instrumental Variables have two critical properties:

1. (IV1) Instrument Exogeneity
   - $\frac{1}{N}Z'\epsilon = \frac{1}{N}\sum_{i=1}^{N} z_i\epsilon_i \xrightarrow{p} E(z_i\epsilon_i) = 0$
   - In words: The instruments are uncorrelated with the error
     - ★ Usually ensured by assuming $E(\epsilon_i|z_i) = 0$
     - ★ And evaluated by thinking about $Cov(z_i, \epsilon_i)$

Instrumental Variables IV

- NOTE: Instrument Exogeneity also implies the instruments don't *themselves* belong in the structural equation.
- Suppose they did, i.e. suppose:

$$\begin{aligned} \text{True model: } y_i &= x_i'\beta + z_i'\gamma + \epsilon_i^* \\ \text{You estimate: } y_i &= x_i'\beta + \epsilon_i \\ & \quad \text{(using } z_i \text{ as an instrument)} \end{aligned}$$

Instrumental Variables V

$$\begin{aligned} \text{True model: } y_i &= x_i'\beta + z_i'\gamma + \epsilon_i^* \\ \text{You estimate: } y_i &= x_i'\beta + \epsilon_i \\ &\quad \text{(using } z_i \text{ as an instrument)} \end{aligned}$$

- Just because you don't include z_i in your model doesn't mean it's not actually *in there*
  - It's clearly true that $\epsilon_i = z_i'\gamma + \epsilon_i^*$
- Thus *even if* $Cov(z_i, \epsilon_i^*) = 0$, you have...

$$\begin{aligned} Cov(z_i, \epsilon_i) &= Cov(z_i, z_i'\gamma + \epsilon_i^*) \\ &= V(z_i)'\gamma \neq 0 \end{aligned}$$

  - ...and you've violated Instrument Exogeneity

Instrumental Variables VI

Two critical IV properties, cont.:

2. (IV2) Instrument Relevance
   - $\frac{1}{N}Z'X = \frac{1}{N}\sum_{i=1}^{N} z_i x_i' \xrightarrow{p} E(z_i x_i') \equiv \Sigma_{ZX} \neq 0$
   - In words: The instruments are correlated with the RHS variables
     - Evaluated by estimating $Cov(z_i, x_i)$
     - (In a "first-stage" regression)
     - (To be introduced soon)

Instrumental Variables VII

- As well as one final regularity assumption,
  Assumption 6': Regular Z's:
  - $\frac{1}{N} Z'Z = \frac{1}{N} \sum_{i=1}^{N} z_i z_i' \xrightarrow{p} E(z_i z_i') \equiv \Sigma_{ZZ}$
    - ★ Where Σ_{ZZ} is a finite $L \times L$ matrix
    - ★ (In words: A standard regularity condition analogous to OLS's (A6, Regular X's))

Instrumental Variables Intuition I

- OK, fine for the math. But what is the intuition for IV?
- It's a sequence of logic that goes like this:
  1. In essence, the OLS estimate $\hat{\beta}_k$ measures the correlation between y_i and x_{ik}
     - ★ (Controlling for the other x's)
  2. Under the CLRM Assumptions, this measures the *causal* effect of x_{ik} on y_i
     - ★ (Especially (A1, Linearity) and (A2, Mean-zero error))

Instrumental Variables Intuition II

Sequence of logic, cont.:

3. Sometimes, however, we have endogeneity and (A2, Mean-zero error) *doesn't* hold
   - $(E(\epsilon_i|x_i) \neq 0)$
   - (The bias formula shows that $E(\hat{\beta}_k) \neq \beta_k$)
   - (Thus OLS *cannot* recover the causal effect of x_{ik} on y_i)

Instrumental Variables Intuition III

Sequence of logic, cont.:

4. IV provides a solution: find an instrument that, as it varies...
    - ...causes x_i to vary...
        - [Assumption (IV2, Instrument relevance)]
    - ...in a way that *isn't* correlated with ϵ_i...
        - [Assumption (IV1, Instrument exogeneity)]
    - ...and let that "clean" variation in x_{ik} measure the causal effect of β_k

That's It!

Fitting our example into this framework

- We can fit our earlier example into this framework
- Our demand curve was written as:

$$q_i \quad = \alpha_0 + \alpha_1 p_i + \alpha_2 inc_i + \epsilon_{i1}$$

- Thus:
  - $y_i = q_i$,
  - $x'_{1i} = p_i, \;\; x'_{2i} = [1 \quad inc_i], \;$ thus $x'_i = [p_i \quad 1 \quad inc_i]$
  - $z'_{1i} = w_i, \;\; x'_{2i} = [1 \quad inc_i], \;$ thus $z'_i = [w_i \quad 1 \quad inc_i]$

Instrumental Variables Notes I

Note:

1. Once we move into instrumental variables, we focus on large-sample properties, i.e.
   - Consistency rather than Unbiasedness
     - Indeed taking the expectation of the IV estimator is a *pain in the neck*
     - Whereas taking its probability limit is easy
   - Asymptotic Normality rather than Normality
   - We'll discuss the properties of the IV estimator after defining it

Instrumental Variables Notes II

Note:

2. z_i is always constructed so that its exogenous components, x_{2i}, "instrument for themselves"
   - So $z_i' = [z_{1i}' \quad x_{2i}']$
     - (where z_{1i} are the instruments for the RHS endog variables, x_{1i})
   - The real trick is therefore to find the z_{1i},
     - In our example, $z_{1i}' = [w_i]$, $x_{2i}' = [1 \quad inc_i]$
     - And thus $z_i' = [w_i \quad 1 \quad inc_i]$

Instrumental Variables Notes III

Note, cont:

3. What might qualify as instruments? The two key IV conditions suggest the answer:
    1. Instrument Exogeneity $\Rightarrow$ variables uncorrelated with the error term
        - ★ (i.e. Exogenous variables)
    2. Instrument Relevance $\Rightarrow$ variables *in the (some!) model*
        - ★ (*Even if the full model isn't specified*)
        - ★ (Without this, there is no justification for the claim that $\Sigma_{ZX} \neq 0$)

Instrumental Variables Notes IV

- All the exogenous variables (across all equations) are candidates.
  - Thus cost/competition shifters are good candidates in our example:
    - They both plausibly influence our (unspecified) supply equation...
    - (And - hopefully - can be assumed to be exogenous)
    - (And don't belong in the demand curve directly)
  - What if we haven't written down a full system of simultaneous equations?
    - Where do instruments come from in this case???
    - Answer:* __________

IV & 2SLS: Just- versus Over-identified

- The exact form of the IV estimator depends on the number of
  - Instruments versus RHS variables (parameters)
- If $L = K$, we say we are just-identified.
  - We can estimate by (simple) IV
    - ★ Note the condition $L = K$ is the same as the condition $L_1 = K_1$
- If $L > K$, we say we are over-identified.
  - We can estimate by Two-Stage Least Squares (2SLS)
- And finally if $L < K$, we say we are* __________
  - *__________

OLS and IV as moment estimators

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Aside: An alternative way to derive the OLS estimator I

- You've been taught OLS minimizes the sum of squared residuals
  - It can also be rationalized another - completely different - way
- Recall our key assumption that $E(\epsilon_i|x_i) = 0 \Rightarrow Cov(x_i, \epsilon_i) = 0$
  - This is a "population moment condition", i.e.
  - It is a *moment condition* that we assume *holds in the population*
- Moment conditions play an important role in econometrics
  - One of the most widely used estimation methods is the Generalized Method of Moments (GMM)

Aside: An alternative way to derive the OLS estimator II

- The general idea is to solve a system of equations for parameters that satisfy the moment condition in one's sample
  - The sample analog to $Cov(x_i, \epsilon_i) = 0$ is $\frac{1}{N}X'e = 0 \ \Leftrightarrow X'e = 0$
- Suppose we imposed (A1, Linearity) and (A2, Mean-zero error)

$$\begin{aligned} y &= X\beta + \epsilon \qquad \text{is the PopRegFn} \\ &= X\tilde{\beta} + e \qquad \text{is the SampRegFn} \end{aligned}$$

  - ...and decided to choose an estimator, $\tilde{\beta}$ to satisfy the moment condition, $X'e = 0$

Aside: An alternative way to derive the OLS estimator III

$$\begin{aligned} X'e &= 0 \\ X'(y - X\tilde{\beta}) &= 0 \\ X'y &= X'X\tilde{\beta} \\ \tilde{\beta} &= (X'X)^{-1}X'y \end{aligned}$$

- That looks familiar! $\tilde{\beta} = \hat{\beta}$, the OLS estimator of β!

$\boxed{\hat{\beta}^{OLS} \text{ solves the sample analog of the population moment condition, } Cov(x_i, \epsilon_i) = 0}$

IV as the solution to a moment condition

- This same principle underlies all IV estimators
  - i.e., both the just- and over-identified cases
- Instead of imposing the OLS population moment condition, $Cov(x_i, \epsilon_i) = 0$,
  - IV does so with the instrument matrix, z_i, i.e. $Cov(z_i, \epsilon_i) = 0$
- Why using z_i and not x_i?
  - Because of endogeneity, $Cov(x_i, \epsilon_i) \neq 0$...
  - ... but with a valid instrument matrix $Cov(z_i, \epsilon_i) = 0$)

Just-identified IV

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Just-identified IV Estimation I

- In the simplest case, we are just-identified, i.e.
  - $L = K$, i.e.
    - the number of instruments we have is equal to the number of parameters, OR
  - $L_1 = K_1$, i.e.
    - the number of instruments for our RHS endogenous variables equals the number of our RHS endogenous variables
  - (These are the same because the K_2 elements in x_{2i} are present in both X and Z...
    - ...and therefore don't influence whether or not you're just-identified)

Just-identified IV Estimation II

- In the just-identified case, the IV estimator just sets the sample analog of the (identifying) moment condition to 0

$$\begin{aligned} Z'\epsilon &= 0 \Rightarrow \\ Z'(y - X\beta) &= 0 \Rightarrow \end{aligned}$$

$$\boxed{\hat{\beta}_{IV} = (Z'X)^{-1}Z'y}$$

- Being just-identified is important, as it ensures that $(Z'X)$ is square
  - (Assumption (IV2) plus a final assumption that rank(Z) = rank(X) = K ensures $Z'X$ is invertible)

Just-identified IV Properties: Consistency

- As suggested, we focus on the consistency rather than the unbiasedness of the IV estimator
- By the properties of the probability limit operator:

$$
\begin{aligned}
\hat{\beta}_{IV} &= \beta + (Z'X)^{-1} Z'\epsilon \\
plim\ \hat{\beta}_{IV} &= plim\ \beta + plim\ (Z'X)^{-1} plim\ Z'\epsilon \\
plim\ \hat{\beta}_{IV} &= \beta + plim\ (\frac{1}{N} Z'X)^{-1} plim\ \frac{1}{N} Z'\epsilon \\
&= \beta + \Sigma_{ZX}^{-1}\ 0 \quad \text{by (IV1) + (IV2)} \\
&= \beta
\end{aligned}
$$

  - Thus the IV estimator is consistent

Just-identified IV Properties: Asymptotic Normality I

- We turn next to the asymptotic distribution of $\hat{\beta}_{IV}$
- From the last slide, we can write

$$\begin{aligned}\hat{\beta}_{IV} &= \beta + (Z'X)^{-1}Z'\epsilon \\ \Rightarrow \sqrt{N}(\hat{\beta}_{IV} - \beta) &= (\tfrac{1}{N}Z'X)^{-1}\tfrac{1}{\sqrt{N}}Z'\epsilon\end{aligned}$$

  - (This should look familiar - it's very similar to the OLS formulas...
    - ...but with Z' in place of X')

Just-identified IV Properties: Asymptotic Normality II

$$\boxed{\sqrt{N}(\hat{\beta}_{IV} - \beta) = (\tfrac{1}{N}Z'X)^{-1}\tfrac{1}{\sqrt{N}}Z'\epsilon}$$

- We know from earlier that

$$\begin{aligned} \tfrac{1}{N}(Z'X)^{-1} &\xrightarrow{p} E(Z'X) = \Sigma_{ZX} \text{ and} \\ \tfrac{1}{N}(Z'Z)^{-1} &\xrightarrow{p} E(Z'Z) = \Sigma_{ZZ} \end{aligned}$$

- What of $\frac{1}{\sqrt{N}}Z'\epsilon$?

Just-identified IV Properties: Asymptotic Normality III

- We could (but won't) show that

$$\frac{1}{\sqrt{N}} Z'\epsilon \xrightarrow{d} N(E(x_i\epsilon_i), V(z_i\epsilon_i))$$

where

- $E(z_i\epsilon_i) =$__________
- The form of $V(z_i\epsilon_i) = E(\epsilon_i^2 z_i z_i') = E(\epsilon_i^2)E(z_i z_i')$ depends on further assumptions about the variance-covariance matrix of ϵ

Just-identified IV Properties: Asymptotic Normality IV

- If we're willing to make our earlier assumptions (A3, Homoskedasticity) and (A4, No Correlation), then

$$\begin{aligned} V(z_i\epsilon_i) &= E(\epsilon_i)^2 E(z_i z_i') \\ &= \sigma^2 E(z_i z_i') && \text{under (A3) and (A4)} \\ &= \sigma^2 \Sigma_{zz} && \text{under (our new) (A6')} \end{aligned}$$

- By Slutsky's Theorem, then:

$$\begin{aligned} \sqrt{N}(\hat{\beta}_{IV} - \beta) &\xrightarrow{d} \Sigma_{ZX}^{-1} N(0, \sigma^2 \Sigma_{ZZ})) \\ &= N(0, \sigma^2 \Sigma_{ZX}^{-1} \Sigma_{ZZ} \Sigma_{ZX}^{-1}) \end{aligned}$$

Just-identified IV Properties: Asymptotic Normality V

$$\boxed{\sqrt{N}(\hat{\beta}_{IV} - \beta) \xrightarrow{d} N(0, \sigma^2 \Sigma_{ZX}^{-1} \Sigma_{ZZ} \Sigma_{ZX}^{-1})}$$

- We can approximate this in small samples with

$$\sqrt{N}(\hat{\beta}_{IV} - \beta) \quad \overset{a}{\sim} \quad N(0, s^2(\tfrac{1}{N}Z'X)^{-1}(\tfrac{1}{N}Z'Z)(\tfrac{1}{N}Z'X)^{-1})$$

$$\hat{\beta}_{IV} \quad \overset{a}{\sim} \quad N(\beta, \tfrac{s^2}{N}(\tfrac{1}{N}Z'X)^{-1}(\tfrac{1}{N}Z'Z)(\tfrac{1}{N}Z'X)^{-1})$$

$$\boxed{\hat{\beta}_{IV} \overset{a}{\sim} N(\beta, s^2(Z'X)^{-1}(Z'Z)(Z'X)^{-1})}$$

  - where $e_i = y_i - x_i'\hat{\beta}_{IV}$ and $s^2 = \frac{1}{N}\sum_{i=1}^{N} e_i^2$
    - ★ (Note for IV, we always get the "sandwich formula", even if (A3) and (A4) hold)

Just-identified IV Properties: Summary

- Based on these two properties, we can say that the IV estimator is consistent and asymptotically normally distributed:

$$\hat{\beta}_{IV} \overset{a}{\sim} N(\beta, s^2(Z'X)^{-1}(Z'Z)(Z'X)^{-1})$$

- If the variance-covariance matrix of the errors is heteroskedastic,* then the asymptotic distribution of $\hat{\beta}_{IV}$ is

$$\sqrt{N}(\hat{\beta}_{IV} - \beta) \xrightarrow{d} N(0, \Sigma_{ZX}^{-1} \lim_{N\to\infty} E\left[\left(\frac{1}{N}\sum_{i=1}^{N} \epsilon_i^2 z_i z_i'\right)\right] \Sigma_{ZX}^{-1})$$

and we can approximate it with

$$\hat{\beta}_{IV} \overset{a}{\sim} N(\beta, (Z'X)^{-1}(\textstyle\sum_{i=1}^{N} e_i^2 z_i z_i')(Z'X)^{-1})$$

Over-identified IV / 2SLS

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Two-Stage Least Squares (2SLS)

- If one has more instruments than parameters ($L > K$), then our moment conditions are:

$$Z'\epsilon = Z'(y - X\beta) = 0$$

- This is of dimension $L \times 1$, but we only have K unknowns in β
  - There generally will not be any way to exactly satisfy *all* the L moments
- Q: So what do we do? Which $\hat{\beta}_{IV}$ should we choose?*
  - ____
  - ____
  - ____

2SLS - Two-Stage Least Squares I

If interpreted *literally*, Two-Stage Least Squares proposes:

1. Stage 1: Regress the included RHS endogenous variables, x_{1i}, on all the instruments, z_i

$$\begin{aligned} X_1 &= Z\pi + v \quad \Rightarrow \\ \hat{\pi} & \Rightarrow \\ \hat{X}_1 &= Z\hat{\pi} \end{aligned}$$

  - Where we assume for simplicity that we have just a single endogenous variable, i.e. X_1 is $N \times 1$
    - (If we had more, we would run regress each column of X_1 on the full matrix of instruments, Z, getting a $\hat{\pi}$ for each column)
  - This is (These are) called the "First-Stage Regression(s)"

2SLS: The First Stage I

- Economists play close attention to the results of the first stage when evaluating an IV regression
- They pay particular attention to the joint significance of the instruments for X_1, Z_1, in $Z = [Z_1 \quad X_2]$
  - i.e. in the reduced-form equation,

$$x_{1i} = z_{1i}'\pi_1 + x_{2i}'\pi_2 + \nu_{i1}$$

  - We calculate the F-statistic for the hypothesis, $H_0 : \pi_1 = 0$.
    - ★ This has L_1 numerator degree of freedom and $N - K_2 - 1$ denominator degrees of freedom

2SLS: The First Stage II

- A typical rule of thumb is that the F-test on the instruments *should exceed 10*.
  - (If you have just one instrument, the F-statistic is the square of the t-statistic...
  - ...so this means a t-statistic in excess of $\sqrt{10} = 3.16$)
    - [This is from the paper by Staiger and Stock (1997)]
- We'll discuss why once I give the intuition of the IV estimator

2SLS - Two-Stage Least Squares II

2. Stage 2: "Regress" y on $\hat{X}_1$ and X_2, i.e.

$$\begin{aligned} y &= \hat{X}_1\beta_1 + X_2\beta_2 + \epsilon \\ &= \hat{X}\beta + \epsilon \qquad\qquad \text{where} \quad \hat{X} = [\hat{X}_1 \quad X_2] \end{aligned}$$

3. That's it!
4. Those are the two stages in Two-Stage Least Squares, but ...
   - How does that compare with the intuition we covered earlier?

2SLS Intuition I

No surprise: it's the same intuition:

- We'd like to estimate

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

  - but X_1 is endogenous, i.e. $E(\epsilon|X_1) \neq 0$
  - (And even if $E(\epsilon|X_2) = 0$, we showed that correlation between ϵ and *any* X's biases *all* coefficients)

- We estimate instead:

$$y = \hat{X}_1\beta_1 + X_2\beta_2 + \epsilon$$

  - where $\hat{X}_1 = Z\hat{\pi}$ from the First-Stage regression
    - ★ Critically: $\hat{X}_1$ is that part of X_1 that *can be explained by* Z

2SLS Intuition II

$$\begin{aligned} y &= X_1\beta_1 + X_2\beta_2 + \epsilon \\ y &= \hat{X}_1\beta_1 + X_2\beta_2 + \epsilon \end{aligned}$$

2SLS Intuition, cont.

- We've essentially replaced the "problematic" (i.e. endogenous) X_1 with its prediction as a function of Z
  - Since it's a function of Z, *all of which are exogenous*, $\hat{X}_1$ isn't correlated with ϵ... even if X_1 is
    - (Because under Assumption (IV1), $E(\epsilon|Z) = 0$)
- Thus no more endogeneity problem!
  - $\Rightarrow$ consistent estimation of β_1 and β_2!

IV Intuition Redux I

- So the intuition of 2SLS is:
  - "Replace the problematic variables with their fitted values (which are a function of exogenous instruments)"
- Another way to say the same thing goes like this:
- Imagine taking one of the elements of the problematic (endogenous) X_1
- And dividing it into two parts:
  1. A problematic part
     - Which is correlated with ϵ_i
  2. A non-problematic part
     - Which is uncorrelated with ϵ_i

IV Intuition Redux II

- We'd like to rely only on the *non-problematic* variation in X_1 to identify β
  - Leaving aside the problematic variation
- This is imperfect of course
  - As relying on less than the full variation in any variable means we'll get noisier estimates of $\hat{\beta}$
  - But imprecision is better than bias!

IV Intuition Redux III

- This is exactly what IV does:
  - It relies on variation in X_1 *induced by variation in the instruments, Z*
  - Since these instruments are uncorrelated with ϵ, this is "non-problematic" variation
    - You can imagine...

See Figure in Class

2SLS - Two-Stage Least Squares III

- Important note:
  - You should never literally run the 2 regressions in 2SLS as it gives the wrong standard errors for $\hat{\beta}_{2SLS}$
    - ★ Why? Because we are using a fitted value in the second stage regression ($\hat{X}_1$)
    - ★ This introduces additional error into $V(\hat{\beta}_{2SLS})$, but Stata and R don't know you are using a fitted value
    - ★ (and therefore don't account for it)
- Of course, you could bootstrap the standard errors, but
  - I didn't teach you how to bootstrap standard errors. [:-)]
  - There is an easier solution

2SLS - Two-Stage Least Squares IV

- Instead: Simply use $\hat{X}_1$ as part of a just-identified IV specification,
  - i.e. instrument for $X = [X_1 \quad X_2]$ with $Z_{2SLS} = [\hat{X}_1 \quad X_2]$
    - ★ This is $N \times K$ no matter how many instruments were in Z_1

$\Rightarrow$ Thus, from our earlier equation: $\hat{\beta}_{2SLS} = \hat{\beta}_{IV}$ using $Z_{2SLS} = [\hat{X}_1 \quad X_2]$ as the instrument matrix

$$\hat{\beta}_{2SLS} = (Z'_{2SLS}X)^{-1}Z'_{2SLS}y$$

- We're all set to show the properties of the 2SLS estimator...
  - Except to do so we need a convenient way to represent $\hat{X}_1$

2SLS using P_Z I

- Recall our projection matrix from earlier, $P_X = X(X'X)^{-1}X'$
  - Which gives the fitted value of a regression of <anything> on X
- Earlier, we said the first-stage regression is the regression of x_{1i} on z_i:

$$\begin{aligned} X_1 &= Z\pi + v && \Rightarrow \\ \hat{\pi} &= (Z'Z)^{-1}Z'X_1 && \Rightarrow \\ \hat{X}_1 &= Z\hat{\pi} \\ &= Z(Z'Z)^{-1}Z'X_1 \\ \Rightarrow \hat{X}_1 &= P_Z X_1 \end{aligned}$$

2SLS using P_Z II

- Recall that a general instrument vector is written as $Z = [Z_1 \quad X_2]$
  - So it includes the instrument(s) for X_1 (Z_1) as well as the exogenous variables, X_2
- We can use this fact and the properties of the projection matrix to write Z_{2SLS} in a convenient form:
  - (1) $P_Z X_2 =$???
  - (2) Last slide we showed that $P_Z X_1 = \hat{X}_1$
  - (1) + (2) $\Rightarrow \; Z_{2SLS} \equiv [\hat{X}_1 \quad X_2] = P_Z[X_1 \quad X_2] = \; P_Z X$
- Thus:

$$\boxed{Z_{2SLS} = P_Z X \qquad \text{and} \qquad Z'_{2SLS} = X' P_Z}$$

2SLS using P_Z III

$$\boxed{Z_{2SLS} = P_Z X \qquad \text{and} \qquad Z'_{2SLS} = X'P_Z}$$

- These allow us to write the formula for the 2SLS estimator in terms of its fundamental elements

$$\begin{aligned} \hat{\beta}_{2SLS} &= (Z'_{2SLS}X)^{-1}Z'_{2SLS}y \\ &= (X'P_Z X)^{-1}X'P_Z y \end{aligned}$$

$$\boxed{\hat{\beta}_{2SLS} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y}$$

  - (This representation is much easier to work with to show consistency and asymptotic normality)

2SLS Properties: Summary I

- One can derive the properties of the 2SLS estimator in much the same way as we did for the just-identified IV estimator
  - So I'll just summarize the formulas for you here:
    - ★ The 2SLS estimator is consistent:

$$\hat{\beta}_{2SLS} \quad \xrightarrow{p} \quad \beta$$

    - ★ Under (A3) and (A4) - homoscedasticity and no serial correlation - the 2SLS estimator is asymptotically normal with approximate distribution:

$$\begin{aligned} \hat{\beta}_{2SLS} \quad & \overset{a}{\sim} \quad N(\beta, \sigma^2 (X' P_Z X)^{-1}) \\ & \overset{a}{\sim} \quad N(\beta, \sigma^2 (X' Z (Z' Z)^{-1} Z' X)^{-1}) \end{aligned}$$

2SLS Properties: Summary II

$$\boxed{\begin{array}{rcl} \hat{\beta}_{2SLS} & = & (X'P_ZX)^{-1}X'P_Zy \\ \hat{\beta}_{2SLS} & \stackrel{a}{\sim} & N(\beta, \sigma^2(X'P_ZX)^{-1}) \end{array}}$$

- Compare the formulas above to the OLS formulas:
  - $\hat{\beta} = (X'X)^{-1}X'y$ and $\hat{\beta} \stackrel{a}{\sim} N(\beta, \sigma^2(X'X)^{-1})$
- Endogeneity requires us to instrument for "problematic" X_1 with Z_1, with two effects:
  1. We can only rely on that variation in X induced by Z (in blue)
     - ★ (i.e. $X'P_ZX$ instead of $X'X$)
  2. And this induces *higher variance* in $V(\hat{\beta}_{2SLS})$ relative to $V(\hat{\beta})$ (in red)
     - ★ Due to relying only on that part of the variation in X induced by Z, $X'P_ZX$

Instrumental Variables Comments I

- As mentioned earlier, in the over-identified case, we have more moments than parameters to estimate
  - So we may not be able to find a single β that solves all of them
- The 2SLS estimator is the most efficient one
  - (Under our assumptions, particularly (A3) and (A4))
- More generally, one must weight the L moments to solve for the K parameters in β
  - The next - optional - slides show the form of the IV estimator for an arbitrary weighting matrix, W
  - (Where the 2SLS weighting matrix is $W = (Z'Z)^{-1}$)

Instrumental Variables Comments II*

- Formally, let M be an $(L \times L)$ matrix of weights to give to each of our moments
- Let Q be the quadratic form (the matrix analog to a sum of squares & cross products) in the sample moments:

$$Q = \underbrace{(y - X\beta)'Z}_{(1\times L)} \underbrace{W}_{(L\times L)} \underbrace{Z'(y - X\beta)}_{(L\times l)}$$

Note:

- Q is a scalar
- The optimal weighting matrix in general is proportional to the inverse of the variance of the moment conditions.
  - ★ Under (A3) and (A4), $V(Z'\epsilon) = E(Z'\epsilon\epsilon'Z) = (Z'(\sigma^2 I_N)Z) = \sigma^2(Z'Z)$
  - ★ Thus the 2SLS estimator uses the weighting matrix $W = (Z'Z)^{-1}$

Instrumental Variables Comments III*

- For an arbitrary W, taking derivatives w.r.t β and solving yields

$$\boxed{\hat{\beta} = (X'ZWZ'X)^{-1}X'ZWZ'y}$$

- You can also show:

$$\begin{aligned} \hat{\beta} &\overset{a}{\sim} N(\beta, AV_{\beta}) \\ AV_{\beta} &\approx (X'ZWZ'X)^{-1}X'ZW\Omega WZ'X(X'ZWZ'X)^{-1} \\ \Omega &= V(Z'\epsilon) = E(Z'\epsilon\epsilon'Z) \end{aligned}$$

Show yourself you get the 2SLS formula if $\Omega = \sigma^2 Z'Z$ and $W = (Z'Z)^{-1}$

  - (What a complicated mess!)
  - (But useful if you take further econometrics...)

Instrumental Variables Comments IV

- While these formulas are superficially complicated, efficient weighting mechanisms pop up frequently in econometrics:
  - Weighting moments for efficient estimation (GMM)
    - What we've done here
  - Weighting observations for heteroscedasticity correction (GLS)
  - Weighting equations for cross-equation correlations (SUR)

IV: Practical Considerations

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IV: Practical considerations

- We close our discussion of IV with some practical considerations:
  1. Avoiding the need for instruments
  2. Finding instruments
  3. IV standard errors
  4. Weak Instruments
  5. Testing
     - ★ Endogeneity
     - ★ Overidentification

Avoiding the need for instruments I

- We've talked to this point as if the presence of a "problematic" RHS variable is a given
  - Induced by there being correlation between at least one x_i and ϵ_i
  - And focused on finding an instrument(s) z_i to shift x_i in a way that's uncorrelated with ϵ_i
- But I have to emphasize that there is another - indeed easier - solution if the endogeneity is due to a correlated unobservable (if it's possible):
  - *Take the problematic variation out of the error term*

Avoiding the need for instruments II

- In particular, suppose there is some *potentially measurable* economic variable that
  - Isn't currently in the econometric model
    - ★ And is thus in ϵ_i
  - Is correlated with one of our x's
    - ★ Causing endogeneity
- If we can just include that variable in the econometric model, all is good:
  - We'll be picking up its effect (taking it *out* of ϵ_i)
  - (Hopefully) leaving whatever is left in ϵ_i uncorrelated with x_i
    - ★ Permitting consistent estimation with OLS

Avoiding the need for instruments III

- Some support for this line of argument in the academic literature
  - e.g. Rossi (2014, *Marketing Science*), "Even the Rich Can Make Themselves Poor: A Critical Examination of IV Methods in Marketing Applications"
- Thus always keep this strategy in the back of your mind!
  - If it's possible, it is very likely to be a better strategy than IV as retains all the variable in x_i to estimate β
  - (Example: Ruralness in Marilyn's Bachelor's thesis)

Finding instruments I

- One of the greatest challenges with IV estimation is *finding instruments*
- We need something that's both exogenous w.r.t the error in the equation of interest *and* correlated with an endogenous variable of interest
- This can be very, very challenging

Finding instruments II

- For example, imagine modeling a family's decisions:
  - Whether and how much to work, whether and for how long to go to school, whether and how many children to have, where to live, etc.
  - Each plausibly impacts each other (structural equations), requiring instruments for each in the appropriate equation
  - Good luck!!!
- In practice, researchers usually focus on a subset of the decisions
  - And/or try to jointly model them all as a function of (hopefully!) exogenous "initial conditions"
  - (Where born, parent's characteristics, "innate" attributes, etc.)
- We'll go through many examples of this in class and on your problem set(s)

Finding instruments III

- Indeed, let's return to Marilyn's bachelor's thesis seeking to measure the impact of cross-border workers on anti-immigrant vote shares among a panel of 200 Swiss cities over 10 years:

$$v_{it} \quad = \quad \beta_1 + \beta_2 CBW_{it} + \tilde{x}_i' \tilde{\beta} + \epsilon_{it}$$

- Recall what caused our concerns about endogeneity:
  - (I think: correlated unobservables. Possibly also endogeneity, but probably not)
- What would be the qualitative features of a good instrument?
  - (Hint: Relevance:) Something correlated with the number of cross-border workers...
  - (Hint: Exogeneity:) ... that is uncorrelated with unobserved determinants of anti-immigrant vote shares (ϵ_{it})
- Any ideas???

IV Standard Errors I

- It's very common for standard errors on IV estimates to be larger - and perhaps much larger - than their OLS counterparts
- We've hinted at why this is a few slides ago

IV Standard Errors II

- IV "works" by having an instrument, z_{1i}, "move around" the included RHS endogenous variable, x_{1i}
  - Recall $\hat{\beta}_{2SLS} = (X'P_Z X)^{-1} X'P_Z y$
- This is necessarily imperfect, as z_{1i} isn't perfectly correlated with x_{1i}
  - (And if it was, then it *wouldn't* satisfy (IV1), Instrument Exogeneity)
- As a result, there is less instrument-induced-variation in x_{1i} than x_{1i} could provide by itself (if it weren't endogenous)
  - And less x-variation means higher standard errors

Weak Instruments I

- Many applied researchers often worry about *Weak Instruments*
- What does this mean?
  - Weak instruments have no precise definition
  - A useful working definition is instruments for which the F-test on the joint significance of the instruments in the first-stage regression is "small"
  - (Recall we wanted an F-test of about 10 when we had a single RHS endogenous variable)

Weak Instruments II

- Weak instruments are problematic because they can exacerbate *finite-sample bias*
  - i.e. IV may be consistent in large samples, but is in general still biased
    - (The bias is generally towards the OLS estimate)
  - And this bias increases the weaker is the correlation between instruments and included endogenous variables

Weak Instruments III

- With a single endogenous regressor, Staiger and Stock (1997, *Econometrica*) estimate the size of the bias as $1/F$
  - Where F is the value of the first-stage F-statistic introduced earlier
- With multiple endogenous regressors, Stock, Wright, and Yogo (2002) calculate the minimum eigenvalue of a matrix-analogue to the first-stage F-statistic
  - And provide critical values for this eigenvalue for different bias tolerances

Weak Instruments III

- Well, this is all quite depressing
  - What should you do???
- No great options, but here's what we have:
  1. Try to include covariates that soak up the source of endogeneity
  2. If the problem is due to too many instruments, some of which are weak, then just drop the weak ones and keep the strong ones
     - (Can test with individual t-values in the first-stage regression(s))
  3. If problem is small-sample bias, can use estimators other than 2SLS
     - e.g, LIML, Split-sample IV, Jackknife IV
     - But all have downsides (harder to implement, less efficiency, etc.)
  4. Consider alternative research designs
     - Particularly (field) experiments

Testing

- There are two types of tests that often arise in the context of IV estimation
  1. Testing endogeneity
  2. Testing overidentifying restrictions

Endogeneity Test I

- Since we've spent so much time worrying about endogeneity...
  - Finding them, higher standard errors, weak instruments, etc.
- It would be nice if we could test for it!
- Indeed, there is a test as long as you have at least one potential instrument
  - (So we're not completely out of the woods)
  - (And note: we are testing for endogeneity...
  - ... *conditional* on the validity of our instrument)
- How?

Endogeneity Test II

- Consider both structural and reduced-form equations with scalar x_{1i}:

$$y_i = x_{1i}\beta_1 + x_{2i}'\beta_2 + \epsilon_{i1} \tag{1}$$

$$x_{1i} = z_{1i}'\pi_1 + x_{2i}'\pi_2 + \nu_{i1} \tag{2}$$

- We'd like to test whether x_{1i} is correlated with ϵ_{i1} in (1)
  - Perhaps we can use equation (2) to do so!
  - Since ϵ_{i1} is (by assumption) uncorrelated with both z_{1i} and x_{2i}...
  - ...then the only way x_{1i} could be correlated with ϵ_{i1} is if ν_{i1} is
    - ★ Why is this true?

Endogeneity Test III

- We don't observe ν_{i1}, but we can estimate it as the residual in the equation (2), $\hat{\nu}_{i1}$
- We can test whether this correlated with ϵ_{1i} by simply putting it into Equation (1).
  - If the parameter on it is significant, then we have endogeneity
- Formally estimate the following *by OLS*:

$$y_i = x_{1i}'\beta_1 + x_{2i}'\beta_2 + \delta\hat{\nu}_{i1} + \epsilon_{i1}$$

and test $H_0 : \delta = 0$

- This is called (Hausman) endogeneity test
  - Note you might be surprised by the familiarity of the estimated coefficients (other than $\hat{\delta}$ in this regression)

Overidentifying Test I

- In general, we would *love* to test the validity of an instrument
  - In other words, is it really the case that $Z'\epsilon = 0$???
- In principle, we have an estimate of ϵ_i, e_i, so can't we use this as a proxy?
  - Unfortunately not
  - We showed earlier that the $Z'e$ is *exactly* zero when we are just-identified
    - (Thus no test is possible)

Overidentifying Test II

- But... we can test whether $Z'\epsilon = 0$ when we are *over-identified*
  - As $Z'\epsilon = 0$ in the population, but $Z'e$ won't (exactly) equal zero in any sample
  - So the question is whether the deviation of $Z'e$ from zero is statistically large
- Tho note: this way we can test the the validity of *some* of the instruments
  - But may not know which one(s) are causing trouble if we reject!
  - (So a bit frustrating...)

Overidentifying Test III

- The formal test has three steps:
  1. Estimate (1) by 2SLS $\Rightarrow e_{i1}$
  2. Regress e_{i1} on z_{1i} and x_{2i} (i.e. all the exogenous variables)
     - ★ And calculate the R^2
  3. Then, under H_0 : *Instrument validity*, $NR^2 \overset{a}{\sim} \chi^2_{L_1 - K_1}$
     - ★ where the degrees of freedom equal the number of "extra" (i.e. overidentifying) instruments you have

- Last note: the overidentifying test is often also called the "J-Test" (from Sargan (1958))

Examples Redux

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Examples Redux

- Recall that we said there are three common sources of endogeneity:
  1. Correlated unobservables
  2. Measurement error (in X)
  3. Reverse causality / Simultaneous equations
- I also showed three examples whose results suggested bias of the type we had anticipated
  - Can we "solve" these issues using IV estimation?

Correlated Unobservables Example Redux I

- For the first, correlated unobservables/wage-ability example, we use data on married women's wages
  - And found (perhaps implausibly) high estimates of the returns to education (of 10.9%)
- Authors have proposed using as an instrument for a woman's education the *education of her parents*
- Do you think these are likely to satisfy the conditions for a good instrument?*
  - Relevance: ______________________________
  - Exogeneity:

    ____

Correlated Unobservables Example Redux II

- Remember what we said was the likely sign of the bias?
  - ______________________________
- Let's see what happens when we instrument with father's education

See Stata Example in Class

- Do these results seem (more) reasonable to you?*
  - ____
  - ____

Measurement Error Example Redux I

- For this second, measurement-error example, we used data from average health insurance coverage across the 50 US states (and the District of Columbia) in 2007
  - And (perhaps) found evidence of attenuation bias (0.23% pretty small)
- The author of the study from which I grabbed the data suggested using average high-school completion rate as an IV for their permanent income
- Do you think these are likely to satisfy the conditions for a good instrument?*
  - Relevance: ______________________________
  - Exogeneity:

    ____

Measurement Error Example Redux II

- Let's see what happens when we instrument with whether someone got an advanced degree

See Stata Example in Class

- Do these results seem (more) reasonable to you?*
  - ____
  - ____
- Which do you believe (most)?*
  - ____

Simultaneous Equations Example Redux I

- For our final, simultaneous equations example, we use annual data on the purchase of truffles
- This is a classic Supply and Demand environment
  - So it's natural to use a supply shifter as an instrument for a demand equation
  - In this data we have the rental price of...

Simultaneous Equations Example Redux II

- Do you think this is likely to satisfy the conditions for a good instrument?*
  - Relevance: ____
  - Exogeneity:

    ____
- Remember what we said was the likely sign of the bias?
  - ______________________

Simultaneous Equations Example Redux III

- Let's see what happens when we instrument with the rental price of truffle pigs

See Stata Example in Class

- Do these results seem (more) reasonable to you?*
  - ____

IV Conclusions I

- Understanding the potential sources of bias in regression...
  - ...and how to resolve it using Instrumental Variables...
  - ...*cannot* be overestimated as a key skill in your "Econometrician's toolkit"
- In my view it's what separates "Careful Econometricians" from "Regression Runners"
  - Please be one of the former and not one of the latter!

IV Conclusions II

- In the remainder of the course, we will cover the second-most common method to address endogeneity bias
  - Panel Data Methods

Coming Up Next!

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