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Assignment 9

Problem 1) For $(n\log n+1)^2+(\log n+1)(n^2+1)=O(n^t)$ to hold true, n^t must be an upper bound of $(n\log n+1)^2+(\log n+1)(n^2+1)$. For this to be true, $\lim_{n\to\infty}\frac{(n\log n+1)^2+(\log n+1)(n^2+1)}{n^t}\le d \mid d\in\mathbb{R}_{>0}$. This statement is true unless d is ∞, so t must be an integer such that n^t grows at a faster rate than $(n\log n+1)^2+(\log n+1)(n^2+1)$. Since $(n\log n+1)^2+(\log n+1)(n^2+1)$ grows at a rate of $n^2(\log n)^2$, t must be ≥ 3 for the limit to approach 0, which would make the original statement hold true. For example, if $t=2$, then the limit would approach ∞ and the original statement would not hold true. Thus, 3 is the least integer $t \mid (n\log n+1)^2+(\log n+1)(n^2+1)=O(n^t)$.

Problem 2) Counter-example: Let $f(n)=n+1$; $g(n)=n-1$; $h(n)=n$

First, prove $f(n)$ and $g(n)$ are both $\Theta(h(n))$.

$c\le\lim_{n\to\infty}\frac{n+1}{n}\le d$	$c\le\lim_{n\to\infty}\frac{n-1}{n}\le d$
$c\le\lim_{n\to\infty}1+\frac{1}{n}\le d$	$c\le\lim_{n\to\infty}1-\frac{1}{n}\le d$
$c\le\lim_{n\to\infty}1+0\le d$	$c\le\lim_{n\to\infty}1-0\le d$
$c\le 1\le d$; $c=1$, $d=1$	$c\le 1\le d$; $c=1$, $d=1$
$f(n)=\Theta(h(n))$ holds true.	$g(n)=\Theta(h(n))$ holds true.

Consider $(f-g)(n)=\Theta(h(n))$

$$c\le\lim_{n\to\infty}\frac{n+1-(n-1)}{n}\le d$$
$$c\le\lim_{n\to\infty}\frac{n+1-n+1}{n}\le d$$
$$c\le\lim_{n\to\infty}\frac{2}{n}\le d$$
$$c\le 0\le d,\ \neg\exists c \text{ such that } c\in\mathbb{R}_{>0}.$$

As depicted in this counter-example, even if $f(n)$ and $g(n)$ are both $\Theta(h(n))$, $(f-g)(n)\ne\Theta(h(n))$.

Problem 3) Consider $f(n)=O(g(n))$ and $g(n)=O(f(n))$. This means that:

(i) $\frac{f(n)}{g(n)}\le a \mid a\in\mathbb{R}_{>0}$ AND (ii) $\frac{g(n)}{f(n)}\le b \mid b\in\mathbb{R}_{>0}$ which implies $\frac{f(n)}{g(n)}\ge\frac{1}{b}$

$\frac{1}{3}\le\frac{1}{2}\rightarrow\frac{3}{1}\ge\frac{2}{1}$ $\quad$ $5,\frac{1}{5}\in\mathbb{R}_{>0}$

Taking (i) and (ii) into account, we get $\frac{1}{b}\le\frac{f(n)}{g(n)}\le a$. If $b\in\mathbb{R}_{>0}$, $\frac{1}{b}\in\mathbb{R}_{>0}$

Since $a,\frac{1}{b}\in\mathbb{R}_{>0}$, $\frac{f(n)}{g(n)}$ is upper and lower bounded by constants.

Therefore, if $f(n)=O(g(n))$ and $g(n)=O(f(n))$, then $f(n)=\Theta(g(n))$.