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Problem 1

If 36 days are picked at random from a 2018 calendar, then at least 6 days must fall on the same day of the week. Prove by contradiction.

Assume: 36 days are picked at random from a calendar & it is not true that 6 days fall on the same day of the week.

Proof: (1) There are 7 days in one week. So there are 7 different cases when a day is picked.
(2) If the days picked are spread out evenly across the week, then the eighth day will fall on a day of the week that already has a pick.
(3) Using this logic, since $5 \times 7 = 35$, 35 days spread out evenly will result in each day of the week having 5 picks.
(4) Now, once the 36^{th} day is picked, all possible cases will result in at least one day of the week having 6 picks.
(5) This is inconsistent with the second part of the assumption.

$\therefore$ by contradiction, if 36 days are picked at random from a 2018 calendar, then at least 6 days must fall on the same day of the week. ■

Problem 2

Prove: $\min(x,y) = \frac{x+y-|x-y|}{2}$; $x,y \in \mathbb{R}$

Proof: Assume x and y are real numbers. Show $\min(x,y) = \frac{x+y-|x-y|}{2}$. Consider the following cases.

Case 1: $x-y > 0$

This implies $x > y$

$\min(x,y) = \frac{x+y-|x-y|}{2}$

$y = \frac{x+y-(x-y)}{2}$; definition of absolute value, $|x-y| = x-y$ (since $(x-y)>0$)

$y = \frac{2y}{2}$

$y = y$ □

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Problem 2 cont.

Case 2: $x-y<0$

This implies $x<y$

$\min(x,y) = \frac{x+y-|x-y|}{2}$

$x = \frac{x+y-(-(x-y))}{2} = \frac{x+y+(x-y)}{2}$; definition of absolute value, $|x-y| = -(x-y)$ (since $x-y<0$)

$x = \frac{2x}{2}$

$x = x$ □

Case 3: $x-y=0$

This implies $x=y$

$\min(x,y) = \frac{x+y-|x-y|}{2}$

$x = \frac{x+y-\|0\|}{2}$ $(\|x-y\| = \|0\|)$	$y = \frac{x+y-\|0\|}{2}$ $(\|x-y\| = \|0\|)$
$x = \frac{x+x-0}{2}$ (sub y with x)	$y = \frac{y+y-0}{2}$ (sub x with y)
$x = \frac{2x}{2}$	$y = \frac{2y}{2}$
$x = x$ □	$y = y$ □

OR

$\therefore$ for all possible cases, $\min(x,y) = \frac{x+y-|x-y|}{2}$ ■

Problem 4

Prove by contrapositive: If $x \notin \mathbb{Q}$, then $\frac{1}{x} \notin \mathbb{Q}$

Contrapositive: If $\frac{1}{x} \in \mathbb{Q}$, then $x \in \mathbb{Q}$

Assume: $\frac{1}{x} \in \mathbb{Q}$

Proof: Since $\frac{1}{x} \in \mathbb{Q}$, then by definition, $\frac{1}{x} = \frac{a}{b}$ for some $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $b \neq 0$

Let $1 = a$, and $x = b$

Then, $1 \in \mathbb{Z}$, $x \in \mathbb{Z}$, $x \neq 0$.

$x = \frac{x}{1}$

Since $x \in \mathbb{Z}$, $1 \in \mathbb{Z}$, and $1 \neq 0$, then by definition, $\frac{x}{1} \in \mathbb{Q}$.

Since $\frac{x}{1} = x$, $x \in \mathbb{Q}$.

We have proven that if $\frac{1}{x} \in \mathbb{Q}$, then $x \in \mathbb{Q}$

$\therefore$ by contrapositive, if $x \notin \mathbb{Q}$, then $\frac{1}{x} \notin \mathbb{Q}$. ■

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Problem 3:

Prove: Even(n+3) <=> Odd(5n+8)

Must show Even(n+3) → Odd(5n+8)

Odd(5n+8) → Even(n+3)

Prove: Even(n+3) → Odd(5n+8)

$n+3 = 2k,\ k \in \mathbb{Z}$ - by def. of Even

So, $5n+8 = 5n+15-7$

$= 5(n+3)-7$

$= 5(2k)-7$

$= 10k-7$

$= 10k-8+1$

$= 2(5k-4)+1$

Since $k \in \mathbb{Z}$ by closure $5k-4 \in \mathbb{Z}$

So $2(5k-4)+1$ is in $2k+1$ form

which is def. of odd.

Therefore, Even(n+3) → Odd(5n+8) holds

Prove: Odd(5n+8) → Even(n+3)

Proof by contrapositive: Odd(n+3) → Even(5n+8)

$n+3 = 2k+1,\ k \in \mathbb{Z}$ - by def. of Odd

So, $5n+8 = 5n+15-7$

$= 5(n+3)-7$

$= 5(2k+1)-7$

$= 10k+5-7$

$= 10k-2$

$= 2(5k-1)$

Since $k \in \mathbb{Z}$ by closure $5k-1 \in \mathbb{Z}$

So $2(5k-1)$ is in $2k$ form

which is def. of Even

Therefore, by contrapositive

Odd(5n+8) → Even(n+3) holds

∴ Since both Even(n+3) → Odd(5n+8) and Odd(5n+8) → Even(n+3) holds Even(n+3) <=> Odd(5n+8) holds ∎

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Problem 5:

Prove: $(\forall n \in \mathbb{N})(5 \mid 6^n - 1)$

Proof by Induction: $P(n): 5 \mid 6^n - 1$ where $6^n - 1 = 5a,\ a \in \mathbb{Z}$

Base Case $P(0): 5 \mid 6^0 - 1$, $6^0 - 1 = 5a$

So $P(0)$ holds $\quad 1 - 1 = 5a$

$0 = 5a$

$0 = a,\ 0 \in \mathbb{Z}$

Inductive Step: Prove: $P(k) \rightarrow P(k+1)$

Induction Hypothesis: $P(k)$ holds, i.e. $5 \mid 6^k - 1$ where $6^k - 1 = 5a\ a \in \mathbb{Z}$

Want to Prove: $P(k+1)$ holds, i.e. $5 \mid 6^{k+1} - 1$ where $6^{k+1} - 1 = 5b\ b \in \mathbb{Z}$

$5 \mid 6^{k+1} - 1 = 6^{k+1} - 1 = 5b$

$= 6^k \cdot 6 - 1 = 5b$

$= (6^k - 1)(6+1) = 5b$ Expanded out

$= 5a(6+1) = 5b$ Induction Hypothesis

$= 35a = 5b$

Since $a \in \mathbb{Z}$, by closure $35a \in \mathbb{Z}$ and

$b \in \mathbb{Z}$, by closure $5b \in \mathbb{Z}$

Since $6^{k+1} - 1 = 5b$ holds

Therefore $5 \mid 6^{k+1} - 1$ holds

$\therefore$ By induction $5 \mid 6^n - 1$ for all $n \in \mathbb{N}$