Methods of Applied Stats, Generalized Linear Models

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Outline

- Examples
- Generalized Linear Models
- Likelihood-based inference
- Applications

Motivating example: Shuttle data

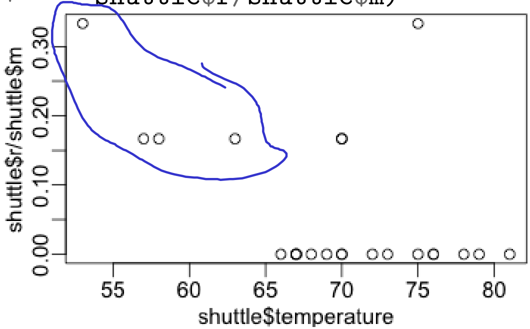
```
> data("shuttle", package = "SMPracticals")
> rownames(shuttle) = as.character(rownames(shuttle))
> shuttle[1:4, ]

  m r temperature pressure
1 6 0          66       50
2 6 1          70       50
3 6 0          69       50
4 6 0          68       50
```

- m: number of rings
- r: number of damaged rings

Questions and models

- Are shuttle rings more likely to get damaged in cold weather?
- Pressure is a confounder
- The data aren't Gaussian
- Binomial distribution r failures from m trials
- Model failure probability as a function of temperature, pressure

```
> plot(shuttle$temperature,
+      shuttle$r/shuttle$m)
```



Motivating example: Fiji birth data

- Fiji Fertility Survey, 1974 opr.princeton.edu/archive/wfs/FJ.aspx
- pbrown.ca/teaching/appliedstats/data/fiji.RData created from pbrown.ca/teaching/appliedstats/data/fiji.R

```
> fiji[1:4, ]

  age ageMarried monthsSinceM failedPreg pregnancies children sons
1  25     18to20           72          0           0        0    0
2  31     15to18          184          0           6        6    2
3  40     15to18          269          0           2        2    1
4  46     15to18          206          0           9        9    6
  firstBirthInterval residence literacy ethnicity
1             60-Inf     rural      yes    fijian
2              12-23     rural      yes    fijian
3                0-7     rural      yes    fijian
4                lt0     rural      yes    fijian
```

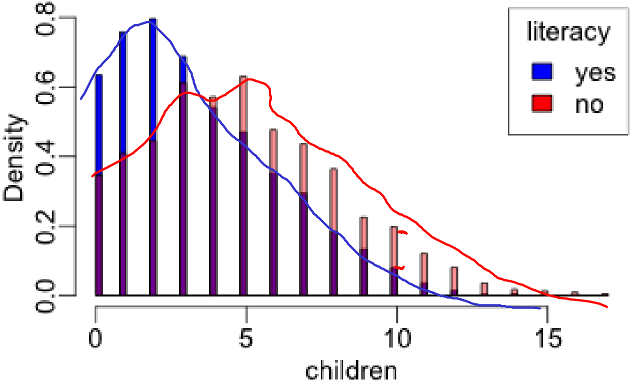
Questions and models

- Do literate women tend to have smaller families?
- Is this only because illiterate women marry earlier?

```
> table(fiji$ageMarried=='0to15
+   fiji$literacy)

        yes   no
FALSE 3483  778
TRUE   323  333
```

- children $\sim$ Poisson

```
> literate = fiji$literacy == "yes"
> hist(fiji[literate, "children"], breaks = 1
+      prob = TRUE, main = "", xlab = "childre
```



Motivating example: Smoking

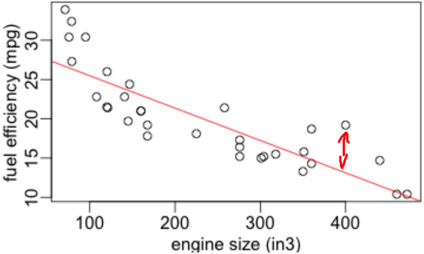
- 2014 American National Youth Tobacco Survey

```
> smoke[1:4, c("Age", "Sex", "Race", "RuralUrban", "state",
+   "ever_cigarettes")]

  Age Sex     Race RuralUrban state ever_cigarettes
1  13   M hispanic      Urban    AZ           FALSE
2  12   F hispanic      Urban    AZ           FALSE
3  14   M   native      Urban    AZ           FALSE
4  13   M hispanic      Urban    AZ           FALSE

> table(smoke$ever_cigarettes, smoke$Race)


        white black hispanic asian native pacific
  FALSE  7576  2626     4543   844    250      56
  TRUE   2243   731     1397   118     83      26
```

Ordinary Least Squares

noise
error
random variation

Car data



The wrong way

$$Y_i = X_i^{\mathsf{T}}\beta + Z_i$$
$$Z_i \sim \mathrm{N}(0, \sigma^2)$$

The right way

$$Y_i \sim N(\mu_i, \sigma^2)$$
$$\mu_i = X_i^{\mathsf{T}}\beta$$

- Y_i: fuel efficiency of car i – dependent, response
- $X_i = (X_{i1}, X_{i2})$: covariates – independent
  - $X_{i1} = 1$, intercept
  - X_{i2} engine size

parameters
β
σ^2

OLS in R

```
> data("mtcars", package = "datasets")
> mtcars[1:3, c("mpg", "disp")]
               mpg disp
Mazda RX4     21.0  160
Mazda RX4 Wag 21.0  160
Datsun 710    22.8  108
```

1 +

```
> mtFit = lm(mpg ~ disp, data = mtcars)
> knitr::kable(summary(mtFit)$coef, digits = 3)
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	29.600	1.230	24.070	0
disp	-0.041	0.005	-8.747	0

$$Y_i \sim N(\beta_0 + X_i\beta_1, \sigma^2)$$

- mpg is the *response variable*
- disp is the *explanatory variable* or *covaraite*

Maximum Likelihood Estimation

- Model parameters: β, σ
- Likelihood function:

$$L(\beta,\sigma|Y) = \pi(Y;\beta,\sigma)$$

- MLE's:

$$(\hat{\beta},\hat{\sigma}) = \text{argmax}_{\beta,\sigma} L(\beta,\sigma|Y)$$

- Here `myLik` calculates $-\log L(\beta,\sigma|Y)$
- and `optim` minimizes it
- `lm` knows the exact answer is

$$\hat{\beta} = (X^\mathsf{T}X)^{-1}X^\mathsf{T}y$$

closed form solution

```
> myLik = function(param) {
+   mu = param[1] + param[2] * mtcars$disp
+   - sum(dnorm(mtcars$mpg, mean=mu,
+         sd=param[3], log=TRUE))}
> myLik(c(1,1,1))
[1] 974915.4
> optim(c(10,0.1,1), myLik,
+   control = list( parscale = c(1,0.01,1))
+   )$par
[1] 29.58930278 -0.04117094  3.14674642
> mtFit$coef
(Intercept)        disp
29.59985476 -0.04121512
```

β_0 β_1 σ

Generalized Linear Models

$X_i^T \beta$ = linear predictor

$$
\begin{aligned}
Y_i \sim & G(\mu_i, \theta) \\
h(\mu_i) = & X_i^\mathsf{T} \beta
\end{aligned}
$$

- G is the distribution of the response variable
- μ_i is a location parameter for observation i
- θ are additional parameters for the density of G. non-linear parameters
- h is a link function
- X_i are covariates for observation i
- β is a vector of regression coefficients - fixed effects

- Ch 6 of Wakefield 2013
- Ch 4,5 of Davison 2003 http://books1.scholarsportal.info/viewdoc.html?id=/ebooks/ebooks1/cambridgeonline/2012-11-08/1/9780511815850
- Ch 6 of Faraway 2005 http://www.tandfebooks.com/isbn/9780203492284

Ordinary Least Squares again

$X_i = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \Rightarrow T$

?

$X_i = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$

$$\begin{aligned} Y_i &\sim G(\mu_i, \theta) \\ h(\mu_i) &= X_i^{\mathsf{T}}\beta \end{aligned} \qquad \begin{aligned} Y_i &\sim N(\mu_i, \sigma^2) \\ \mu_i &= X_i^{\mathsf{T}}\beta \end{aligned}$$

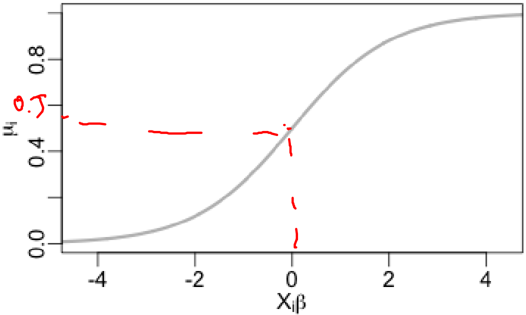
- G is a Normal distribution
- θ is the variance parameter, denoted σ^2
- h is the identity function

Binomial (or logistic) regression

$$Y_i \sim \text{Binomial}(N_i, \mu_i)$$

$$\log\left(\frac{\mu_i}{1-\mu_i}\right) = X_i^{T}\beta$$

- G is a Binomial distribution
- ...or a Bernoulli if $N_i = 1$
- h is the logit link



- $X_i^{\mathsf{T}}\beta$ can be negative
- μ_i is between 0 and 1.

Are shuttle rings more likely to get damaged in cold weather?

```
  m r temperature pressure
1 6 0          66       50
2 6 1          70       50
3 6 0          69       50
4 6 0          68       50
```

$$
\begin{aligned}
Y_i \sim & \text{Binomial}(\overset{6}{N_i}, \mu_i) \\
\log\left(\frac{\mu_i}{1-\mu_i}\right) = & X_i\beta
\end{aligned}
$$

- m: number of rings, N_i
- r: number of damaged rings Y_i
- pressure, temperature: covariates X_i
- μ_i: probability of a ring becoming damaged given X_i
- $\beta_{\text{pressure}}, \beta_{\text{intercept}}$: confounders
- $\beta_{\text{temperature}}$: parameter of interest

Inference: parameter estimation

$$
\begin{aligned}
Y_i &\overset{\text{ind}}{\sim} G(\mu_i, \theta) \\
h(\mu_i) &= X_i \beta
\end{aligned}
$$

$$\pi(Y_1 \dots Y_N; \beta, \theta) = \prod_{i=1}^{N} f_G(Y_i; \mu_i, \theta)$$

$$\log L(\beta, \theta; y_1 \dots y_N) = \sum_{i=1}^{N} \log f_G(y_i; \mu_i, \theta)$$

- The Y_i are *independently distributed*
- **Joint density** π of random variables $(Y_1 \dots Y_N)$ is the product of the marginal densities f_G
- **Likelihood function** L given observed data $y_1 \dots y_N$ is a function of the parameters
- **Maximum Likelihood Estimation**:

$$\hat{\beta}, \hat{\theta} = \text{argmax}_{\beta,\theta} \overset{\text{log}}{} L(\beta, \theta; y_1 \dots y_N)$$

min − log L

- The best parameters are those which are most likely to produce the observed data

Shuttle example in R

- `glm` works like `lm` with a `family` argument
- Binomial models in GLM require `y` to be a matrix with two columns
- ... `y` and `N-y`, which is unfortunate

```
> shuttle$notDamaged = shuttle$m - shuttle$r
> shuttle$y = as.matrix(shuttle[,c('r','notDamaged')])
> shuttleFit = glm(y ~ temperature + pressure, +1
+   family=binomial(link='logit'), data=shuttle)
> shuttleFit$coef

 (Intercept)  temperature     pressure
 2.520194641 -0.098296750  0.008484021
```

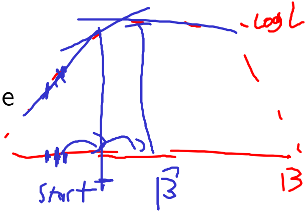
Shuttle example the hard way

$\leftarrow -\log L(\beta;Y)$

```
> logLikShuttle = function(param) {
+   muLogit = param[1] + param[2] * shuttle$temperature +
+     param[3] * shuttle$pressure
+   mu = exp(muLogit)/(1+exp(muLogit))
+   - sum(dbinom(shuttle$r, size=shuttle$m, prob=mu, log=TRUE))}
> optim(c(2,0,0), logLikShuttle,
+   control = list( parscale = c(1,0.01,0.01)))$par

[1]  2.51894107 -0.09828029  0.00848237

> shuttleFit$coef

 (Intercept)  temperature     pressure
 2.520194641 -0.098296750  0.008484021
```

Efficient maximization

- Iterative Reweighted Least Squares is the 'classic' algorithm when G is in the exponential family
- ... but GLM's are easy for any density which is differentiable
- The derivatives wrt β are easy to compute with the chain rule



$$\frac{\partial}{\partial \beta_p} \log L(\beta, \theta; y_1 \dots t_N) =$$

$$\sum_{i=1}^{N} \left[\frac{d}{d\mu} \log f_G(Y_i; \mu, \theta) \right]_{\mu = h^{-1}(X_i^\intercal \beta)} \left[\frac{d}{d\eta} h^{-1}(\eta) \right]_{\eta = X_i^\intercal \beta} \cdot X_{ip}$$

- Analytical expressions exist for the derivatives of $\log f_G$ and h^{-1}
- Second derivatives are also tractable
- Numerical maximization to find $\hat{\beta}$ is fast when derivatives are available

Numerical maximizers

- There are hundreds of them
- `optim` is the standard `R` optimizer, which has 6 methods available
  - some methods will use gradients if you provide them
- `TrustOptim` uses derivatives and ‘trust regions’, the method used in INLA
- `ipopt` is probably the cutting edge
- Statisticians don’t make enough use of of-the-shelf optimizers

Automatic differentiation

$$\sum_{i=1}^{N} \left[\frac{d}{d\mu} \log f_G(Y_i; \mu, \theta) \right]_{\mu = h^{-1}(X_i^\intercal \beta)} \left[\frac{d}{d\eta} h^{-1}(\eta) \right]_{\eta = X_i^\intercal \beta} \cdot X_{ip}$$

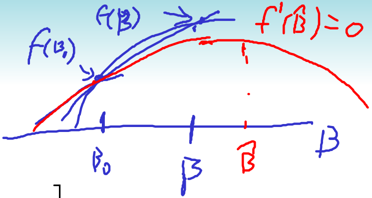
- Overkill for most GLM's, but infinitely extensible
- computers evaluate logs, sines, and other functions through some Taylor-series like polynomial thing.
- ... which are easy to differentiate
- AD programs can take computer code and figure out how to differentiate it
- used in Neural Nets, Hamiltonian MCMC, optimization, and many more

```
> shuttleFitAD <- glmmTMB::glmmTMB(
+   y ~ temperature + pressure,
+   family=binomial(link='logit'),
+   data=shuttle)
> shuttleFitAD$fit$par
       beta         beta         beta
 2.520195003 -0.098296758  0.008484022
```

Multivariate

Taylor series expansion

of order 2



$$\log L(\beta, \theta; \mathbf{y}) \approx \log L(\beta_0, \theta; \mathbf{y}) + (\beta - \beta_0)^{\mathsf{T}} \left[\frac{\partial}{\partial \beta} \log L(\beta_0, \theta; \mathbf{y}) \right] + \left(\frac{1}{2}\right) (\beta - \beta_0)^{\mathsf{T}} \left[\frac{\partial^2}{(\partial \beta)^2} \log L(\beta_0, \theta; \mathbf{y}) \right] (\beta - \beta_0)$$

- Set $\beta_0 = \hat{\beta}$ and the first derivative is zero

$$L(\beta, \theta; \mathbf{y}) \approx L(\hat{\beta}, \theta; \mathbf{y}) \exp \left\{ \left(\frac{1}{2}\right) (\hat{\beta} - \beta)^{\mathsf{T}} \left[\frac{\partial^2}{(\partial \beta)^2} \log L(\hat{\beta}, \theta; \mathbf{y}) \right] (\hat{\beta} - \beta) \right\}$$

- Looks like a Normal distribution $\hat{\beta} \sim \mathrm{N}\left[\beta, I(\hat{\beta})^{-1}\right]$ information matrix
- Roots of diagonal elements of $I(\hat{\beta})^{-1}$ are standard errors

Inference

- Information Matrix:

$$I(\hat{\beta}|Y) = \frac{\partial}{\partial\beta\partial\beta^\mathsf{T}} - \log L(\beta|Y)\Big|_{\hat{\beta}}$$

- MLE's are approximately Normal

$$\hat{\beta} \sim \mathsf{MVN}(\beta, I(\hat{\beta}|Y)^{-1})$$

- standard errors are roots of diagonals of inverted Information Matrix

```
> (infMat = numDeriv::hessian(
+   f=logLikShuttle, x=shuttleFit$coef))
           [,1]       [,2]       [,3]
[1,]    7.81222   504.4844   1415.968
[2,]  504.48442 33096.7647  90802.259
[3,] 1415.96790 90802.2589 274388.164
> sqrt(diag(solve(infMat)))
[1] 3.486838277 0.044890480 0.007677616
```

```
> knitr::kable(summary(shuttleFit)$coef, digits = 4)
```

— glm

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	2.5202	3.4868	0.7228	0.4698
temperature	-0.0983	0.0449	-2.1897	0.0285
pressure	0.0085	0.0077	1.1051	0.2691

```
> knitr::kable(summary(shuttleFitAD)$coef$cond, digits = 4)
```

← TMB

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	2.5202	3.4753	0.7252	0.4684
temperature	-0.0983	0.0448	-2.1948	0.0282
pressure	0.0085	0.0077	1.1076	0.2681

```
> sqrt(diag(solve(infMat)))

[1] 3.486838277 0.044890480 0.007677616
```

Some notes

- Don't confuse
  - *Models*,
  - *Inference methodologies*, and
  - *Algorithms*.
- **Models**: Generalized linear models, ARIMA time series, population sampling
- **Inference methodologies**: Frequentist or Likelihood-based, Bayesian, Method-of-moments, partial likelihood.
- **Algorithms**: Least Squares, IRWLS, Markov Chain Monte Carlo, INLA, the Lasso

Likelihood ratio tests

$$2[\log L(\hat{\beta}; \mathbf{y}) - \log L(\beta; \mathbf{y})] \sim \chi^2_P$$

where P is the number of parameters in β

Comparing nested models

- H_0: $\beta_k = C_k$ for all $k \in \Omega$
  - $\Omega \subset \{1 \dots P\}$
- H_1: β unconstrained.
- **Nested**: H_0 is a special case of H_1
- Write $\hat{\beta}^{(C)}$ as the constrained MLE's under H_0

$$2[\log L(\hat{\beta}; \mathbf{y}) - \log L(\hat{\beta}^{(C)}; \mathbf{y})] \sim \chi^2_{|\Omega|}$$

What about θ?

$$
\begin{aligned}
Y_i \sim & G(\mu_i, \theta) \\
h(\mu_i) = & X_i^\mathsf{T} \beta
\end{aligned}
$$

- When G is Poisson or Binomial, θ isn't used
- When G is exponential family θ factors out
  - e.g. σ^2 for Gaussian models

- Suppose G is Weibull and θ is a shape parameter?
- Derivatives of $\log L$ wrt θ are still straightforward
  - but more complicated than wrt β
- The Taylor series expansion and information matrix are still valid
  - but not used much in practice
- Most software packages estimate a $\hat{\theta}$ with numerical optimization
  - and treat it as 'known' and fixed when making inference on β

What you need to know

- No closed form MLE's for GLM's
- Derivatives are easy so maximization is quick
- There are nice parameters and nasty parameters
  - Easy standard errors for $\hat{\beta}$ based on Normal/2nd order Taylor approximation.
  - The θ parameters are non-linear and more challenging.

Interpreting logistic models

$$Y_i \sim \text{Binomial}(N_i, \mu_i)$$
$$\log\left(\frac{\mu_i}{1-\mu_i}\right) = \sum_{p=1}^{P} X_{ip}\beta_p$$
$$\left(\frac{\mu_i}{1-\mu_i}\right) = \prod_{p=1}^{P} \exp(\beta_p)^{X_{ip}}$$

- μ_i is a probability
- $\log[\mu_i/(1-\mu_i)]$ is a log-odds
- $\mu_i/(1-\mu_i)$ is an odds
- If $\mu_i \approx 0$, then $\mu_i \approx \mu_i/(1-\mu_i)$

Suppose $X_{1p} = X_{2p}$ for all p except $X_{2q} = X_{1q} + 1$

$$\beta_q = \log\left(\frac{\mu_2}{1-\mu_2}\right) - \log\left(\frac{\mu_1}{1-\mu_1}\right)$$
$$\exp(\beta_q) = \left(\frac{\mu_2}{1-\mu_2}\right) \Bigg/ \left(\frac{\mu_1}{1-\mu_1}\right)$$

- β_q is the log-odds ratio
- $\exp(\beta_q)$ is the odds ratio
- $\exp(\text{intercept})$ is baseline odds, when $X_{i2} \ldots X_{iP} = 0$.

Centring parameters

```
> quantile(shuttle$temperature)
  0%  25%  50%  75% 100%
  53   67   70   75   81
> quantile(shuttle$pressure)
  0%  25%  50%  75% 100%
  50   75  200  200  200
```

- Currently the intercept is log-odds when temperature = 0 and pressure = 0
- centre the covariates so the intercept refers to
  - temperature = 70 (degrees Farenheit)
  - pressure = 200 units of something

```
> shuttle$temperatureC = shuttle$temperature - 70
> shuttle$pressureC = shuttle$pressure - 200
> shuttleFit2 = glm(y ~ temperatureC + pressureC, family = "binomial",
+   data = shuttle)
```

Pmisc

```
> install.packages("Pmisc", repos = "http://r-forge.r-project.org")
```

Shuttle odds parameters

```
> (theCiMat = Pmisc::ciMat(0.95))
            est       2.5     97.5
Estimate      1  1.000000 1.000000
Std. Error    0 -1.959964 1.959964
> parTable = summary(
+   shuttleFit2)$coef[,
+   rownames(theCiMat)] %*% theCiMat
> rownames(parTable)[1]= "Baseline"
```

```
> knitr::kable(exp(parTable),
+   digits=3)
```

	est	2.5	97.5
Baseline	0.070	0.028	0.176
temperatureC	0.906	0.830	0.990
pressureC	1.009	0.993	1.024

Table 1: MLE's of baseline odds and odds ratios, with 95% confidence intervals.

Interpreting shuttle parameters

	est	2.5	97.5
Baseline	0.070	0.028	0.176
temperatureC	0.906	0.830	0.990
pressureC	1.009	0.993	1.024

- The odds of a ring being damaged when temperature = 70 and pressure = 200 is 0.0697, which corresponds to a probability of

```
> signif(exp(parTable[1,'est']) / (1+exp(parTable[1,'est'])), 3)
[1] 0.0651
```

- Each degree increase in temperature (in Yankee units) decreases the odds of damage by (in percent)

```
> signif(100 * (1 - exp(parTable[2, "est"])), 3)
[1] 9.36
```

Inter-quartile range

```
> quantile(shuttle$temperature)
  0%   25%   50%   75%  100%
  53    67    70    75    81
> quantile(shuttle$pressure)
  0%   25%   50%   75%  100%
  50    75   200   200   200
```

- A one-unit change in pressure means less than a one-unit change in temperature
- **Inter-quartile range**: odds ratio between 75th and 25th percentiles of a variable

```
> shuttleIQR = apply(shuttle[,c('temperatureC','pressureC')], 2,
+          function(xx) diff(quantile(xx, probs=c(0.25, 0.75))) )
> shuttleIQR

temperatureC    pressureC
           8          125
```

Odds ratios for interquartile ranges

```
> parTableIQR = exp(
+   diag(c(1,shuttleIQR[c('temperatureC','pressureC')]) ) %*% parTable)
> parTableIQR[1,] = parTableIQR[1,]  / (1+parTableIQR[1,])
> rownames(parTableIQR) = gsub("C$", "", rownames(parTable))
> Pmisc::mdTable(parTableIQR, digits=3, caption = "MLEs of baseline probab
```

Table 6: MLEs of baseline probabilities and odds ratios for interquartile ranges, with 95pct confidence intervals

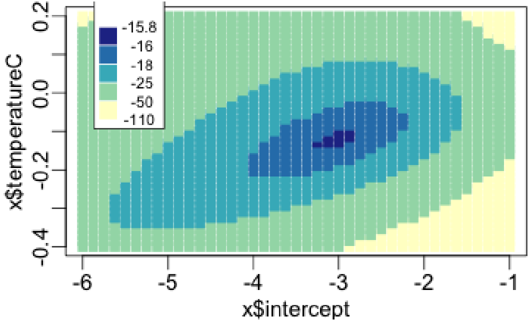
	est	2.5	97.5
Baseline	0.065	0.027	0.149
temperature	0.455	0.225	0.921
pressure	2.888	0.440	18.942

: MLEs of baseline probabilities and odds ratios for interquartile ranges, with 95pct

Plotting the likelihood function

```
> x = expand.grid(intercept = seq(-6, -1, len=41),
+   temperatureC = seq(-0.4, 0.2, len=41))
> x$lik = mapply(function(intercept, temperatureC) {
+   meanLogit = intercept + shuttle$temperatureC * temperatureC
+   meanProb = exp(meanLogit)/(1+exp(meanLogit))
+   sum(dbinom(shuttle$r, shuttle$m, meanProb, log=TRUE))
+ }, intercept = x$intercept, temperatureC = x$temperatureC)
```

```
> head(x)
  intercept temperatureC       lik
1    -6.000         -0.4 -28.65212
2    -5.875         -0.4 -28.55750
3    -5.750         -0.4 -28.53083
4    -5.625         -0.4 -28.57420
5    -5.500         -0.4 -28.68961
6    -5.375         -0.4 -28.87898
```



Likelihood function 3d

Evaluate the log-likelihood surface in the vicinity of the MLE

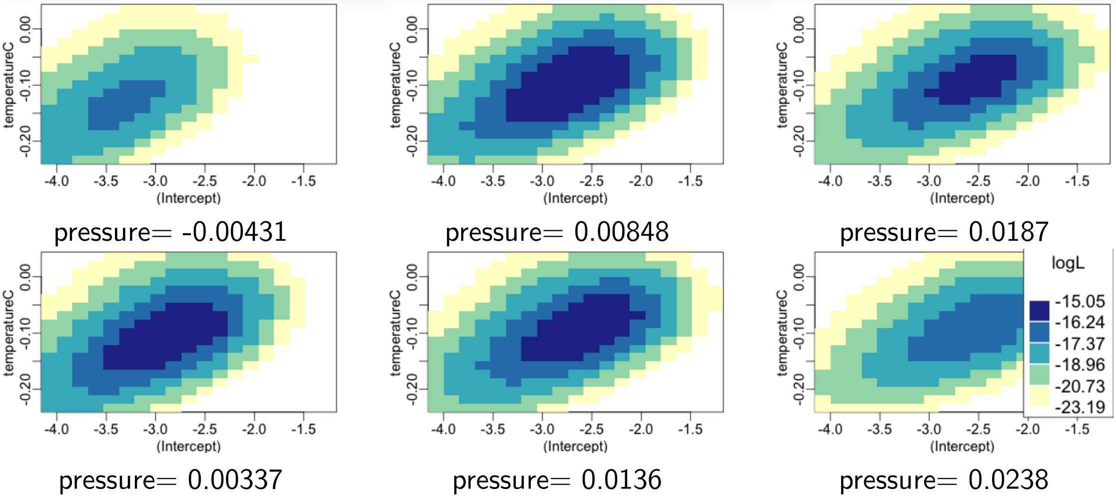
```
> shuttleL = Pmisc::likSurface(shuttleFit2)
>
> dim(shuttleL$logLik)
```

```
[1] 19 19 19
```

```
> head(shuttleL$parameters)
```

```
     (Intercept) temperatureC    pressureC
[1,]   -4.079471   -0.2329679 -0.014547274
[2,]   -3.922171   -0.2180044 -0.011988241
[3,]   -3.764872   -0.2030410 -0.009429208
[4,]   -3.607572   -0.1880775 -0.006870175
[5,]   -3.450272   -0.1731140 -0.004311143
[6,]   -3.292972   -0.1581506 -0.001752110
```

Shuttle likelihood function



Likelihood as a function of the intercept and temperature coefficient, for different values of the pressure coefficient.

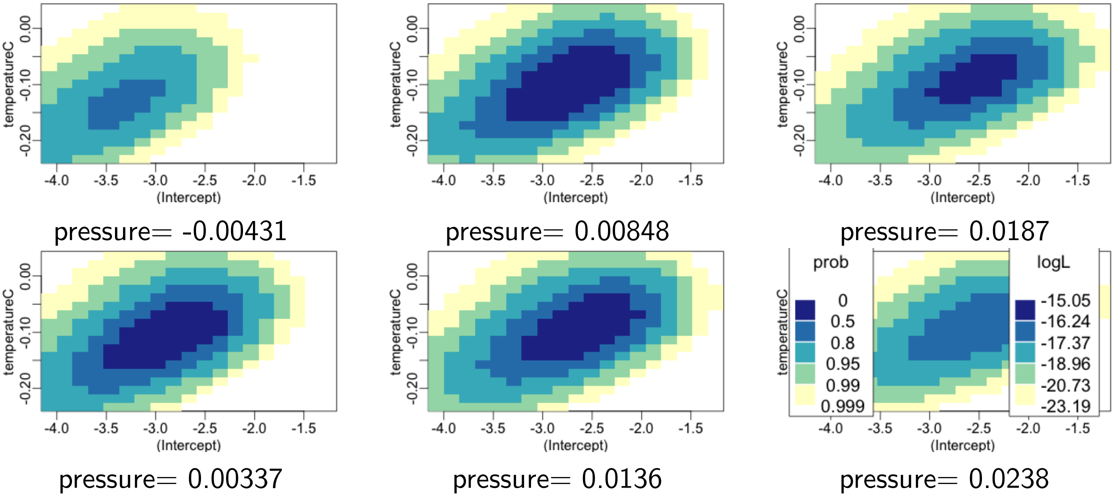
Shuttle LR test

are neither temperature nor pressure important?

```
> shuttleFitNoCovariates = glm(y ~ 1, family = "binomial",
+   data = shuttle)
> lmtest::lrtest(shuttleFit, shuttleFitNoCovariates)
```

```
Likelihood ratio test

Model 1: y ~ temperature + pressure
Model 2: y ~ 1
  #Df  LogLik Df  Chisq Pr(>Chisq)
1   3 -15.053
2   1 -18.895 -2 7.6847    0.02144 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Shuttle likelihood function revisited



Colours are differences of χ^2_3 quantiles from the maximum (-15.0529).

Profile likelihood

- The ‘pressure’ covariate is a confounder (or a nuisance variable).
- It isn’t significant, but we’ll keep it in the model to be conservative
- Suppose we want to test $\beta_{\text{intercept}} = \text{logit}(0.1) \approx -2.2$ and $\beta_{\text{temperature}} = 0$?
- **Unconstrained model**: maximize $L(\beta_{\text{intercept}}, \beta_{\text{temperature}}, \beta_{\text{pressure}})$
- **Constrained model**: maximize $L(-2.2, 0, \beta_{\text{pressure}})$
- **Profile likelihood function** of the intercept and temperature:

$$\tilde{L}(a, b) = \max_{\beta_{\text{pressure}}} L(a, b, \beta_{\text{pressure}})$$

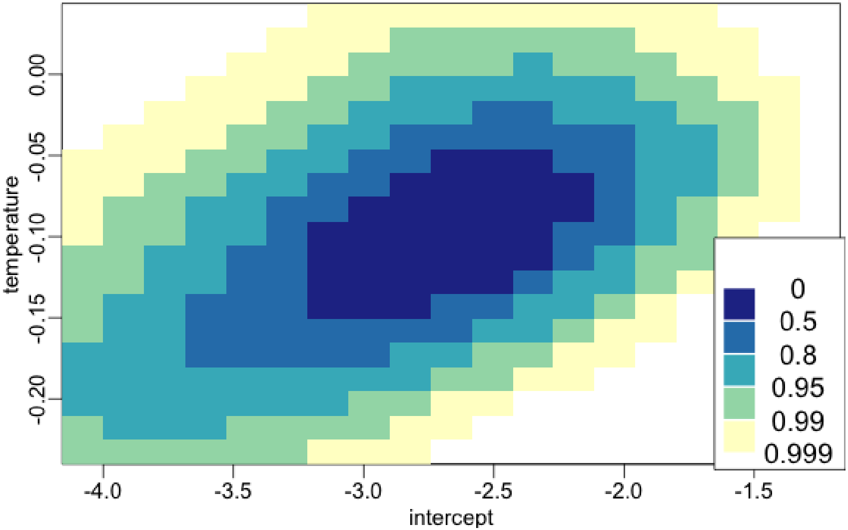
- More generally, $\theta = \{\theta_1 \ldots \theta_P\}$ and $C \subset \{1 \ldots P\}$

$$\tilde{L}_C(\{\theta_q; q \in C\}) = \max_{\theta_r; r \notin C} L(\{\theta_q; q \in C\} \cup \{\theta_r; r \notin C\})$$

Profile likeihood for the shuttle data

```
> dim(shuttleL$logL)

[1] 19 19 19

> shuttleProfL = apply(shuttleL$logL, 1:2, max)
> dim(shuttleProfL)

[1] 19 19

> breaks2df = max(shuttleL$logL) - qchisq(shuttleL$breaks,
+   df = 2)/2
> image(shuttleL$parameters[, 1], shuttleL$parameters[, 2],
+   shuttleProfL, breaks = breaks2df, col = shuttleL$col,
+   xlab = "intercept", ylab = "temperature")
> mapmisc::legendBreaks("bottomright", shuttleL, inset = 0)
```

Profile likelihood for the shuttle data



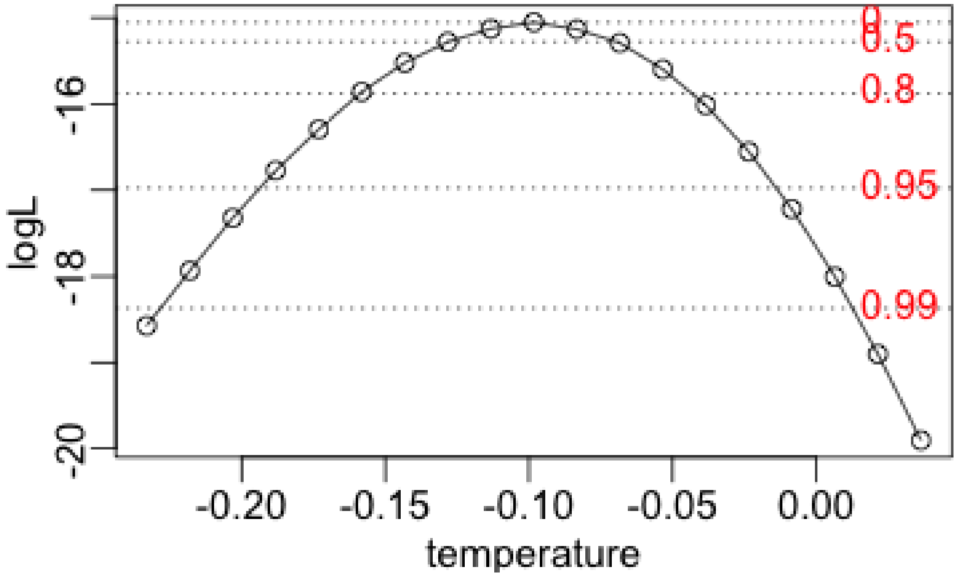
Profile likelihoods

- The `apply` function is used to find the maximum of the 19 likelihoods computed for each pair of intercept and temperature variables.
- Strictly speaking, a numerical optimizer should be used, varying the the pressure parameter continuously.
- The ξ^2 quantiles are computed for 2 degrees of freedom (intercept and temperature parameters)
- A 1-D profile likelihood for the temperature parameter takes the maximum of the $19 \cdot 19$ likelihoods computed for each temperature parameter value.

1-D Profile likelihood

```
> dim(shuttleL$logL)
[1] 19 19 19
> colnames(shuttleL$parameters)
[1] "(Intercept)"  "temperatureC" "pressureC"
> shuttleProfTemp = apply(shuttleL$logL, 2, max)
> breaks1df = max(shuttleL$logL) - qchisq(shuttleL$breaks,
+   df = 1)/2
> plot(shuttleL$parameters[, 2], shuttleProfTemp, type = "o",
+   xlab = "temperature", ylab = "logL")
> abline(h = breaks1df, lty = 3)
> text(par("usr")[2] - 4 * par("cxy")[1], breaks1df, shuttleL$breaks,
+   pos = 4, col = "red")
```

1-D Profile likelihood



Information-based v profile likelihoods

- Profile likelihoods are not often used in Applied Statistics
- Is this for historical reasons, or are they not very useful?
- Information-based confidence regions are regarded as good enough.
- Joint confidence regions from 2-D profile likelihoods are interesting (?)

Where we are

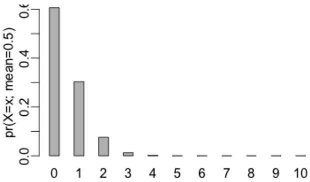
- discussed applied statistics
- reviewed GLM's
- profile likelihoods and LR tests
- saw exaples in R

Next

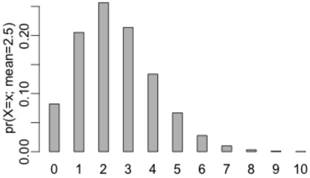
- GLM's with continuous variables
- conclusions
- practical in R

The Poisson Distribution

$$X_i \sim \text{Poisson}(\lambda_i)$$
$$pr(X_i = x) = \lambda_i^x \exp(-\lambda_i)/x!$$
$$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$



$\lambda = 0.5$



$\lambda = 2.5$

Poisson regression

$$
\begin{aligned}
Y_i &\sim \text{Poisson}(\mu_i) \\
\log(\mu_i) &= X_i\beta
\end{aligned}
$$

$$
\begin{aligned}
Y_i &\sim G(\mu_i, \theta) \\
h(\mu_i) &= X_i^{\mathsf{T}}\beta
\end{aligned}
$$

- G is a Poisson distribution
- h is the log link

Infinitely Divisible Laws

Normal

$$
\begin{aligned}
Y_i &\sim \mathsf{N}(\mu_i, \sigma^2) \\
Y_i &= \sum_{k=1^K} X_{ik} \\
X_{ik} &\sim \mathsf{N}(\mu_i/K, \sigma^2/K)
\end{aligned}
$$

Poisson

$$
\begin{aligned}
Y_i &\sim \text{Poisson}(\lambda_i) \\
Y_i &= \sum_{k=1^K} X_{ik} \\
X_{ik} &\sim \text{Poisson}(\lambda_i/K)
\end{aligned}
$$

Gamma

$$
\begin{aligned}
Y_i &\sim \text{Gamma}(\alpha, \beta_i) \\
Y_i &= \sum_{k=1^K} X_{ik} \\
X_{ik} &\sim \text{Gamma}(\alpha/K, \beta_i)
\end{aligned}
$$

- When our observed Y_i are actually the sum of many unobserved X_{ij} ...
  - Number of cigarettes smoked last month is sum of daily counts
  - Height of a child is sum of inches grown each month
- ... we're justified in considering only distributions which are infinitely divisible

Not infinitely divisible

Uniform

$$X_{ik} \sim \text{Unif}(a_i, b_i)$$

- $\sum_k X_{ij}$ isn't uniform.
- Can't construct a uniform Y_i from a sum.

Binomial

$$Y_i \sim \text{Binom}(\rho_i, N)$$

$$Y_i = \sum_{k=1}^{K} X_{ik}$$

$$X_{ij} \sim \text{Binom}(\rho_i, N/K)$$

- Not true for all K
- Only valid when N is a multiple of K

Log-Normal

$$X_{ik} \sim \text{LN}(\mu_i/K, \sigma/K)$$
$$\log(X_{ik}) \sim \text{N}(\mu_i/K, \sigma^2/K)$$

- $\sum_k X_{ik}$ isn't log-Normal
- but

$$\prod_k X_{ik} \sim \text{LN}(\mu_i, \sigma)$$

- a good model when Y_i is a share price?

Do literate Fijians tend to have fewer children?

$$
\begin{aligned}
Y_i \sim & \text{Poisson}(O_i \mu_i) && (1)\\
\log(\mu_i) = & X_i \beta
\end{aligned}
$$

or

$$
\begin{aligned}
Y_i \sim & \text{Poisson}(\rho_i) && (2)\\
\log(\rho_i) = & X_i \beta + \log(O_i)
\end{aligned}
$$

	monthsSinceM	literacy	children
1	72	yes	0
2	184	yes	6
3	269	yes	2
4	206	yes	9

- Y_i: number of children of women i
- μ_i: rate of children born, per month
- O_i: **Offset term**, number of months since first married
- X_i: intercept, literacy

- Write (1) when communicating with humans.
- Write (2) when communicating with computers.

Poisson Processes in time

- Suppose P_i are a sequence of event times

$$\{P_1 \dots P_N\} \sim \text{Poisson process}(\lambda)$$

Definition

$\{P_1 \dots P_N\}$ is a homogeneous Poisson process with intensity λ if

1. $|\{P_i \in A\}| \sim \text{Poisson}(\lambda|A|)$ for any time interval A
2. Given $P_i, P_j \in A$, P_i and P_j are independent and uniformly distributed in A

A property

$|\{i; P_i \in A_k\}| \sim \text{Poisson}(\lambda|A_k|)$ for any non-overlapping regions A_1 and A_2.

Fijian births as a Poisson process

- μ_i is the intensity, per month, of children born from woman i
- $O_i = |A_i|$ where A_i is the time interval since first married to the date the survey was given
- $Y_i \sim \text{Poisson}(\mu_i O_i)$
- **Infinitely divisible**: divide A_i into trillions of one nanosecond long time intervals δ_{ik}
- X_{ik} is the number of children born to woman i in interval δ_{ik}

$$\begin{aligned} X_{ik} &\sim \text{Poisson}\left(\mu_i \frac{1}{10^6 \cdot 60 \cdot 60 \cdot 24 \cdot 30}\right) \\ \sum_k X_{ik} &\sim \text{Poisson}(\mu_i O_i) \end{aligned}$$

Is this a good model

Yes

- The data look Poisson: non-negative counts, right skewed
- μ_i can be interpreted independently of O_i
- It's simple because it's a GLM
- other model could be constructued, but this one has a rigorous mathematical foundation
- A good balance betwen being right and being useful
- ... at least as a starting point

No

- Dividing our data into nanosecond-long intervals isn't necessary
- **Dependence**: births can't be closer than about 8 months apart
- **Overdispersion**: fertility and desired family size vary from person to person
- **Inhomogeneity**: fertility decreases with age

The log-log link function

- Suppose shuttle rings during launch i break following a Poisson process with intensity λ_i.
- During launch i, ring j has failure times P_{ijk} for $k = 1 \dots K_{ij}$
- $K_{ij} \sim$ Poisson$(\delta\lambda_i)$ where δ is the duration of a mission.
- We only observe $Z_{ij} = 0$ if $K_{ij} = 0$ and $Z_{ij} = 0$ if $K_{ij} > 0$

$$
\begin{aligned}
\{P_{ijk}; k = 1 \dots K_{ij}\} \sim & \text{Poisson process}(\lambda_i) \\
\log(\lambda_i) =& X_i\beta \\
pr(K_{ij} = 0) =& \exp(-\delta\lambda_i) \\
pr(Z_{ij} = 1) =& \mu_i \\
\mu_i =& 1 - \exp[-\delta \exp(X_i\beta)] \\
\log[-\log(1 - \mu_i)] =& \log(\delta) + X_i\beta
\end{aligned}
$$

Notes on the log-log link

$$\log[-\log(1-\mu_i)] = \log(\delta) + X_i\beta$$

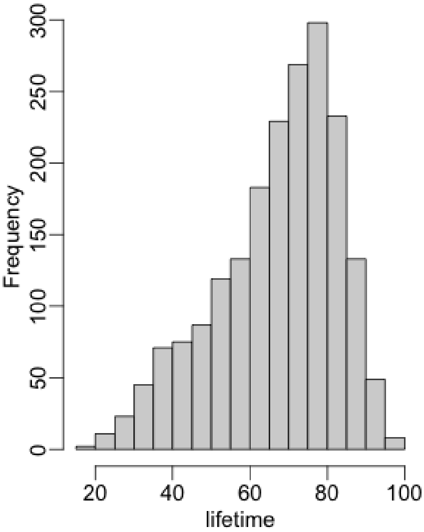
- If $Y_i = \sum_{j=1}^{N_i} Z_{ij}$, then Y_i is Binomial(N_i, μ_i).
- If we ignore δ, the intercept parameter becomes $\beta_0 + \log(\delta)$
- β parameters with log-log links are rate ratios rather than odds ratios
- The logit link function is the 'standard' approach. In practice, there would need to be a reason given when using a log-log link.
- It is rarely possible for the data to tell you which link function provides the better fit.

Cricket data

```
> data('cricketer', package='DAAG')
> cricketer[4000:4003,]
      left year life dead acd kia inbed cause
4073 right 1888   27    1   1   1     0   acd
4074 right 1890   86    1   0   0     1 inbed
4075 right 1896   90    1   0   0     1 inbed
4076 right 1916   72    1   0   0     1 inbed
```

Remove cricketers born after 1890 and those killed as soldiers.

```
> dat = cricketer[cricketer$year < 1890 &
+    cricketer$kia == 0,]
> hist(dat$life, xlab='lifetime', main='')
```



The problem

- Doug Altman and Martin Bland (2005). "Do the left-handed die young?" In: *Significance* 2.4, pp. 166–170. DOI: 10.1111/j.1740-9713.2005.00130.x
- Do left handed people live less long than right-handers?
- The hypothesis is yes, they die in accidents using right-handed equipment.
- This is really a 'survival analysis' problem requiring a proportional-hazards model, but we'll use GLM's for illustrative purposes.

Gamma distribution

$$X \sim \text{Gamma}(\phi, \nu)$$

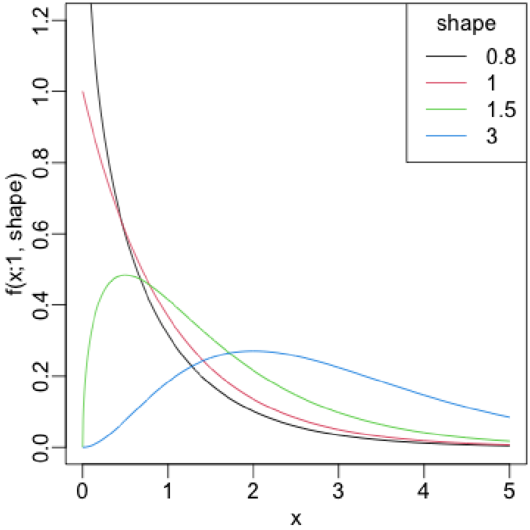
$$f(x; \phi, \nu) = \frac{(x/\phi)^{\nu-1} \exp(-x/\phi)}{\Gamma(\nu)\phi}$$

- ϕ is the range parameter
- ν is the shape parameter

$$\mathrm{E}(X) = \phi\nu, \ \ \mathrm{var}(X) = \phi^2\nu$$

- Coefficient of variation:

$$1/\sqrt{\nu} = \text{sd}(X) / \mathrm{E}(X)$$



Gamma regression

$$Y_i \sim \text{Gamma}(\mu_i/\nu, \nu)$$
$$\log(\mu_i) = X_i\beta$$

- $\mathrm{E}(Y_i) = \mu_i$
- glm reports a 'dispersion' parameter $1/\nu$.
- and the log-link isn't the default

```
> dat$decade = (dat$year - 1850)/10
> cFit = glm(life ~ decade + left, data=dat, family=Gamma(link='log'))
> knitr::kable(rbind(summary(cFit)$coef,
+   shape=c(1/summary(cFit)$dispersion, NA, NA, NA)), digits=4)
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.1751	0.0085	490.3097	0.0000
decade	0.0242	0.0038	6.3081	0.0000
leftleft	-0.0162	0.0136	-1.1901	0.2341
shape	18.7245	NA	NA	NA

Is this a good model?

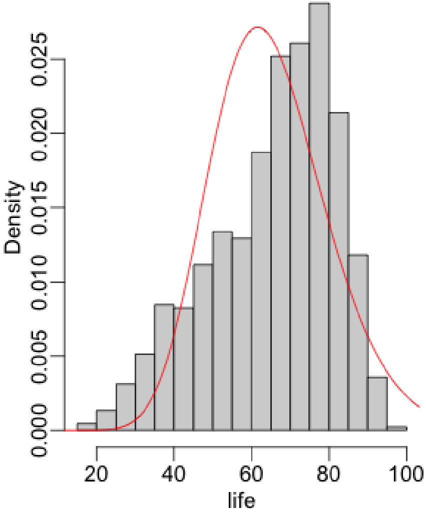
- Let's see if a histogram of the data looks Gamma distributed.
- Confine ourselves to right-handers born near 1850 to limit the effect of year and handedness on the distribution
- and overlay the density of the fitted Gamma distribution
- for individuals in the baseline group, lifetimes are Gamma distributed with the following parameters

```
> shape = 1/summary(cFit)$dispersion
> scale = exp(cFit$coef["(Intercept)"])/shape
```

Modelled and empirical distribution

```
> hist(dat$life[dat$left == "right" &
+      abs(dat$decade) < 2], prob = TRUE,
+      main = "", xlab = "life")
> xSeq = seq(0, 120, len = 1000)
> lines(xSeq, dgamma(xSeq, shape = shape,
+      scale = scale), col = "red")
```

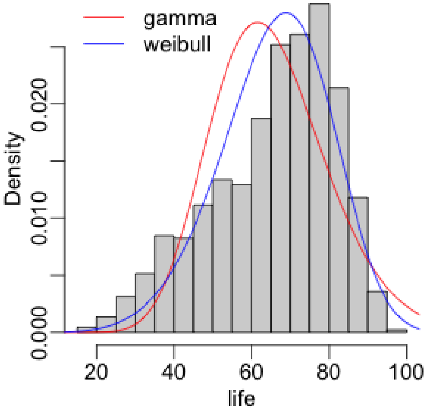
- the `prob=TRUE` argument plots empirical densities instead of frequencies.
- `xSeq` is a vector of ages
- `dgamma` is the density of a Gamma distribution
- the Gamma hasn't capture the right-skewness.



Weibull regression

```
> library('survival')
> cFitS = survreg(Surv(life) ~ decade + left,
+   data=dat, dist='weibull')
> knitr::kable(summary(cFitS)$table,digits=2)
```

	Value	Std. Error	z	p
(Intercept)	4.27	0.01	605.35	0.00
decade	0.02	0.00	5.20	0.00
leftleft	-0.01	0.01	-0.65	0.51
Log(scale)	-1.68	0.02	-90.20	0.00

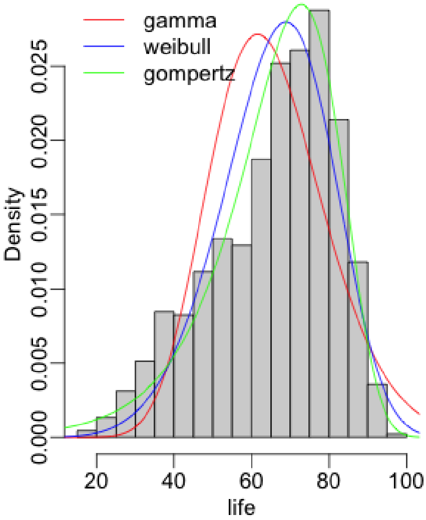


- Weibull is the 'standard' for event times
- `survreg`'s `scale` is the Weibull shape parameter
- shape $= 0.187$, right skewed.

Gompertz regression

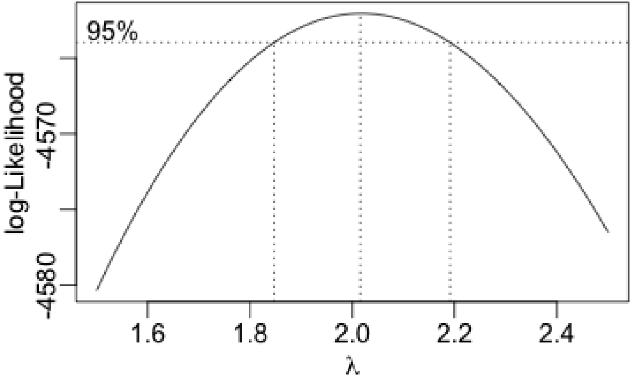
```
> library('flexsurv')
> cFitG = flexsurvreg(Surv(life)~decade+left,
+   data = dat, dist = 'gompertz')
> knitr::kable(cFitG$res, digits = 3)
```

	est	L95%	U95%	se
shape	0.079	0.076	0.082	0.001
rate	0.000	0.000	0.000	0.000
decade	-0.092	-0.125	-0.060	0.017
leftleft	0.033	-0.082	0.149	0.059

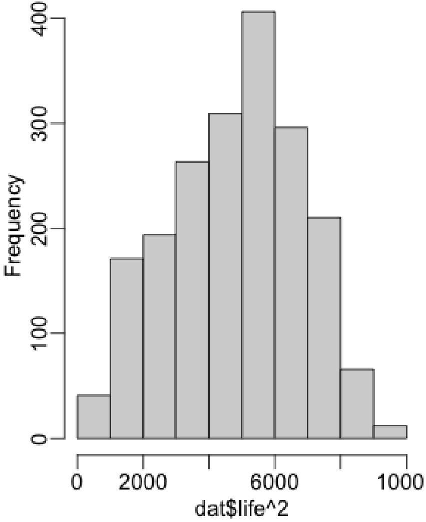


Box-Cox transformation

```
> library("MASS")
> boxcox(life ~ decade + left, data = dat,
+       lambda = seq(1.5, 2.5, len = 20))
```



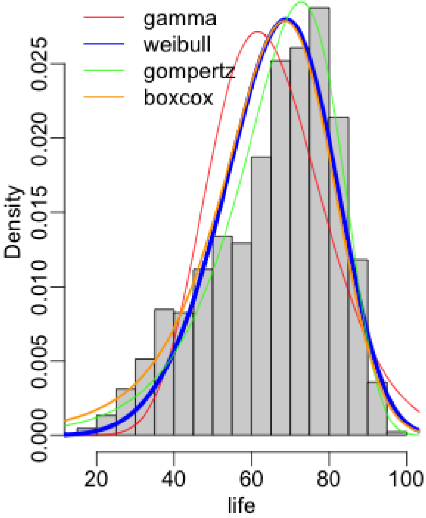

```
> hist(dat$life^2, main = "")
```



Ordinary Least Squares with transformed data

```
> dat$lifesq = dat$life^2
> fitBC = lm(lifesq~decade+left, data=dat)
> densBc = dnorm(xSeq^2,
+         mean = fitBC$coef['(Intercept)'],
+         sd = summary(fitBC)$sigma
+ ) / c(0,diff(sqrt(xSeq)))
> knitr::kable(summary(fitBC)$coef, digits=2)
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4488.34	72.20	62.17	0.00
decade	208.48	32.50	6.41	0.00
leftleft	-135.52	115.52	-1.17	0.24



Notes on GLM's with continuous data

- It's not easy to test which distribution is best, since they're not nested.
- GLM's are not often used with continuous data
  - they're almost always Binomial or Piosson
  - with the notable exception of the Weibull for event times
- 'standard practice' is to transform continuous data to normality (logs, Box-Cox)
- The squared transform model is simplest (?)
- The Weibull is easier to interpret. The percentage change in expected lifetime for left-handers (with 95% confidence interval) is

```
> 100*(exp(cFitS$coef['leftleft'] +
+    c(0, -2, 2) * summary(cFitS)$table['leftleft','Std. Error']) - 1)

[1] -0.7163102 -2.8803498  1.4959491
```

Diagnostics for GLM's

- Assessing model fit is difficult for binary and count data!
- residuals don't have nice properties
- Histograms can be useful
- as can exploratory plots
- For the homework question on fruit flies, it will suffice to show that (or if) the Gamma is a good fit.

Where we are

- discussed applied statistics
- binary, Poisson and continuously-valued GLM's

Next

- contrasts

A more complex glm

```
> cFitCause = glm(life~decade+left*cause,
+   data = dat, Gamma(link='log'))
> summary(cFitCause)$coef[,1:2]
                       Estimate  Std. Error
(Intercept)          3.81442882 0.057493774
decade               0.02379132 0.003789957
leftleft            -0.15479897 0.103518957
causeinbed           0.36433991 0.057388596
leftleft:causeinbed  0.14360209 0.104423075
> table(dat$cause)
alive   acd inbed
    0    23  1945
```

$$
\begin{aligned}
Y_i \sim & \text{Gamma}(\mu_i/\nu, \nu) \\
\log(\mu_i) = & X_i\beta \\
\mathrm{E}(Y_i) = & \mu_i
\end{aligned}
$$

- β_1 = time trend
- β_2 = left v right contrast
- β_3 = right in bed v accident contrast
- β_4 = contrast of left cause v right cause
- β_0 = intercept 1850, right, accidental death

Looking inside

```
> cFitCause$model[c(1,2,15,604),]
     life decade  left cause
2576   43   -0.4 right   acd
2577   47    3.7 right inbed
2606   68   -0.1  left inbed
3689   40    1.0  left   acd
> model.matrix(cFitCause$formula, cFitCause$model)[c(1,2,15,604),]
     (Intercept) decade leftleft causeinbed leftleft:causeinbed
2576           1   -0.4        0          0                   0
2577           1    3.7        0          1                   0
2606           1   -0.1        1          1                   1
3689           1    1.0        1          0                   0
```

On the natural scale

$$
\begin{aligned}
f(Y_i;\phi,\nu) =& \frac{(x/\phi)x^{\nu-1}\exp(-x/\phi)}{\Gamma(\nu)\phi}\\
Y_i \sim& \text{Gamma}(\mu_i/\nu,\nu)\\
\log(\mu_i) =& X_i\beta
\end{aligned}
$$

	Estimate
(Intercept)	3.81442882
decade	0.02379132
leftleft	-0.15479897
causeinbed	0.36433991
leftleft:causeinbed	0.14360209

- when i is a lefty, j is a righty, $X_{ip} = X_{jp}$ for $p \neq 2$

$$\exp(\beta_2) = \mu_i/\mu_j$$

- k died in bed, ℓ accidental, both righties, same birth year

$$\exp(\beta_3) = \mu_k/\mu_\ell$$

- m, n are lefties died in bed and accidentally respectively.

$$\exp(\beta_4) = \frac{\mu_m/\mu_n}{\mu_k/\mu_\ell}$$

- β_2 is the *contrast* between log-expected lifetimes of a lefty and righty

Suppose I want to report different contrasts?

```
                   Estimate
(Intercept)      3.81442882
decade           0.02379132
leftleft        -0.15479897
causeinbed       0.36433991
leftleft:causeinbed  0.14360209
```

- Rate ratio for cause, with left-handers
- $\mu_m/\mu_n = \exp(\beta_3 + \beta_4)$
- What's the standard error for this thing?
- Use $\hat{\beta} \sim \text{N}\left[\beta, I(\hat{\beta})^{-1}\right]$

$$\beta_3 + \beta_4 = A\beta$$
$$\widehat{\beta_3 + \beta_4} \sim \text{N}\left[A\beta, AI(\hat{\beta})^{-1}A^{\mathsf{T}}\right]$$

```
> Avec = c(0, 0, 0, 1, 1)
> c(est=crossprod(Avec, cFitCause$coef),
+   stderr=sqrt(crossprod(Avec, summary(cFitCause)$cov.scaled) %*% Avec))

       est     stderr
0.50794200 0.08722727
```

Conclusions

- GLM's are easy
- Before 1990 one could be forgiven for modelling r/m as Gaussian with the shuttle data
- ...now there's no excuse.

Exercise: Fiji data

- glm with ‘family=‘poisson’
- create a variable ‘logMonthsSinceM’
- put `offset(logMonthsSinceM)` in the model formula

References I

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