Actuarial, Statistical, and Financial Formulas Reference Document

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Aligned with IFOA Standards

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1. Financial Mathematics

Time Value of Money

1. **Compound Interest (Accumulation Factor)**

$$FV = PV \cdot (1+i)^n \quad \text{or} \quad FV = PV \cdot e^{\delta n}$$

   - i: Annual effective interest rate.
   - δ: Force of interest ($\delta = \ln(1+i)$).

2. **Discount Factor**

$$v = \frac{1}{1+i} = e^{-\delta}$$

3. **Present Value (PV) of Annuity-Certain**

   - *Annuity-immediate* (payments at end of period):

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

   - *Annuity-due* (payments at start of period):

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} \quad \text{where } d = \frac{i}{1+i}$$

4. **Future Value (FV) of Annuity-Certain**

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}, \quad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

Additional Financial Mathematics Formulas

1. **Perpetuity**

   - Annuity-immediate:

$$a_{\langle\infty\rangle} = \frac{1}{i}$$

- Annuity-due:

$$\ddot{a}_{\langle\infty\rangle} = \frac{1}{d}$$

2. **Varying Annuities**

- Increasing Annuity-Immediate:

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

- Increasing Annuity-Due:

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

3. **Continuous Annuities**

- Present Value:

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$$

- Future Value:

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

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2. Actuarial Mathematics

Life Contingencies

1. **Present Value of Whole Life Annuity**

- *Annuity-due* (IFOA notation):

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x$$

- *Continuous annuity*:

$$\bar{a}_x = \int_0^\infty v^t \cdot {}_tp_x \, dt$$

2. **Present Value of Term Life Insurance**

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \cdot {}_kp_x \cdot q_{x+k}$$

   - q_x: Probability of death within 1 year for age x.

3. **Net Premium Reserve**

$${}_tV = \ddot{a}_{x+t} - P \cdot \ddot{a}_{x+t:\overline{n-t}|}$$

   - P: Premium payment.

4. **Endowment Insurance**

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + v^n \cdot {}_np_x$$

5. **Deferred Life Annuity**

$${}_m\ddot{a}_x = \sum_{k=m}^{\infty} v^k \cdot {}_kp_x$$

6. **Accumulated Cost of Insurance**

$$\bar{A}_x = \int_0^\infty v^t \cdot {}_tp_x \cdot \mu_{x+t} \, dt$$

Additional Life Contingencies Formulas

1. **Present Value of Whole Life Insurance**
   - Discrete:

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_kp_x \cdot q_{x+k}$$

   - Continuous:

$$\bar{A}_x = \int_0^\infty v^t \cdot {}_tp_x \cdot \mu_{x+t} \, dt$$

2. **Variance of the Present Value of a Whole Life Insurance**

$$Var(Z) = {}^2A_x - (A_x)^2$$

where

$${}^2A_x = \sum_{k=0}^{\infty} v^{2(k+1)} \cdot {}_kp_x \cdot q_{x+k}$$

3. **Temporary Life Annuity**
   - Annuity-due:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_kp_x$$

   - Continuous:

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t \cdot {}_tp_x \, dt$$

4. **m-thly Life Annuities**
   - Annuity-due:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

5. **m-thly Life Insurance**
   - $$A_x^{(m)} \approx A_x + \frac{m-1}{2m} i$$

Mortality and Survival Functions

1. **Survival Probability**

$${}_tp_x = e^{-\int_0^t \mu_{x+s} \, ds}$$

   - μ_x: Force of mortality.

2. **Deferred Mortality Probability**

$${}_{t|u}q_x = {}_tp_x \cdot {}_uq_{x+t}$$

3. **Complete Expectation of Life**

$$\mathring{e}_x = \int_0^{\infty} {}_tp_x \, dt$$

4. **Select Mortality**

$$ {}_tp_{[x]+s} = \frac{l_{x+s+t}}{l_{x+s}} $$

Additional Mortality and Survival Functions

1. **Curtate Expectation of Life**

$$ e_x = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=0}^{\infty} {}_kp_x - 1 = \ddot{a}_x - 1 $$

2. **De Moivre's Law**

$$ {}_tp_x = \frac{\omega - x - t}{\omega - x}, \quad 0 \leq t \leq \omega - x $$

where ω is the limiting age.

3. **Uniform Distribution of Deaths (UDD)**

$$ {}_tp_x = 1 - t \cdot q_x, \quad 0 \leq t \leq 1 $$

4. **Mortality Rate**

$$ q_x = 1 - p_x $$

Commutation Functions (IFOA)

1. **Discounted Lives**

$$ D_x = v^x l_x $$

2. **Accumulated Annuity Factors**

$$ N_x = \sum_{k=0}^{\infty} D_{x+k} $$

3. **Death Benefit Factors**

$$ C_x = v^{x+1} d_x, \quad M_x = \sum_{k=0}^{\infty} C_{x+k} $$

4. **Annuity-Due Commutation**

$$\ddot{a}_x = \frac{N_x}{D_x}$$

5. **Term Insurance Commutation**

$$A^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}$$

Additional Commutation Functions

1. **Immediate Annuity**

$$a_x = \frac{N_{x+1}}{D_x}$$

2. **Whole Life Insurance**

$$A_x = \frac{M_x}{D_x}$$

3. **Temporary Annuity**

$$\ddot{a}_{x:n} = \frac{N_x - N_{x+n}}{D_x}$$

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3. Probability and Statistics

Distributions

1. **Normal Distribution**

$$X \sim N(\mu, \sigma^2) \Rightarrow f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

2. **Binomial Distribution**

$$P(X = k) = (\genfrac{}{}{0pt}{}{n}{k}) p^k (1-p)^{n-k}$$

3. **Poisson Distribution**

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

4. **Exponential Distribution**

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

5. **Gamma Distribution**

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0$$

Additional Distributions

1. **Chi-Squared Distribution**

$$f(x; k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$$

where k is the degrees of freedom.

2. **t-Distribution**

$$f(t; v) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\Gamma(\frac{v}{2})} \left(1 + \frac{t^2}{v}\right)^{-\frac{v+1}{2}}$$

where v is the degrees of freedom.

3. **Weibull Distribution**

$$f(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, \quad x \geq 0$$

Expectation and Variance

1. **Law of Total Expectation**

$$E[X] = E[E[X|Y]]$$

2. **Variance Formula**

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

3. **Covariance**

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Additional Expectation and Variance Formulas

1. **Law of Total Variance**

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

2. **Correlation**

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Hypothesis Testing

1. **Z-Test Statistic**

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

2. **Chi-Square Test**

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

3. **t-Test Statistic**

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

Additional Hypothesis Testing Formulas

1. **F-Test Statistic**

$$F = \frac{s_1^2}{s_2^2}$$

   where s_1^2 and s_2^2 are the sample variances of two populations.
2. **p-value** *The probability of obtaining test results at least as extreme as the results actually observed during the test, assuming that the null hypothesis is correct.*

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4. Financial Derivatives

1. **Black-Scholes Option Pricing**

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

  - $d_1 = \frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$.
  - *Macaulay Duration*:

$$D = \frac{\sum t \cdot PV(CF_t)}{\text{Price}}$$

  - *Convexity*:

$$C = \frac{\sum t(t+1) \cdot PV(CF_t)}{(1+i)^2 \cdot \text{Price}}$$

Additional Financial Derivatives Formulas

1. **Put-Call Parity**

$$C - P = S_0 - Ke^{-rT}$$

2. **Delta**

$$\Delta = \frac{\partial C}{\partial S}$$

3. **Gamma**

$$\Gamma = \frac{\partial^2 C}{\partial S^2}$$

4. **Vega**

$$v = \frac{\partial C}{\partial \sigma}$$

5. **Theta**

$$\Theta = \frac{\partial C}{\partial t}$$

6. **Rho**

$$\rho = \frac{\partial C}{\partial r}$$

Additional Actuarial Formulas

1. **Joint Life Annuity**

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot {}_k p_{xy}$$

2. **Last Survivor Annuity**

$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$

3. **Net Premium for Whole Life Insurance**

$$P_x = \frac{A_x}{\ddot{a}_x}$$

4. **Thiele's Differential Equation**

$$\frac{d}{dt} {}_t V = \delta \cdot {}_t V + P - \mu_{x+t}(S - {}_t V)$$

5. **Variance of Present Value**

$$\text{Var}(PV) = E[PV^2] - (E[PV])^2$$

5. Regulations and Standards (IFRS 17)

Key Considerations under IFRS 17

1. **Measurement Models**:
   - **General Measurement Model (GMM)**: Used for most insurance contracts. Requires estimating future cash flows, discounting them, and adding a risk adjustment for non-financial risk.
   - **Premium Allocation Approach (PAA)**: A simplified approach for short-duration contracts or when the GMM is too burdensome. Revenue is recognized as

premiums are received, and a simplified liability for incurred claims is used.
    - **Variable Fee Approach (VFA)**: Used for contracts with direct participation features. The CSM is adjusted to reflect the changes in the value of the underlying items.
2. **Contractual Service Margin (CSM)**:
    - Represents the unearned profit that the insurer will recognize over the coverage period.
    - Initially measured as the difference between the fulfillment cash flows and the premiums received.
    - Recognized in profit or loss as services are provided.
3. **Risk Adjustment**:
    - Reflects the compensation an insurer requires for bearing the uncertainty about the amount and timing of future cash flows.
    - Can be determined using techniques such as Confidence Level, Cost of Capital, or Quantile techniques.
4. **Discount Rates**:
    - Should reflect the current market rates and the characteristics of the insurance contract liabilities.
    - Adjustments may be needed to reflect liquidity premiums.
5. **Presentation**:
    - Insurers must present insurance revenue separately from insurance service expenses.
    - Disclose significant judgments and assumptions made in applying IFRS 17.
6. **Transition**:
    - IFRS 17 provides different transition approaches: Full Retrospective, Modified Retrospective, and Fair Value.
    - The choice of transition approach can significantly impact the opening balance sheet.
7. **Relevant Formulas/Concepts for IFRS 17 Calculations**:
    - **Best Estimate of Future Cash Flows**:

    $$BE = \sum_{t=1}^{n} \frac{E[CF_t]}{(1+r)^t}$$

    Where $E[CF_t]$ is the expected cash flow at time t, and r is the discount rate.
    - **Risk Adjustment (RA)**:
        - Cost of Capital Approach:

        $$RA = \sum_{t=1}^{n} \frac{CF_t \times CoC}{(1+r)^t}$$

Where CoC is the cost of capital rate.

- **Contractual Service Margin (CSM) Amortization**:

$$\text{Amortization} = \frac{\text{CSM at start of period}}{\text{Coverage Units}} \times \text{Coverage Units Provided}$$

6. IFRS 17 Insurance Contracts

Measurement Models

1. **General Measurement Model (GMM) - Building Blocks Approach**

$$Insurance\ Contract\ Liability = FCF + RA + CSM$$

   where:

   - FCF: Fulfilment Cash Flows
   - RA: Risk Adjustment
   - CSM: Contractual Service Margin

2. **Premium Allocation Approach (PAA) - Liability for Remaining Coverage**

$$LRC = Premiums\ Received - Acquisition\ Cash\ Flows + Amortization$$

3. **Loss Component Calculation**

$$Loss\ Component = max(0, -CSM)$$

Risk Adjustment Calculation Methods

1. **Cost of Capital Method**

$$RA = \sum_{t=1}^{T} \frac{CoC \cdot SCR_t}{(1 + r_f)^t}$$

   - CoC: Cost of Capital rate
   - SCR: Solvency Capital Requirement

2. **Confidence Level Approach**

$$RA = VaR_\alpha(Losses) - E(Losses)$$

where α is the confidence level (e.g., 75%)

CSM Adjustments

1. **CSM Roll-forward**

$$CSM_t = CSM_{t-1} \cdot (1+i) + New\ Business - Amortization \pm Experience\ Adjustments$$

2. **Coverage Units Recognition**

$$CSM\ Release = CSM_t \cdot \frac{CU_t}{\sum_{i=t}^{T} CU_i}$$

where CU represents Coverage Units

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7. Risk Measures

Value at Risk and Related Measures

1. **Value at Risk (VaR)**

$$P(X \leq VaR_\alpha) = \alpha$$

2. **Conditional Tail Expectation (CTE) / Expected Shortfall**

$$CTE_\alpha(X) = E[X|X > VaR_\alpha]$$

3. **Risk-Based Capital**

$$RBC\ Ratio = \frac{Total\ Adjusted\ Capital}{Required\ Risk\ Based\ Capital}$$

Risk Adjusted Performance Measures

1. **Sharpe Ratio**

$$SR = \frac{R_p - R_f}{\sigma_p}$$

2. **Risk-Adjusted Return on Capital (RAROC)**

$$RAROC = \frac{Expected\ Return}{Economic\ Capital}$$