According to the Ref.[1], there are $N = 128$ neurons coupled in a one-dimensional ring. Dynamics of membrane voltages of the jth neuron is:

$$C\frac{dv^j}{dt} = -g_l(v^j - V_l) - g_e^j(t)(v^j - V_E) - g_i^j(t)(v^j - V_I), \tag{1}$$

where $C = 1$ is the (nondimensionalized) total capacitance, $g_l = 50$ is the (nondimensionalized) leakage conductance, $V_l = 0$ is the (nondimensionalized) leakage=rest=reset potential, $V_E = \frac{14}{3}$ and $V_I = -\frac{2}{3}$ are the (nondimensionalized) excitatory and inhibitory synaptic reversal potentials, respectively. Only time t retains dimension (S).

The time-dependent conductances, i.e., $g_e^j(t)$ and $g_i^j(t)$, arising from external forcing and the network activity of this population of neurons, are given by the following equation:

$$\begin{aligned} \mathrm{g}_e^j(t) &= g_{e0}^j(t) + \textstyle\sum_k A_{j,k} \sum_l G_e(t - t_l^k) \\ \mathrm{g}_i^j(t) &= g_{i0}^j(t) + \textstyle\sum_k B_{j,k} \sum_l G_i(t - T_l^k), \end{aligned} \tag{2}$$

where t_l^k (T_l^k) denotes the time of the lth spike of the kth excitatory (inhibitory) neuron, and $A_{j,k}$ and $B_{j,k}$ are coupling strengths, which are isotropic Gaussians in the angle separation of neurons along the ring. Both the excitatory and inhibitory postsynaptic conductance functions, i.e., G_e and G_i, are given by the following equation but with different time constants:

$$\begin{aligned} G(t - t_{spike}) &= \frac{(\frac{t - t_{spike}}{\tau})^m}{e^{\frac{t - t_{spike}}{\tau}}}, t \geq t_{spike} \\ &= 0, t < t_{spike}, \end{aligned} \tag{3}$$

where $m = 5$ and τ is a time constant. $\tau_e = 0.6mS$. $\tau_i = 1.0mS$. The input excitation conductance is $g_{e0}^j(t) = 25\sin(t)$. Effects of inhibition is ignored, i.e., $g_i^j(t) = 0$. In my implementation, only g_{i0}^j is set to 0. The model stops at $t = 1S$.

Fixed time-step ($\Delta t = t_{n+1} - t_n$ is a const) schemes, including first-order Euler, RK2 and RK4, are used to solve the model. To obtain a high precision, a linear interpolant should be used to find spikes times for Euler and RK2; a cubic interpolant should be used for RK4. Below is the modified RK2 scheme with linear interpolant.

Whenever a new v_{n+1} is greater than the (nondimensionalized) threshold $\overline{v} = 1$ and $v_n < \overline{v}$, the neuron has fired at some time during this last time-step. A linear interpolant can be used to estimate the membrane potential between discretized times v_n and v_{n+1}:

$$v(t) = v_n + \frac{(v_{n+1} - v_n)}{\Delta t}(t - t_n). \tag{4}$$

Let $v(t_{spike}) = \overline{v}$, then the spike time can be approximated by solving the above equation at $t = t_{spike}$:

$$t_{spike} = t_n + \frac{\overline{v} - v_n}{v_{n+1} - v_n} \Delta t. \tag{5}$$

Once t_{spike} is found, the values of v_n and v_{n+1} should also be updated, so that the (interpolated) solution trajectory passes through the reset potential V_l at time t_{spike}.

Firstly, Eq. (1) can be rewriten as

$$\frac{dv}{dt} = -f(t, v) = -\alpha(t)v + \beta(t) \tag{6}$$

where $\alpha(t) = \frac{g_l + g_e(t) + g_i(t)}{c}$ is the total (nondimensionalized) conductance, and $\beta(t) = \frac{g_l V_l + g_e(t) V_E + g_i(t) V_i}{c}$ is the (nondimensionalized) "difference current". The superscript j has been dropped in Eq. (6) for clarity.

Then, a single RK2 step approximates the changes in the membrane potential by

$$\begin{aligned} v_{n+1} &= v_n + \tfrac{k_1 + k_2}{2} \Delta t, \\ k_1 &= f(t_n, v_n) = -\alpha_0 v_n + \beta_0, \\ k_2 &= f(t_n + \Delta t, v_n + k_1 \Delta t) = -\alpha_1 [v_n + (-\alpha_0 v_n + \beta_0)\Delta t] + \beta_1, \end{aligned} \tag{7}$$

where $\alpha_0 = \alpha(t_n), \alpha_1 = \alpha(t_n + \Delta t)$, $\beta_0 = \beta(t_n), \beta_1 = \beta(t_n + \Delta t)$.

The updated v_n and v_{n+1} are denoted by $\tilde{v}_n$ and $\tilde{v}_{n+1}$, respectively. We have $\tilde{v}_n$ and $\tilde{v}_{n+1}$ are related by a single RK2 step, $\tilde{v}(t_{spike}) = V_l$, and the points $(t_n, \tilde{v}_n)$, (t_{spike}, V_l), and $(t_{n+1}, \tilde{v}_{n+1})$ are in a line. Thus, by solving a binary linear equation group, we can obtain:

$$\tilde{v}_n = \frac{2V_l - (t_{spike} - t_n)(\beta_0 + \beta_1 - \alpha_1 \beta_0 \Delta t)}{2 - (t_{spike} - t_n)(\alpha_0 + \alpha_1 - \alpha_1 \alpha_0 \Delta t)}, \tag{8}$$

Finally, we can update $\tilde{v}_{n+1}$:

$$\tilde{v}_{n+1} = \tilde{v}_n + \frac{\Delta t}{2}(\tilde{k}_1 + \tilde{k}_2), \tag{9}$$

where $\tilde{k}_1 = -\alpha_0 \tilde{v}_n + \beta_0$ and $\tilde{k}_2 = -\alpha_1(\tilde{v}_n + \tilde{k}_1 \Delta t) + \beta_1$, respectively.

References

1. M J Shelley, L Tao. Efficient and accurate time-stepping schemes for integrate-and-fire neuronal networks. Journal of Computational Neuroscience, 11, 111, 2001