Day-2 Practical Session, 26 May 2021

Part 1: Incidence Weighting Estimator (IWE) under Bipartite Incidence Graph Sampling (BIGS)

Li-Chun Zhang[1,2,3] and *Melike Oguz-Alper*[2]

[1]*University of Southampton (L.Zhang@soton.ac.uk)*, [2]*Statistics Norway*, [3]*University of Oslo*

Illustration I: BIGS-IWE strategy: small data set

In this illustration, we will compare the efficiencies of several IWE estimators including the priority-rule estimators under BIG sampling. The first example is based on a small graph the node labels of which are the same as those in the graph described in Section 2.4.1 in Lecture Notes. Edges are created randomly by using the R-function **skthBIG**.

Description of the population and sampling strategies

- Population BIG: $\mathcal{B} = (F, \Omega; H)$, H consists of edges between *sampling units* $i \in F$ and *study units* $\kappa \in \Omega$
- Sample BIG: $\mathcal{B}_s = (s_0, \Omega_s; H_s)$ with $s_0 \in F$, $\Omega_s = \alpha(s_0)$, and $s_{ref} = s_0 \times \Omega$ such that $H_s = H \cap s_{ref} = H \cap (s_0 \times \Omega)$
- β_κ: *ancestry* set of $\kappa \in \Omega_s$ and α_i: *successors* of $i \in s_0$
- s_0 of size n selected with SRSWOR from sampling frame F of size N

Formula sheet

- The parameter of interest: size of Ω:

  $\theta = \sum_{\kappa \in \Omega} y_\kappa$, where $y_\kappa = 1$ for all $\kappa \in \Omega$

- IWE based on $\mathcal{B}_s = (s_0, \Omega_s; H_s)$ by BIGS

  $\hat{\theta} = \sum_{(i\kappa) \in H_s} W_{i\kappa} \frac{y_\kappa}{\pi_i}$

- Hansen-Hurwitz (HH) type estimators: special case of IWE, *constant* weights

  $\hat{\theta} = \sum_{i \in s_0} \frac{z_i}{\pi_i}$, where $z_i = \sum_{\kappa \in \alpha_i} w_{i\kappa} y_\kappa$, with $\sum_{i \in \beta_\kappa} w_{i\kappa} = 1$

  - HH-type estimator with *equal* weights: *multiplicity* estimator (Birnbaum and Sirken 1965)

    $w_{i\kappa} \equiv \frac{1}{|\beta_\kappa|}$

  - HH-type estimator with *unequal* weights: *probability and inverse degree-adjusted (PIDA) weights*

    $w_{i\kappa} \propto \frac{\pi_i}{|\alpha_i|^\gamma}$, $\gamma > 0$

NB. Under SRS of s_0 and when $\gamma = 0$, the HH-type estimator with PIDA weights become equivalent to the multiplicity estimator above

- HTE: a special case of IWE

  $\hat{\theta}_{HT} = \sum_{\kappa \in \Omega_s} \frac{y_\kappa}{\pi_{(\kappa)}}$

The first-order inclusion probabilities $\pi_{(\kappa)} = \Pr(\kappa \in \Omega_s)$ can be calculated, under SRS of s_0, by

$\pi_{(\kappa)} = 1 - \bar{\pi}_{\beta_\kappa} = 1 - \binom{N-|\beta_\kappa|}{n} / \binom{N}{n}$, where $|\beta_\kappa|$ is the size of the ancestor set of κ

- Priority-rule estimators with *priority rule* to the sample edges H_s: $I_{i\kappa} = 1$ if $i = \min(s_0 \cap \beta_\kappa)$, and $I_{i\kappa} = 0$ otherwise

  $\hat{\theta}_p = \sum_{(i\kappa) \in H_s} \left(\frac{I_{i\kappa}\omega_{i\kappa}}{p_{(i\kappa)}} \right) \frac{y_\kappa}{\pi_i}$, where $p_{(i\kappa)} = \Pr(I_{i\kappa} = 1 | \kappa \in \Omega_s)$

The probabilities $p_{(i\kappa)}$ can be calculated, under SRS of s_0, by

$p_{(i\kappa)} = \binom{N-1-d_{i(\kappa)}}{n-1} / \binom{N-1}{n-1}$, where $d_{i(\kappa)}$ the number of nodes with higher probability than i for each $\kappa \in \Omega$ and $i \in \beta_\kappa$ for the priority-rule $\min(s_0 \cap \beta_\kappa)$

NB. R-package **igraph** has to be installed before running R-functions below that generates random graphs.

Description of R-function **skthBIG**

1. Function parameters

- **showplot**: Use **TRUE** to get BIG illustration. Default value: **FALSE**

2. Main steps of the function

- A random bipartite graph generated with $F = \{1, 2, 3, 4\}$ and $\Omega = \{5, 6, 7, 8, 9, 10, 11\}$
- Out and in-degrees initiliased based on $|\alpha_i|$ and $|\beta_\kappa|$, for $i \in F$ and $\kappa \in \Omega$, in the example presented in Section 2.4.1. However, final values may differ from these initial value due to random generation of edges. The total number of degrees may differ as well, since multiple edges and loops are removed if they exist in the initial random graph generated

3. Main outputs of the function

- BIG plot shown if **showplot**= **TRUE**
- A list of random graph generated: Use **$G** to get the graph

Description of R-function **zFun**

1. Function parameters

- **popgraph**: population graph to be used: outputs of either **skthBIG** or **mainrndBIG**
- **coefgamma**: coefficient to be used in the HH-type estimator with PIDA weights; default value 0. No effect of the choice if **multiplicity**= **TRUE**
- **n**: sample size of initial sample s_0; default value 2
- **multiplicity**: Use **TRUE** to get z_i values based on equal weights, i.e. $w_{i\kappa} = |\beta_\kappa|^{-1}$; default **FALSE**

2. Main steps of the function

- Edge set derived from the population graph, as well as the labels of the vertices in F and Ω
- $|\alpha_i|$ and $|\beta_\kappa|$ calculated based on the edge set
- z_i values calculated for all $i \in F$ for chosen values of γ

3. Main outputs of the function

- z_i values returned

Description of R-function **mainBIGSIWE**

1. Function parameters

- **popgraph**: population graph to be used: use the output of the function **skthBIG**
- **coefgamma**: coefficient to be used in the HH-type estimator with PIDA weights; default value 0
- **n**: sample size of initial sample s_0; default value 2

2. Main steps of the function

- Edge set derived from the population graph, as well as the labels of the vertices in F and Ωv
- $|\alpha_i|$ and $|\beta_\kappa|$ calculated based on the edge set
- Inclusion probabilities $\pi_{(\kappa)}$ calculated based on $|\beta_\kappa|$
- All possible samples of size n selected with SRS from F
- For each random sample, estimates obtained from the HTE, the HH-type estimator and the priority-rule estimator. For the last one, three random orderings of out-degrees, i.e. α_i, considered: *random*, *ascending* and *descending*

3. Main outputs of the function

- Expected values and the sampling variances of the estimators calculated based on the sample estimates over all possible samples

---