The fundamental group of a graph

02.12.2020

Sources: • "Foundations of a connectivity theory for simplicial complexes" [Barcelo, Kramer, Laubenbacher, Weaver]
• "Higher discrete homotopy groups of graphs" [Lutz]

1. Precise definition
2. Homotopy invariance
3. Examples

§ Precise definition

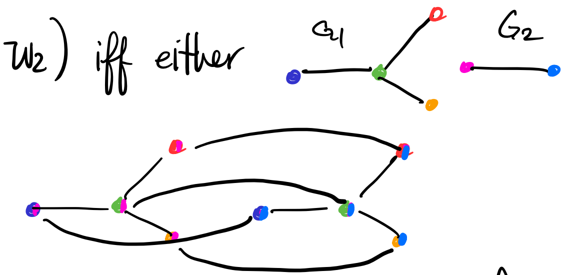
Recall: Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be simple graphs.

- The graph product $G_1 \otimes G_2$ is the graph with $V(G_1 \otimes G_2) = V_1 \times V_2$ and there is an edge between (v_1, v_2) and (w_1, w_2) iff either

$$v_1 = w_1 \quad \text{and} \quad v_2 \sim w_2$$

OR

$$v_2 = w_2 \quad \text{and} \quad v_1 \sim w_1$$



- A graph map $f: G_1 \to G_2$ is a set map on vertices, $V_1 \to V_2$ such that if $v \sim w$ in E_1, then either

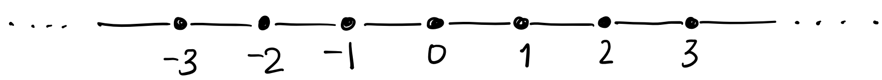
$$f(v) = f(w)$$

OR

$$f(v) \sim f(w) \text{ in } E_2.$$

either contract edges down to a vertex or map edges according to where adjacent vertices are mapped.

- Let I_∞ denote the graph with vertex set $V(I_\infty) = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and an edge $i \sim j$ iff $|i-j| = 1$.



- Two graph maps $f, g: G_1 \to G_2$ are homotopic if for some $n \geq 0$, there is a graph map $h: G_1 \otimes I_n \longrightarrow G_2$ such that

"a homotopy"

$$h(-, 0) = f$$

and

$$h(-, n) = g$$

Def Given $f: G_1 \to G_2$ and subgraphs $H_1 \subseteq G_1$, $H_2 \subseteq G_2$, write it as a **POINTED GRAPH MAP** $f: (G_1, H_1) \to (G_2, H_2)$ if $v \in H_1 \Rightarrow f(v) \in H_2$.

Let G be a graph and $v_0 \in V(G)$.

Consider I_∞ and a graph map $f: I_\infty \to G$. If there exists some $r \geq 0$ such that for all $i \in V(I_\infty)$ with $|i| \geq r$, $f(i) = v_0$, then call the smallest such r the **RADIUS of f** and so we have a pointed map

$$f: (I_\infty, I_{\geq r}) \longrightarrow (G, v_0)$$

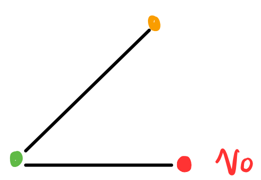
where $r = \min\{ r_0 : f(i) = v_0 \quad \forall \ |i| \geq r_0 \}$

and $I_{\geq r} = \{ i \in I_\infty : |i| \geq r \}$

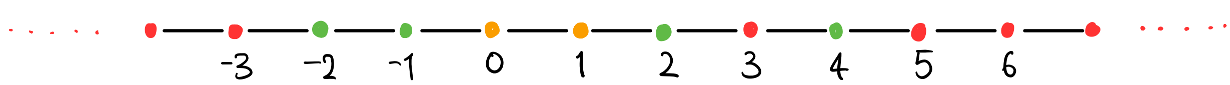
some denote $I_{\geq r}$ by $\partial \mathbb{Z}$ (with r implicit)

$\rightsquigarrow$ From last time, think of this "radius" as being the length of the "active area" of the "loop" on G.

Example from last time:

$G =$ 

Let $f: (I_\infty, I_{\geq r}) \to (G, v_0)$ be the map given by the following colouring:



What is the radius?

$r = 5$

$\boxed{\text{Def}}$ Graph maps $f, g: (I_\infty, I_{\geq r}) \longrightarrow (G, v_0)$ are **BASED HOMOTOPIC** if there is a homotopy $h: I_\infty \otimes I_n \to G$ from f to g such that $h_i = h(-, i)$ is a graph map $h_i: (I_\infty, I_{\geq r'}) \longrightarrow (G, v_0)$ $\forall i$.
Denote the homotopy class by $[f]$.

<u>Remark</u>: 2 maps f, g do <u>not</u> have to have the <u>same</u> radius to be homotopic. We choose the larger radius in the above definition. (so that the maps have a chance at being homotopic)

$\boxed{\text{Def}}$ Let $v_0 \in G$. The set

$$\pi_1(G) := \left\{ [f] \;:\; f: (I_\infty, I_{\geq r}) \to (G, v_0) \text{ a graph map} \right\}$$

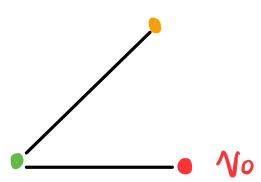
is a group with "concatenation" as multiplication. (i.e),
Let $f: (I_\infty, I_{\geq r_f}) \to (G, v_0)$ and $g: (I_\infty, I_{\geq r_g}) \to (G, v_0)$ be graph maps with radii r_f and r_g. Then,

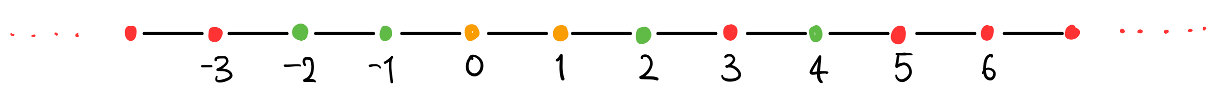
$$[f] \cdot [g] = [p] \quad \text{where}$$

$p: (I_\infty, I_{\geq r}) \to (G, v_0)$ is defined by

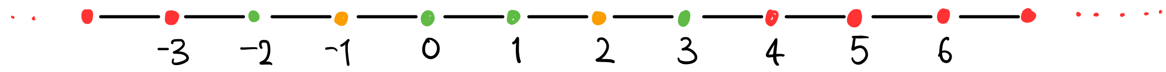
$$f \cdot g(i) = p(i) = \begin{cases} f(i) & \text{if } i \leq r_f \\ g(i - (r_f + r_g)) & \text{if } i > r_f \end{cases}$$

well-defined: same as topological space: take $h_i \cdot h'_i$

eg $G =$ 

$f:$ 

g :

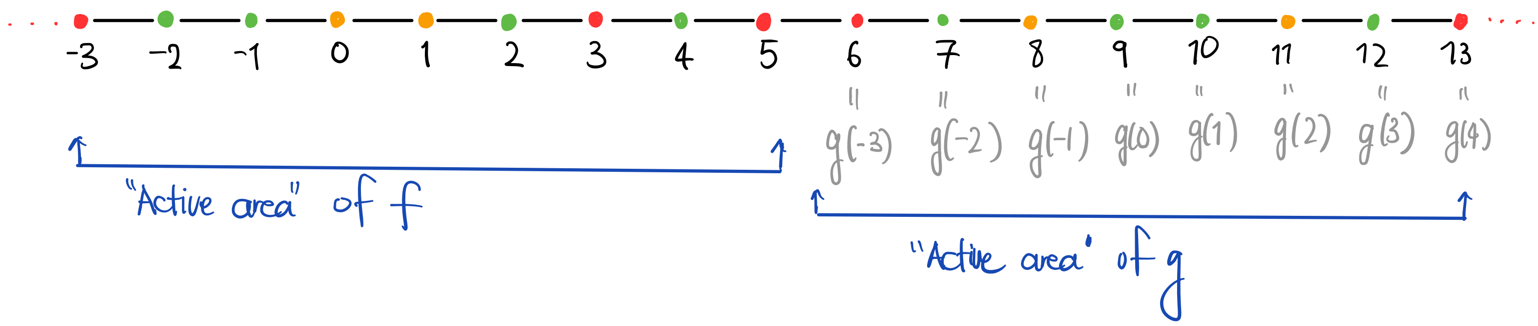


$r_f = 5$

$r_g = 4$

$$f \cdot g\,(i) = \begin{cases} f(i) & \text{if } i \leq 5 \\ g(i-9) & \text{if } i > 5 \end{cases}$$

$f \cdot g$:

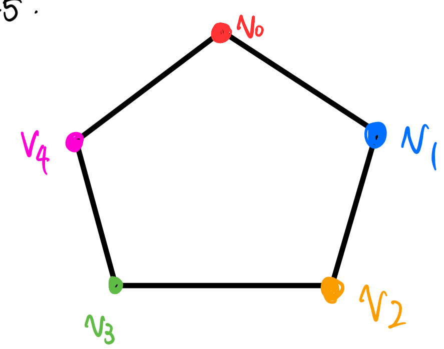


How do inverses work?

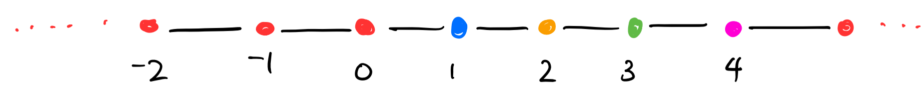
For $f: (I_\infty, I_{\geq r}) \to (G, v_0)$, $[f] \in \pi_1(G, v_0)$, take $\bar{f}: (I_\infty, I_{\geq r}) \to (G, v_0)$ to be the loop f traversed in the opposite direction.

$\leadsto$ we obtain symmetry and homotope out from the "middle" eg C_5.

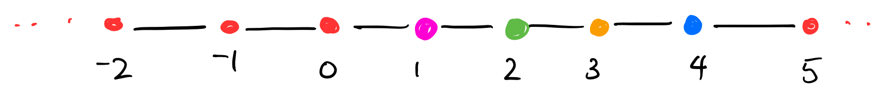
$C_5 =$



f :

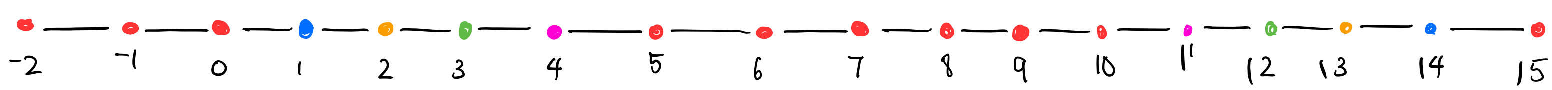


$\bar{f}$:



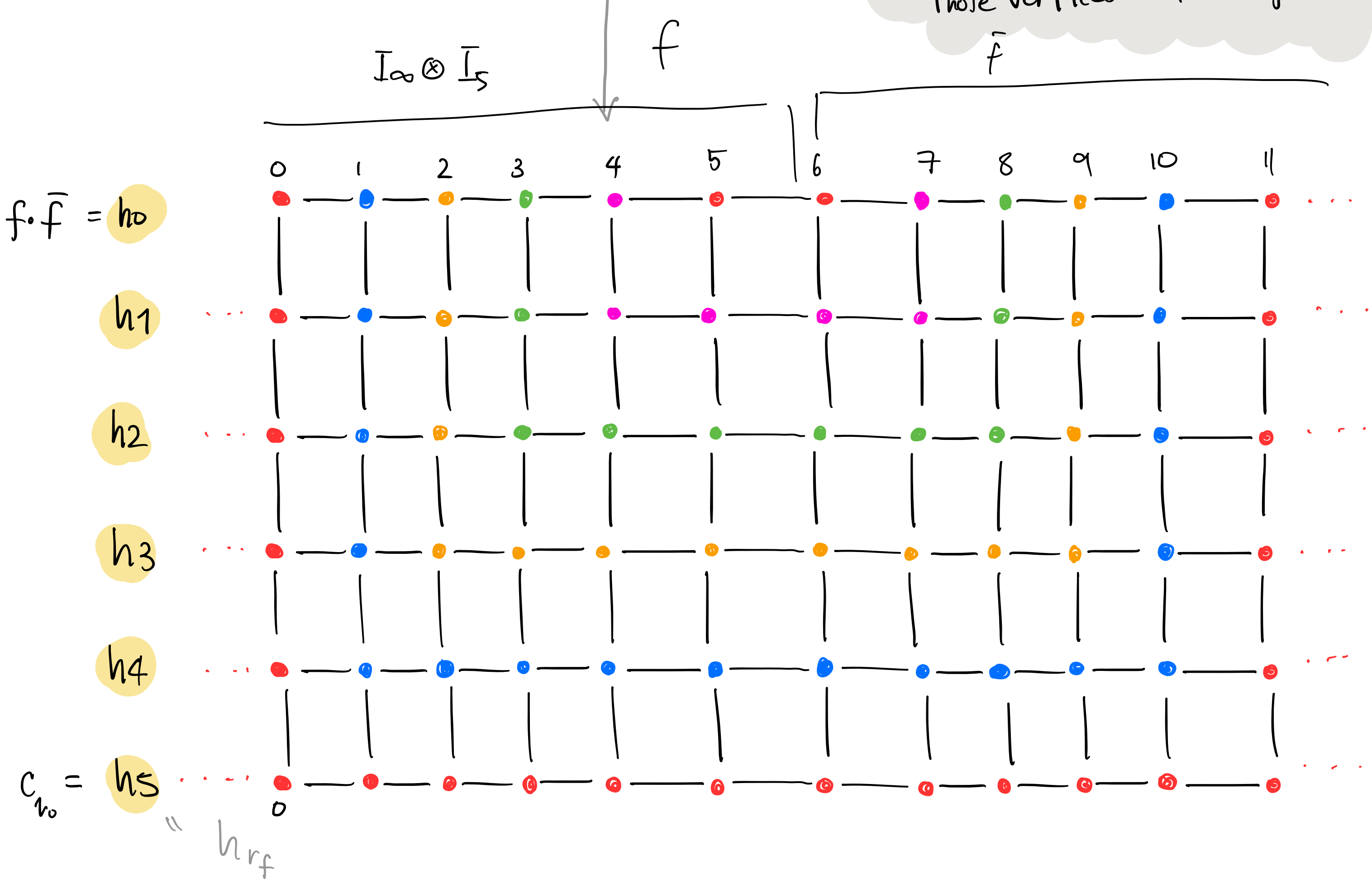
$$f\cdot\bar{f}(i) = \begin{cases} f(i) & \text{if } i \leq 5 \\ \bar{f}(i-10) & \text{if } i > 5 \end{cases}$$

$f\cdot\bar{f}$:



$h : I_\infty \otimes I_5 \longrightarrow G$

whenever there is a vertical/horizontal edge, those vertices must be adjacent



In general...

What is the homotopy $h: I_\infty \otimes I_{r_f} \longrightarrow G$?

(shift so that $f \cdot \bar{f}$ "starts" at $j=0$)

Let $h_i = f_i \cdot \bar{f}_i$ where

$$f_i(j) = \begin{cases} f(j) & \text{if } 0 \leq j \leq r_f - i \\ f(r_f - i) & \text{if } r_f - i < j \leq r_f \\ v_0 & \text{else} \end{cases}$$

⇝ similar to topological spaces: f_i is equal to f on $[0, r_f - i]$ and stationary at $f(r_f - i)$ on the remaining interval, then concatenate it with $\bar{f}_i$ to get h_i

§ Homotopy invariance

$\boxed{\text{Prop}}$ A based G-homotopy equivalence of based graphs induces an isomorphism of graph homotopy groups.

$\boxed{\text{Proof}}$ The proof is "identical" to the corresponding proof for homotopic maps of topological spaces. [Barcelo, Kramer, Laubenbacher, Weaver]

First, we define the induced group homomorphism of a based graph map.

Suppose $f: (G_1, v_1) \to (G_2, v_2)$ is a based graph map. Then, define

$$f_*: \pi_1(G_1, v_1) \longrightarrow \pi_1(G_2, v_2) \quad \text{by}$$

$$f_*: [\alpha] \longmapsto [f \circ \alpha]$$

where $\alpha: (I_\infty, I_{\geq r}) \to (G_1, v_1)$.

If $\alpha \sim \beta$, then $f \circ \alpha \sim f \circ \beta$

$f \circ h_t$ (compare with homotopy)

- f_* is a well-defined group homomorphism and $(f \circ g)_* = f_* \circ g_*$

Now, let $f:(G_1, v_1) \to (G_2, v_2)$ be a homotopy equivalence, with homotopy inverse $g:(G_2, v_2) \to (G_1, v_1)$.

Then,

$$f \circ g \simeq id_{G_2}$$

$\Rightarrow f \circ g$ is an isomorphism (take homotopy from $f \circ g$ to id_{G_2} and take the path traced out by the basepoint)

Also,

$\Rightarrow (f \circ g)_* = f_* \circ g_*$ is an isomorphism

$\Rightarrow g_*$ is injective, f_* is surjective.

Then, use $g \circ f \simeq id_{G_1}$ to get that g_* is surjective, f_* is injective

$\Rightarrow f_*, g_*$ isomorphisms, so

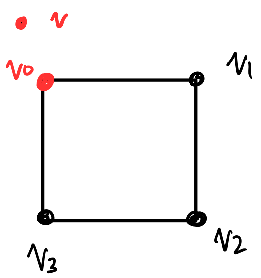
$$\pi_1(G_1, v_1) \cong \pi_1(G_2, v_2)$$

$\square$

§ Examples

① Let G_1 :

$C_4 = G_2$:



$$f : G_1 \longrightarrow C_4$$
$$v \longmapsto v_0$$

<u>Claim:</u> f is a homotopy equivalence (ie, C_4 is contractible) → homotopy from identity to constant map

↳ homotopy equiv to point

Let $g : C_4 \xrightarrow{!} G_1$

Then, $gf : G_1 \to G_1$

$$gf(v) = g(v_0) = v$$

$$\Rightarrow gf = id_{G_1}$$

<u>want to show:</u> $fg \simeq id_{C_4}$ $\quad$ $fg : C_4 \to C_4$, $id_{C_4} : C_4 \to C_4$

Let $h : C_4 \otimes I_2 \to C_4$ be defined by

$$h\big|_{C_4 \otimes \{0\}} = id_{C_4}$$

$$h\big|_{C_4 \otimes \{1\}} = \varphi \quad \text{where } \varphi(v_0, 1) = \varphi(v_3, 1) = v_0$$
$$\varphi(v_1, 1) = \varphi(v_2, 1) = v_1$$

And on $C_4 \otimes \{2\}$:

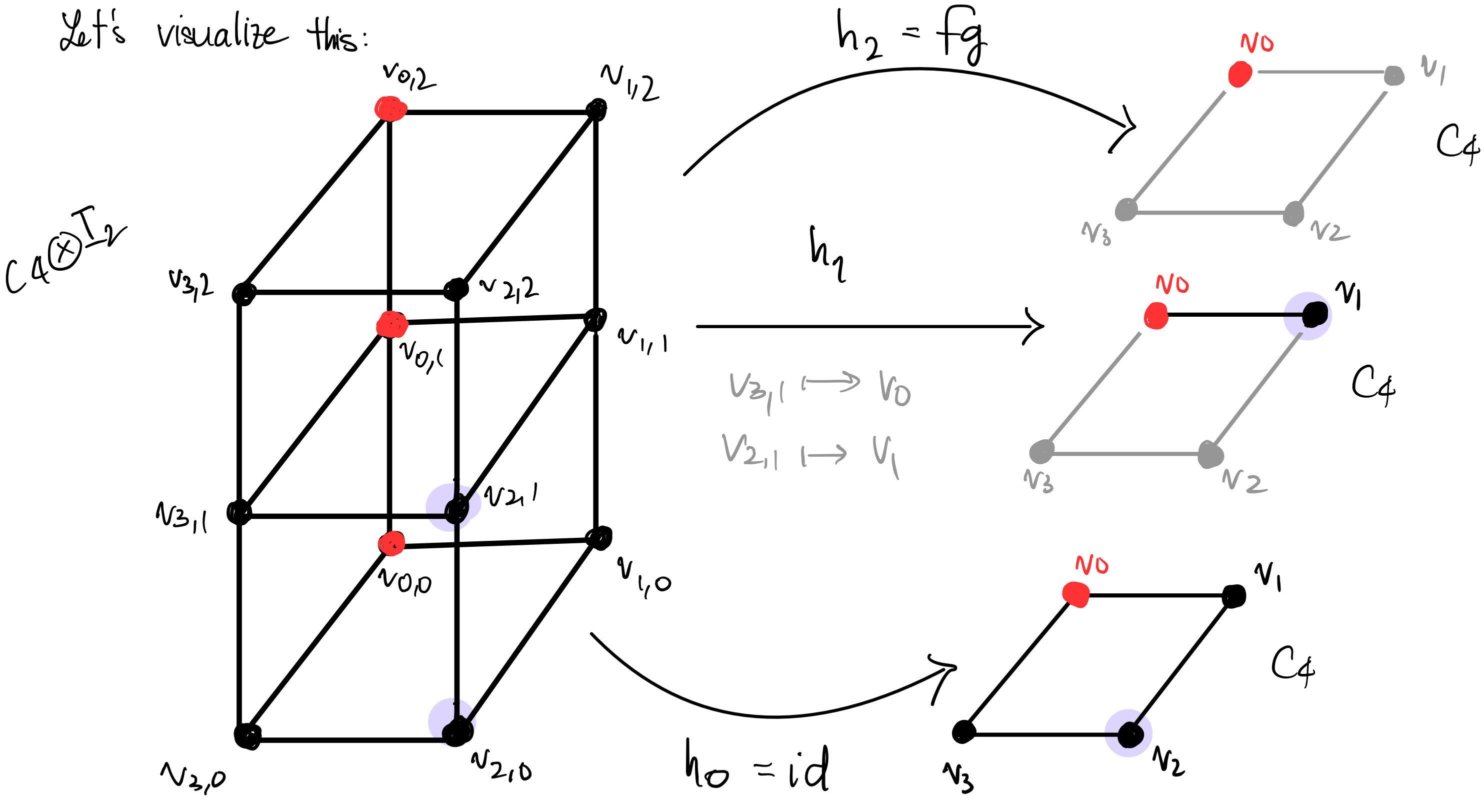
$$h(v_i, 2) = v_0 \quad \forall i.$$

$fg(v_i) = f(v_0) = v_0$

Notice that $h(-, 0) = id_{C_4}$
$h(-, 2) = fg$

To show that h is a homotopy, it remains to show that h is a graph map.

Let's visualize this:



Denote $\underset{\substack{\uparrow \\ C_4 \otimes I_2}}{(v_i, j)} = v_{i,j}$

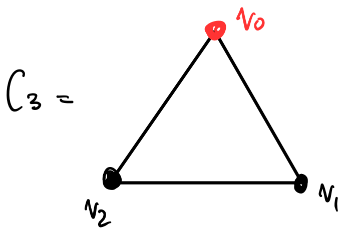
Then, h is a graph map. (edges either incident to mapped vertices or contracted down to a vertex)

$\Rightarrow$ h is a homotopy.

$\Rightarrow$ $fg \simeq id_{C_4}$

Thus, $\pi_1(C_4, v_0) \cong \pi_1(\bullet, v) = \{e\}$

② Similar to the above example, $C_3 \simeq *$



$f: * \to C_3$

$v \mapsto v_0$

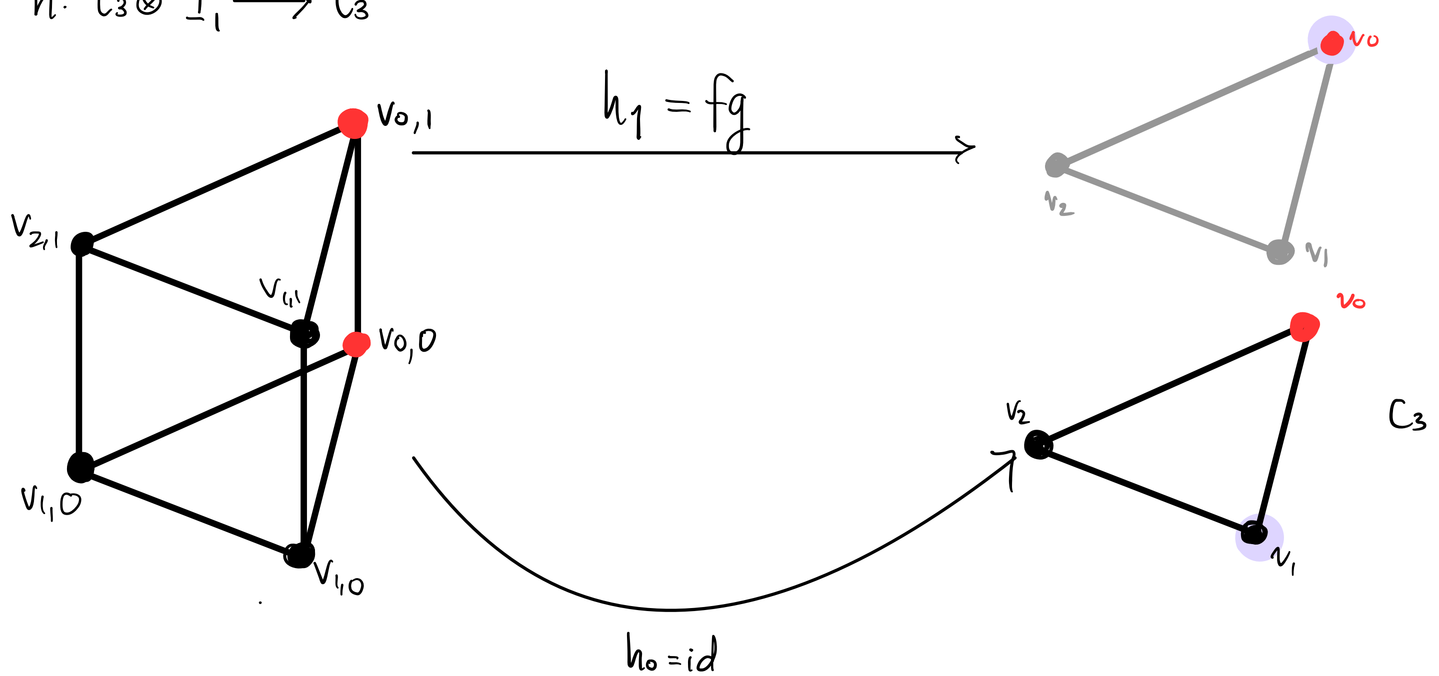
$fg(v_i) = f(*) = v_0$

$g: C_3 \longrightarrow *$

$gf = id_*$

and $fg \simeq id_{C_3}$ via

$h: C_3 \otimes I_1 \longrightarrow C_3$



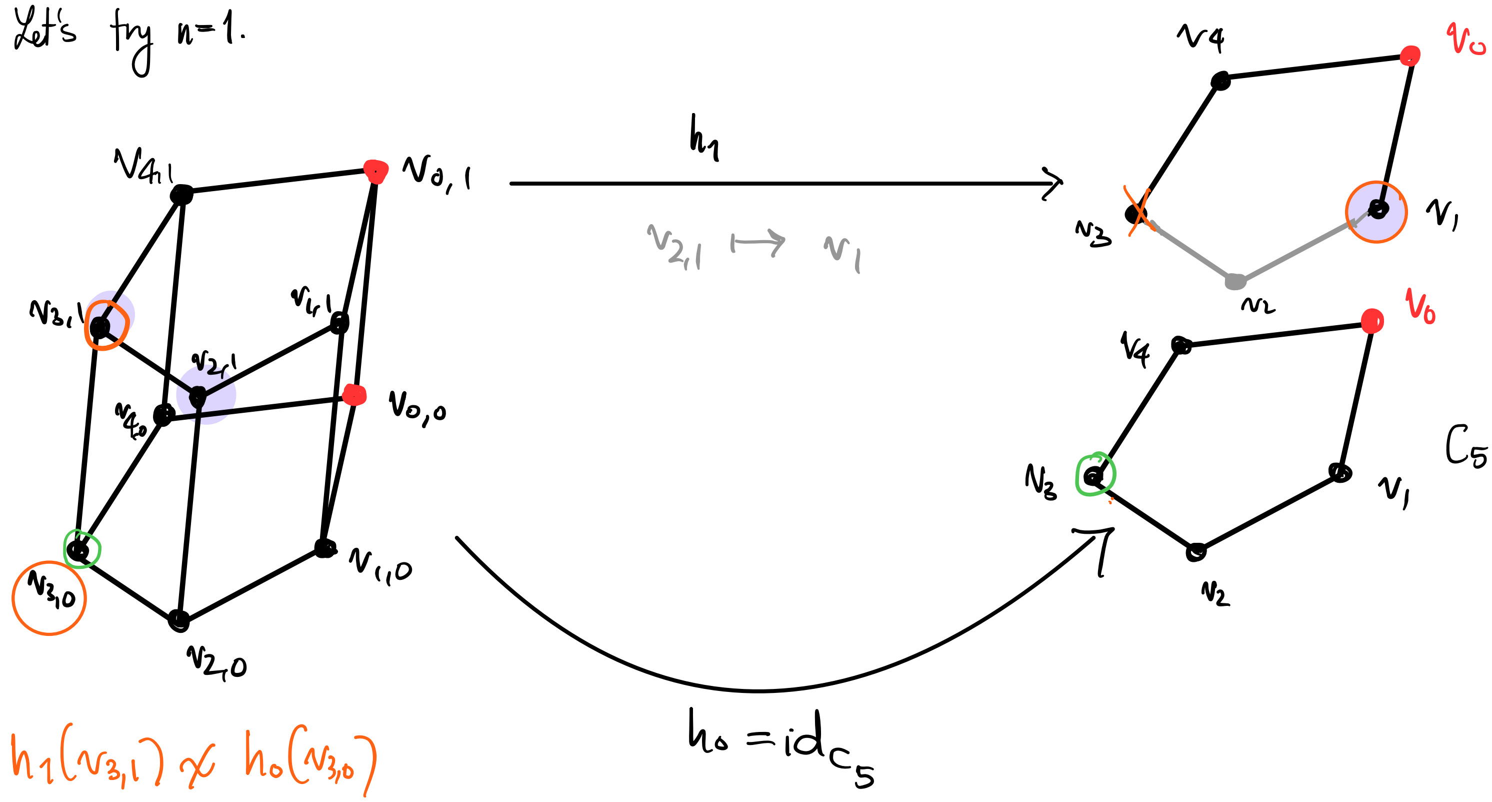
③ $C_5 \not\simeq *$

Fix some $n > 0$. Let $h : C_5 \otimes I_n \longrightarrow C_5$ be such that

$$h(-,0) = \mathrm{id}_{C_5}$$

<u>Claim:</u> $h(-,i)$ is surjective for all $1 \leq i \leq n$.

Let's try $n = 1$.



$h_1(v_{3,1}) \not\sim h_0(v_{3,0})$

↯ not possible because $v_{3,1} \sim v_{2,1}$

But $h_1(v_{3,1}) \neq h_1(v_{2,1})$

Same phenomenon occurs in higher dimensions ⇝ cannot contract an edge down.

<u>Remark:</u> It can be shown that $\pi_1(C_5) \cong \mathbb{Z}$

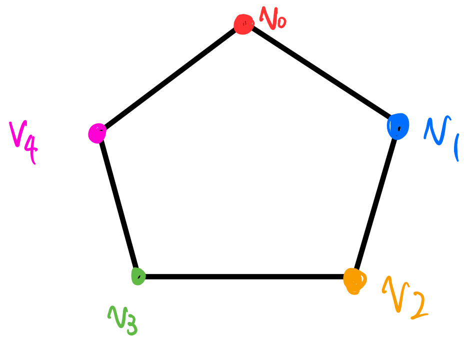
⇝ construct X_{C_5} (cell complex) by regarding C_5 as a 1-complex

$\Rightarrow X_{C_5} \cong S^1$

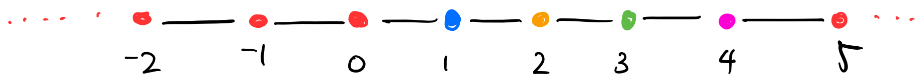
Explicit generator of $\pi_1(C_5) = [f]$ where

$$f : (I_\infty, I_{\geq r}) \to (C_5, v_0)$$

$$f(i) = \begin{cases} v_i & 0 \leq i < 5 \\ v_0 & \text{else} \end{cases}$$

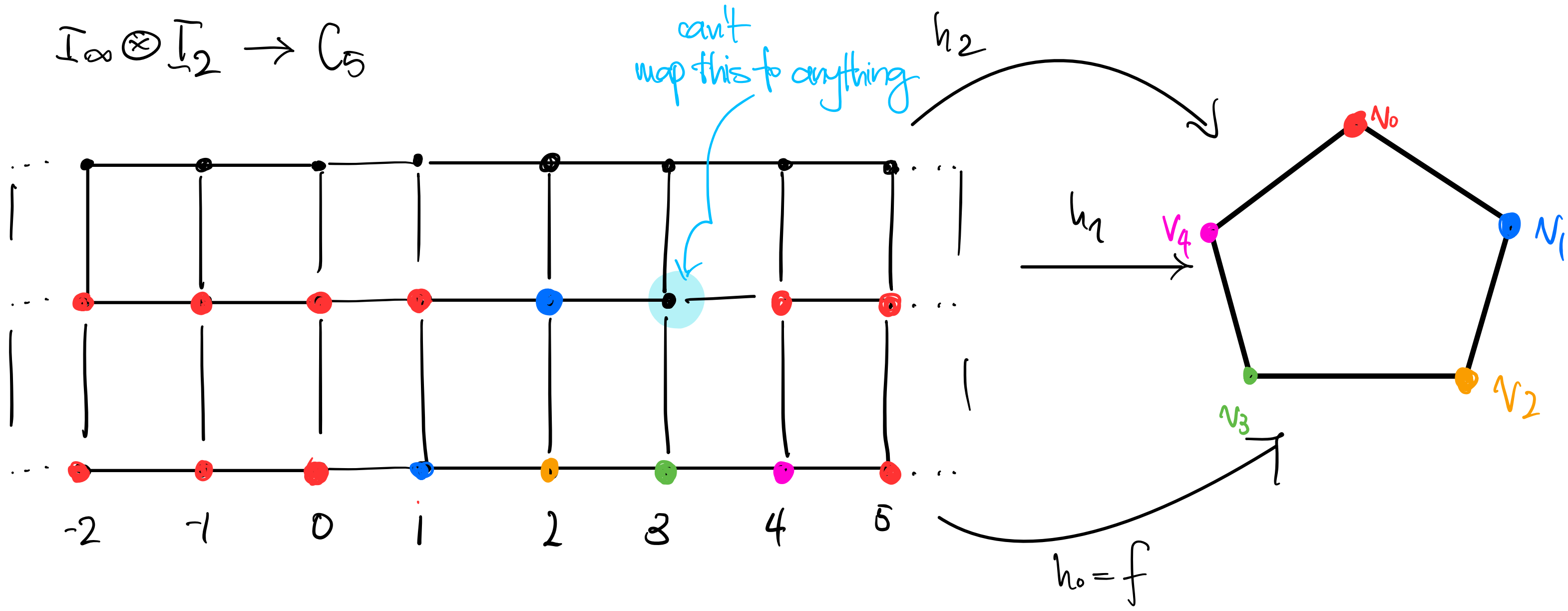


f :



$f \not\simeq c$ where c is the constant map.

$I_\infty \otimes I_2 \to C_5$



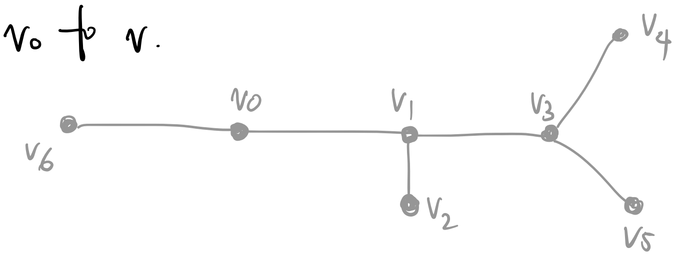
④ Trees

Let T be a finite tree (ie., no cycles).

unique because T is a tree.

Let $v_0 \in V$. For each $v \in V \setminus v_0$, let $p(v)$ denote the unique neighbour of v that lies on the simple path from v_0 to v.

Let $p(v_0) := v_0$.



ie $\varphi : T \to T$

$$\varphi(v) := \begin{cases} p(v) & \text{if } v \neq v_0 \\ v_0 & \text{if } v = v_0 \end{cases}$$ is a graph map.

Let $n := \operatorname{diam}(T) = \max\{ \#\text{ of edges between } v_i \text{ and } v_j : v_i, v_j \in V \}$

"longest path between two vertices of the tree"

Define $c : T \otimes I_n \longrightarrow T$ by

$$c(v,i) = \begin{cases} v & \text{if } i = 0 \\ \varphi\big(c(v,i-1)\big) & \text{if } 1 \leq i \leq n \end{cases}$$

"At each time step, take the next neighbour on the path to v_0"

$$c(v,1) = \varphi(v)$$
$$c(v,2) = \varphi\big(c(v,1)\big) = \varphi(\varphi(v))$$

<u>Claim:</u> c is a homotopy.

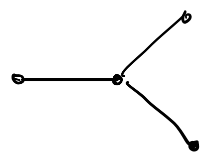
First, $c(-,0) = \mathrm{id}_T$ and

$$c(v,n) = \begin{cases} \varphi(c(v,n-1)) & \text{if } \varphi^{n-1}(v) \neq v_0 \\ v_0 & \text{if } \varphi^{n-1}(v) = v_0 \end{cases}$$

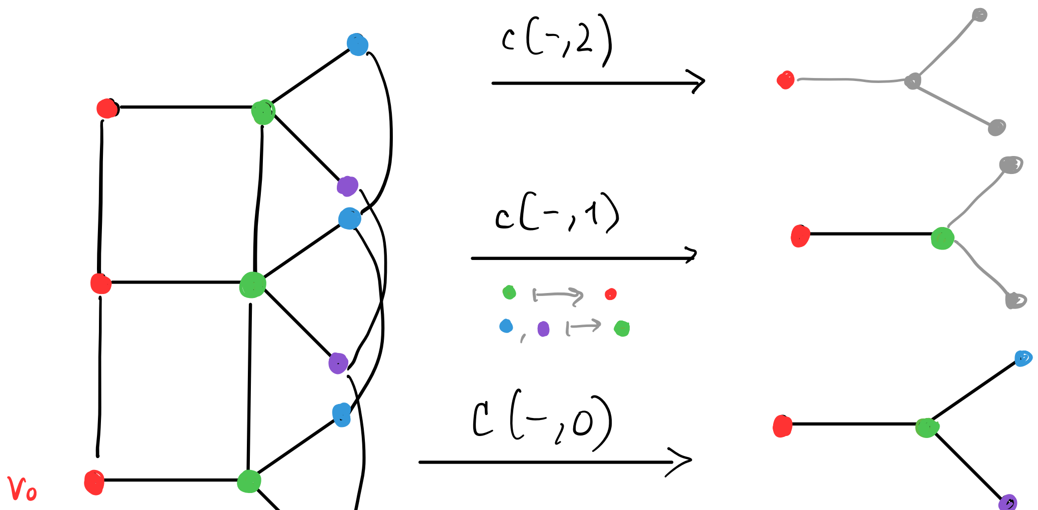
$= v_0$ because n is the diameter (each time you take one step closer to v_0 and you do this the maximum number of times by definition of the diameter of a tree)

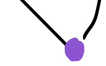
Now, we have to verify that c is a graph map.

<u>Motivating example:</u>

$T =$  $n = 2$

$T \otimes I_2$





$$c(v,1) = \varphi(c(v,0))$$

$$c(v,0) = v$$

$$\Rightarrow c(v,1) = \varphi(v) = \begin{cases} \rho(v) & \text{if } v \neq v_0 \\ v_0 & \text{if } v = v_0 \end{cases}$$

closest vertex on the path from v_0 to v_1.

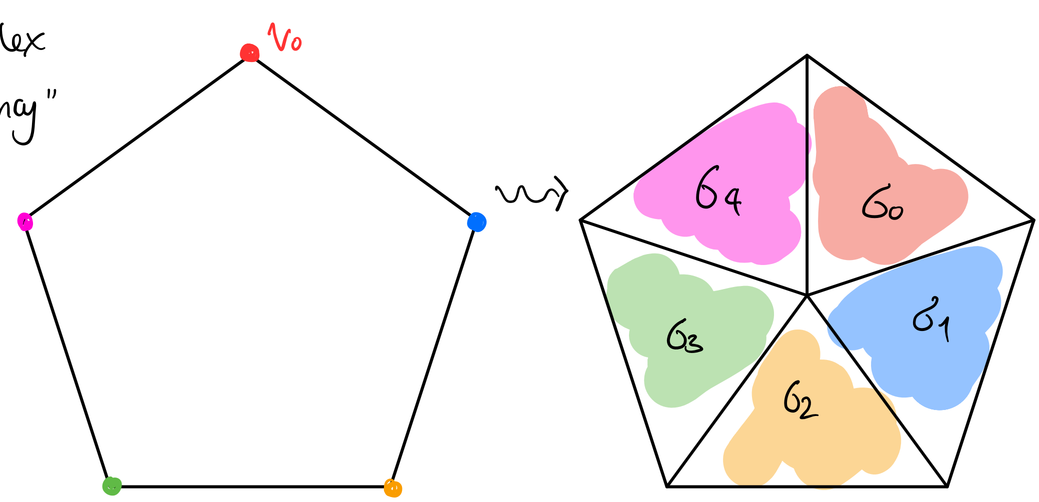
$$c(v,2) = \varphi(c(v,1))$$ = unique vertex from v_0 to v at the next step.

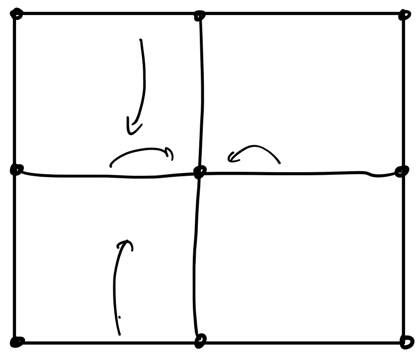
<u>In general</u>: c is either mapping a vertex to itself or to the nearest vertex at each step; so it must be a graph map.

<u>Remarks</u>:

- discrete homotopy theory ⇝ sometimes called A-homotopy theory.
- The theory for graphs is stated as a specific instance of the general theory for simplicial complexes.
  ⇝ Barcelo et al. introduce it in this way

eg C_5: A 2-dimensional complex where we regard "adjacency" of two 2-cells as sharing an edge (1-cell)





• When is a graph contractible?

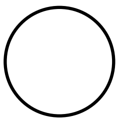
↳ characterization ?

<u>Challenge</u>: An example of:

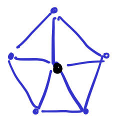
• A graph with only 3- or 4-cycles that is not contractible

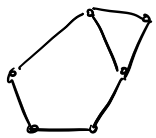
↳ If it only has 3-cycles, it is contractible?

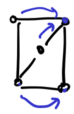
$C_5 \cong S^1$

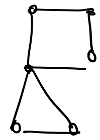


The presence of a 5-cycle doesn't characterize contractibility

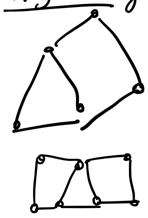




 $\simeq \bullet$



<u>Sharing an edge</u>:



$$\pi_1(G \otimes H) \overset{?}{=} \pi_1(G) \times \pi_1(H)$$

$$\pi_1(X) = \mathrm{HoTop}_*(S^1, X)$$

($S^1 \to X$ in homot. cat of spaces.)

• π_1 is represented by S^1.

<u>Q</u>: Is π_1 of graphs represented by something in the homot. cat?

<u>Intuition/conjecture</u>: no, it isn't

• cone = homotopy colimit

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d] & X \\ * & \end{tikzcd}$$