

The fundamental group of a graph

1. Precise definition
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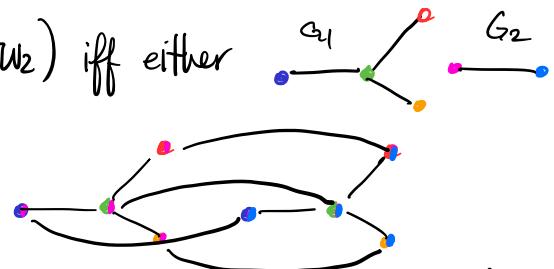
Sources:

- "Foundations of a connectivity theory for simplicial complexes" [Barcelo, Kramer, Laubenbacher, Weaver]
- "Higher discrete homotopy groups of graphs" [Lutz]

§ Precise definition

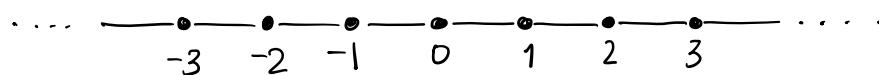
Recall: Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be simple graphs.

- The graph product $G_1 \otimes G_2$ is the graph with $V(G_1 \otimes G_2) = V_1 \times V_2$ and there is an edge between (v_1, v_2) and (w_1, w_2) iff either
 - $v_1 = w_1$ and $v_2 \sim w_2$
 - $v_2 = w_2$ and $v_1 \sim w_1$
- A graph map $f: G_1 \rightarrow G_2$ is a set map on vertices, $V_1 \rightarrow V_2$ such that if $v \sim w$ in E_1 , then either
 - $f(v) = f(w)$
 - $f(v) \sim f(w)$ in E_2 .



either contract edges down to a vertex or map edges according to where adjacent vertices are mapped.

- Let I_∞ denote the graph with vertex set $V(I_\infty) = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ and an edge $i \sim j$ iff $|i-j|=1$.



- Two graph maps $f, g: G_1 \rightarrow G_2$ are homotopic if for some $n \geq 0$, there is a graph map $h: G_1 \otimes I_n \rightarrow G_2$ such that
 - "a homotopy" ↗
 - $h(-, 0) = f$
 - and $h(-, n) = g$

Def Given $f: G_1 \rightarrow G_2$ and subgraphs $H_1 \subseteq G_1$, $H_2 \subseteq G_2$, write it as a **POINTED GRAPH MAP** $f: (G_1, H_1) \rightarrow (G_2, H_2)$ if $v \in H_1 \Rightarrow f(v) \in H_2$.

Let G be a graph and $v_0 \in V(G)$.

Consider I_∞ and a graph map $f: I_\infty \rightarrow G$. If there exists some $r > 0$ such that for all $i \in V(I_\infty)$ with $|i| > r$, $f(i) = v_0$, then call the smallest such r the RADIUS of f and so we have a pointed map

$$f: (I_\infty, I_{\succ}) \longrightarrow (G, \succ_0)$$

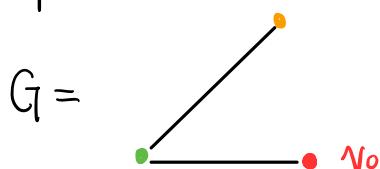
$$\text{where } r = \min \left\{ r_0 : f(i) = v_0 \quad \forall \quad |i| \geq r_0 \right\}$$

$$\text{and } I_{\geq r} = \left\{ i \in I_\infty : |i| \geq r \right\}$$

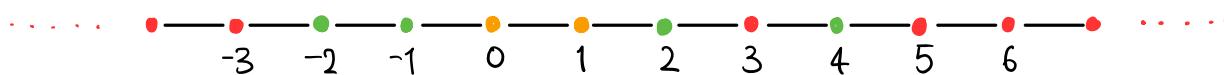
some denote
 $I_{\pi r}$ by ∂Z
(with r implicit)

From last time, think of this "radius" as being the length of the "active area" of the "loop" on G.

Example from last time:



Let $f: (I_\infty, I_{\text{irr}}) \rightarrow (G, v_0)$ be the map given by the following colouring:



What is the radius?

$$r=5$$

Def Graph maps $f, g: (I_\infty, I_{\geq r}) \rightarrow (G, v_0)$ are BASED HOMOTOPIC if there is a homotopy $h: I_\infty \otimes I_n \rightarrow G$ from f to g such that $h_i = h(-, i)$ is a graph map $h_i: (I_\infty, I_{\geq r}) \rightarrow (G, v_0) \quad \forall i$. Denote the homotopy class by $[f]$.

Remark: 2 maps f, g do not have to have the same radius to be homotopic. We choose the larger radius in the above definition. (so that the maps have a chance at being homotopic)

Def Let $v_0 \in G$. The set

$$\Pi_r(G) := \left\{ [f] : f: (I_\infty, I_{\geq r}) \rightarrow (G, v_0) \text{ a graph map} \right\}$$

is a group with "concatenation" as multiplication. i.e,

Let $f: (I_\infty, I_{\geq r_f}) \rightarrow (G, v_0)$ and $g: (I_\infty, I_{\geq r_g}) \rightarrow (G, v_0)$ be graph maps with radii r_f and r_g . Then,

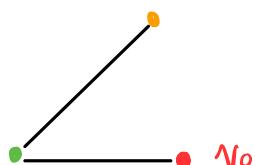
$$[f] \cdot [g] = [p] \quad \text{where}$$

$p: (I_\infty, I_{\geq r}) \rightarrow (G, v_0)$ is defined by

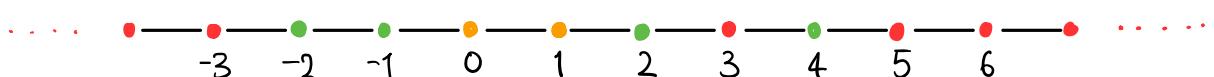
$$f \cdot g(i) = p(i) = \begin{cases} f(i) & \text{if } i \leq r_f \\ g(i - (r_f + r_g)) & \text{if } i > r_f \end{cases}$$

well-defined:
same as
topological
space: take
 $h_i \cdot h'_i$

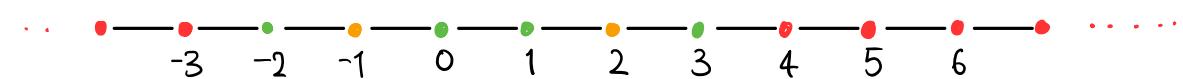
(eq) $G =$



$f:$



$g:$

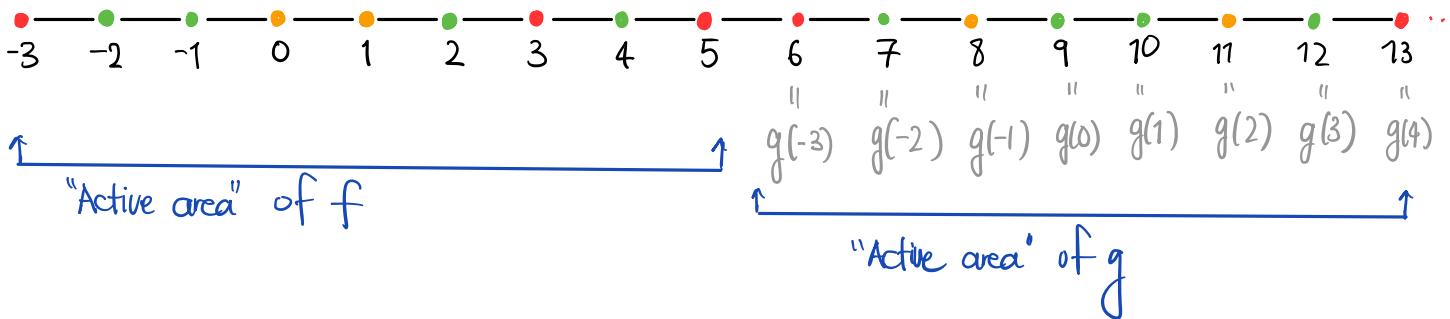


$$r_f = 5$$

$$r_g = 4$$

$$f \circ g(i) = \begin{cases} f(i) & \text{if } i \leq 5 \\ g(i-9) & \text{if } i > 5 \end{cases}$$

$f \circ g:$

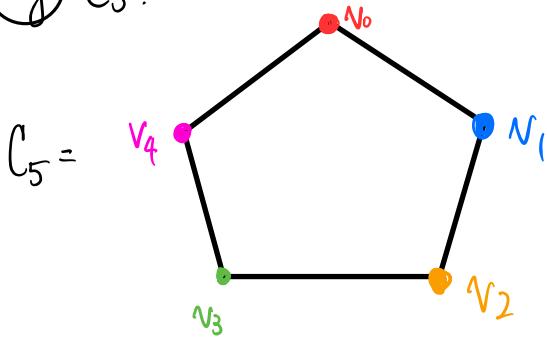


How do inverses work?

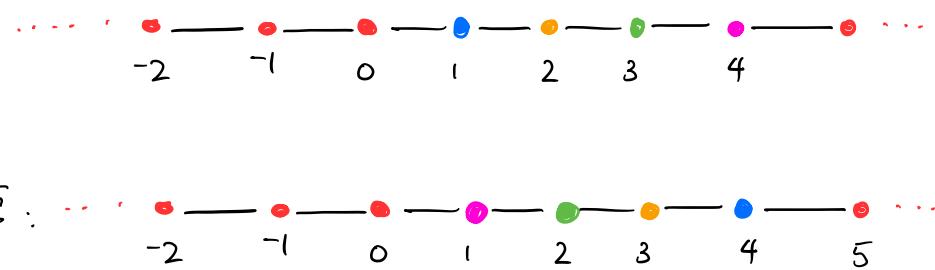
For $f: (I_\infty, I_{\geq r}) \rightarrow (G, v_0)$, $[f] \in \Pi_1(G, v_0)$, take $\bar{f}: (I_\infty, I_{\geq r}) \rightarrow (G, v_0)$ to be the loop f traversed in the opposite direction.

we obtain symmetry and homotope out from the "middle"

eg C_5 .



$f :$

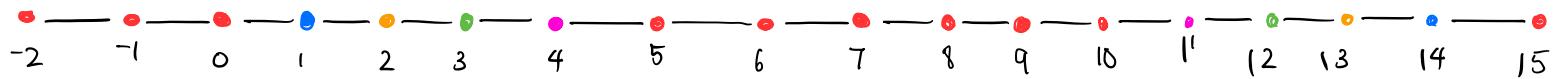


$\bar{f} :$



$$f \cdot \bar{f}(i) = \begin{cases} f(i) & \text{if } i \leq 5 \\ \bar{f}(i-10) & \text{if } i > 5 \end{cases}$$

$f \cdot \bar{f} :$



$h : I_\infty \otimes I_5 \rightarrow G$

whenever there is
a vertical/horizontal edge,
those vertices must be adjacent

$I_\infty \otimes I_5$



\bar{f}

$f \cdot \bar{f} = h_0$

h_1

h_2

h_3

h_4

$C_{n_0} = h_5$

$\approx h_{rf}$

In general...

What is the homotopy $h: I_\infty \otimes I_{r_f} \rightarrow G$?

Let $h_i = f_i \circ \bar{f}_i$ where

$$f_i(j) = \begin{cases} f(j) & \text{if } 0 \leq j \leq r_f - i \\ f(r_f - i) & \text{if } r_f - i < j \leq r_f \\ v_0 & \text{else} \end{cases}$$

\Rightarrow similar to topological spaces : f_i is equal to f on $[0, r_f - i]$ and stationary at $f(r_f - i)$ on the remaining interval, then concatenate it with \bar{f}_i to get h_i

§ Homotopy invariance

Prop A based G -homotopy equivalence of based graphs induces an isomorphism of graph homotopy groups.

\hookrightarrow [Barcelo, Kramer, Laubenbacher, Weaver]

Proof The proof is "identical" to the corresponding proof for homotopic maps of topological spaces.

First, we define the induced group homomorphism of a based graph map.

Suppose $f: (G_1, v_1) \rightarrow (G_2, v_2)$ is a based graph map. Then, define

$$f_*: \pi_1(G_1, v_1) \longrightarrow \pi_1(G_2, v_2) \quad \text{by}$$

$$f_*: [\alpha] \longmapsto [f \circ \alpha]$$

where $\alpha: (I_\infty, I_{r_f}) \rightarrow (G_1, v_1)$.

• f_* is a well-defined group homomorphism and

$$(f \circ g)_* = f_* \circ g_*$$

(shift so that
 $f \circ \bar{f}$ "starts"
at $j=0$)

If $\alpha \sim \beta$, then
 $f \circ \alpha \sim f \circ \beta$

folk (compare with homotopy)

Now, let $f: (G_1, v_1) \rightarrow (G_2, v_2)$ be a homotopy equivalence, with homotopy inverse $g: (G_2, v_2) \rightarrow (G_1, v_1)$.

Then,

$$f \circ g \simeq \text{id}_{G_2}$$

$\Rightarrow f \circ g$ is an isomorphism (take homotopy from $f \circ g$ to id_{G_2} and take the path traced out by the basepoint)

Also,

$$\Rightarrow (f \circ g)_* = f_* \circ g_* \text{ is an isomorphism}$$

$\Rightarrow g_*$ is injective, f_* is surjective.

Then, use $g \circ f \simeq \text{id}_{G_1}$ to get that g_* is surjective, f_* is injective

$\Rightarrow f_*, g_*$ isomorphisms, so

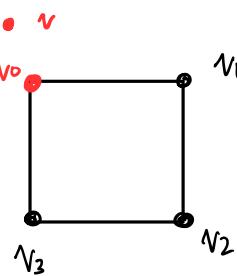
$$\pi_1(G_1, v_1) \cong \pi_1(G_2, v_2)$$

□

Examples

① Let G_1 :

$$C_4 = G_2:$$



$$f: G_1 \rightarrow C_4$$

$$v \mapsto v_0$$

↗ homotopy from identity

Claim: f is a homotopy equivalence (ie, C_4 is contractible) ↗ constant map

$$\text{Let } g: C_4 \xrightarrow{!} G_1$$

↪ homotopy equiv + pt

$$\text{Then, } gf: G_1 \rightarrow G_1$$

$$\begin{aligned} gf(v) &= g(v_0) \\ &= v \end{aligned}$$

$$\Rightarrow gf = \text{id}_{G_1}$$

$$\underline{\text{want to show: }} fg \simeq \text{id}_{C_4}$$

$$fg: C_4 \rightarrow C_4, \quad \text{id}_{C_4}: C_4 \rightarrow C_4$$

Let $h: C_4 \otimes I_2 \rightarrow C_4$ be defined by

$$h|_{C_4 \otimes \{0\}} = \text{id}_{C_4}$$

$$h|_{C_4 \otimes \{1\}} = \varphi \quad \text{where } \varphi(v_0, 1) = \varphi(v_3, 1) = v_0$$

$$\varphi(v_1, 1) = \varphi(v_2, 1) = v_1$$

And on $C_4 \otimes \{2\}$:

$$h(v_i, 2) = v_0 \quad \forall i.$$

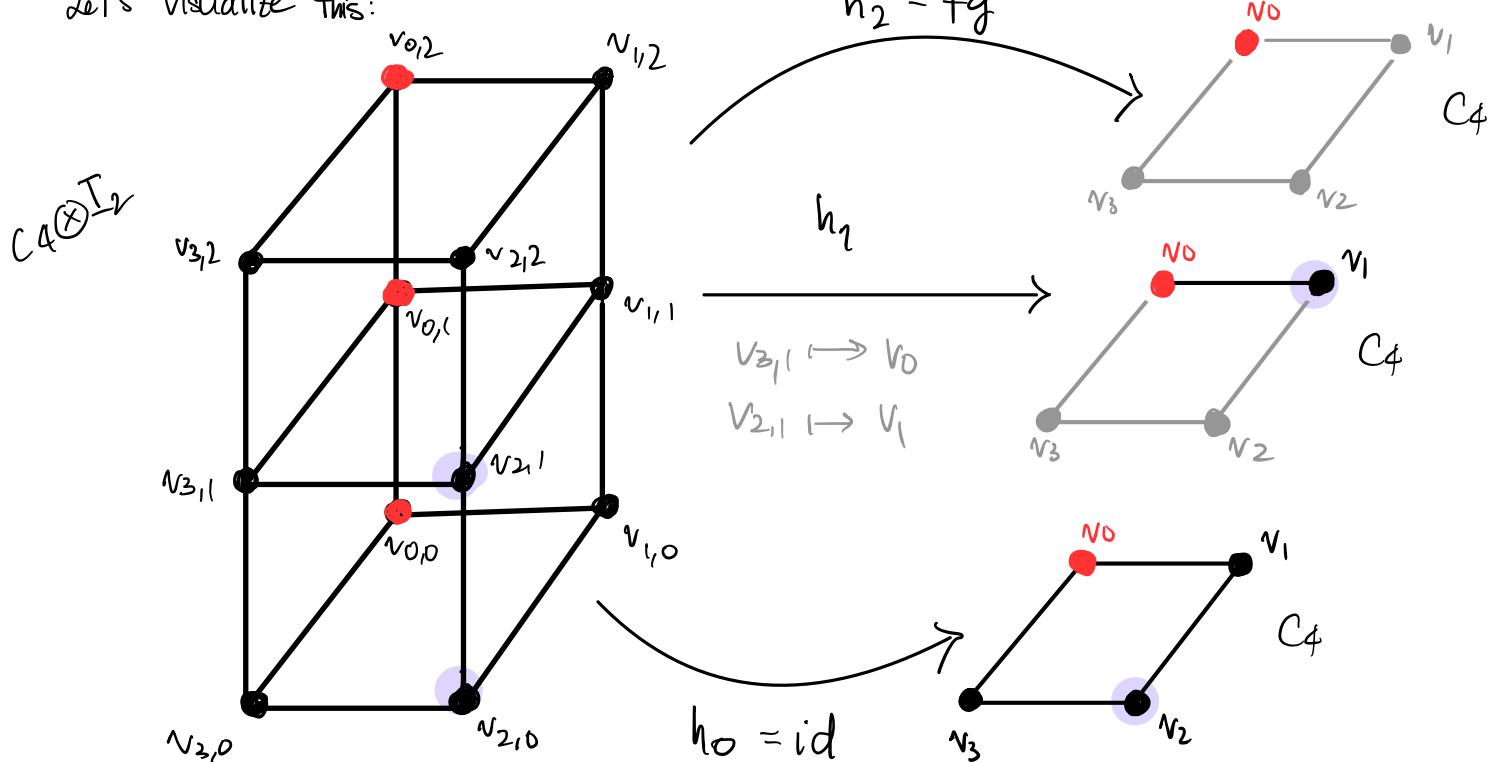
$$fg(v_i) = f(v_0) = v_0$$

$$\text{Notice that } h(-, 0) = \text{id}_{C_4}$$

$$h(-, 2) = fg$$

To show that h is a homotopy, it remains to show that h is a graph map.

Let's visualize this:



$$\text{Denote } (v_{i,j}) = v_{i-j}$$

\Downarrow

$$C4 \otimes I_2$$

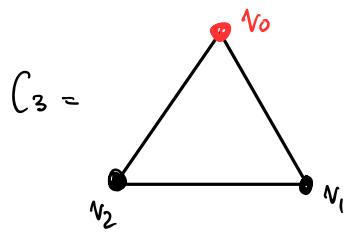
Then, h is a graph map. (edges either incident to mapped vertices or contracted down to a vertex.)

$\Rightarrow h$ is a homotopy.

$$\Rightarrow fg \simeq \text{id}_{C4}$$

$$\text{Thus, } \pi_1(C4, v_0) \cong \pi_1(\bullet, v) = \{e\}$$

② Similar to the above example, $C_3 \simeq *$



$$f: * \rightarrow C_3$$

$$v \mapsto v_0$$

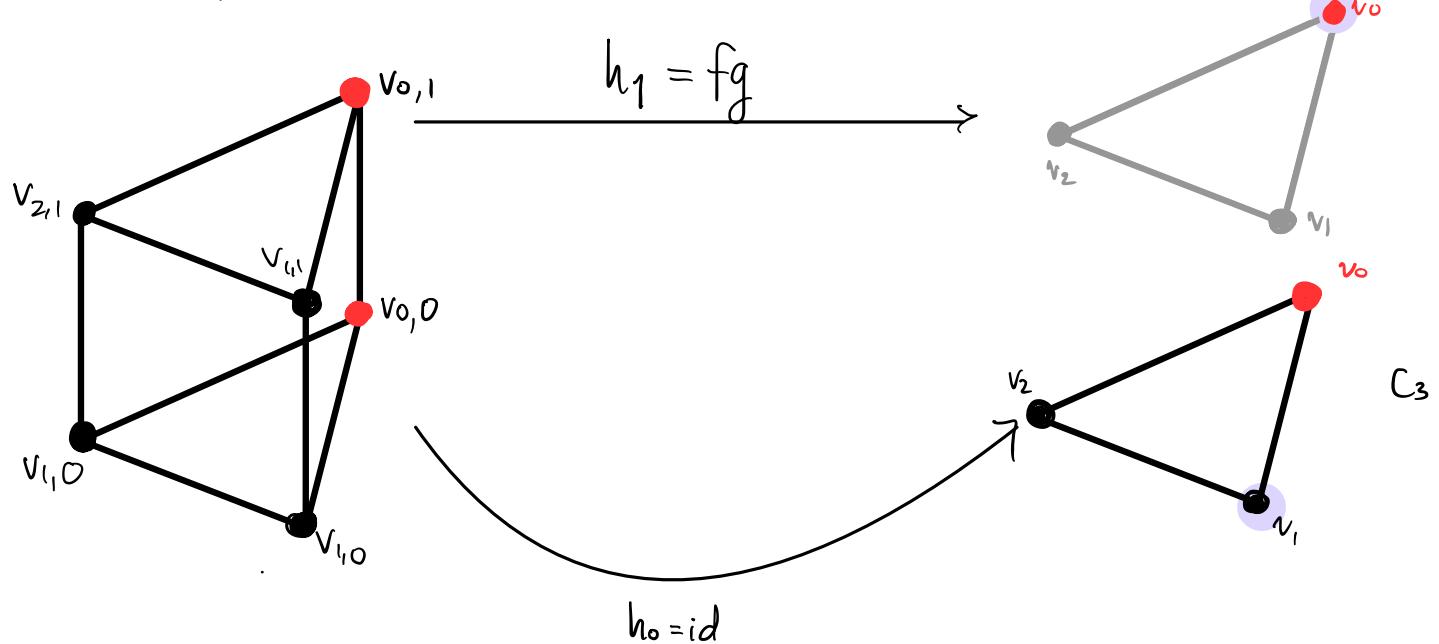
$$fg(v_i) = f(*) = v_0$$

$$g: C_3 \rightarrow *$$

$$gf = id_*$$

and $fg \simeq id_{C_3}$ via

$$h: C_3 \otimes I_1 \rightarrow C_3$$



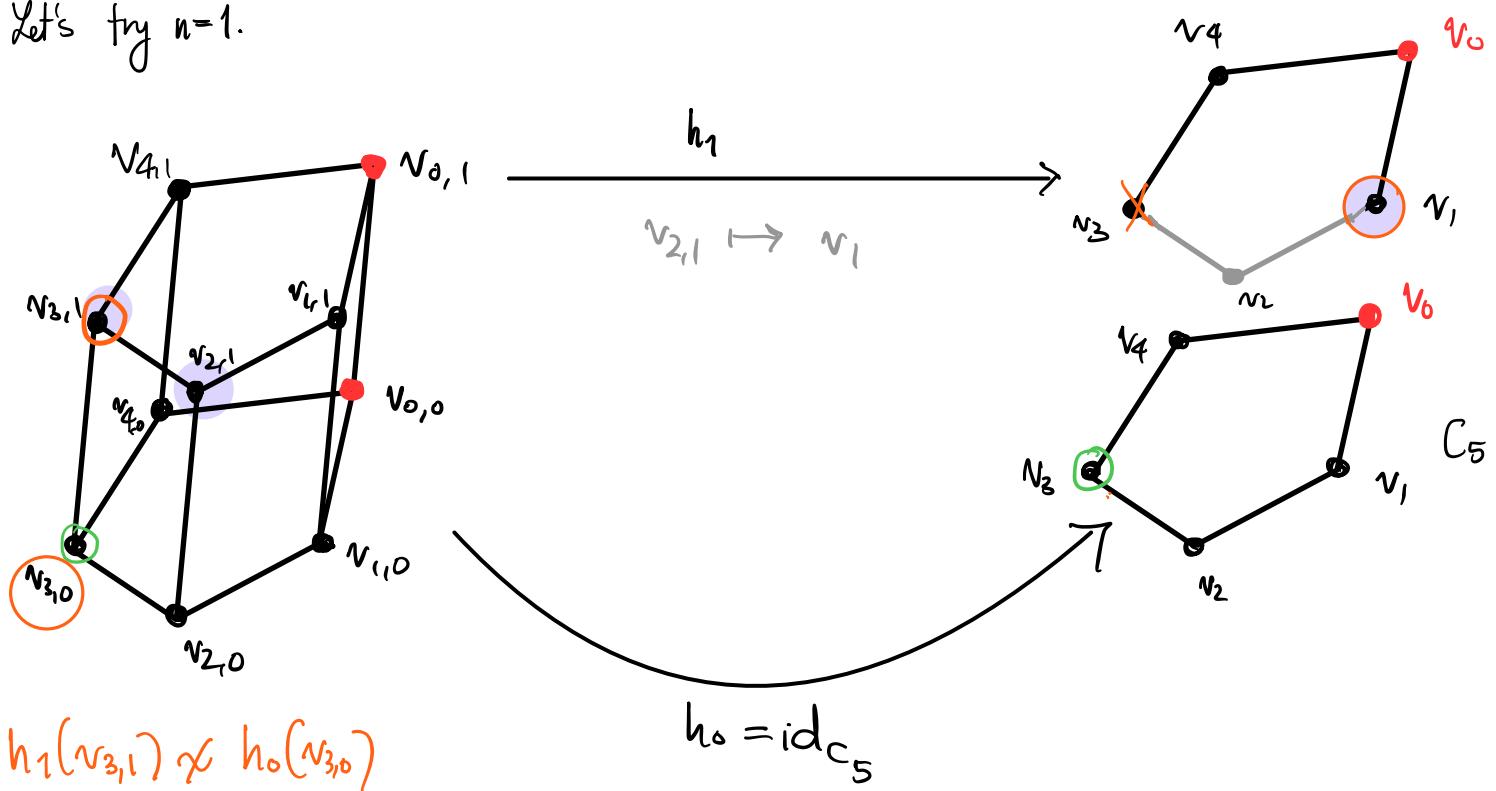
③ $C_5 \neq *$

Fix some $n > 0$. Let $h: C_5 \otimes I_n \rightarrow C_5$ be such that

$$h(-, 0) = \text{id}_{C_5}$$

Claim: $h(-, i)$ is surjective for all $1 \leq i \leq n$.

Let's try $n=1$.



↙ not possible because $v_{3,1} \sim v_{2,1}$

But $h_1(v_{3,1}) \neq h_1(v_{2,1})$

Some phenomenon occurs in higher dimensions \rightsquigarrow cannot contract an edge down.

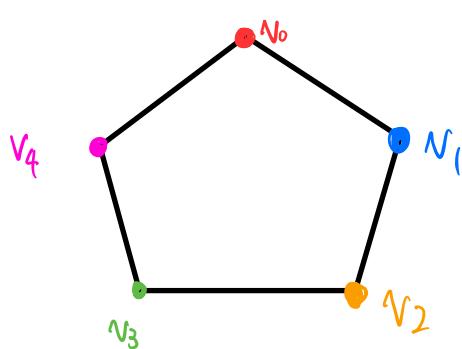
Remark: It can be shown that $\pi_1(C_5) \cong \mathbb{Z}$

\rightsquigarrow construct X_{C_5} (cell complex) by regarding C_5 as a 1-complex
 $\Rightarrow X_{C_5} \cong S^1$

Explicit generator of $\pi_1(C_5) = [f]$ where

$$f: (I_\infty, I_{\geq r}) \rightarrow (C_5, v_0)$$

$$f(c_i) = \begin{cases} v_i & 0 \leq i < 5 \\ v_0 & \text{else} \end{cases}$$

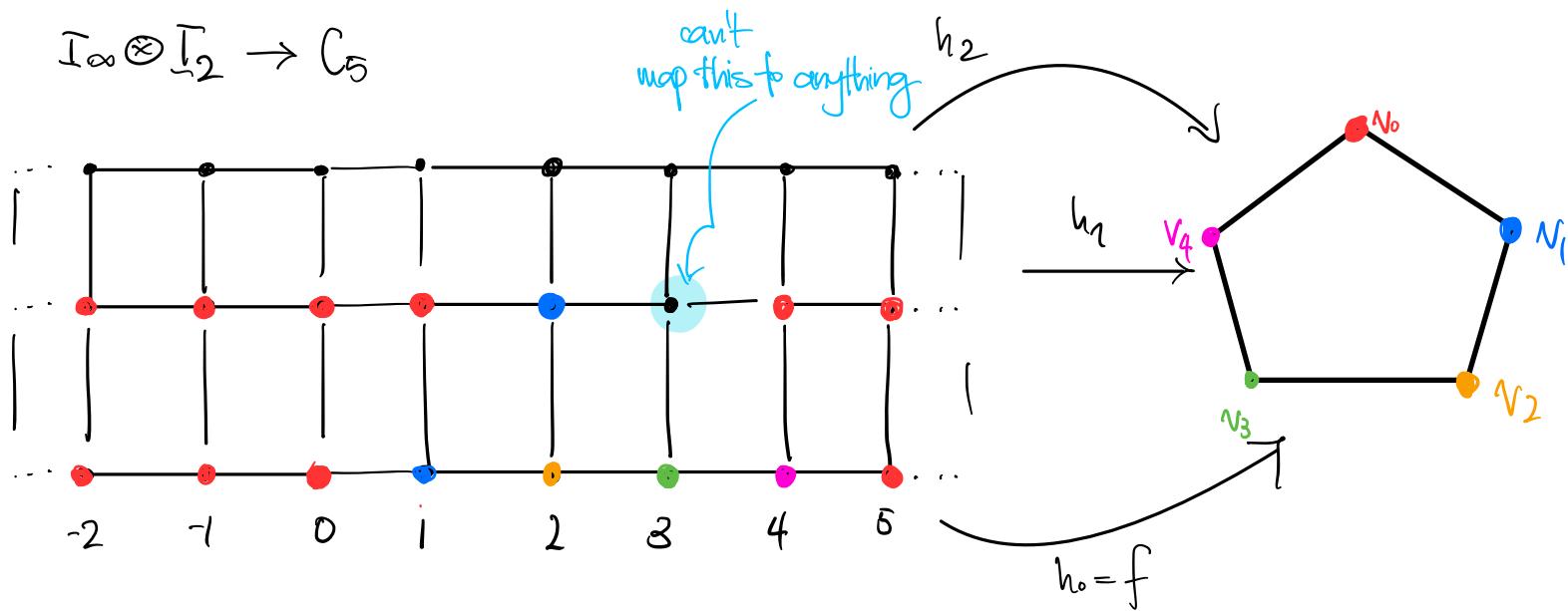


$f :$

A horizontal number line with tick marks at integer intervals from -2 to 5. Red dots are placed at every integer from -2 to 4. A blue dot is at 1. An orange dot is at 2. A green dot is at 3. A magenta dot is at 4. The number 5 is written below the tick mark for 5.

$f \neq c$ where c is the constant map.

$$I_\infty \circledast I_2 \rightarrow C_5$$



④ Trees

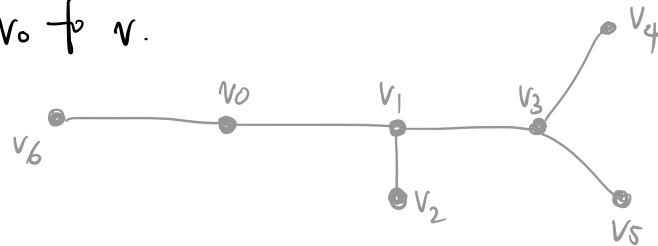
Let T be a finite tree (ie., no cycles).

Let $v_0 \in V$. For each $v \in V \setminus v_0$, let $p(v)$ denote the unique neighbour of v that lies on the simple path from v_0 to v .

Let $p(v_0) := v_0$.

(1e) $\varphi: T \rightarrow \overline{T}$

$$\varphi(v) := \begin{cases} p(v) & \text{if } v \neq v_0 \\ v_0 & \text{if } v = v_0 \end{cases} \quad \text{is a graph map.}$$



Let $\eta := \text{diam}(T) = \max \left\{ \begin{array}{l} \text{"# of edges between } v_i \text{ and } v_j : v_i, v_j \in V \\ \text{"longest path between two vertices of the tree"} \end{array} \right\}$

Define $c: T \otimes I_n \rightarrow T$ by

$$c(v, i) = \begin{cases} v & \text{if } i = 0 \\ \varphi(c(v, i-1)) & \text{if } 1 \leq i \leq n \end{cases}$$

"At each time step, take the next neighbour on the path to v_0 "

Claim: c is a homotopy.

First, $c(-, 0) = \text{id}_T$ and

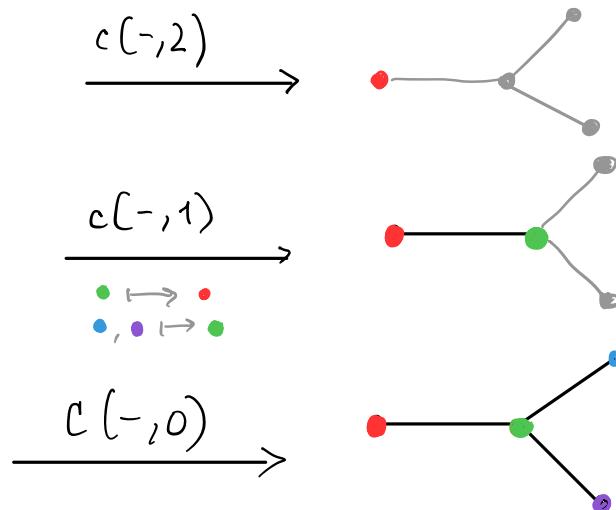
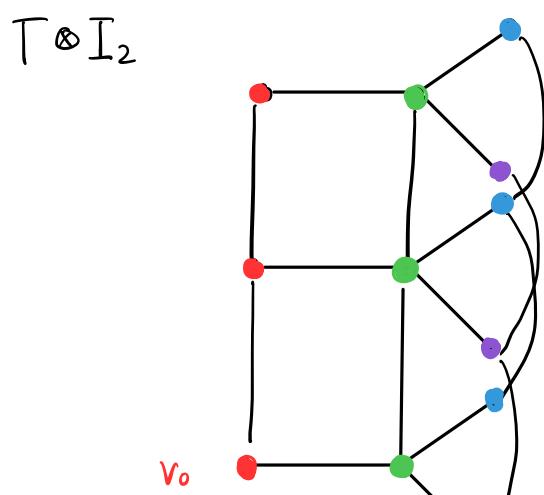
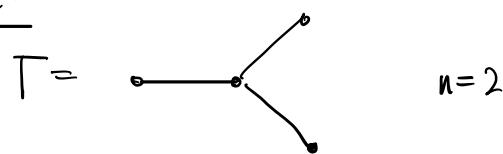
$$\begin{aligned} c(v, 1) &= \varphi(v) \\ c(v, 2) &= \varphi(\varphi(v)) \\ &= \varphi(\varphi(v)) \end{aligned}$$

$$c(v, n) = \begin{cases} \varphi(c(v, n-1)) & \text{if } \varphi^{n-1}(v) \neq v_0 \\ v_0 & \text{if } \varphi^{n-1}(v) = v_0 \end{cases}$$

$= v_0$ because n is the diameter (each time you take one step closer to v_0 and you do this the maximum number of times by definition of the diameter of a tree)

Now, we have to verify that c is a graph map.

Motivating example:



$$c(v, 1) = \varphi(c(v, 0))$$

$$c(v, 0) = v$$

$$\Rightarrow c(v, 1) = \varphi(v) \quad \text{closest vertex on the path from } v_0 \text{ to } v_1.$$

$$= \begin{cases} p(v) & \text{if } v \neq v_0 \\ v_0 & \text{if } v = v_0 \end{cases}$$

$$c(v, 2) = \varphi(c(v, 1)) = \text{unique vertex from } v_0 \text{ to } v \text{ at the next step.}$$

In general: c is either mapping a vertex to itself or to the nearest vertex at each step; so it must be a graph map.

Remarks:

- discrete homotopy theory \rightsquigarrow sometimes called A-homotopy theory.
- The theory for graphs is stated as a specific instance of the general theory for simplicial complexes.
 \rightsquigarrow Barcelo et al. introduce it in this way

C_5 : A 2-dimensional complex

(eq)
where we regard "adjacency"

of two 2-cells as

sharing an edge (1-cell)

