EEN1034 - Digital Signal Processing (Digital Filters and DFT)

Acknowledgment
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those given by
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- Introduction to the Discrete Time Fourier Transform (DTFT)
- Magnitude/Phase spectrum
- Energy spectral density
- Properties of the DTFT
- Analysis of digital filters using the DTFT

Discrete-Time Fourier Transform

Definition: The Discrete Time Fourier Transform of a digital signal $x[n]$ is given by:

$$X(jf) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n T_s}$$

$\omega = 2\pi f$ so the Fourier transform can also be written as:

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n T_s}$$

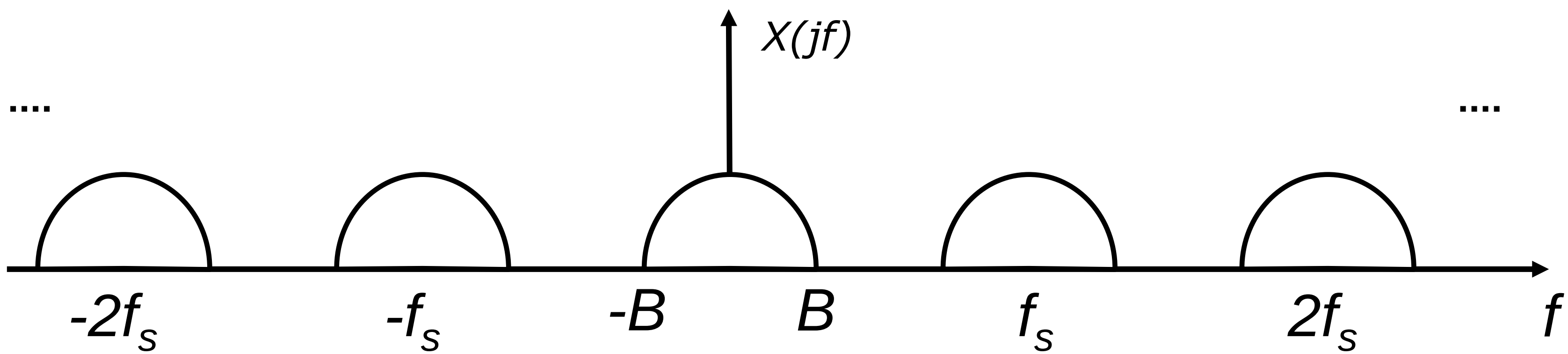
If the signal has a finite duration, the summation is finite. It is infinite for a signal that has an infinite duration.

Discrete-Time Fourier Transform

Let $x[n]$ be the discrete signal obtained from samples of an analog signal $x_a(t)$.
The spectrum of the discrete signal is related to that of the analog signal.

$$X(jf) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(j(f - kf_s)) = \sum_{k=-\infty}^{\infty} x[n] e^{-j2\pi f n T_s}$$

It is a replication of the analog spectrum but scaled by $\frac{1}{T_s}$



Discrete-Time Fourier Transform

In order for the DTFT to exist, the discrete-time signal should be absolutely summable. That is:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Discrete-Time Fourier Transform

E.g.: What is the Discrete-time Fourier Transform of $x[n] = (0.45)^n u[n]$?

$$\sum_{n=0}^{\infty}(0.45)^n e^{-j\omega n T_s} = \sum_{n=0}^{\infty}\left(0.45e^{-j\omega T_s}\right)^n$$

$$X(j\omega) = \sum_{n=0}^{\infty}\left(0.45e^{-j\omega T_s}\right)^n = \frac{1}{1 - 0.45e^{-j\omega T_s}}$$

Spectrum of Discrete Signals

The DTFT, $X(j\omega)$, is periodic with a period, $\frac{2\pi}{T_s}$.

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(f - k f_s)$$

The DTFT defined above is also known as the spectrum of $x[n]$. Since $X(j\omega)$ is complex, there are two parts to the spectrum - the magnitude spectrum and phase spectrum:

Spectrum of discrete signals

Example

What is the magnitude spectrum of the following?

$$X(j\omega) = \frac{1}{1 - 0.45e^{-j\omega T_s}}$$

Expand $X(j\omega)$ and find the magnitude of the resulting complex function.

$$X(j\omega) = \frac{1}{(1 - 0.45cos\omega T_s) + j0.45sin\omega T_s}$$

$$|X(j\omega)| = \frac{1}{\sqrt{1.2025 \ - 0.9cos\omega T_s}}$$

Inverse Fourier Transform

To find the original sequence if $X(j\omega)$ is known, one uses the inverse Fourier transform for discrete signals.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\omega)e^{jn\omega T_s}\, d(\omega T_s)$$

What is the inverse transform of $X(j\omega) = 0.25e^{-j\omega T_s}$?

$$x[n] = \frac{0.25}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega T_s}e^{j\omega n T_s}\, d(\omega T_s) = \frac{0.25}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-1)T_s}\, d(\omega T_s)$$

$$x[n] = 0.25\,\frac{\sin(n-1)\,\pi}{(n-1)\pi}$$

$$x[n] = \{0, 0.25, 0, 0\ , 0\ , \ldots\} \ \text{ for } \ n = \{0, 1, 2, 3, 4, \ldots.\}$$

Energy Spectral Density

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(j\omega)|^2 \, d(\omega T_s)$$

$|X(j\omega)|^2$ represents the distribution of energy as a function of frequency. Hence, the energy density spectrum of a discrete energy signal $x[n]$ is given by:

$$S_{xx}(\omega) = |X(j\omega)|^2 = X(j\omega)X^*(j\omega)$$

Example

Find the energy spectral density of the discrete energy signal $x[n] = (0.5)^n u(n)$?

$$S_{xx}(\omega) = X(j\omega)X^*(j\omega) = \frac{1}{(1-0.5e^{-j\omega T_s})(1-0.5e^{j\omega T_s})}$$

$$= \frac{1}{1.25 - cos\omega T_s}$$

DTFT and z-Transform

The z-Transform of a DT signal $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

If $z = e^{j\omega T_s}$, $X(z)$ reduces to $X(j\omega)$. So

$$X(j\omega) = X(z = e^{j\omega T_s})$$

If one knows the z-transform of a signal, one can find the spectrum and spectral densities.

Properties of the DTFT

Since the DTFT can be assumed to be a special case of z-Transform, the DTFT possesses the majority of the properties of z-Transform.

Linear Property:

$$\mathcal{F}(a_1 x_1[n] + a_2 x_2[n]) = a_1 \mathcal{F}(x_1[n]) + a_2 \mathcal{F}(x_2[n])$$

Time Shifting Property: If

$$x[n] \leftrightarrow X(\omega)$$

$$x[n-k] \leftrightarrow X(j\omega)e^{-j\omega k T_s}$$

Properties of the DTFT

Time Reversal Property:

$$x[n] \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$$

then

$$x[-n] \leftrightarrow X(-j\omega)$$

Convolution Property:

$$x_1[n] \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega)$$

$$x_2[n] \overset{\mathcal{F}}{\leftrightarrow} X_2(j\omega)$$

$$x[n] = x_1[n] * x_2[n] \overset{\mathcal{F}}{\leftrightarrow} X(j\omega) = X_1(j\omega)X_2(j\omega)$$

Properties of the DTFT

Frequency shifting property:

$$x[n] \leftrightarrow X(j\omega)$$

$$e^{j\omega_0 n T_s} x[n] \leftrightarrow X(j(\omega - \omega_0))$$

Frequency Response of a Discrete-time Filter

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

The output, y[n], of a linear time invariant (LTI) system is:

$$y[\mathrm{n}] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Based on the convolution property of the DTFT,

$$y[n] = x[n] * h[n] \leftrightarrow Y(j\omega) = X(j\omega)H(j\omega)$$

Frequency Response of a Digital Filter

The frequency response of a digital filter is defined as
$H(j\omega)$

$$H(j\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n T_s}$$

If the digital filter is stable, $h[n]$ is absolutely summable. Hence $H(\omega)$ always exists for stable filters.

Example

Find the frequency response of the following simple digital filter:
$y[n] = 0.5x[n] + 0.5x[n-1]$?

$H(z) = 0.5 + 0.5z^{-1}$.

$$H(j\omega) = 0.5e^{j\omega 0} + 0.5e^{-j\omega T_s} = 0.5(1 + e^{-j\omega T_s})$$

Frequency Response of a Digital Filter

If $x[n] = Ae^{j\omega_0 nT_s}$ is input to a digital filter, the steady-state output can be given by

$$y[n]= A|H(j\omega_0)|\ e^{j(\omega_0 nT_s+\angle H(\omega_0))}$$

Frequency Response of a Digital Filter

$$x[n] = \cos(\omega_0 n T_s) \longrightarrow \boxed{\text{DSP}} \longrightarrow y[n] = \alpha \cos(\omega_0 n T_s + \beta)$$

Observations

a) DSP does not change the frequency, ω_0

b) The filter alters the amplitude of the sinusoidal input: $1 \rightarrow \alpha$

c) The filter introduces a phase: β

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are dependent on the filter coefficients and $\boldsymbol{\omega_0}$.

Frequency Response of a Digital Filter

Example

Determine the steady-state output of the system

$$y[n] = 0.9y[n-1] + 0.1\,x[n]$$

The input is

$$x(t) = \cos(20t)\,, T_s = 0.05, x[n] = \cos(n)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.1z}{z - 0.9}$$

$$z = e^{j\omega T_s} = e^{j1}$$

$$H\left(z=e^{j}\right)=\frac{Y(z)}{X(z)}=\frac{0.1}{1-0.9e^{-j}}$$

$$=\frac{0.1}{1-0.9(\cos(1)-jsin(1))}$$

$$\left|H\left(z=e^{j}\right)\right|=1.0927, \angle H\left(z=e^{j}\right)=-0.9748$$

$$y[n]=1.0927\cos(n-0.9748)$$

Summary

If $x[n]$ is a discrete-time (DT) signal, its DTFT is defined as

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega T_s}, \qquad -\pi \leq \omega T_s \leq \pi$$

$X(j\omega)$ is periodic with a period of 2π

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(j(\omega - 2k\pi f_s))$$

$X_a(j\omega)$ is the Fourier transform of the underlying analog signal

$|X(j\omega)|$ is the magnitude spectrum of $x[n]$
$\angle X(j\omega)$ is the phase spectrum of $x[n]$

Summary

One can characterise digital filters in the frequency domain by obtaining the DTFT of the unit sample response, h[n].

$$\mathrm{H}(j\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega T_s}, \qquad -\pi \leq \omega T_s \leq \pi$$

$\mathrm{H}(j\omega)$ is periodic with a period of 2π.

$|H(j\omega)|$ is the magnitude spectrum of $h[n]$
$\angle H(j\omega)$ is the phase spectrum of $h[n]$

The frequency response of the output of the filter is
$Y(j\omega) = H(j\omega)X(j\omega)$