

Improving the Pole Relocating Properties of Vector Fitting

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Abstract—This paper describes a modification of the vector fitting (VF) procedure for rational function approximation of frequency-domain responses. The modification greatly improves the ability of VF to relocate poles to better positions, thereby improving its convergence performance and reducing the importance of the initial pole set specification. This is achieved by replacing the high-frequency asymptotic requirement of the VF scaling function with a more relaxed condition. Calculated results demonstrate a major improvement of performance when fitting responses that are contaminated with noise. The procedure is also shown to be advantageous for wideband modeling of transmission lines, network equivalents, and transformers.

Index Terms—Macromodel, rational approximation, system identification, vector fitting (VF).

I. INTRODUCTION

VECTOR FITTING (VF) [1], [2] has become a popular tool for system identification of linear systems in the frequency domain. The application has typically been the modeling of devices and subsystems for the purpose of transient analysis in power systems [3]–[5] and signal integrity characterization of microwave systems [6], [7]. VF has also been used for shielding analysis in electromagnetic-compatibility (EMC) studies [8], Green's functions representation [9], and optimal sample calculations [10]. VF is essentially a robust reformulation of the Sanathanan–Koerner iteration [11] using rational basis functions (partial fractions) instead of polynomials and pole relocation instead of weighting [12]. In addition, VF gives a fitting with guaranteed stable poles, is directly applicable to multiterminal systems, and a computer code is freely available [13]. New formulations of VF have been developed that utilize time-domain responses [14] and frequency derivatives [15].

VF is based on iteratively relocating an initial pole set to better locations. When fitting the frequency-domain response of a rational function using the correct order, the poles can often be relocated to their final locations in a single step with near machine precision. However, in practical applications, one will fit the response using a lower order function and it then turns out that a few iterations are needed. The situation may deteriorate when the frequency response contains a nonrational element (for instance, noise), and the convergence of VF can, in some cases, even stall. In this paper, it is shown that the convergence properties can be greatly improved by making a small modification to VF. The formulation of VF involves a scaling function

$\sigma(s)$ which approaches unity at high frequencies. It is shown that this high-frequency asymptotic condition can have a very undesirable effect on the convergence. This problem is overcome by replacing the asymptotic condition by a more relaxed condition which only serves to produce a nontrivial solution for the least-squares (LS) problem. Calculated results demonstrate that the modification provides major improvements to the convergence properties.

II. VECTOR FITTING

We start by reviewing the original formulation of VF. The objective is to approximate a frequency response $f(s)$, (generally, a vector; hence, the designation VF) with a rational function

$$f(s) = \sum_{m=1}^N \frac{r_m}{s - a_m} + d + se \quad (1)$$

where the terms d and e are optional. As explained in [1], [2], the VF first identifies the poles of $f(s)$ by solving in the least-squares sense, the linear problem

$$\sigma(s)f(s) = p(s) \quad (2)$$

where

$$\sigma(s) = \sum_{m=1}^N \frac{\tilde{r}_m}{s - q_m} + 1 \quad (3)$$

$$p(s) = \sum_{m=1}^N \frac{r_m}{s - q_m} + d + se \quad (4)$$

where $\sigma(s)$ is a scalar while $p(s)$ is generally a vector, and $\{q_m\}$ is a set of initial poles. (All poles and residues in (3) and (4) are real or come in complex conjugate pairs while d and e are real). It can then be shown [1] that the poles of $f(s)$ must be equal to the zeros of $\sigma(s)$ which can be calculated as the eigenvalues of a matrix [16, p. 612]

$$\{a_m\} = \text{eig}(\mathbf{A} - \mathbf{b} \cdot \mathbf{c}^T). \quad (5)$$

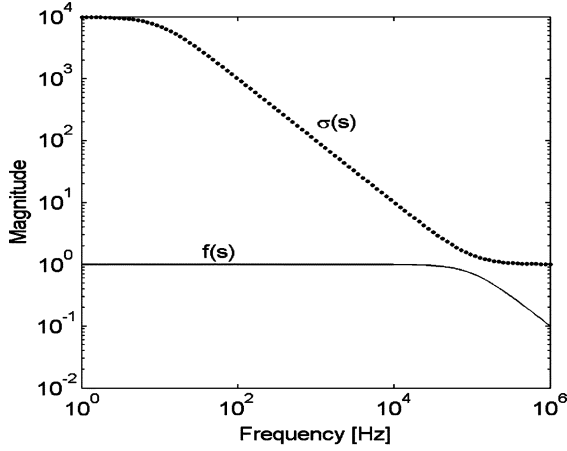
In (5), \mathbf{A} is a diagonal matrix holding the initial poles $\{q_m\}$, \mathbf{b} is a column vector of ones, and \mathbf{c}^T is a row vector holding the residues $\{\tilde{r}_m\}$.

This procedure can be applied in an iterative manner where (2)–(5) are solved repeatedly with the new poles $\{a_m\}$ replacing the previous poles $\{q_m\}$. This pole relocation procedure usually converges in 2–3 iterations. After the poles have been identified, the residues of (1) are finally calculated by solving the corresponding LS problem with known poles.

Manuscript received March 28, 2005; revised June 11, 2005. Paper no. TPWRD-00174-2005.

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Digital Object Identifier 10.1109/TPWRD.2005.860281

Fig. 1. Rational approximation of $f(s)$, No noise.

III. CONVERGENCE PROBLEM

The rationale of the pole identification of VF is that the solution of (2) will produce the poles of $f(s)$ as the zeros of $\sigma(s)$. As explained in [1], this will indeed be the case when $f(s)$ is rational and the number of initial poles $\{q_m\}$ equals the number of poles $\{a_m\}$ of $f(s)$.

However, when $f(s)$ contains a nonrational contribution (e.g., noise), or the number of initial poles is lower than the order of $f(s)$, it will not be possible to satisfy (2) exactly. In that situation, the solution of (2) will simply be the one that has the smallest LS error. Since $f(s)$ is multiplied with $\sigma(s)$ and the right side of (2) is an unknown (rational) function, there is an incentive to produce a $\sigma(s)$ which is small in magnitude since this allows to reduce the magnitude of both the left and right side of (2). At the same time, relocating poles a long distance may require a $\sigma(s)$ with a large variation in magnitude. Since $\sigma(s)$ is enforced to approach unity at high frequencies by (3), a large dynamic variation in $\sigma(s)$ will generally lead to a large magnitude of $\sigma(s)$ in some frequency interval(s) and, thus, an increase of the LS error.

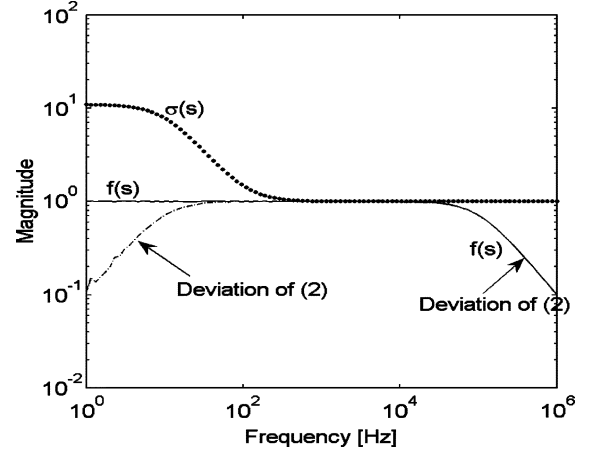
We will demonstrate this problem for a simple example. Consider the fitting of the first-order rational function

$$f(s) = \frac{-a}{s - a} \quad (6)$$

with $a = -2\pi \cdot 10^5$ using an initial pole at 10 Hz: $q = -2\pi \cdot 10$.

Fig. 1 shows $f(s)$ and $\sigma(s)$ obtained by solving (2). As expected, $\sigma(s)$ gets a zero at 100 kHz (which is taken as the pole of $f(s)$). The fitting error of (2) is close to machine precision ($\approx 1E-14$). It is noted that $\sigma(s)$ has a very large dynamic variation, approaching 10000 at low frequencies.

Fig. 2 shows the same result after adding $\pm 1\%$ random noise to $f(s)$. The new pole now appears at 109.6 Hz instead of at 100 kHz, and the dynamic variation in $\sigma(s)$ is very much reduced. In this case, it is not possible to obtain zero fitting error of (2) due to the noise. If the pole had been relocated to the correct position (100 kHz) as in Fig. 1, the magnitude of $\sigma(s)$ would have approached 10 000 at low frequencies which would have resulted in a magnification of the deviation of (2) as compared

Fig. 2. Rational approximation of $f(s)$, 1% noise. Original VF.

to the smaller $\sigma(s)$ in Fig. 2. Increasing the noise level further reduces the dynamic variation in $\sigma(s)$ and, thus, the ability of relocating the pole.

On the other hand, if we look at the example where $f(s)$ has a pole at 10 Hz and we use an initial pole at 100 kHz, then $\sigma(s)$ becomes in the noiseless case very small at low frequencies (and approaches unity at high frequencies). In the noisy case, the pole becomes relocated to nearly its correct value in a single iteration since the dynamic variation of $\sigma(s)$ now leads to a reduction of the fitting error of (2). It can thus be concluded that the asymptotic requirement that $\sigma(s)$ approaches unity at high frequencies represents an asymmetry of the LS problem (2) which can significantly impair the pole relocation process.

IV. MODIFIED VF

A modification of the VF formulation (MVF) is shown in the following which gives improved convergence by the removal of the asymptotic requirement of the LS problem (2). This is achieved by replacing (3) with

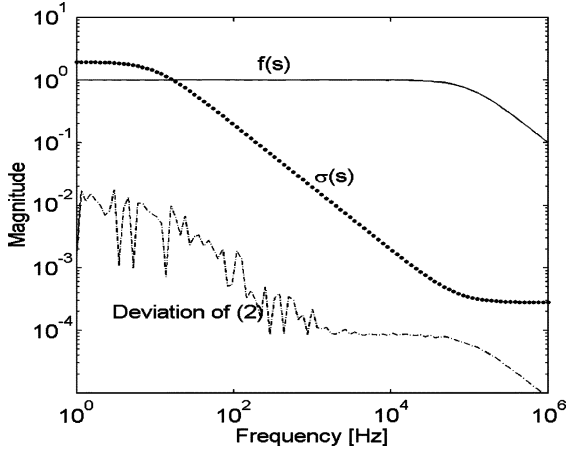
$$\sigma(s) = \sum_{m=1}^N \frac{\tilde{r}_m}{s - q_m} + \tilde{d} \quad (7)$$

where \tilde{d} is real.

In order to avoid the trivial (null) solution, we add one equation to the resulting LS problem

$$\text{Re} \left\{ \sum_{k=1}^{N_s} \left(\sum_{m=1}^N \frac{\tilde{r}_m}{s_k - q_m} + \tilde{d} \right) \right\} = N_s. \quad (8)$$

Equation (8) enforces that the sum of the real part of $\sigma(s)$ over the given frequency samples equals some nonzero value, without fixing any of the free variables. As MVF converges, $\sigma(s)$ will approach unity at all frequencies ($\{\tilde{r}_m\} = 0$, $\tilde{d} = 1$) similarly as in the original VF formulation. Note that the procedure will also work when using a different right side in (8) since the resulting scaling of $\sigma(s)$ will cause an identical scaling of $p(s)$ in (2). From this, it follows that the criterion (8) does not impose any constraint on the LS problem (2) other than preventing $\sigma(s)$ from becoming zero.

Fig. 3. Rational approximation of $f(s)$. 1% noise. Modified VF.

Equation (8) should be weighted in relation to the size of $f(s)$ in the LS problem, for example

$$\text{weight} = \frac{\|w(s) \cdot f(s)\|_2}{N_s} \quad (9)$$

where w is the specified weight for the fitting of $f(s)$.

Since $\sigma(s)$ during iterations does not approach unity at high frequencies, (5) must be replaced with [16, p. 612]

$$\{a_m\} = \text{eig}(\mathbf{A} - \mathbf{b} \cdot \tilde{d}^{-1} \cdot \mathbf{c}^T). \quad (10)$$

The zeros calculation by (10) is only applicable with a non-zero \tilde{d} . If the absolute value of \tilde{d} is found to be smaller than $\text{tol} = 1E-8$, the solution is discarded and the LS problem is solved again with a fixed value for \tilde{d} in (7): $\tilde{d} = \text{tol} \cdot (\tilde{d}/\text{abs}(\tilde{d}))$.

We now demonstrate the new formulation for the same problem as in Fig. 2 (Fig. 3). It is seen that the dynamic variation in $\sigma(s)$ is now much bigger and that $\sigma(s)$ approaches a value that is much smaller than unity at high frequencies ($3E-4$). The pole was in a single step relocated from 10 Hz to 70.9 kHz (target: 100 kHz), compared to only 109.6 Hz with the original VF formulation (Fig. 2).

V. EXAMPLE: FREQUENCY-DEPENDENT NETWORK EQUIVALENT (FDNE)

A. Case

The advantages of the MVF formulation will be demonstrated by an example from FDNE identification (Fig. 4).

The 3×3 admittance matrix \mathbf{Y} was calculated with respect to the feeding bus between 50 Hz and 1 MHz and we will be fitting one of the diagonal elements. Normally, the initial poles for VF would be selected as complex conjugate pairs that are linearly distributed over the full frequency range (i.e., between 50 Hz and 1 MHz). To better test the pole relocating properties, we will instead distribute the initial poles (complex pairs) in the lower range only, between 50 Hz and 500 kHz. This requires VF to relocate many poles over long distances.

B. Fitting the Response

Fig. 5 shows the resulting fitting by the original VF when using $N = 50$ poles and 15 iterations. It is seen that the two resonance peaks around 950 kHz have not been captured. Fig. 6

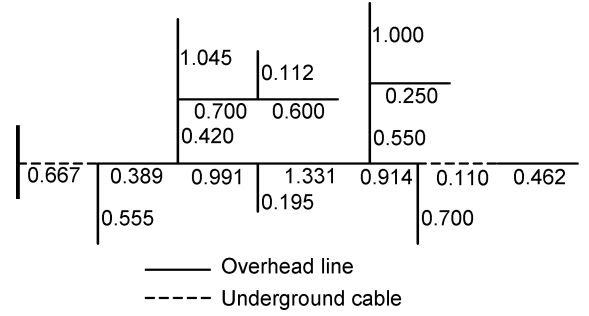
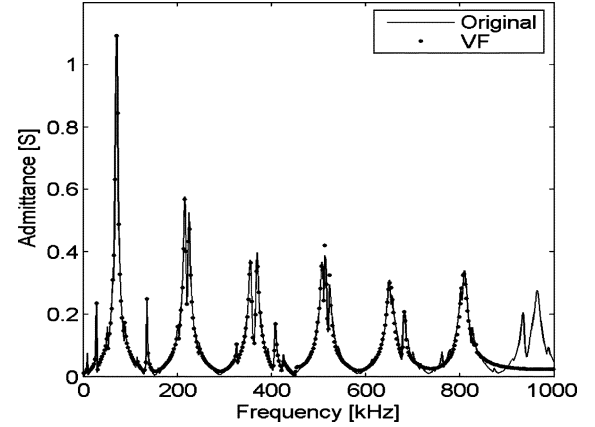
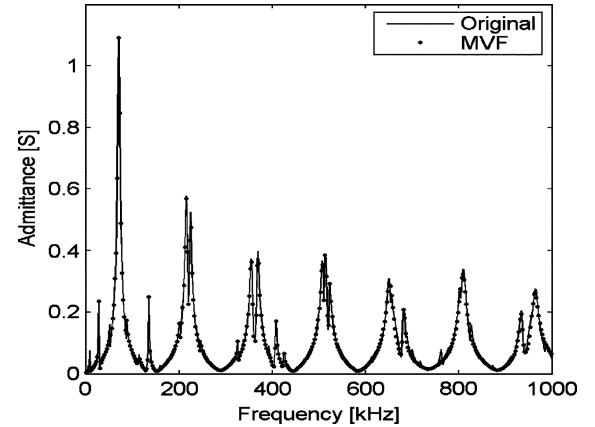


Fig. 4. Three-phase distribution system. All lengths are in kilometers.

Fig. 5. Rational approximation. $N = 50$. Original VF (15 iterations).Fig. 6. Rational approximation. $N = 50$. Modified VF (15 iterations).

shows that by using the modified VF, all dominant resonance peaks are fitted.

Fig. 7 shows the root-mean-square (rms) error as a function of the iteration number, when using VF and MVF. It is seen that with $N = 50$ poles, MVF gives a much faster convergence and higher accuracy for the end result. With $N = 100$, the improvement is small. In the latter case, the fitting error is small with both approaches and so the poles can in each iteration be relocated in longer steps.

C. Effect of Noise

We now repeat the fitting after adding real-valued random noise between ± 0.10 to the response. Fig. 8 shows the rms error as a function of the iteration number, similarly as in Fig. 7. It is

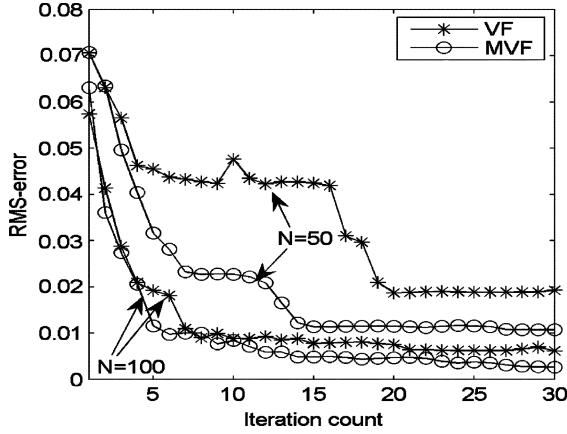


Fig. 7. RMS error as a function of iteration number.

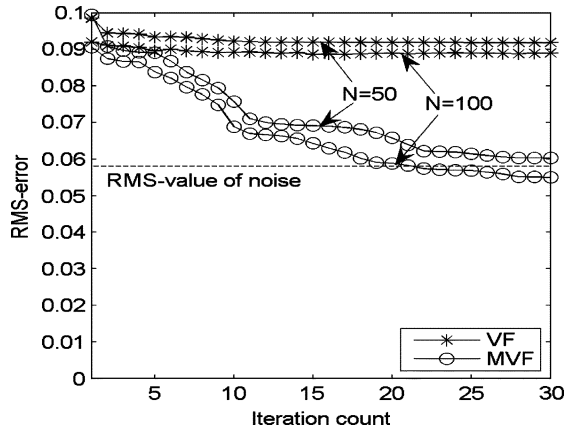
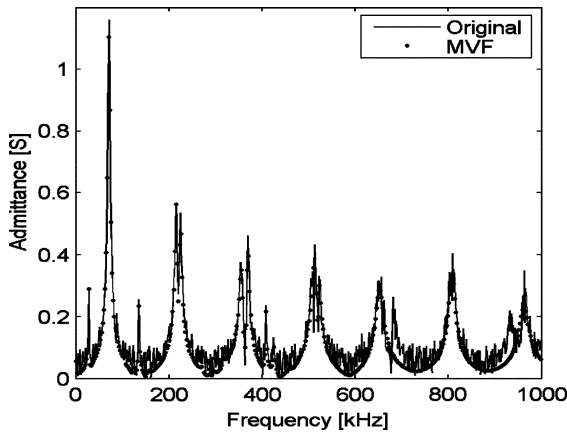


Fig. 8. RMS error as a function of iteration number.

Fig. 9. Rational approximation by MVF ($N = 50, 30$ iterations).

seen that with the original VF formulation, the iterations hardly reduce the fitting error at all, even with an order as high as 100. With MVF, the rms error decreases with an iteration to a level which is close to the rms value of the noise itself (0.0058). The resulting approximation by MVF is compared to the original response in Fig. 9 ($N = 50$).

A comparison was also made when the initial poles were placed at high frequencies: complex conjugate pairs linearly distributed between 500 kHz and 1 MHz. Again, VF stalled whereas MVF gave a convergence even faster than in Fig. 8.

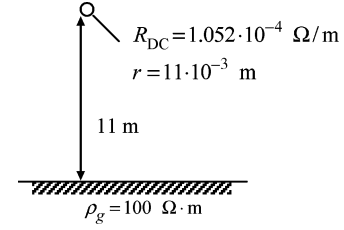
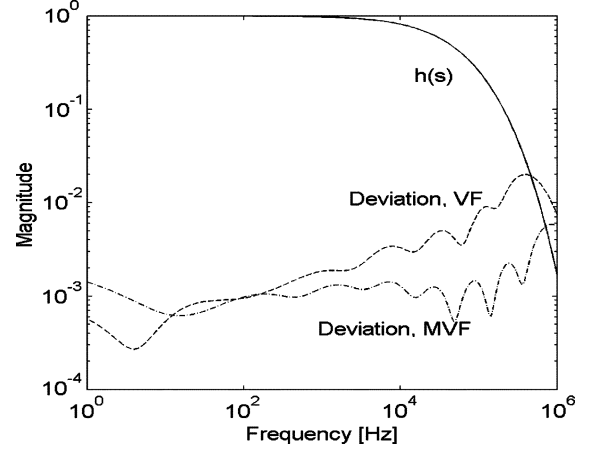


Fig. 10. Overhead line.

Fig. 11. Rational approximation of $h(s)$: Deviation by VF and MVF (five iterations $N = 7$).

VI. EXAMPLE: TRANSMISSION-LINE MODELING

In this example, we consider the fitting of the propagation function h of a 25-km single conductor overhead line over lossy ground (Fig. 10).

For the configuration in Fig. 10, the series impedance z was calculated, taking into account the skin effect in the conductor and ground, and the shunt admittance $y = sC$. The propagation function was obtained as

$$h(s) = \exp\left(-\sqrt{z(s)y(s)}l\right). \quad (11)$$

Finally, the lossless time delay was removed from h by multiplying (11) with the factor

$$\exp\left(\frac{sl}{c}\right) \quad (12)$$

where l is the line length and c is the speed of light.

A rational approximation was calculated by fitting h in the frequency range 1 Hz–1 MHz using a seventh-order approximation. The initial poles were taken as real and logarithmically spaced over the fitting range, which is the “recommended” choice for transmission-line modeling. Parameters d and e in (1) were specified to be zero. Fig. 11 shows the magnitude function of $h(s)$ and the magnitude of the complex deviation from the rational approximation, after five iterations with VF and MVF, respectively. It is seen that MVF gives a significantly smaller deviation (note the logarithmic ordinate axis).

Fig. 12 shows the rms error as a function of the iteration number. It can be seen that MVF gives a much faster convergence, although the final result (ten iterations) is, in this case, quite similar. A small increase of the fitting error is observed with MVF after iteration 4.

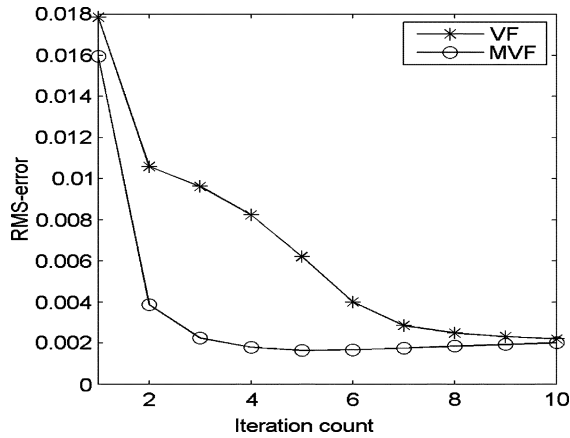
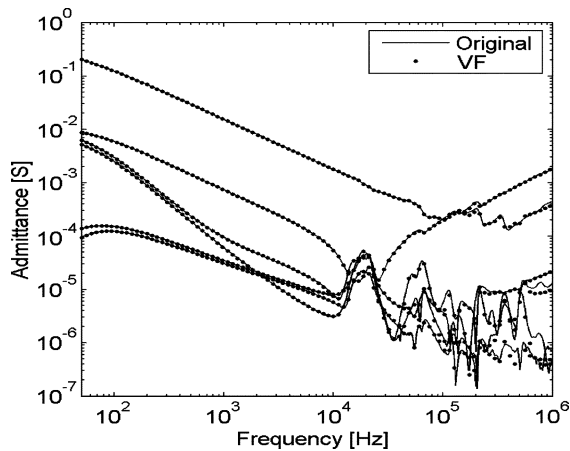


Fig. 12. RMS error as a function of iteration number.

Fig. 13. Fitting by VF ($N = 40$, 15 iterations).

VII. EXAMPLE: TRANSFORMER MODELING

In this example, we consider the modeling of a two-winding power transformer from measured terminal responses. The 6×6 admittance matrix \mathbf{Y} was measured from 50 Hz to 1 MHz [4]. A state-space model is to be established by fitting \mathbf{Y} by its columns [16].

When including low frequencies in the measurements, it becomes difficult to specify the initial poles to be used by VF. The smooth behavior at low frequencies suggests to use logarithmically spaced poles while the resonant behavior at high frequencies suggests to use linearly spaced poles. In practice, one can get away from this difficulty by using a combination of logarithmically and linearly spaced poles, but this is a complication.

Fortunately, with MVF, the significance of the selected initial poles is greatly reduced. Fig. 13 shows the fitting of the first column of \mathbf{Y} obtained by VF after 15 iterations when using as initial poles 20 complex pairs that are logarithmically spaced between 50 Hz and 1 MHz. (Inverse LS weighting by [4, eq. (18)] was used in order to improve the accuracy where elements are small). However, a closer look at Fig. 13 reveals that the quality of the fitting is significantly poorer at high frequencies (> 100 kHz) than at low frequencies, which is a consequence of the high concentration of initial poles at low frequencies.

The calculations were repeated when using MVF. The expanded view in Fig. 14 makes a direct comparison between the

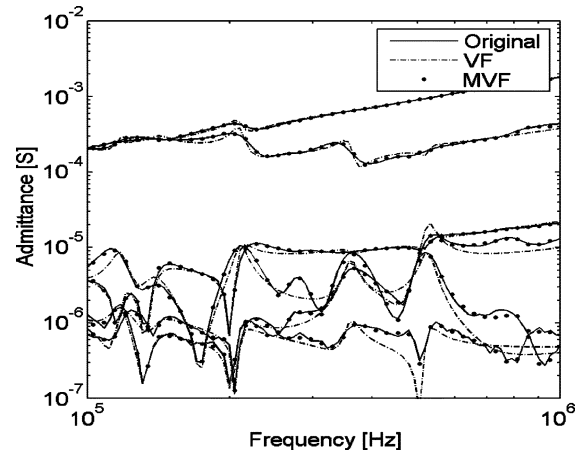


Fig. 14. Expanded view: VF and MVF at high frequencies.

result by the two approaches at high frequencies. It is clearly seen that MVF gives a more accurate result than VF, which is a direct result of the better pole relocating properties of MVF. The accuracy at lower frequencies appeared similar for the two approaches.

VIII. DISCUSSION

The original formulation of VF involves a scaling function $\sigma(s)$ which is enforced to approach unity at infinite frequency. It was shown in Section III that this asymptotic requirement reduces the ability to relocate initial poles to better positions because the asymptotic requirement combined with the required large dynamic variation of $\sigma(s)$ can result in an increase of the LS fitting error of (2). This situation was greatly improved when replacing the asymptotic requirement by a more relaxed criterion (Section IV) which requires the sum of the real part of $\sigma(s)$ over the frequency samples to be nonzero. This new criterion allows $\sigma(s)$ to freely vary while avoiding the trivial (null) solution. A further discussion on the convergence properties of VF versus MVF can be found in [17]. It is shown that VF is biased to relocating the poles towards low frequencies due to a down-scaling phenomenon, in particular when fitting noisy responses and when using too low orders. As a result, VF will, in general, produce a less accurate end result than MVF.

The MVF formulation gives one additional row and one additional column in the LS equation for the solution of (2) but the additional computation time is negligible.

The modified VF (MVF) was in Section V applied to an example with the initial poles placed in the lower frequency band and with added noise. In this case, the convergence of VF was very poor while the performance of MVF remained acceptable. One could argue that this is a contrived example but it serves well in demonstrating the convergence problems with VF and to show the improvement achieved by MVF.

In Section VI, the MVF was applied to transmission-line modeling by the method of characteristics where one of the tasks is to model the propagation function. Here, MVF was shown to converge much faster than VF when using a “recommended” choice of initial poles. One might think that this result is insignificant because low-order fitting of scalar functions is in any case very fast. However, some transmission-line models

rely on optimizing the time delay used for compensation of (11). This is the case for the phase-domain transmission line/cable models available in EMTDC and EMTP-RV, see also [18]. Such procedures result in a large number of calls to the fitting procedure, thus making the faster convergence of MVF advantageous.

In Section VII, VF was applied to columnwise fitting of measured transformer responses (Y -parameters). In this case, it was difficult to specify a good set of initial poles and the slow convergence of VF gave a somewhat inaccurate result at high frequencies, even after 15 iterations. A significantly better result was achieved by MVF. It is remarked that the behavior of transformer models can be highly sensitive to small perturbations of the model parameters [4], thus making accurate fitting very important.

Finally, it is remarked that an alternative approach for improving the VF convergence was recently introduced in [19] by a "hard relocating" procedure

IX. CONCLUSION

This paper has presented a modification of VF by replacing the high-frequency asymptotic constraint of the scaling function $\sigma(s)$ with a milder summation requirement. This replacement greatly improves the ability of VF to relocate poles to better positions, thus reducing the significance of the choice of initial poles. The improvement is particularly significant when highly accurate approximations are impossible; for instance, when fitting responses that are contaminated with noise. Calculated results have also demonstrated significant advantages of the modified VF when applied to transmission-line modeling and wideband transformer modeling. The modification to the VF algorithm is straightforward and does not increase the computational cost. A Matlab implementation is available in [13].

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