EEN1047
CONTROL SYSTEMS ANALYSIS

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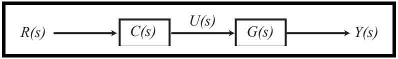
Section 4

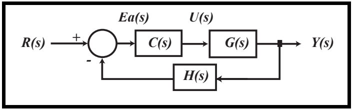
Sensitivity

4.1 Linear Feedback Control Systems

- A **Control System** is a set of interconnected systems designed to provide a desired system response.
- **Open Loop System**



  - Has no feedback.
  - The output is generated directly by the input.
- **Closed Loop System**



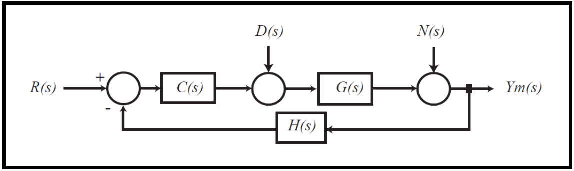
  - Has a feedback loop; is error based.
  - Compares the output with the desired response.
  - Often $H(s) = 1$, called a **unity feedback** system.
  - Feedback reduces the **sensitivity** of the control system to model inaccuracies and disturbances.

4.2 A Benefit of Feedback

- Feedback facilitates the reduction of **System Sensitivity**.
  - A process, represented by a transfer function $G(s)$, undergoes changes over time due to aging and/or variations in its environment.
  - The original transfer function will give an inaccurate result in an **open–loop** configuration.
  - A **closed–loop** system will measurement changes in the output due to the changes in $G(s)$ and will attempt to correct the output.
  - The degree to which changes in system parameters affect the transfer function is called **sensitivity**.
  - If changes in system parameters have no effect on the transfer function, the system is said to have zero sensitivity (ideal).
  - The greater the system sensitivity, the larger the effect that system parameter changes will have on the transfer function.

4.3 Disturbance Signals

- The general system model for looking at **system sensitivity** is:



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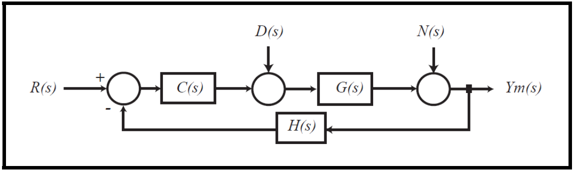

- Note that we have assumed that there is no pre-conditioning filter $P(s)$ in the system.
- There are two **signals** indicated in the model.
  - $D(s)$ is a disturbance that affects the input of the process.
  - $N(s)$ is a disturbance that affects the output of the process, it is usually a measurement noise introduced by the output sensor.

4.4 Open Loop Variations

- In order to examine the effects of parameter variations on the transfer function, we will assume that $D(s) = N(s) = 0$.
- As the system is linear, for simplicity, assume that $C(s) = 1$.
- Taking the open–loop case and letting Δ denote 'a change in':

Open–loop

$$Y(s) = G(s).R(s)$$
$$G_{alt}(s) = G(s) + \Delta G(s)$$
$$\Rightarrow Y_{new}(s) = Y(s) + \Delta Y(s) = [G(s) + \Delta G(s)].R(s)$$
$$\Delta Y(s) = \Delta G(s).R(s)$$

- The change in output is directly related to the change in the process; if there is a big change in the process, there will be a big change in the output (may be undesirable!).

4.5 Closed Loop Variations

- If there is a change in a model $G(s)$ that is in a closed-loop configuration:

Closed–loop

$$Y(s) = \frac{G(s)}{1 + GH(s)}.R(s)$$

$$G_{alt}(s) = G(s) + \Delta G(s)$$

$$Y_{new} = Y(s) + \Delta Y(s) = \frac{G(s) + \Delta G(s)}{1 + GH(s) + \Delta G(s)H(s)}.R(s)$$

$$\Delta Y(s) = \left[\frac{G(s) + \Delta G(s)}{1 + GH(s) + \Delta GH(s)} - \frac{G(s)}{1 + GH(s)}\right].R(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s))(1 + GH(s) + \Delta GH(s))}.R(s)$$

$$\Delta Y(s) \approx \frac{\Delta G(s)}{[1 + GH(s)]^2}R(s) \qquad \text{If } GH(s) >> \Delta GH(s)$$

4.5 Closed Loop Variations

- The change in output depends on the change in the model **and** on $G(s)$ and $H(s)$.

Closed–loop

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$$G_{alt}(s) = G(s) + \Delta G(s)$$

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4.6 Closed-Loop Variations - Equation

Closed-loop

$$\Delta Y(s) \approx \frac{\Delta G(s)}{[1 + GH(s)]^2}.R(s)$$

- Comparing the open-loop and closed-loop expressions for the change in output, it can be seen that the closed-loop expression features elements of the original system dynamics.
- If the expression $[1 + GH] > 1$, then the effect of the change in $G(s)$ is reduced.
- If $[1 + GH] > 1$, then $[1 + GH]^2 \gg 1$ and the reduction effect is greater.
- This factor *is* usually much greater than unity in most practical cases.
- Thus, the feedback loop (even with $H(s) = 1$) reduces the effect of changes in system dynamics on the output.

4.7 Parameter Variations - Example Changes

- Take an example of a first oder motor model with gain $K_{,} = 20$ and time constant $\tau_m = 0.6$ (s).

$$G_m(s) = \frac{K_m}{\tau_m s + 1} = \frac{20}{0.6s + 1}$$

- Assume that $C(s) = 1$ and $H(s) = 0.45$ in the closed-loop configuration:
  - Calculate the open-loop steady-state gain of the system:

$$G_m(0) = 20$$

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  - Calculate the open-loop steady-state gain of the system:

$$G_m(0) = 20$$

- Calculate the CLTF of the system:

$$T(s) = G_{CL}(s) = \frac{G(s)}{1 + GH(s)} = \frac{20}{0.6s + 10}$$

  - Calculate the closed-loop steady-state gain of the system:

$$T(0) = G_{CL}(0) = \frac{20}{10} = 2$$

4.8 Parameter Variations - Example Changes

- It is now assumed that some change has occurred in the motor model; the effect of this change is that the motor gain value changes to $K_m^* = 15$, i.e. a 25% reduction.
  - Calculate the open-loop steady-state gain of the new system:

$$G_m^*(0) = 15 \quad \text{versus } G_m(0) = 20$$

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$$G_m^*(0) = 15 \quad \text{versus } G_m(0) = 20$$

- Calculate the CLTF of the changed system:

$$T^*(s) = G_{CL}^*(s) = \frac{G^*(s)}{1 + G^*H(s)} = \frac{15}{0.6s + 7.75}$$

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  - Calculate the closed-loop steady-state gain of the system:

$$T^*(0) = G_{CL}*(0) = \frac{15}{7.75} = 1.9355 \quad \text{versus } T(0) = 2$$

- The closed–loop system does not experience a change of 25% - this is due to feedback.

4.9 Sensitivity - Definition

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- The sensitivity function of a function F to a change in parameter p is:

Definition

$$S_p^F = \lim_{\Delta p \to 0} \frac{\%\ change\ in\ F}{\%\ change\ in\ p}$$
$$= \lim_{\Delta p \to 0} \frac{\Delta F/F}{\Delta p/p} = \lim_{\Delta p \to 0} \frac{p\Delta F}{F\Delta p}$$

$$S_p^F = \frac{p}{F}\frac{\delta F}{\delta p}$$

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- The system sensitivity of a CLTF T to a change in process G is:

Definition

$$S_G^T = \lim_{\Delta G \to 0} \frac{\Delta T / T}{\Delta G / G}$$
$$= \lim_{\Delta G \to 0} \frac{G \Delta T}{T \Delta G}$$

$$S_G^T = \frac{G}{T} \frac{\delta T}{\delta G}$$

4.11 System Sensitivity - Expression

- Taking the general closed-loop system with $P(s) = C(s) = 1$ and $N(s), D(s) = 0$:

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General Expression

$$
\begin{aligned}
T(s) &= \frac{G(s)}{1 + GH(s)} \\
S_G^T &= \frac{G}{T}\frac{\delta T}{\delta G} = \frac{G}{\frac{G}{1+GH}} . \frac{\delta G\,(1 + GH)^{-1}}{\delta G} \\
&= \frac{G\,(1 + GH)}{G} . \frac{(1 + GH)\,.1 - GH}{(1 + GH)^2} \\
&= \frac{G\,(1 + GH)}{G} . \frac{1}{(1 + GH)^2} \\
&= \frac{1}{1 + GH}
\end{aligned}
$$

4.12 Sensitivity - CLTF to H

- For the same closed–loop configuration, the sensitivity of $T(s)$ to changes in $H(s)$ can be calculated:

General Expression

$$T(s) = \frac{G(s)}{1 + GH(s)}$$

$$S_H^T = \frac{H}{T}\frac{\delta T}{\delta H} = \frac{H}{\frac{G}{1+GH}} . \frac{\delta G\,(1+GH)^{-1}}{\delta H}$$

$$= \frac{H\,(1+GH)}{G} . \left(-G\,(1+GH)^{-2}\,.G\right)$$

$$= \frac{H\,(1+GH)}{G} . \frac{-G^2}{(1+GH)^2}$$

$$= \frac{-GH}{1+GH}$$

4.13 Sensitivity - CLTF to p

- The sensitivity of the closed-loop system to a particular parameter of the open-loop system can be defined using the chain rule:

Sensitivity to parameter α

$$
\begin{aligned}
S_{\alpha}^{T} &= \frac{\alpha}{T}\frac{\delta T}{\delta \alpha} \\
&= \frac{G}{T}\frac{\alpha}{G}\cdot\frac{\delta T}{\delta G}\frac{\delta G}{\delta \alpha} \\
&= \frac{G}{T}\frac{\delta T}{\delta G}\cdot\frac{\alpha}{G}\frac{\delta G}{\delta \alpha} \\
&= S_{G}^{T} S_{\alpha}^{G}
\end{aligned}
$$

- Even if the chain rule is not used, the same result should be found using the general definition

4.14 Sensitivity Definition Example

- Recall the first order motor model Section 4.7 with gain K_m and time constant τ_m. Assume that $C(s) = 1$ and $H(s) = K_h$ in the closed-loop configuration.

$$G_m(s) = \frac{K_m}{\tau_m s + 1}$$

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$$G_m(s) = \frac{K_m}{\tau_m s + 1}$$

Example

$$T(s) = G_{CL} = \frac{K_m}{\tau_m s + 1 + K_m K_h}$$

$$S_{K_m}^{T} = \frac{K_m}{T}\frac{\delta T}{\delta K_m}$$

$$= \frac{K_m}{\frac{K_m}{\tau_m s+1+K_m K_h}} \cdot \left[\frac{(\tau_m s + 1 + K_m K_h).1 - K_m.K_h}{(\tau_m s + 1 + K_m K_h)^2}\right]$$

$$= \frac{\tau_m s + 1}{\tau_m s + 1 + K_m K_h}$$

4.15 Sensitivity Example - Chain Rule

- The sensitivity function can also be calculated using the chain rule representation.

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Example

$$
\begin{aligned}
S_{K_m}^{T} &= S_{G}^{T}.S_{K_m}^{G} \\
&= \frac{1}{1+GH}.\frac{K_m}{G}.\frac{\delta G}{\delta K_m} \\
&= \frac{\tau_m s+1}{\tau_m s+1+K_m K_h}.\frac{K_m}{\frac{K_m}{\tau_m s+1}}.\frac{1}{\tau_m s+1} \\
&= \frac{\tau_m s+1}{\tau_m s+1+K_m K_h}.\left(\tau_m s+1\right).\frac{1}{\tau_m s+1} \\
&= \frac{\tau_m s+1}{\tau_m s+1+K_m K_h}
\end{aligned}
$$

4.16 Sensitivity Example with Values

- Checking what this means with numerical values,
  $\Delta K_m = 25\%, \tau = 0.6\ s, K_h = 0.45$:

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Example

$$S_{K_m}^T = \frac{\tau_m s + 1}{\tau_m s + 1 + K_m K_h}$$
$$= \frac{0.6s + 1}{0.6s + 1 + 0.45(15)}$$
$$= \frac{1}{7.75}|_{s=0} = 0.129$$
$$S_{K_m}^T = \lim_{\Delta K_m \to 0} \frac{\%\ change\ in\ T}{\%\ change\ in\ K_m}$$
$$= \frac{\frac{2-1.9355}{2} * 100}{25} \quad \text{(from earlier)}$$
$$= \frac{3.225}{25} = 0.129$$