Continuous-time solution – Laplace Techniques

- The general linear Single Input Single Output (SISO) continuous-time state-space model is given by:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

An alternative approach to the time-domain methods is to use of the Laplace Transform.
Remember

$$L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0), \qquad L\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \frac{df}{dt}(0), \quad \text{etc.}$$

The solution to the state-space equation with the Laplace Transform is calculated as follows:

$$s\boldsymbol{I}\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}U(s)$$
$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{x}(0) + \boldsymbol{B}U(s)$$
$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s)$$

So

$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s) + \boldsymbol{D}U(s)$$

- **Example:** Determine the unit-step response for the system:

$$\frac{d\boldsymbol{x}(t)}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}(t)$$

if the input $u(t)$ is a step and the initial value of the states is

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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Solution
The input is a unit step so

$$U(s) = \frac{1}{s}$$

The initial state is given

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s) + \boldsymbol{D}U(s)$$

$$Y(s) = [1 \quad 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + [1 \quad 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}$$

$$Y(s) = [1 \quad 0]\left(\begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + [1 \quad 0]\left(\begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}$$

$$Y(s) = [1 \quad 0]\frac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + [1 \quad 0]\frac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}$$

$$Y(s) = \frac{s+3}{s^2+5s+6} + \frac{2}{s^2+5s+6}\frac{1}{s}$$

$$= \left(\frac{s^2+3s}{s^2+5s+6}\right)\frac{1}{s} + \left(\frac{2}{s^2+5s+6}\right)\frac{1}{s}$$

$$= \left(\frac{s^2+3s+2}{s^2+5s+6}\right)\frac{1}{s}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$s^2 + 3s + 2 = A(s^2 + 5s + 6) + Bs(s+3) + Cs(s+2)$$

Equate coefficients of powers of s

$$1 = A + B + C$$
$$3 = 5A + 3B + 2C$$
$$2 = 6A$$

$$A = \frac{1}{3}, B = 0, C = 2/3$$

$$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} = \frac{1}{3s} + \frac{2}{3(s+3)}$$

$$y(t) = \frac{1}{3}(1 + 2e^{-3t})u(t)$$

Note that as should be the case, this matches the result in example 5.3.

- **Example:** Determine the unit-impulse response for the system:

$$\frac{d\boldsymbol{x}(t)}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t)$$

  and the initial values of the states are zero.

$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{2}{s^2 + 5s + 6}$$

Now use partial fractions

$$\frac{2}{s^2 + 5s + 6} = \frac{A}{s+2} + \frac{B}{s+3}$$
$$2 = A(s+3) + B(s+2)$$
$$2 = 3A + 2B$$
$$3A + 2B = 0$$
$$A = 2, B = -2$$
$$\frac{2}{s^2 + 5s + 6} = \frac{2}{s+2} - \frac{2}{s+3}$$
$$y(t) = (2e^{-2t} - 2e^{-3t})u(t)$$

Note that as should be the case, this matches the result in example 5.2.