EEN1034- Digital Signal Processing (Digital Filters and DFT)

Acknowledgment
The notes are adapted from those given by
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z-Transform

- Analysis of digital filters using the z-Transform
- Introduction to the Fourier Transform for discrete-time signals

z-Transform Example

Let the input and the output sequence of a digital filter be given respectively by:

$$x[n] = (0.5)^n u[n]$$
$$y[n] = 3(0.5)^n u[n] - 2(0.3)^n u[n].$$

Find the unit sample response of the filter?

$$X(z) = \frac{z}{z - 0.5}$$

$$Y(z) = \frac{3z}{z - 0.5} - \frac{2z}{z - 0.3} = \frac{z^2 + 0.1z}{(z - 0.5)(z - 0.3)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{z^2 + 0.1}{(z-0.5)(z-0.3)}}{\frac{z}{z-0.5}} = \frac{z^2 + 0.1z}{z(z-0.3)} = \frac{z+0.1}{z-0.3}$$

$$H(z) = \frac{z}{z-0.3} + \frac{0.1}{z-0.3}$$

$$h[n] = 0.3^n u[n] + 0.1(0.3)^{n-1} u[n-1]$$

z-Transform example

Consider the filter

$$y[n] = 0.5y[n-1] + 0.5x[n]$$

Find the output $y[n]$ for $n \geq 0$ when this filter is excited by x(t) = $\cos(200t)u(t)$. The sampling interval is 0.01s so x$[n]$ = $\cos(2n)\,u[n]$. $y[-1] = 2$.

Apply the z-transform to both sides:

$$\mathcal{Z}(y[n]) = \mathcal{Z}(0.5y[n-1] + 0.5x[n])$$
$$Y^{+}(z) = 0.5z^{-1}Y(z) + 0.5X^{+}(z)$$
$$Y^{+}(z) = 0.5z^{-1}[Y^{+}(z) + 2z] + 0.5X^{+}(z)$$

Rearrange

$$Y^{+}(z) = \left(\frac{1}{1-0.5z^{-1}}\right) + \left(\frac{0.5}{1-0.5z^{-1}}\right)X^{+}(z)$$
$$Y^{+}(z) = \left(\frac{z}{z-0.5}\right) + \left(\frac{0.5z}{z-0.5}\right)X^{+}(z)$$

From z-Transform tables:

$$A\cos(\omega_0 n T_s)\, u[n] \overset{z}{\leftrightarrow} \frac{A(1 - \cos(\omega_0 T_s)\, z^{-1})}{1 - 2\cos(\omega_0 T_s)\, z^{-1} + z^{-2}}$$

$$A\sin(\omega_0 n T_s)\, u[n] \overset{z}{\leftrightarrow} \frac{A\sin(\omega_0 T_s)\, z^{-1}}{1 - 2\cos(\omega_0 T_s)\, z^{-1} + z^{-2}}$$

$$\cos(2n)\, u[n] \overset{z}{\leftrightarrow} \frac{(1 - \cos(2)\, z^{-1})}{1 - 2\cos(2)\, z^{-1} + z^{-2}}$$

z-Transform

$$Y^+(z) = \left(\frac{z}{z-0.5}\right) + \left(\frac{0.5z}{z-0.5}\right) X^+(z)$$

$$Y^+(z) = \left(\frac{z}{z-0.5}\right) + \left(\frac{0.5z}{z-0.5}\right)\left(\frac{(z^2-\cos(2)z)}{z^2-2\cos(2)\,z+1}\right)$$

Use partial fractions to simplify the second term

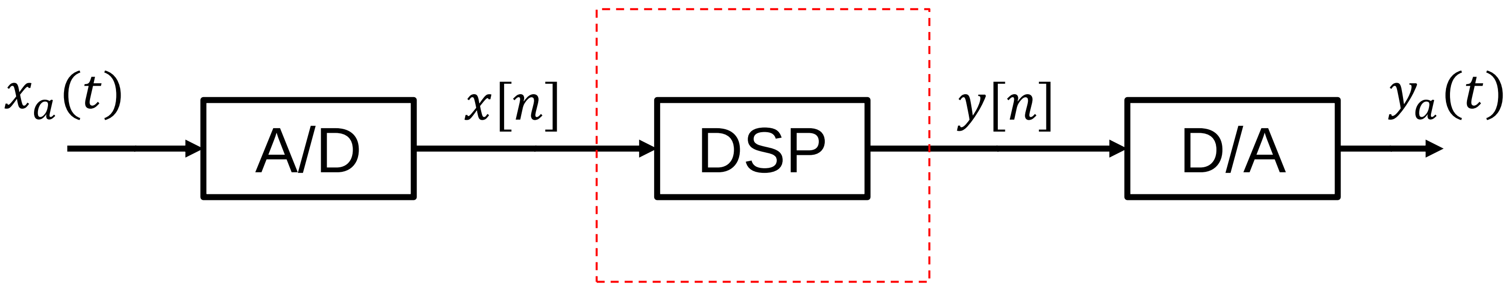
$$\left(\frac{0.5z}{z-0.5}\right)\left(\frac{(z^2-\cos(2)z)}{z^2-2\cos(2)\,z+1}\right)$$
$$=\frac{Az}{z-0.5}+\frac{B(z^2-\cos(2)z)}{z^2-2\cos(2)\,z+1}$$
$$+\frac{C(\sin(2)\,z)}{z^2-2\cos(2)\,z+1}$$

$$\left(\frac{0.5z}{z-0.5}\right)\left(\frac{(z^2-\cos(2)z)}{z^2-2\cos(2)\,z+1}\right)$$
$$=\frac{0.1375z}{z-0.5}+\frac{0.3625(z-\cos(2))}{z^2-2\cos(2)\,z+1}$$
$$+\frac{0.1364(\sin(2))z}{z^2-2\cos(2)\,z+1}$$

$$y[n] = (0.5)^n u[n] + 0.1375(0.5)^n u[n] +$$

$$0.3625\cos(2n)\,u[n] + 0.1364\sin(2n)\,u[n]$$

Signal Reconstruction



D/A conversion is also known as signal reconstruction.
Typically:

- **Sampling**: $x_a(t) \rightarrow x[n]$
- **D/A Conversion/Signal Reconstruction**: $y[n] \rightarrow y_a(t)$

$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n] sinc\left(\frac{t - nT_s}{T_s}\right)$$

This can be obtained by low-pass filtering of $y[n]$ with a brick-wall filter of bandwidth $\frac{f_s}{2}$.

Signal Reconstruction

There are different interpolation schemes to reconstruct the signal from the output samples.

If linear interpolation is used, the interpolation function is

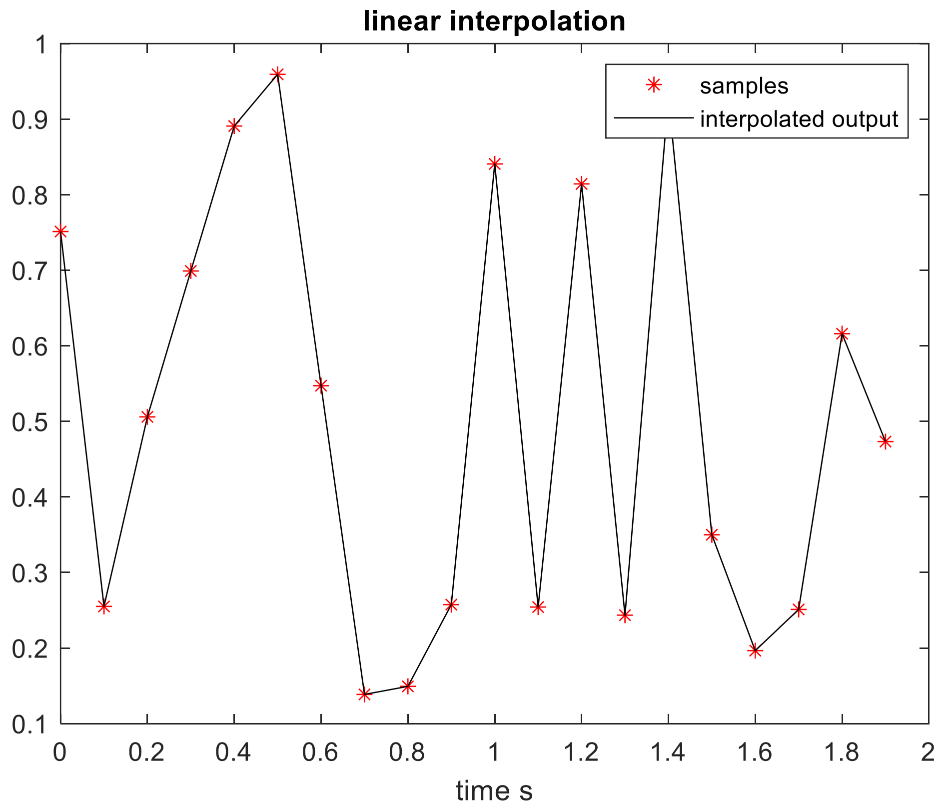
$$h(t) = 1 - \frac{|t|}{T_s}, |t| < T_s$$

Its frequency response is

$$H(j\omega) = \frac{2 - 2\cos(\omega T_s)}{\omega^2 T_s}$$

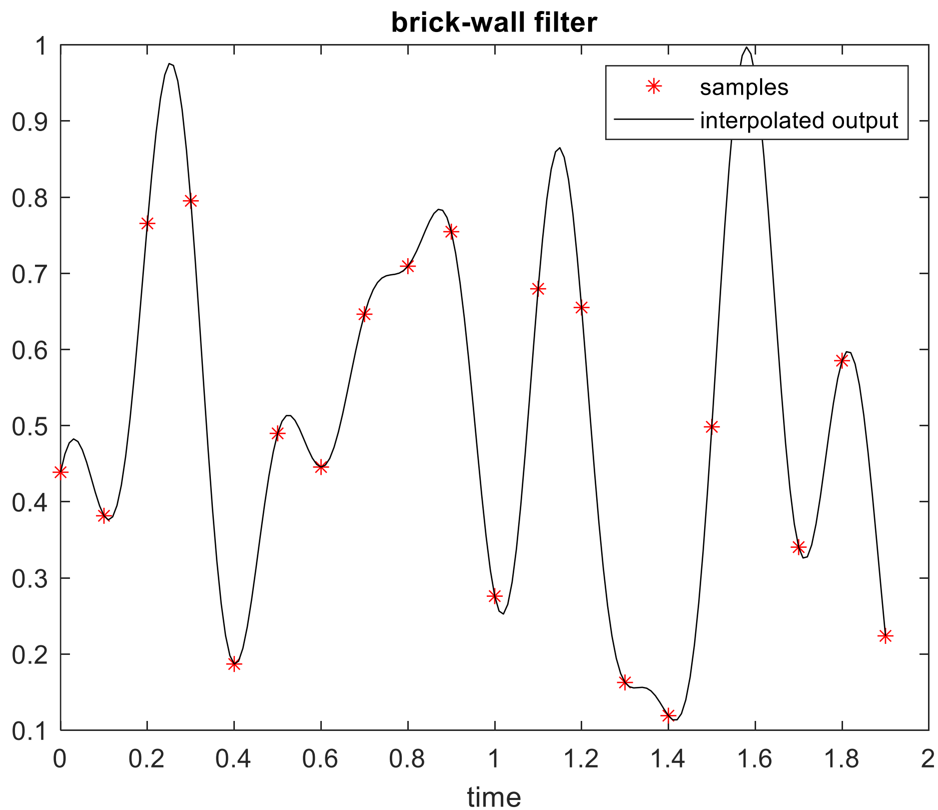
$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n]h(t - nT_s)$$

Signal Reconstruction



Linear Interpolation

Signal Reconstruction



Sinc-function Interpolation