EE401 - Digital Signal Processing (Digital Filters and DFT)

Acknowledgement
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z-Transform

- Introduction to the z–Transform
- Properties of the z-Transform
- Analysis of digital filters using the z-Transform

Definition of z-Transform

z-Transform is defined for a discrete-time signal $x[n]$ as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad ----(A)$$

where z $= e^{sT_s} = e^{\sigma T_s + j2\pi f T_s}$. z is a complex number and f is the frequency.

The infinite summation in right-hand side of (A) may be finite for some signals. The region of the z-plane/complex-plane where $X(z)$ has a finite value is known as **Region of Convergence (ROC)**.

$$X(z) = \mathcal{Z}(x[n]) \quad \text{or} \quad x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$$

Laplace Transform

Laplace Transform is given as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

If this is applied to the sampled signal:

$$X(s) = \int_{-\infty}^{\infty}[\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)]e^{-st}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT_s)e^{-st}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-snT_s}$$

where $s = \sigma + j2\pi f$. By making the substitution: $z = e^{sT_s}$, one can get the z-transform from Laplace transform.

Examples of z-Transform

a) Find the z-Transform of $x[n] = (0.5)^n u[n]$.

Ans: $\sum_{n=0}^{\infty}(0.5)^n z^{-n} = \frac{1}{1-0.5z^{-1}}$

Note: $|0.5z^{-1}| < 1$ for convergence. So ROC: $|z| > 0.5$

b) Find the z-Transform of $x[n] = \{1, 2, 0, 1, 3\}$ for $n = \{0,1,2,3,4\}$?

Ans: $\sum_{n=0}^{\infty} x[n]z^{-n} = 1 + 2z^{-1} + z^{-3} + 3z^{-4}$

ROC is entire z-plane.

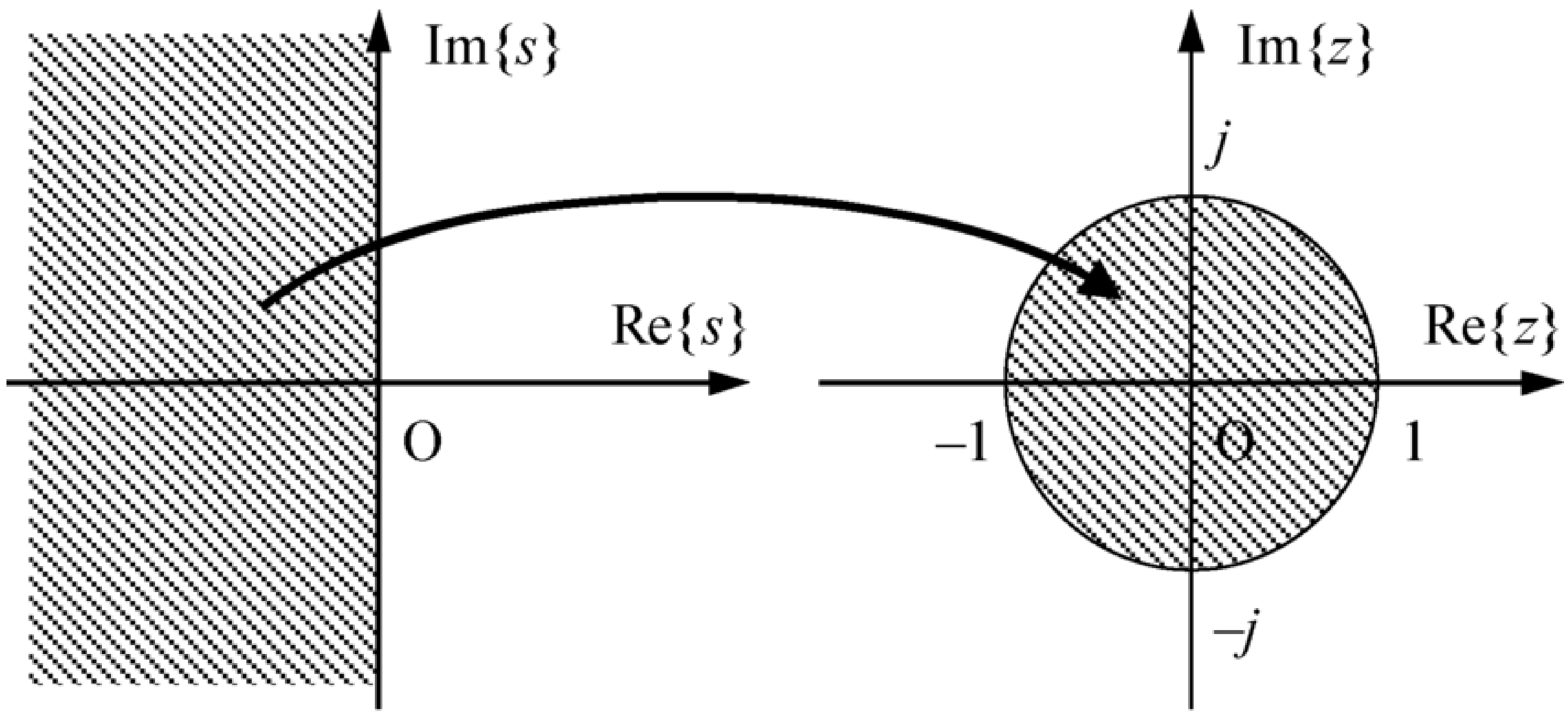
c) Find the z-Transform of $x[n] = \{1, 2, 0, 1, 3\}$ for $n = \{-1,0,1,2,3\}$?

Ans: $\sum_{n=0}^{\infty} x[n]z^{-n} = z + 2 + z^{-2} + 3z^{-3}$

ROC is entire z-plane.

s-plane vs z- plane

$$z = e^{sT_s} = e^{\sigma T_s + j2\pi f T_s}$$



(a) s plane (b) z plane

Properties of z-Transform

Linearity Property

$$\mathcal{Z}(a_1x_1[n] + a_2x_2[n]) = a_1\mathcal{Z}(x_1[n]) + a_2\mathcal{Z}(x_2[n])$$

Time Shifting Property If

$$x[n] \leftrightarrow X(z)$$

$$x[n-k] \leftrightarrow X(z)z^{-k}$$

where ROC of $z^{-k}X(z)$ is the same as that of $X(z)$, but:

a) If $k > 0$, $z \neq 0$

b) If $k < 0$, $z \neq \infty$

Properties of z-Transform

Scaling Property

$$x[n] \overset{z}{\leftrightarrow} X(z) \qquad ROC\colon r_1 \le |z| \le r_2$$

then

$$a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right) \qquad ROC\colon |a|r_1 \le |z| \le |a|r_2$$

If $x[n] = (0.5)^n u[n] \leftrightarrow \frac{1}{1-0.5z^{-1}}$, what is the z-Transform of $x_1[n] = a^n x[n]$?

$$X_1(z) = \frac{1}{1-0.5\left(\frac{z}{a}\right)^{-1}} = \frac{1}{1-0.5az^{-1}} \qquad \text{ROC: } |z| > 0.5a$$

Differentiation in the z-domain

$$x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$$

then

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$ ROC is same as that of $x[n]$

Proof: $$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}$$

System Function

Convolution $y[\mathrm{n}] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$$\begin{matrix} x[n] \\ X(z) \end{matrix} \longrightarrow \boxed{\begin{matrix} h[n] \\ H(z) \end{matrix}} \longrightarrow \begin{matrix} y[n] \\ Y(z) \end{matrix}$$

ROC of $Y(z)$ is the intersection of that of $X(z)$ and $H(z)$

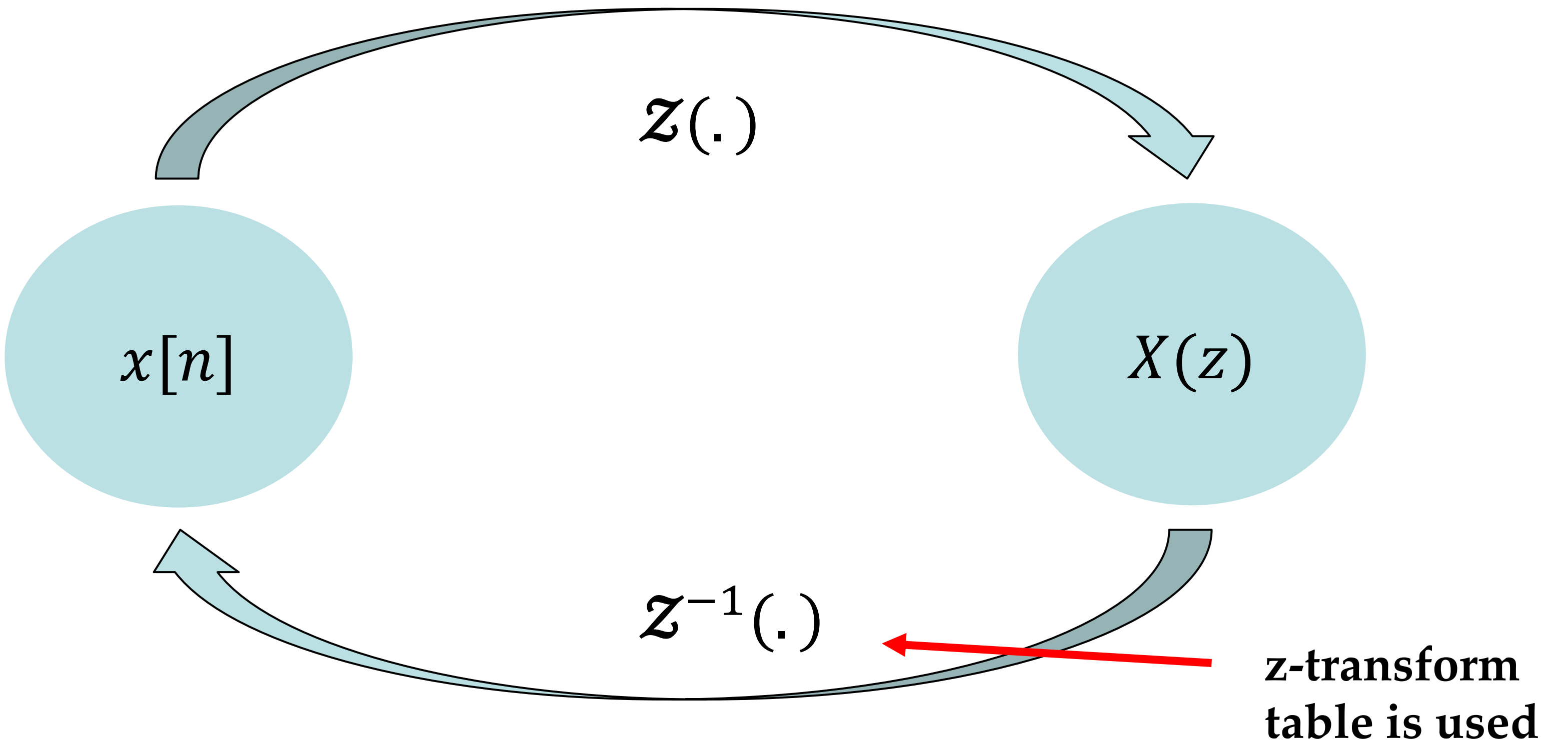
$$y[n] = x[n] * h[n] \overset{z}{\leftrightarrow} Y(z) = X(z)H(z)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\left[\sum_{n=-\infty}^{\infty} x[n-k]z^{-n}\right] = \sum_{k=-\infty}^{\infty} h[k]X(z)z^{-k}$$

$$= X(z)\left[\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right] = X(z)H(z)$$

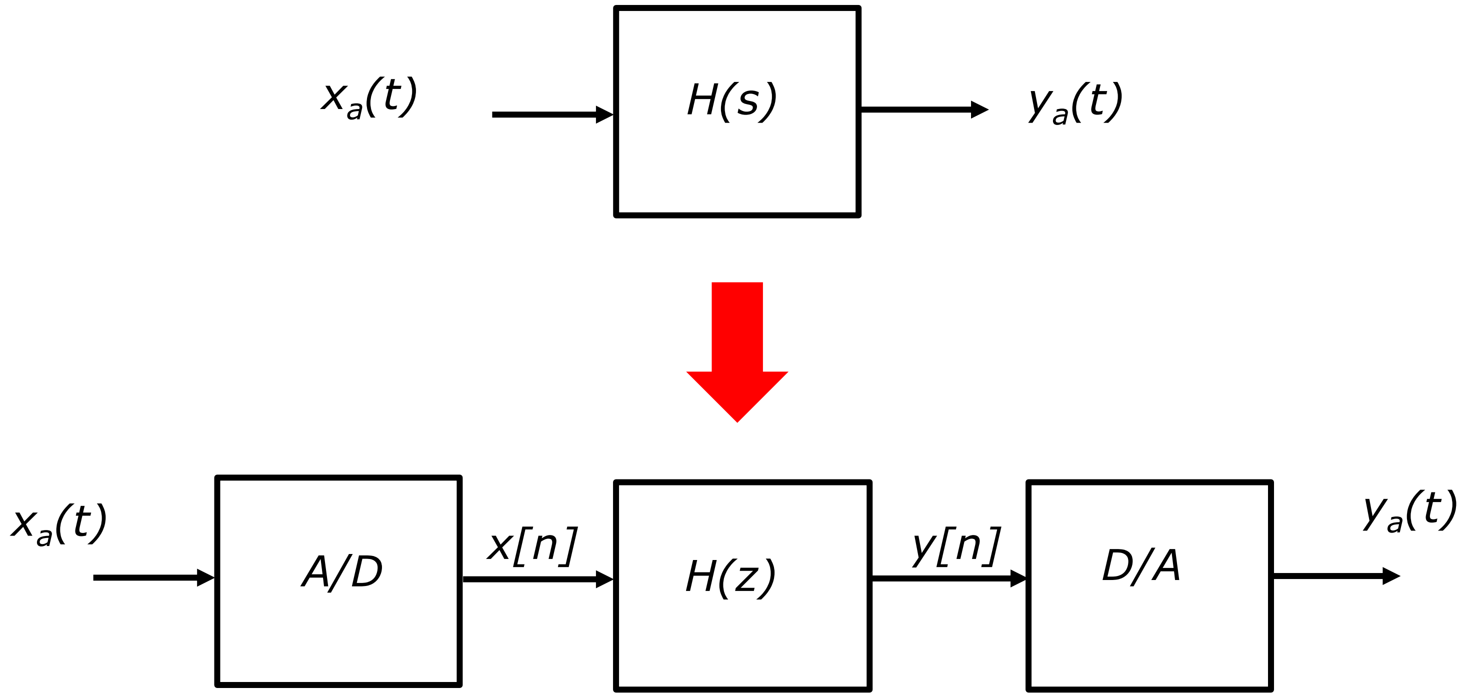
Inverse z-Transform



Similar to other transforms, there is an Inverse Z-Transform integral to evaluate $\mathcal{Z}^{-1}(.)$, but usually the z-transform table is used to find the original signal instead of this inverse integral.

Digital Filters

$$z = e^{sT_s} = e^{\sigma T_s + j2\pi f}$$



$H(z)$ is the z-transform of the unit sample response $h[n]$.

System Function

How to find the transfer function/system function $H(z)$ of a digital filter?

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

E.g.: $y[n] = 0.8y[n-1] + 0.3x[n]$. Find the system function $H(z)$.

Apply the z-transform to both sides:

$$\mathcal{Z}(y[n]) = \mathcal{Z}(0.8y[n-1] + 0.3x[n])$$
$$Y(z) = 0.8z^{-1}Y(z) + 0.3X(z)$$

After rearranging:

This is $H(z)$

$$Y(z) = \left(\frac{1}{1-0.8z^{-1}}\right)X(z)$$

System Function

In general, the system function has the following form and can also be factored.

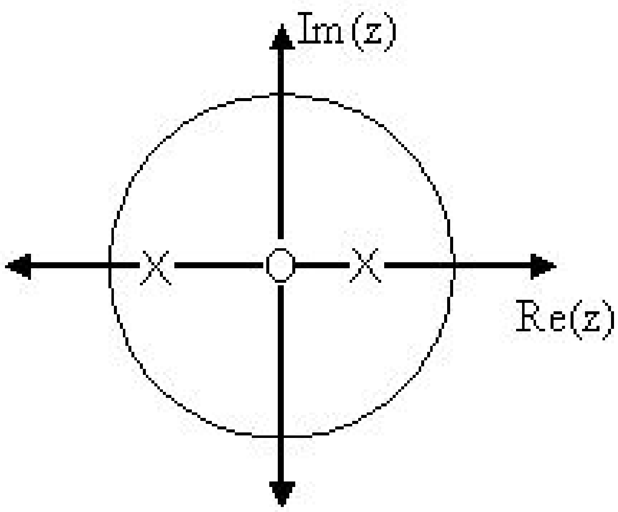
$$H(z) = K \frac{\sum_{k=1}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= K \frac{\prod_{k=1}^{M}(1-z_k z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})} = K \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

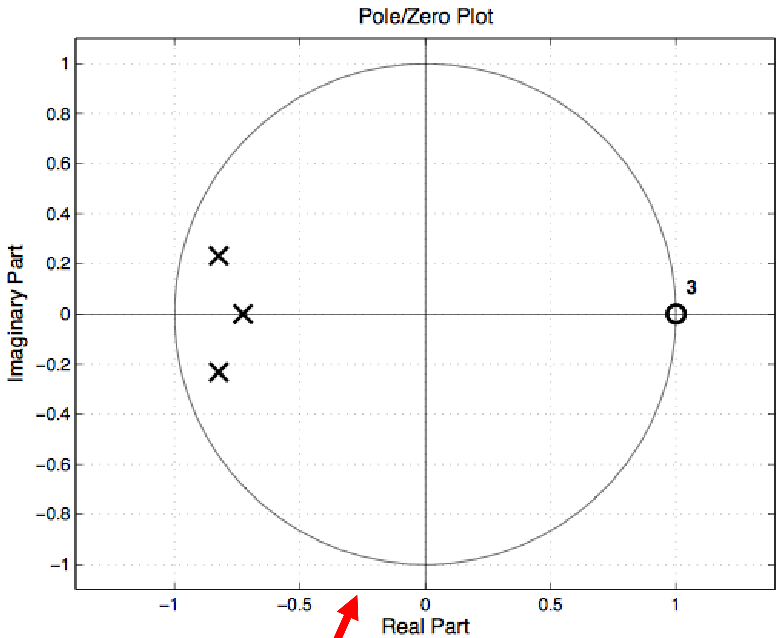
- $H(z)$ has M finite zeros(roots of the numerator polynomial) at $z = \{z_1, \dots, z_M\}$
- N finite poles (roots of the denominator polynomial) at $z = \{p_1, \dots, p_N\}$.
- There may be multiple zeros or poles at $z = 0$ depending on M and N.

Pole-Zero Plot

The Pole-Zero plot is a graphical representation of poles and zeros in the complex z-plane with **crosses (X) for poles** and **circles (o) for zeros**. The multiplicity of poles or zeros is indicated by a number close to the corresponding cross or circle.



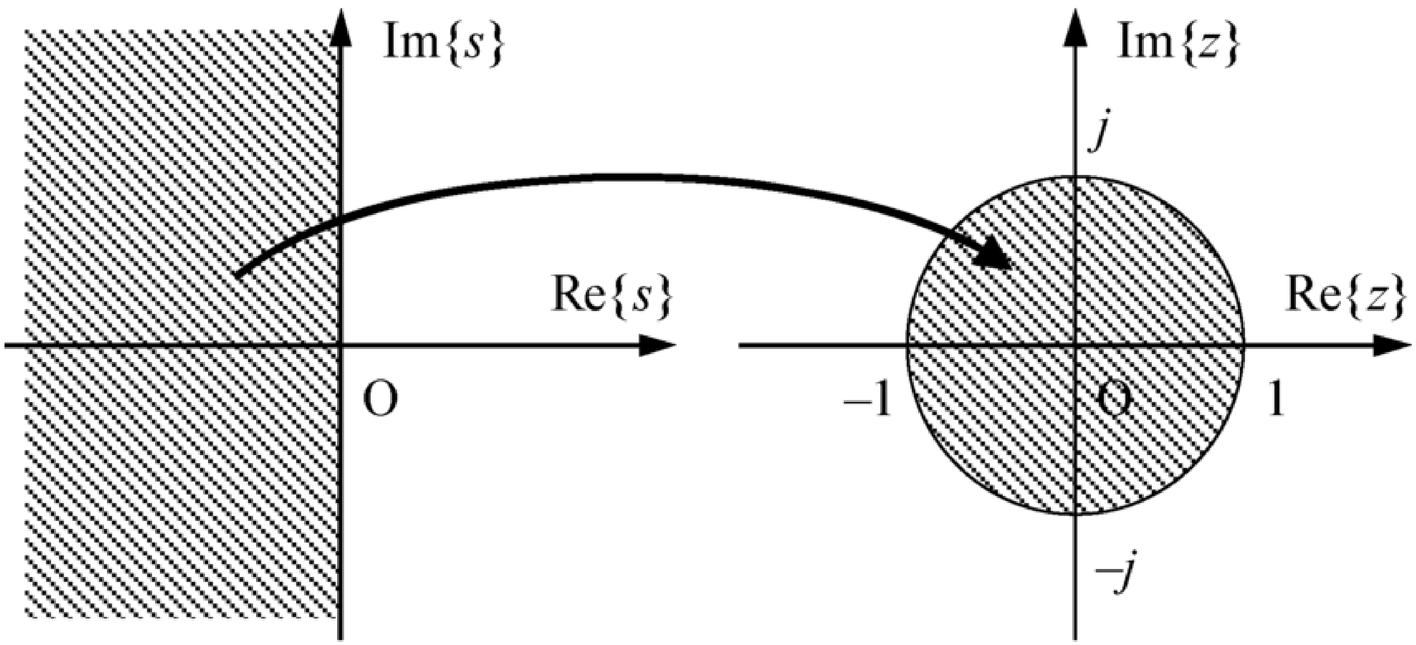
2 poles and one zero



3 poles and 3 zeros

s-plane vs z- plane

$$z = e^{sT_s} = e^{\sigma T_s + j2\pi f}$$



(a) *s* plane (b) *z* plane

A digital filter given by $H(z)$ is asymptotically stable if all of its poles are inside the unit circle.

Relaxed Vs Non-Relaxed

We can categorise digital filters as:

- Relaxed digital filters (all initial conditions are zero)
- Non-relaxed digital filters (filters with non-zero initial conditions)

Handling Initial Conditions

Example

$y[n] = 0.8y[n-1] + 0.3x[n]$.

Find the output $y[n]$ for $n \geq 0$ when the filter is excited by $x[n] = u[n]$ and $y[-1] = b$?

Consider the z-transform of $y[n]$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

$$= bz + \sum_{n=0}^{\infty} y[n]z^{-n}$$

$$= bz + Y^{+}(z)$$

We have split the z-transform of $y[n]$ into two parts. The second part has been denoted by $Y^{+}(z)$. If one can find $Y^{+}(z)$, by inverting it, one can find $y[n]$ for $n \geq 0$.

Handling Initial Conditions

Apply the z-transform to both sides:

$$\mathcal{Z}(y[n]) = \mathcal{Z}(0.8y[n-1] + 0.3x[n])$$
$$Y^+(z) = 0.8z^{-1}Y(z) + 0.3X(z)$$
$$Y^+(z) = 0.8z^{-1}[Y^+(z) + bz] + 0.3X(z)$$

After rearranging:

$$Y^+(z) = \left(\frac{0.8b}{1-0.8z^{-1}}\right) + \left(\frac{0.3}{1-0.8z^{-1}}\right)X(z)$$
$$Y^+(z) = \left(\frac{0.8b}{1-0.8z^{-1}}\right) + \left(\frac{0.3}{1-0.8z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$
$$Y^+(z) = \left(\frac{0.8bz}{z-0.8}\right) + \left(\frac{0.3z}{z-0.8}\right)\left(\frac{z}{z-1}\right)$$

Use partial fractions for the second term

$$\left(\frac{0.3z}{z-0.8}\right)\left(\frac{z}{z-1}\right) = -\frac{1.2z}{z-0.8} + \frac{1.5z}{z-1}$$

$$Y^{+}(z) = \left(\frac{0.8bz}{z-0.8}\right) - \frac{1.2z}{z-0.8} + \frac{1.5z}{z-1}$$

$y[n] = (0.8b + 1.2)(0.8)^n u[n] +$
$1.5u[n].$