EE401 - Digital Signal Processing (Digital Filters and DFT)

Acknowledgment
The notes are adapted from those given by
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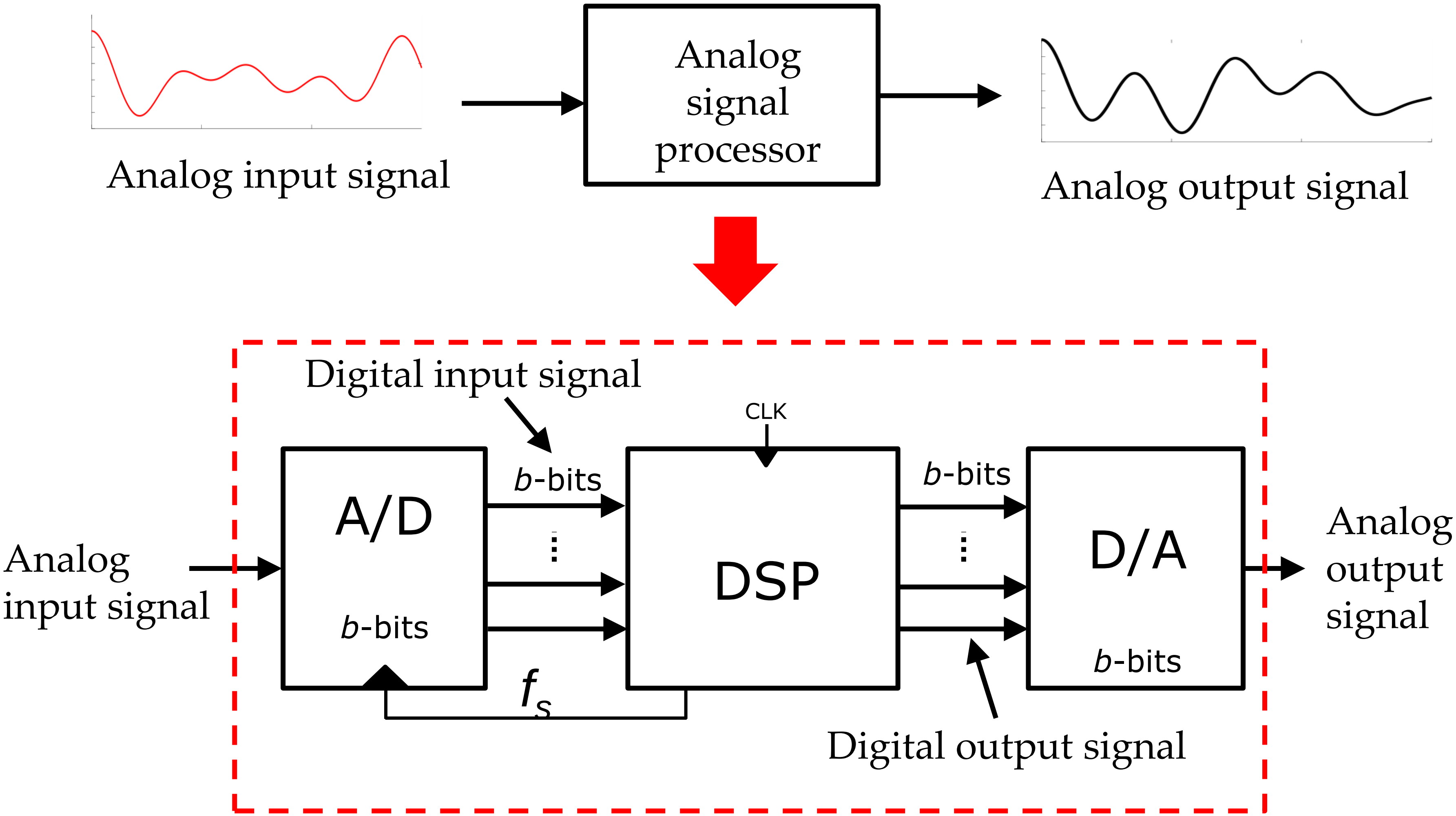
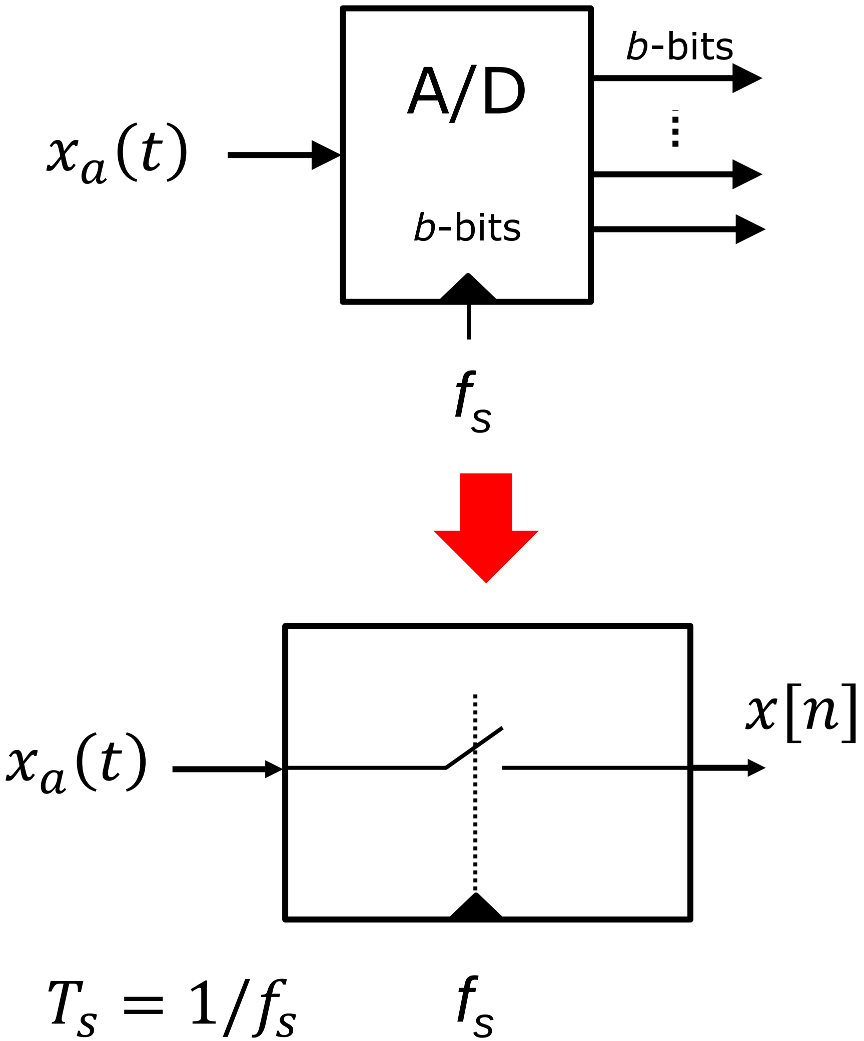
DSP System

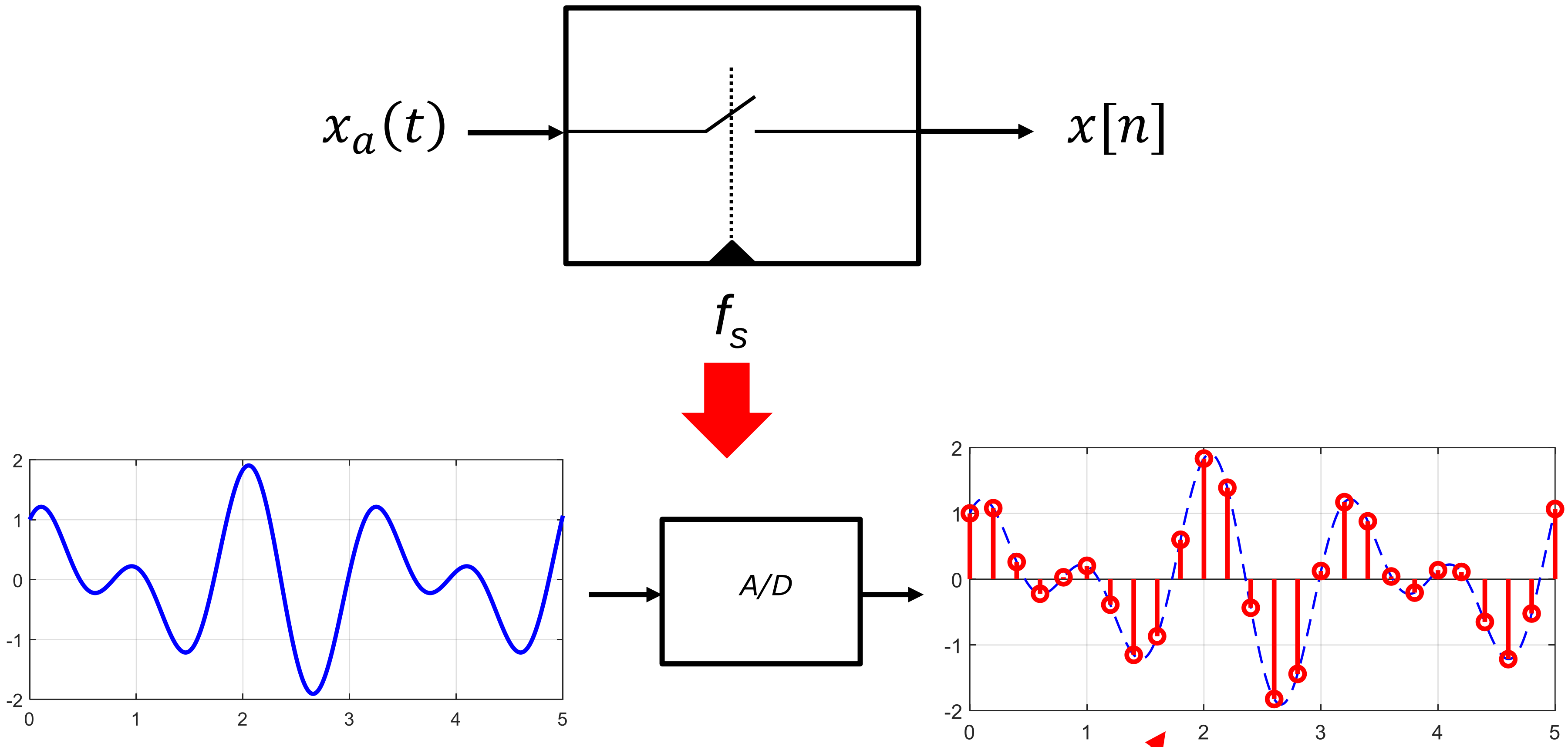

Analog input signal
Analog
signal
processor
Analog output signal
Digital input signal
CLK
A/D
b-bits
b-bits
DSP
b-bits
D/A
b-bits
Analog
input signal
Analog
output
signal
f_s
Digital output signal

Signal Sampling



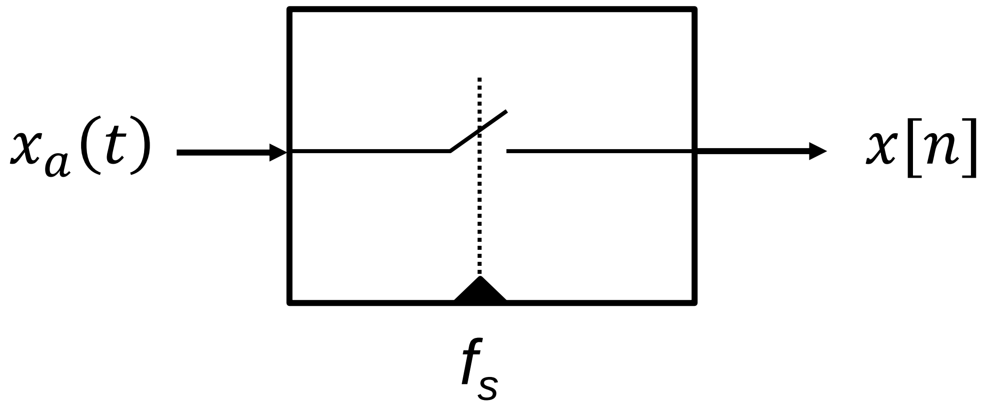
- In practice, the output of A/D converter is a digital signal.
- Digital signals are too complicated for analysis.
- Instead one can use a simplified model (Ideal switch model) and ignore the effects of quantization.
- If $b \geq 12$ *bits* the distortion to the signal is very small.
- $x[n]$ is a discrete-time (DT) signal.

Signal Sampling



This is a discrete-time signal. Samples are separated by sampling interval, T_s.

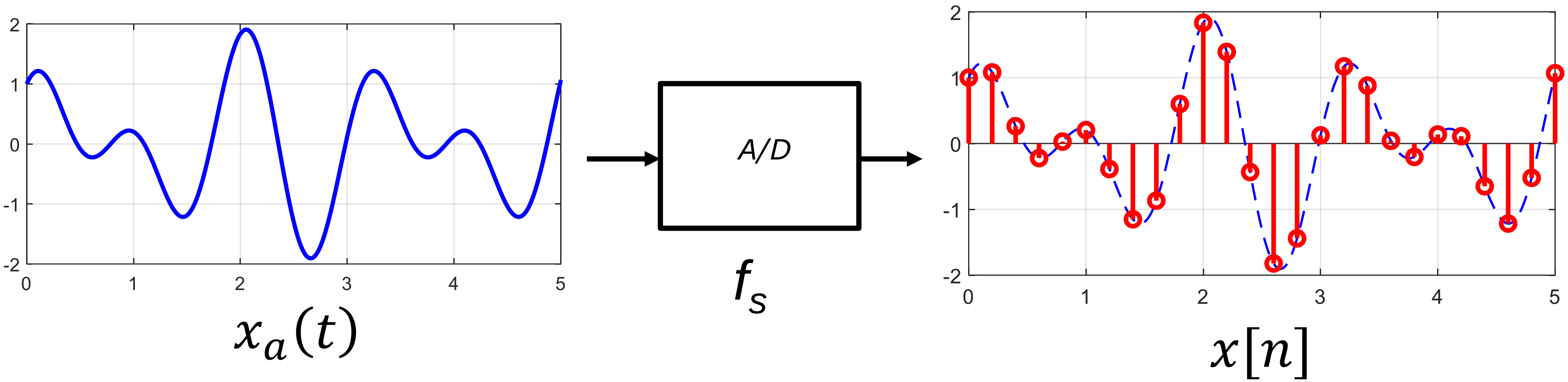
Signal Sampling



$$x_a(t) = \cos(2\pi f t)$$

$$x[n] = \cos(2\pi f n T_s) = \cos\left(2\pi f n \frac{1}{f_s}\right) = \cos\left(2\pi n \frac{f}{f_s}\right)$$

Sampling Theorem

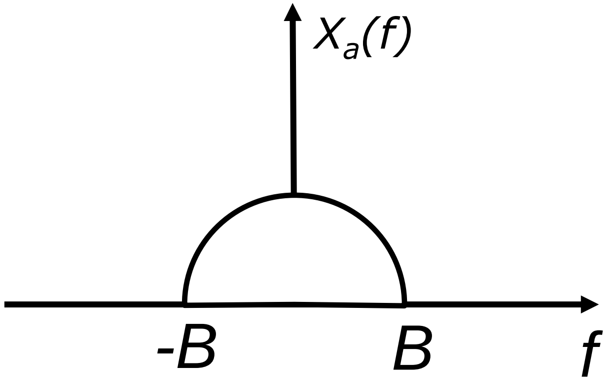


- We know that $x_a(t)$ is band-limited. Can it be sampled at an arbitrary sampling rate? **Answer is NO.**

- Sampling theorem describes the correct sampling frequency for sampling.

Sampling Theorem

Let the analog input signal (after anti-aliasing filtering) be band-limited to B Hz such that $X_a(f) = 0$ for $|f| > B$.



$$x_a(t) \overset{\mathcal{F}}{\Leftrightarrow} X_a(f)$$

$$x_s(t) = x_a(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

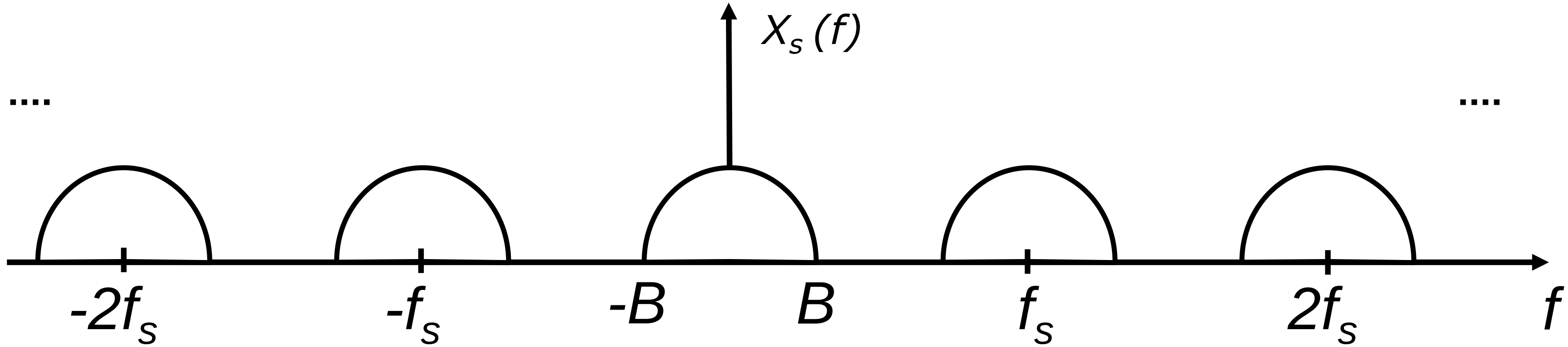
$$= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

Sampling Theorem

$$x_s(t) \overset{\mathcal{F}}{\Leftrightarrow} X_s(f)$$

$$X_s(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(f - kf_s) = \sum_{k=-\infty}^{\infty} x(nT_s)e^{-j2\pi nT_s f}$$

Because of the equation in the middle that $X_s(f)$ can be depicted as



Background theory required to prove the Sampling Theorem

Frequency shift property of Fourier Transforms

$$x_s(t) \overset{\mathcal{F}}{\Leftrightarrow} X_s(\omega)$$

$$e^{jat}x_s(t) \overset{\mathcal{F}}{\Leftrightarrow} X_s(\omega - a) \qquad \omega \ - \mathrm{rad/s}$$

$$e^{jat}x_s(t) \overset{\mathcal{F}}{\Leftrightarrow} X_s\left(f - \frac{a}{2\pi}\right) \qquad f \ - \text{hertz}$$

Fourier series of a pulse train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$X_k = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\, e^{-j2k\pi f_s t} dt$$

$$X_k = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-j2k\pi f_s t} dt = \frac{1}{T_s}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi k f_s t}$$

Sampling Theorem: Proof

Proof

$$X_s(f) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)\right] e^{-j2\pi t f}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-j2\pi n T_s f} \int_{-\infty}^{\infty} \delta(t - nT_s) dt$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-j2\pi n f T_s}$$

$$x_s(t) = x_a(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

$$= x_a(t) \left[\sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi k f_s t}\right]$$

Fourier series for train of impulses

$$= \frac{1}{T_s}\left[\sum_{k=-\infty}^{\infty} x_a(t) e^{j2\pi k f_s t}\right]$$

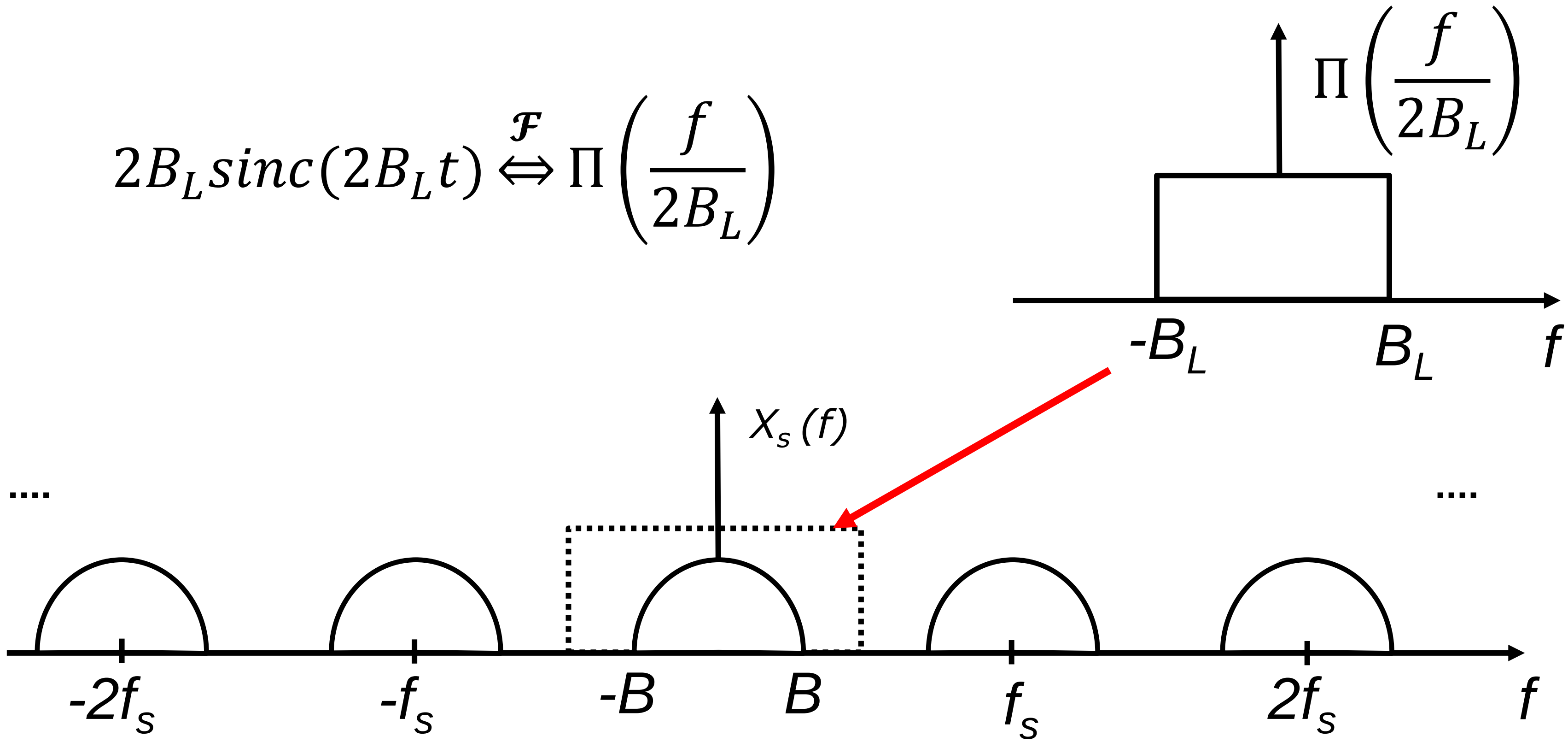
$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - k f_s)$$

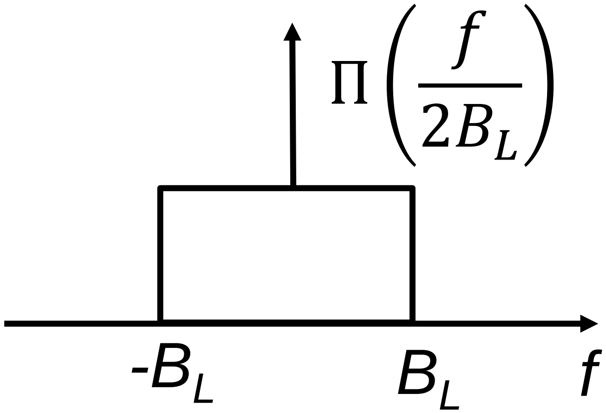
Get the Fourier Transform of both sides and use the frequency shift property

Sampling Theorem

The original analog signal, $x_a(t)$, can be recovered from the sampled signal, $x_s(t)$, by using an appropriately selected low-pass-filter (ideally "Brick-Wall Filter") as follows

$$2B_L sinc(2B_L t) \overset{\mathcal{F}}{\Leftrightarrow} \Pi\left(\frac{f}{2B_L}\right)$$

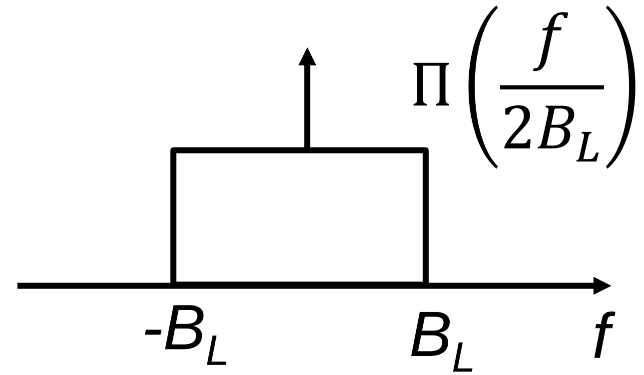


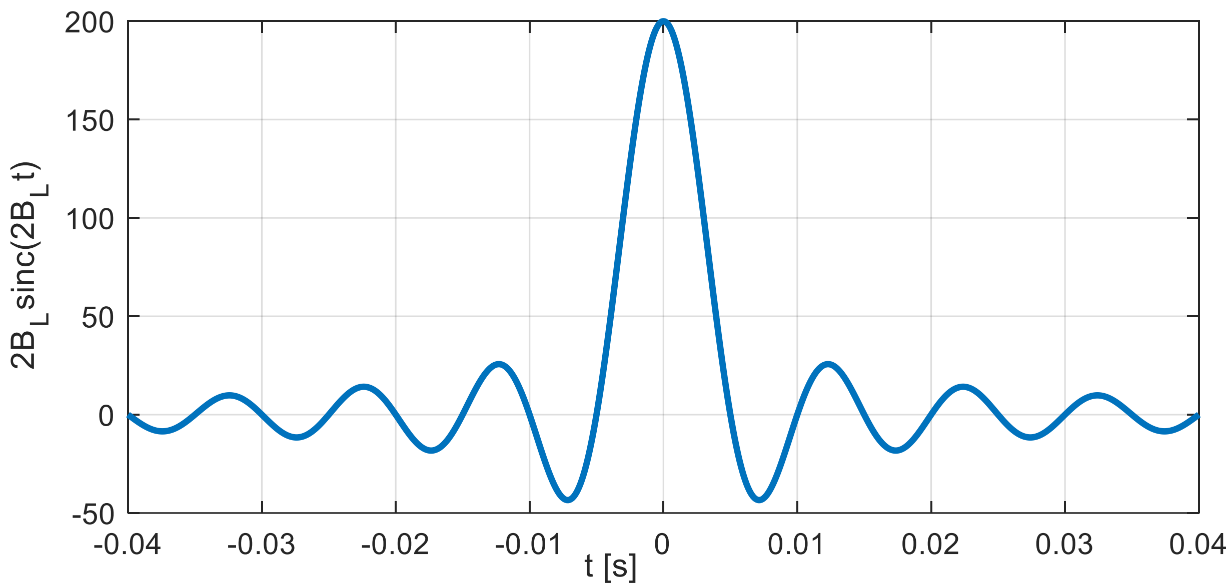


$$\frac{1}{2\pi}\int_{-2\pi B_L}^{2\pi B_L} 1\, e^{j\omega t} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-2\pi B_L}^{2\pi B_L} = 2B_L sinc(2B_L t)$$

Sampling Theorem

$$2B_L sinc(2B_L t) \overset{\mathcal{F}}{\Leftrightarrow} \Pi\left(\frac{f}{2B_L}\right)$$





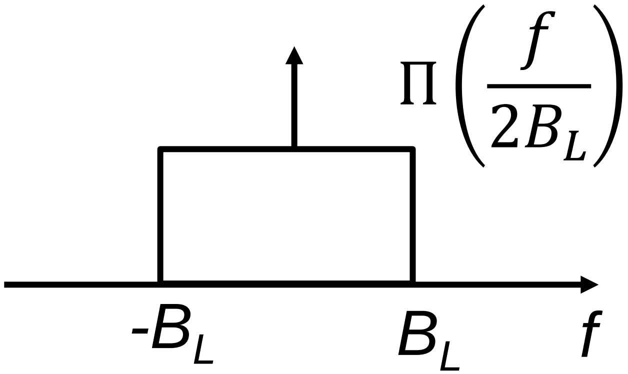
MATLAB:

```
>>
>> B=100;
>> t=-4/B:1/10000:4/B;
>> x=2*B*sinc(2*B*t);
>> plot(t,x)
>>
```

Here $B_L = 100Hz$. Note that the function crosses zero at every $\frac{1}{2B_L}$.

Sampling Theorem

$$2B_L sinc(2B_L t) \overset{\mathcal{F}}{\Leftrightarrow} \Pi\left(\frac{f}{2B_L}\right)$$



$$\bar{x}_a(t) \overset{\mathcal{F}}{\Leftrightarrow} X_s(f)\Pi\left(\frac{f}{2B_L}\right)T_s$$

$$\bar{x}_a(t) \overset{\mathcal{F}}{\Leftrightarrow} \sum_{n=-\infty}^{\infty} x[nT_s]e^{-j2\pi nT_s f} \; \Pi\left(\frac{f}{2B_L}\right)T_s$$

Invoking the frequency shift property for each term:

$$\bar{x}_a(t) = 2B_L T_s \sum_{n=-\infty}^{\infty} x[nT_s] \frac{sin(2\pi B_L(t - nT_s))}{2\pi B_L(t - nT_s)}$$

Sampling Theorem

But typically, $B_L = \frac{f_s}{2}$ is selected. Hence, $\bar{x}_a(t)$ can be simplified to get:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[nT_s] \frac{sin\left(\pi f_s\,(t - nT_s)\right)}{\pi f_s\,(t - nT_s)} = \sum_{n=-\infty}^{\infty} x[nT_s] sinc\left(\frac{t - nT_s}{T_s}\right)$$

Sampling Theorem

Sampling Theorem (Interpretation 1):

If the highest frequency component of an analog signal, $x_a(t)$, is $f_{max} = B$, then the signal should be sampled at a rate $f_s \geq 2f_{max} = 2B$ for the samples to completely (without any information loss) describe the original analog signal. The Nyquist sampling rate is defined as $f_N = 2f_{max}$

Interpretation 2: Any analog signal limited to the bandwidth $f_{max} = B$ and the time interval T can be completely specified by giving at least $2BT$ number of equally spaced samples.

Sampling Theorem

Interpretation 3:

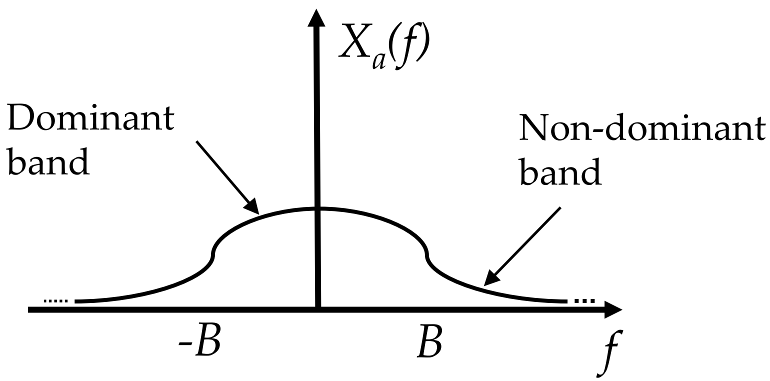
If you are given $2BT$ equally spaced samples spread over a time duration T of a signal with B Hz bandwidth, the underlying analog signal is ***UNIQUE*** and one can find it exactly using:

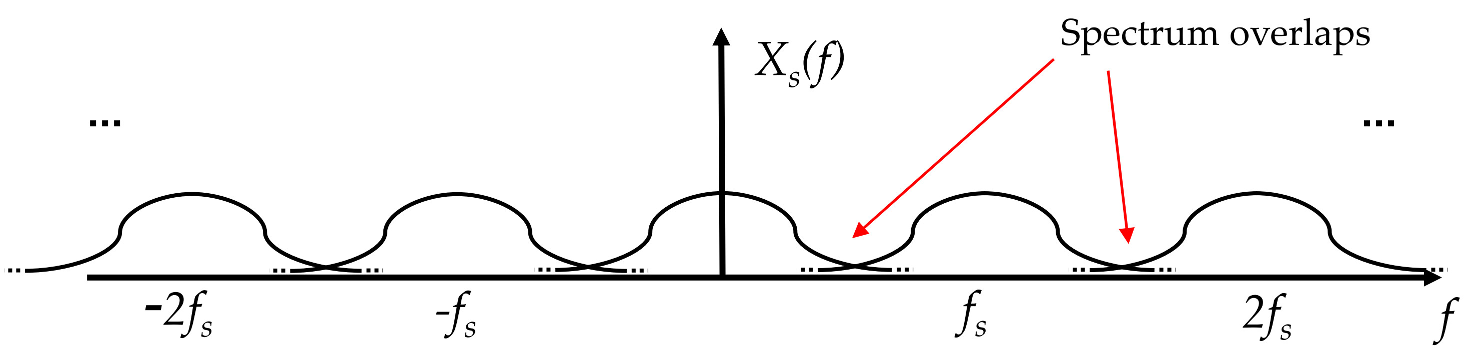
$$x_a(t) = \sum_{n=0}^{2BT-1} x[nT_s] sinc\left(\frac{t - nT_s}{T_s}\right)$$

where $T_s = \frac{1}{2B}$.

Anti-aliasing Filtering Revisited

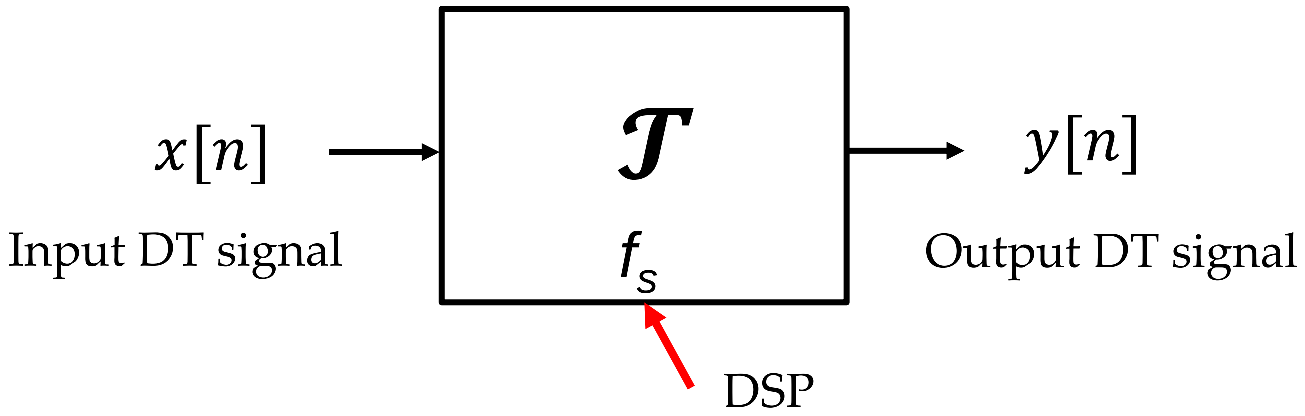
The real analog signals often have a dominant and non-dominant band. The non-dominant band could occupy several times larger a band than the dominant band. Sampling such an analog signal will lead to a sampled signal with following spectrum:





In these circumstances, no finite sampling frequency could avoid spectrum overlaps. Hence, a low-pass filter known as anti-aliasing filter must be used prior to sampling so that the power in the non-dominant band is significantly reduced.

Simplified DSP Model



- This is the simplified model of a DSP system.
- DSP denoted by $\mathcal{T}$ gets an input DT signal and produces an output DT signal.
- $y[n] = \mathcal{T}(x[n])$ or $x[n] \xrightarrow{\mathcal{T}} y[n]$

There is always a sampling frequency f_s associated with a DSP.

Periodic Vs Non-periodic

An analog signal is said to be periodic if $x_a(t + T_p) = x_a(t)$ where T_p is the fundamental period. Similarly a DT signal is said to be periodic if:

$$x[n + N_p] = x[n]$$

where N_p is the fundamental frequency, which has to be an integer.

E.g.: What is the period of $x[n] = \cos(0.4\pi n)$?

$$\cos\left(0.4\pi(n + N_p)\right) = \cos(0.4\pi n)$$

$$\cos(0.4\pi N_p + 0.4\pi n) = \cos(0.4\pi n) \quad - - - - (A)$$

If $0.4\pi N_p = 2\pi$, (A) is satisfied. So $N_p = 5$.

Energy Vs Power

In the context of analog signal processing, the power and energy of an analog signal, $x_a(t)$, are defined respectively as:

Power: $$P = \frac{1}{T_0}\int_0^{T_0} x_a^2(t)dt$$

Energy: $$E = \int_{-\infty}^{\infty} x_a^2(t)dt$$

where T_0 is the time over which the average power is required.

Energy Vs Power (DT)

The energy of a discrete-time energy signal is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \, ,$$

and the power of a discrete-time power signal is defined as:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \, .$$

If the energy E is not finite, the signal is a power signal. If the power P is not finite, the signal is an energy signal.