Asymptotic Waveform Evaluation:

The basics of AWE:

Let the transfer function of a circuit be H(s)

$$H(s) = \frac{A(s)}{B(s)}$$

$$= \sum_{m=0}^{\infty} \frac{k_m}{s - p_m}$$

k_m = residue associated with pole p_m

AWE involves moment matching.

What are moments?

$$H(s) = \int_0^{\infty} h(t) e^{-st} dt \qquad \text{Laplace Transform}$$

Let e^{-st} be expanded around s=0 using a Taylor's series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Then

$$H(s) = \int_0^{\infty} h(t) \left(1 - st + \frac{1}{2} s^2 t^2 - \frac{1}{6} s^3 t^3 + \frac{1}{24} s^4 t^4 \right) dt$$

$$H(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^k \int_0^{\infty} t^k h(t) dt$$

$$\frac{dH(s)}{ds} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} k s^{k-1} \int_0^{\infty} t^k h(t) dt$$

A general definition of moments:

The q^{th} moment is defined as:

$$H(s=0) = \int_0^\infty h(t)dt$$

$$H^{(1)}(s=0) = -\int_0^\infty h(t)tdt$$

$$H^{(2)}(s=0) = \int_0^\infty h(t)t^2dt$$

$$H^{(3)}(s=0) = -\int_0^\infty h(t)t^3dt$$

Application of moments to represent the transfer function $H(s)$

$$\begin{aligned} H(s) &= \int_0^\infty h(t)\left(1 - st + \frac{1}{2}s^2t^2 - \frac{1}{6}s^3t^3 + \frac{1}{24}s^4t^4\right)dt \\ &= H(0) + sH^{(1)}(0) + \frac{1}{2}s^2H^{(2)}(0) + \frac{1}{6}s^3H^{(3)}(0) + \ldots \\ &= m_o + m_1s + m_2s^2 + m_3s^3 + \ldots \\ &= \sum_{k=0}^{\infty}\frac{s^k}{k!}H^{(k)}(s=0) = \sum_{k=0}^{\infty} m_k s^k \end{aligned}$$

Where

$$m_k = \frac{1}{k!}H^{(k)}(s=0) = \frac{(-1)^q}{q!}\int_0^\infty t^q h(t)dt$$

How does AWE work?

Let us assume that the moments are known.

We then want to find the dominant poles and residues of the system.

Let the transfer function be approximated as follows:

$$H(s) \cong \frac{a_o + a_1s + a_2s^2 + \ldots a_{q-1}s^{q-1}}{1 + b_1s + \ldots . + b_qs^q}$$

We don't know the coefficients a_i or b_i.

To illustrate the computation process, we look at the following example – 4th order approximation:

Step 1: Determine the poles

$$H(s) \cong \frac{a_o + a_1s + a_2s^2 + a_3s^3}{1 + b_1s + b_2s^2 + b_3s^3 + b_4s^4}$$

Equate this to: $H(s) = m_o + m_1s + m_2s^2 + m_3s^3 + \ldots$

We get:

$$\left(1+b_1 s+b_2 s^2+b_3 s^3+b_4 s^4\right)\left(m_o+m_1 s+m_2 s^2+m_3 s^3+\ldots\right)=a_o+a_1 s+a_2 s^2+a_3 s^3$$

Multiply and equate powers of s to get

$$\begin{aligned}
s^o &: a_o = m_o \\
s^1 &: a_1 = m_o b_1 + m_1 \\
s^2 &: a_2 = m_o b_2 + m_1 b_1 + m_2 \\
s^3 &: a_3 = m_o b_3 + m_1 b_2 + m_2 b_1 + m_3
\end{aligned}$$

Higher powers of s:

$$\begin{aligned}
s^4 &: 0 = m_o b_4 + m_1 b_3 + m_2 b_2 + m_3 b_1 + m_4 \\
s^5 &: 0 = m_1 b_4 + m_2 b_3 + m_3 b_2 + m_4 b_1 + m_5 \\
s^6 &: 0 = m_2 b_4 + m_3 b_3 + m_4 b_2 + m_5 b_1 + m_6 \\
s^7 &: 0 = m_3 b_4 + m_4 b_3 + m_5 b_2 + m_6 b_1 + m_7
\end{aligned}$$

$$\begin{bmatrix} m_o & m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 & m_4 \\ m_2 & m_3 & m_4 & m_5 \\ m_3 & m_4 & m_5 & m_6 \end{bmatrix} \begin{bmatrix} b_4 \\ b_3 \\ b_2 \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_4 \\ m_5 \\ m_6 \\ m_7 \end{bmatrix}$$

Determination of denominator coefficients and roots:

In general:

$$\begin{bmatrix} m_o & m_1 & m_2 & \ldots & m_{q-1} \\ m_1 & m_2 & m_3 & \ldots & m_q \\ m_2 & m_3 & m_4 & \ldots & m_{q+1} \\ . & . & . & . & . \\ . & . & . & . & . \\ m_{q-1} & m_q & m_{q+1} & \ldots & m_{2q-2} \end{bmatrix} \begin{bmatrix} b_q \\ b_{q-1} \\ . \\ . \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_q \\ m_{q+1} \\ . \\ . \\ m_{2q-1} \end{bmatrix}$$

Gaussian elimination is used to find the coefficients $b_1, b_2 \ldots\ldots b_q$ where q=4 in this case.

Solve $B(s)=0$ to obtain the poles of the system.

That is solve:

$$b_q s^q + b_{q-1} s^{q-1} + b_{q-2} s^{q-2} + \ldots + b_1 s + 1 = 0$$

This gives the poles of the system.

Step 2: Determine the **residues.**

Generalised approach to determining the residues:

$$h(t) = \sum_{j=1}^{q} k_j e^{p_j t}$$

Or in the frequency domain:

$$H(s) = \sum_{j=1}^{q} \frac{k_j}{s - p_j} = \frac{a_o + a_1 s + a_2 s^2 + ... + a_{q-1} s^{q-1}}{1 + b_1 s + b_2 s^2 + ... + b_q s^q}$$

Now:

$$\frac{k_j}{s - p_j} = k_j \left(\frac{1}{s - p_j} \right) = \frac{-k_j}{p_j} \left(\frac{1}{1 - \frac{s}{p_j}} \right)$$

Let: $x = \dfrac{s}{p_j}$

Now: $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...$

Thus: $(1 - \dfrac{s}{p_j})^{-1} = 1 + \left(\dfrac{s}{p_j} \right) + \left(\dfrac{s}{p_j} \right)^2 + \left(\dfrac{s}{p_j} \right)^3 + ...$

Hence:

$$\begin{aligned} H(s) &= \sum_{j=1}^{q} \frac{k_j}{s - p_j} \\ &= \sum_{j=1}^{q} \frac{-k_j}{p_j} \left(1 + \frac{s}{p_j} + \frac{s^2}{p_j^2} + \frac{s^3}{p_j^3} + ... \right) \end{aligned}$$

But: $H(s) = m_o + m_1 s + m_2 s^2 + ... m_{2q-1} s^{2q-1}$

Thus:

The unity terms correspond to m_o

The $\dfrac{s}{p_j}$ terms correspond to $m_1 s$

The $\dfrac{s^2}{p_j^2}$ terms correspond to $m_2 s^2$

The $\dfrac{s^{2q-1}}{p_j^{2q-1}}$ terms correspond to $m_{2q-1} s^{2q-1}$

$$m_o = -\left(\frac{k_1}{p_1} + \frac{k_2}{p_2} + \ldots + \frac{k_q}{p_q}\right)$$
$$m_1 = -\left(\frac{k_1}{p_1^2} + \frac{k_2}{p_2^2} + \ldots + \frac{k_q}{p_q^2}\right)$$
$$m_{2q-1} = -\left(\frac{k_1}{p_1^{2q}} + \frac{k_2}{p_2^{2q}} + \ldots + \frac{k_q}{p_q^{2q}}\right)$$

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \frac{1}{p_1} & \frac{1}{p_2} & \ldots & \frac{1}{p_q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{1}{p_1^{q-1}} & \frac{1}{p_2^{q-1}} & \ldots & \frac{1}{p_q^{q-1}} \end{bmatrix} \begin{bmatrix} \frac{1}{p_1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \frac{1}{p_2} & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & \frac{1}{p_{q-1}} & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{p_q} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ . \\ . \\ . \\ k_q \end{bmatrix} = - \begin{bmatrix} m_o \\ m_1 \\ . \\ . \\ . \\ m_{q-1} \end{bmatrix}$$

Or in symbols: $V\Lambda k = -m$

Thus $\Rightarrow$ $k = -\Lambda^{-1}V^{-1}m$ Residues are now known!

Hence: $h(t) = \sum_{k=1}^{q} k_j e^{p_j t}$

Now how do we calculate the moments!

Consider the state space representation of the system:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = C^T X(s)$$

For an impulse input, $U(s) = 1$

$$Y(s) = C^T (sI - A)^{-1} B$$

Now expand $Y(s)$ about s=0

$Y(s) = C^T (-A)^{-1} B - C^T (-A)^{-2} Bs + C^T (-A)^{-3} Bs^2 \ldots\ldots$

But:

$$Y(s) = m_o + m_1 s + m_2 s^2 + \ldots$$

Hence:

$$m_o = -C^T A^{-1} B$$
$$m_1 = -C^T A^{-2} B$$
$$m_2 = -C^T A^{-3} B$$
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Summary of AWE

1. Form a state – space representation
2. Form the moments
3. Find the poles of the system
4. Find the residues

Hence the impulse response is:

$$h(t) = k_o \delta(t) + k_1 e^{p_1 t} + \ldots + k_n e^{p_n t}$$

Example

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$m_0 = -C^T A^{-1} B = 1$$
$$m_1 = -C^T A^{-2} B = -4$$
$$m_2 = 30$$
$$m_3 = -246$$
$$m_4 = 2037$$
$$m_5 = -16886$$

Case 1: First-order approximation

$$m_0 b_1 = -m_1$$

$$\Rightarrow b_1 = 4$$

The poles are hence the roots of:

$$b_1 p + 1 = 0$$
$$\Rightarrow p = -0.25$$

$$k = -m_0 p$$
$$k = 0.25$$

$$h(t) = 0.25e^{-0.25t}$$

Case 2: Second-order approximation

$$\begin{bmatrix} m_0 & m_1 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} = -\begin{bmatrix} m_2 \\ m_3 \end{bmatrix}$$
$$\Rightarrow b_2 = 6$$
$$b_1 = 9$$

The poles are hence the roots of:

$$b_2 p^2 + b_1 p + 1 = 0$$
$$p_1 = -1.3792$$
$$p_2 = -0.1208$$

To determine the residues:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{p_1} & \frac{1}{p_2} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{bmatrix}$$

$$\Rightarrow \quad k = -\Lambda^{-1} V^{-1} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix}$$
$$k_1 = 0.7809, \quad k_2 = 0.0524$$

$$h(t) = 0.7809e^{-1.3792t} + 0.0524e^{-0.1208t}$$