# Fractional-Order Lossy Transmission Line with Skin Effect using NILT Method

Nawfal Al-Zubaidi R-Smith, Aslihan Kartci, and Lubomír Brančík
Department of Radio Electronics
Brno University of Technology
Brno, Czech Republic
Email: alzubaidi@phd.feec.vutbr.cz, kartci@feec.vutbr.cz, brancik@feec.vutbr.cz

***Abstract*—The paper describes a method of modelling lossy transmission lines with frequency dependent parameters in the fractional-order domain. It is shown that in high frequency systems the frequency dependence of the transmission line (TL) and other related losses are essential to be considered in the model. In this paper fractional-order domain is used to model the lossy transmission line, which compensates for losses and provides higher flexibility and more reality for the TL analysis and simulation. As for the frequency dependence of the TL skin effect is included in the model, due to its high impact in high frequency applications. Analysis of a TL is processed in the Laplace-domain which effectively simplifies the solution. Though, the difficult or even impossible part is to obtain the required original time-domain from the results by analytic means. In order to overcome this difficulty a numerical inverse Laplace transform (NILT) method is introduced in this paper. The NILT method is implemented successfully, and the voltage/current waveform distributions along the lossy fractional-order TL are simulated and the results are displayed. The model is effectively tested and realized in the Matlab language.**

***Keywords*—fractional calculus; frequency dependence; Laplace transform; numerical inversion; transmission line**

## I. Introduction

The study on fractional-order calculus has been introduced more than 300 years ago, although there are still many potential topics in this field that are open for future research. For instance, the historical background of anomalous diffusion in space and time domain using variety of fractional-order derivatives was provided by Metzler and Klafter [1]. The form and properties of fractional-order derivatives as mathematical description can be found in the monographs by Podlubny [2], Herrman [3], and Meerschaert and Sikorskii [4] while the theoretical models and experimental applications are available in [5]. The fractional-order transmission lines (TL) such as T-line, CRLHY-line etc. model including inductor, capacitor, resistor, and conductor is developed in [6], [7] as CMOS on-chip at terahertz frequency band. The tendency of a high-frequency electric current to distribute itself in a conductor so that the current density near the surface is greater than at its core is called the skin effect (SE). The transmission line parameters have obvious frequency-dependent phenomena due to the skin and edge effect on high frequencies. Taking into account the SE and other conditions, the calculation and analysis of frequency-dependent transmission line gives more accurate transmission characteristics. The immittances of fractional-order inductor and fractional-order capacitor are defined according to the fractional calculus of electrical components as $Z_\alpha = s^\alpha L_\alpha$, and $Y_\beta = s^\beta C_\beta$, where $L_\alpha$ is the pseudo-inductance with order $\alpha \in (0, 1]$ and $C_\beta$ is the pseudo-capacitance with order $\beta \in (0.1]$ [8]. While $\alpha$ and $\beta$ are not equal to one, there exist frequency-dependent losses in the real part called dispersion loss of the fractional-order inductance/capacitance expressions. Given these loss descriptions in the fractional-order model, the distributed dispersion loss and non-quasistatic effects are characterized in frequency-dependent terms with much less number of RLGC components as when compared with the integer order model. The aim of this paper is to show the effect of fractional-order parameters on lossy transmission lines with skin effect in the manner of frequency-domain. The voltage (current) wave is composed of the incidence wave and the reflected wave. The voltage of the arbitrary position in the transmission line has two terms corresponding to waves travelling in the opposite directions: the first term represents the incidence wave while the other term represents the reflected wave. Therefore, current/voltage waveforms distributions along the TL are simulated in the time-domain by the use of the hyperbolic NILT method [9]. The paper is organized as follows: Section II shortly introduces the integer and fractional-order lossy TL. Then in section III, the discussion about the frequency-dependency on TL is given. While section IV shows the numerical simulation results, the conclusion is presented in section V.

## II. Fractional-Order Transmission Line

### A. Telegrapher Equations and Laplace domain Model

Transmission line modeling in the time-domain is an ongoing challenge for the simulation of integrated circuits and/or printed circuit boards at high frequencies [10]. A standard uniform lossy transmission line (TL) can be represented mathematically by a pair of telegraphic equations (a set of partial differential equations). As it is well known the analysis and solution of the partial differential equation systems is

This research was financially supported by the Czech Science Foundation under grant 15-18288S, and by Czech Ministry of Education in frame of National Sustainability Program under grant LO1401. For research requirements, infrastructure of the SIX Center was used.

effectively simplified by introducing the Laplace transformation. Therefore when considering zero initial voltage and current distributions and performing Laplace transformation, with respect to time, the result is [11],

$$-\frac{dV(s,x)}{dx} = Z_0(s)I(s,x), \tag{1}$$

$$-\frac{dI(s,x)}{dx} = Y_0(s)V(s,x), \tag{2}$$

where $Z_0(s) = R_0 + sL_0$ and $Y_0(s) = G_0 + sC_0$ are per-unit-length (p.u.l.) series impedance and shunt admittance, respectively, with $R_0, L_0, G_0$, and $C_0$, as p.u.l. primary parameters, representing series resistance, inductance, shunt conductance and capacitance, accordingly. In Fig. 1 the Laplace-domain model of the TL of length l is shown. This model corresponds to the solution of the system of equations (1) and (2), while incorporating boundary conditions.

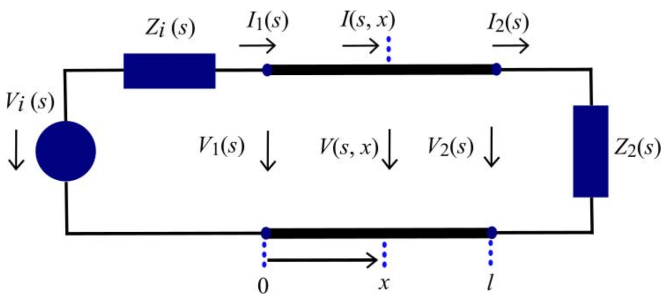


Fig. 1. Laplace-domain model of the TL with linear terminations.

The TL shown in Fig.1 is terminated on both sides with linear networks; these linear networks can be represented with their Thévenin or Norton equivalents, here the former was considered. Accordingly, the Laplace-domain solution results are in the forms:

$$V(s,x) = V_i(s)\frac{Z_c(s)}{Z_i(s)+Z_c(s)}\cdot\frac{e^{-\gamma(s)x}+\rho_2(s)e^{-\gamma(s)[2l-x]}}{1-\rho_1(s)\rho_2(s)e^{-2\gamma(s)l}}, \tag{3}$$

$$I(s,x) = V_i(s)\frac{1}{Z_i(s)+Z_c(s)}\cdot\frac{e^{-\gamma(s)x}-\rho_2(s)e^{-\gamma(s)[2l-x]}}{1-\rho_1(s)\rho_2(s)e^{-2\gamma(s)l}}, \tag{4}$$

with $Z_c(s)$ and $\gamma(s)$ as the characteristic impedance and propagation constant, respectively, given as:

$$Z_c(s) = \sqrt{\frac{Z_0(s)}{Y_0(s)}},\ \gamma(s) = \sqrt{Z_0(s)\cdot Y_0(s)}, \tag{5}$$

where the reflection coefficients $\rho_1(s)$ and $\rho_2(s)$ are described by the following equations:

$$\rho_1(s) = \frac{Z_i(s)-Z_c(s)}{Z_i(s)+Z_c(s)},\ \rho_2(s) = \frac{Z_2(s)-Z_c(s)}{Z_2(s)+Z_c(s)}. \tag{6}$$

In the following section the fractional-order elements (capacitance and inductance) are imposed to the TL model.

*B. Fractional-Order Primary Parameters*

Considering a TL of length l with distributed elements as in Fig.1 the TL has an elementary section as shown in Fig. 2. The primary parameters $L_0', C_0'$ have the fractional orders $\alpha$ and $\beta$, respectively, shown in Fig. 2, where the dash designation represents only the fractional order immittances, keeping into consideration that the numerical value of the parameters is not changed.

As it will be demonstrated imposing fractional parameters give a high degree of freedom for TL modeling, enabling to realize the losses especially resulting from high frequencies [12], and furthermore it builds on the fact that the solution of the TL travelling waveforms continuously depends on the fractional derivative.

Incorporating the fractional-order primary parameters into transmission line systems of equations results in:

$$Z_{cf}(s) = \sqrt{\frac{Z_{0f}(s)}{Y_{0f}(s)}} = \sqrt{\frac{R_0 + s^\alpha L_0'}{G_0 + s^\beta C_0'}}, \tag{7}$$

where $Z_{cf}(s)$ is the fractional-order characteristic impedance, which is replaced instead of the integer-order characteristic impedance $Z_c(s)$ in (5).

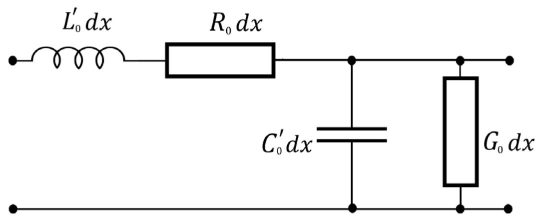


Fig. 2. Fractional-order primary parameters of the TL.

Moreover, the propagation constant $\gamma(s)$ changes accordingly with fractional parameters as:

$$\gamma_f(s) = \sqrt{Z_{0f}(s)\cdot Y_{0f}(s)} = \sqrt{(R_0 + s^\alpha L_0')(G_0 + s^\beta C_0')}. \tag{8}$$

Detailed analysis and discussions of imposed fractional-order parameters will be shown is chapter IV.

## III. Frequency Dependence

Due to continuous rapid increase in operational frequencies in power systems and transmission speed an essential characteristic to be considered is the frequency dependence properties. Generally, skin effect is an important frequency dependent parameter to be considered when dealing with high frequency applications [13]; its impact is usually higher than the polarization effect on the surrounding medium [14]. Mainly, skin effect increases the resultant losses of the TL. By transforming the skin effect into the Laplace-domain the solution is effectively simplified, rather than solving in the time-domain. The fractional-order series impedance is supplemented by the term $K\sqrt{(s)}$, to result in:

$$Z_{0f}(s) = R_0 + s^\alpha L_0' + K\sqrt{s}, \tag{9}$$

where the latter term $K\sqrt{s}$ represents high-frequency internal resistance including high frequency inductive reactance. Consequently, equation (9) is substituted in (7) and (8).

The following chapter presents the possibility of obtaining the time domain simulation of the described TL system of equations via a proposed numerical inverse Laplace transform (NILT) method.

## IV. Analysis and Results

R. Ismail and R. El-Barkouky in [12] have shown the effect of the fractional parameters  and  on the real part of the propagation constant (the attenuation) and the real part of the characteristic impedance (the resistive element) by studying the change in the sign. Basically, in the work done in [12] it illustrates that by checking the sign of the real part of the propagation constant , which should remain negative as it represents attenuation, and as practically $0 < \alpha, \beta < 2$, then the region of the fractional parameters $\alpha$ and $\beta$ are found to be in the range $1 < \alpha + \beta < 3$.

For the analysis let us consider a lossy transmission line of length $l = 2$ m, $R_0 = 0.35$ Ω/m, $L_0 = 265$ nH/m, $G_0 = 0.1$ mS/m, $C_0 = 95$ pF/m, $Z_i = 10$ Ω, $Z_2 = 2.5$ kΩ, and the skin effect parameter $K = 4.5 \cdot 10^{(-4)}\ \Omega\sqrt{\mathrm{s}}/\mathrm{m}$ for frequency dependent line or$K = 0$ is used for the frequency independent line [15]. The fractional-order characteristic impedance with incorporating the skin effect is given as:

$$Z_{cf}(s) = \sqrt{\frac{R_0 + s^{\alpha} L_0' + K\sqrt{(s)}}{G_0 + s^{\beta} C_0'}}\,, \tag{10}$$

consequently, the fractional-order propagation constant with the skin effect is given as:

$$\gamma_f(s) = \sqrt{(R_0 + s^{\alpha} L_0' + K\sqrt{s}) \cdot (G_0 + s^{\beta} C_0')}\,. \tag{11}$$

The relation of $Z_{cf}(s)$ and $\gamma_f(s)$ with the fractional parameters $\alpha$ and $\beta$ are shown in Fig. 3 and Fig. 4, accordingly. The subfigures (a), (b) and (c) illustrate the real value with the regions of interest which bound the selection of $\alpha$, $\beta$. The simulations are done in Matlab where $s = \mathrm{j}\omega$, and $\omega$ is the angular frequency and is chosen as $\omega = 10^9\,\mathrm{rad}\cdot\mathrm{s}^{(-1)}$.

It is interesting to notice the real part of $Z_{cf}$ which represents the resistive element and how it is affected by the fractional parameters $\alpha$ and $\beta$. Mainly, with a constant load, the TL response to applied voltage is resistive rather than reactive, despite being composed of inductive, capacitive and resistive elements. The integer-model case of characteristic impedance can be noticed in the Fig. 3 (a) and that is when $\alpha = 1$ and $\beta = 1$. Interestingly, the real value of the characteristic impedance (resistive element) is positive and increasing when the fractional values are in the range $0 < \alpha, \beta < 1$ and hence we bound our choices in this region; above this range the real value of the characteristic impedance is negative and decreasing as shown in Fig. 3 (c).

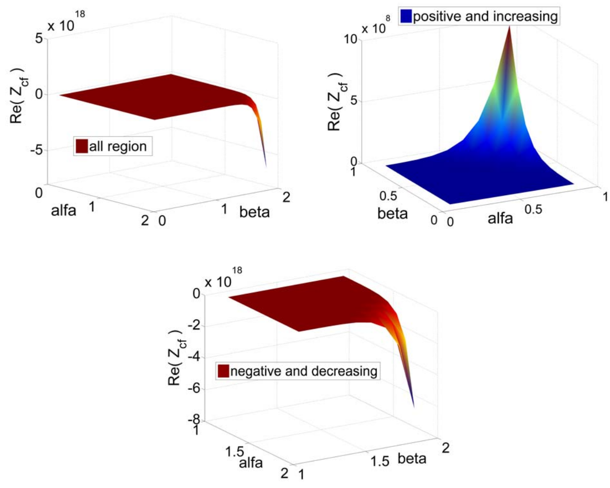

Fig. 3. Real $Z_{cf}$ versus $\alpha$ and $\beta$

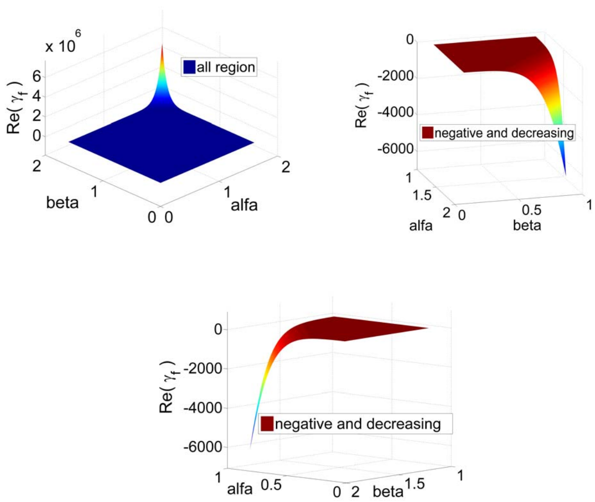

Fig. 4. Real $\gamma_f$ versus $\alpha$ and $\beta$

After the substitutions are done to (3) and (4), the voltage/current waveforms are effectively obtained in the time-domain by undergoing the proposed hyperbolic numerical inverse Laplace transform (NILT) method. The hyperbolic NILT method proposed in authors previous works [9], [16], and [17], is a potential technique to numerically obtain the time-domain simulation of the travelling waveforms. Principally, the method is based on approximating the exponential kernel $\exp(st)$ in the definition Bromwich integral by specific hyperbolic relations. The proposed NILT inversion was tested and performs with high accuracy and computational efficiency. In Fig. 5 and Fig. 6 the voltage and current waveforms propagation at the middle of the TL are shown, respectively. For the analysis the TL was excited with the voltage waveform:

$$v_i(t) = \begin{cases} \sin^2\left(\frac{\pi \cdot t}{2 \cdot 10^{-9}}\right), \text{ for } 0 < t < 2 \cdot 10^{-9}, \\ 0, \quad \text{elsewhere.} \end{cases}$$

The excitation waveform is first converted to the Laplace domain i.e. $V_i(s) = \frac{2\pi^2(1-\exp(-2\cdot10^{-9}s))}{s((2\cdot10^{-9}s)^2+4\pi^2)}$, and then the voltage/current waveforms are simulated for three cases; the integer TL model, the integer TL model with frequency dependence parameters and the fractional-order TL model with frequency dependence parameters, results are illustrated in Fig. 5 and Fig. 6.

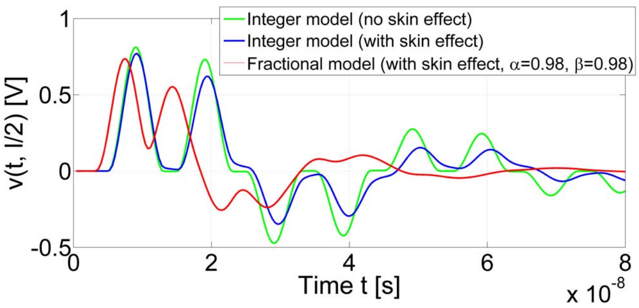


Fig. 5. Voltage waveform in the middle of the TL (x=1 m).

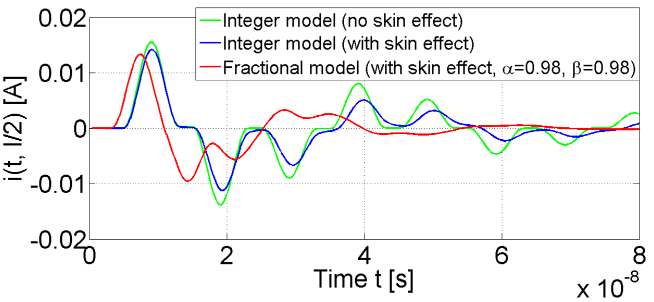


Fig. 6. Current waveform in the middle of the TL (x= 1 m).

By observing the results in Fig. 5 and Fig.6 and comparing the fractional-order model including skin effect with the integer-model it is noticed that the results capture the TL properties which are neglected in the integer model. Though, more optimal choices are currently studied in our future work by comparing our theoretical results to practically measured results of a TL in the laboratory. Fig. 7 and Fig. 8, illustrate the voltage distribution along the TL described above with the choices of fractional parameters $\alpha, \beta = 0.98$ in Fig. 7, and $\alpha, \beta = 0.95$ in Fig. 8, these values are selected to be in the range bounded by checking the sign of the real value of $Z_{cf}$ and $\gamma_f$. It is evident that when $\alpha, \beta = 0.95$ the diffusion is faster than when $\alpha, \beta = 0.98$, such behaviors of fractional parameters were also observed in [18]. This is consistent with the behavior fractional order systems. That, in essence, as the forward wave and reflected wave travel on opposite directions, this leads to a reduction in the range of the voltage wave; specifically, this mainly builds on the fact that the solution continuously depends on the fractional derivative. As a result the fractional transmission line model captures the abnormal diffusion phenomena of the voltage wave in the TL much more precisely than the classical integer model, detailed discussion and illustrations are available in [18], [19].

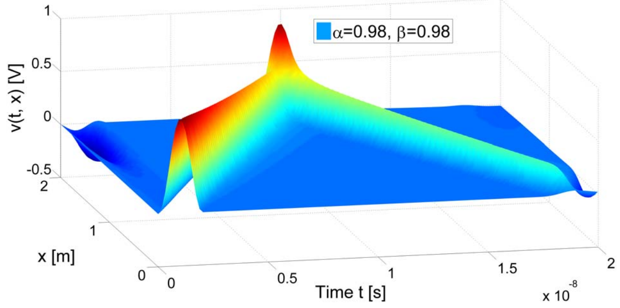


Fig. 7. Voltage distribution along the TL with fractional parameters.

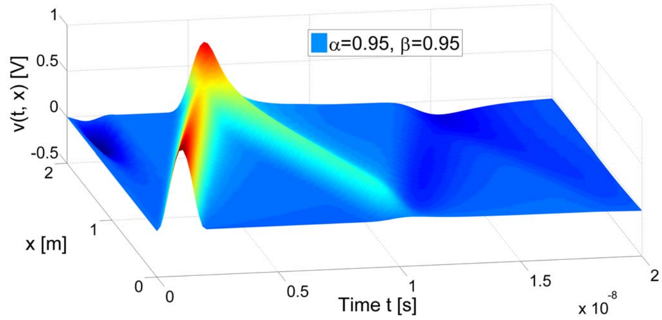


Fig. 8. Voltage distribution along the TL with fractional parameters.

There are vast advantages of applying fractional-order parameters to the TL model, for instance it captures the phenomena and properties that classical integer-order simply neglects, modeling the dynamics of distributed parameter systems using fractional calculus is a useful tool due to its infinite dimensionality [20]. A lossy semi-infinite fractional order TL model considering fractional R and C parameters is shown in [21], and it demonstrates fractional order behavior, where the current into the line is equal to half order derivative of the applied voltage.

The simulation results shown in Fig. 5 and Fig. 6 were computed using the hyperbolic 1D NILT method accelerated with the quotientdifference algorithm [17], on the other hand the 3D voltage distributions in Fig. 7 and Fig. 8 were obtained by the 2D FFT NILT method [11]. The computation time for obtaining the voltage waveform results in Fig. 5 was about 480 milliseconds for the integer-domain model and approximately 790 milliseconds for the fractional-order model. Noticeably, the voltage/current waveform propagation has a high impact caused from the skin effect (Fig. 5 and Fig. 6, in blue).

Generally, skin effect influences the TL by decreasing the cross sectional area, therefore it affects the resistance of the conductor by increasing it. As a result the skin effect causes a TL to heat faster and to a higher temperature at higher frequencies with the same levels of current [22]. Hence, it is an important parameter to be considered especially when working with high frequency systems. As for the fractional-order parameters $\alpha$ and $\beta$, it is noticed that with a slight change in the fractional order (i.e. 0.98) the result (Fig. 5 and Fig. 6 in red) proves that fractional-order modelling provides a high degree of flexibility to model lossy TLs.

## V. Conclusion

In this paper the transmission line modeling was realized by a pragmatic approach considering the fractional-order elements and incorporating frequency dependent parameters. The choice of the fractional parameters was optimized by testing their variation with the real part of the propagation constant and the fractional-order characteristic impedance. Generally, the simulation results are noticed to be more realistic due to the higher degree of freedom for modeling transmission lines with fractional-parameters and the importance of including frequency dependent parameters especially for high frequency applications. Results build on the fact that the solution of the traveling waves continuously depend on the fractional derivative. The 2D time-domain voltage/current waveforms were obtained by utilizing the hyperbolic NILT method and the 3D voltage propagations were obtained via the FFT 2D NILT method. Current work involves a practical measurement of a lossy TL in the laboratory and optimization of fractional-order parameters. Future work will be followed by incorporating fractional-order and frequency dependence for non-uniform lossy transmission lines or more sophisticated multi-conductor transmission lines.

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