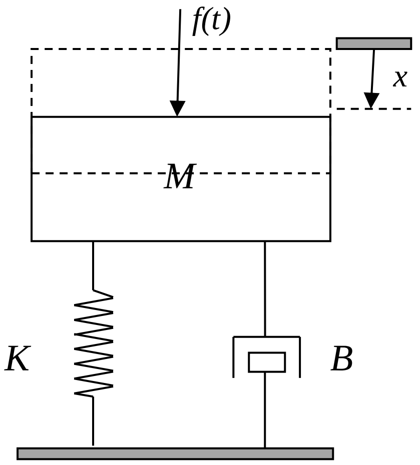
4. State-space representation

4.1 Introduction

- When the differential equations governing the system are linear, the model can be described in terms of a Laplace transfer function.

- Consider for example the spring-mass damper system:



Physical model

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

- While transfer functions are compact and have several advantages in relation to analysis of dynamical systems, they also have a number of weaknesses as follows:
  - They do not handle initial conditions (assumed to be zero).
  - Information about internal variables is lost ($\frac{dx}{dt}$ in the case of the above example).
  - For general *m*-input, *p*-output systems, we would need a total of *m* x *p* transfer functions to fully describe the system.

- An alternative model representation, known as a **state-space model**, overcomes these weaknesses.

- The basic idea behind state-space modelling is to write down a set of first-order differential equations in terms of the system state(s) and input(s).

- **Example 4.1:** Develop a state-space model for the second-order differential equation model of the spring-mass damper system:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$

Solution

We proceed as follows:

Firstly we have a ***second-order*** differential equation; hence we define two states x_1 and x_2:

$$x_1 = x, \quad x_2 = \frac{dx}{dt}$$

When we do it this way we get the CONTROLLABLE CANONICAL FORM state-space model.

We then obtain expressions for $\dfrac{dx_1}{dt}$ and $\dfrac{dx_2}{dt}$:

$$\frac{dx_1}{dt} = \frac{dx}{dt} \qquad\qquad \frac{dx_2}{dt} = \frac{d^2x}{dt^2} = \frac{1}{M}\left(f(t) - B\frac{dx}{dt} - Kx\right)$$

Note that the latter expression is obtained from the original mathematical model.

We can rewrite these equations in terms of the states x_1 and x_2 only, by replacing x and $\dfrac{dx}{dt}$ with their state equivalent. Hence:

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = \frac{1}{M}\left(f(t) - Bx_2 - Kx_1\right)$$

Now, we combine these into matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} f(t)$$

Finally, let us define the output as y = x. Hence y = x_1 giving:

$$[y]=[1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+[0]f(t)\Rightarrow[y]=[1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Our state-space model is expressed, in full, as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}f(t)$$

$$[y]=[1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4.2 Formal definitions

- The commonly used terms associated with state-space representation are as follows:

- **State** – the state of a dynamic system is the smallest set of variables (called state variables) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behaviour of the system for $t \geq t_0$.

- **State variables** – the variables that make up the state as defined above.

- **State vector** – when there is more than one state variable, they are normally collected together into a vector called a state vector.

- **State-space** – the n-dimensional state vector can be viewed as a point moving around in n-dimensional space. This n-dimensional space is known as state-space.

General state-space form

- The dynamics of a general n^{th} order linear dynamical system, with m inputs, is completely described by a n^{th} order state-space equation of the form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\equiv \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \mathbf{Ax} + \mathbf{Bu} \qquad \text{state equation}$$

with a set of initial conditions (one for each state):

$$\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} = \boldsymbol{x_0} \qquad \text{initial conditions}$$

- The model output(s) are given by a linear combination of the states and the inputs. Given p outputs, we obtain:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ d_{21} & \cdots & d_{2m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\equiv \quad \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du} \qquad \text{output equation}$$

- Together, these two equations describe the state-space model of a system.

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu} \qquad , \mathbf{x}(0) = \mathbf{x_0}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

- The matrices ***A***, ***B***, ***C*** and ***D*** are called the state matrix, input matrix, output matrix and direct transmission matrix respectively:

A	-	state matrix (n x n)
B	-	input matrix (n x m)
C	-	output matrix (p x n)
D	-	direct transmission matrix (p x m)

4.3 State – space representations are not unique

State-space representations are NOT unique.

Consider the following system

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2f(t)$$

Controllable canonical form state-space representation

Form the transfer function

$$G(s) = \frac{Y(s)}{F(s)} = \frac{2}{s^2 + 4s + 3}$$

Let $x_{1c}(t) = y(t)$ and $x_{2c}(t) = \frac{dy(t)}{dt}$.

One needs two equations – one for the time derivative of state 1 and one for the time derivative of state 2.

$$\boxed{\frac{dx_{1c}}{dt} = \frac{dy(t)}{dt} = x_{2c}}$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2f$$

$$\frac{d^2y}{dt^2} = -4\frac{dy}{dt} - 3y + 2f$$

Now put in the states

$$\boxed{\frac{d^2y}{dt^2} = \frac{dx_{2c}}{dt} = -4x_{2c} - 3x_{1c} + 2f}$$

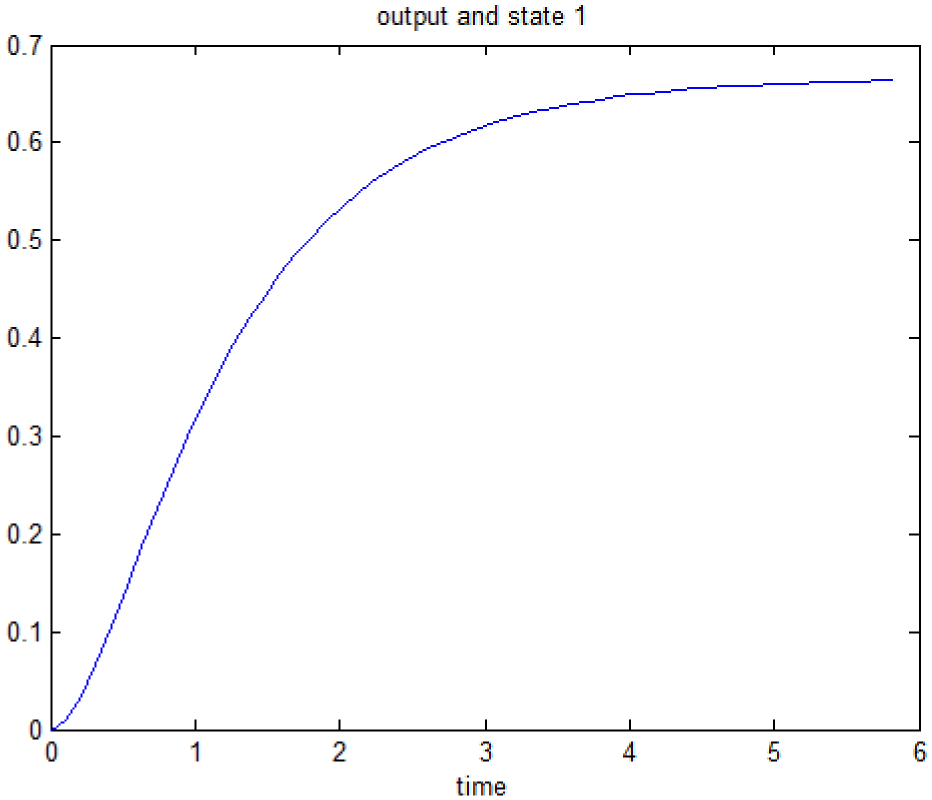
Put in matrix formation

$$\frac{d}{dt}\begin{bmatrix} x_{1c}(t) \\ x_{2c}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} x_{1c}(t) \\ x_{2c}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} f(t)$$

The output is $y(t)$ which is the first state

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_{1c}(t) \\ x_{2c}(t) \end{bmatrix}$$

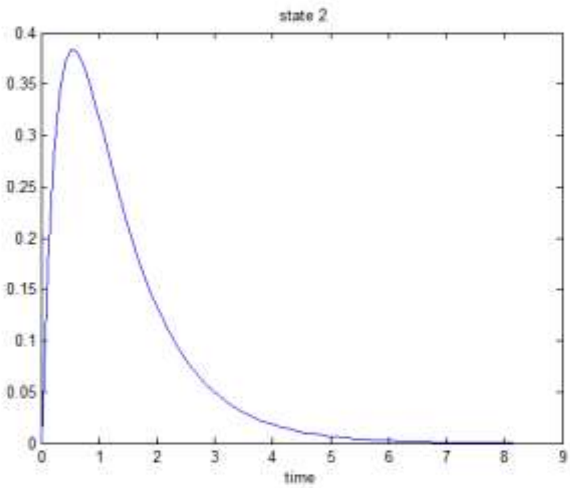
The unit-step response is as follows



Suppose you wanted to see what the second state looks like

$$o(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1c}(t) \\ x_{2c}(t) \end{bmatrix}$$

Then you could plot this.

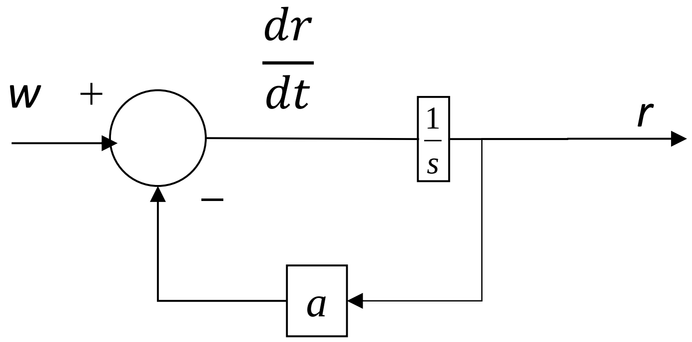


Series form representation

Express $G(s)$ as the product of $\frac{1}{s+a}$ terms.

$$G(s) = \frac{2}{s^2 + 4s + 3} = 2\left(\frac{1}{s+1}\right)\left(\frac{1}{s+3}\right)$$

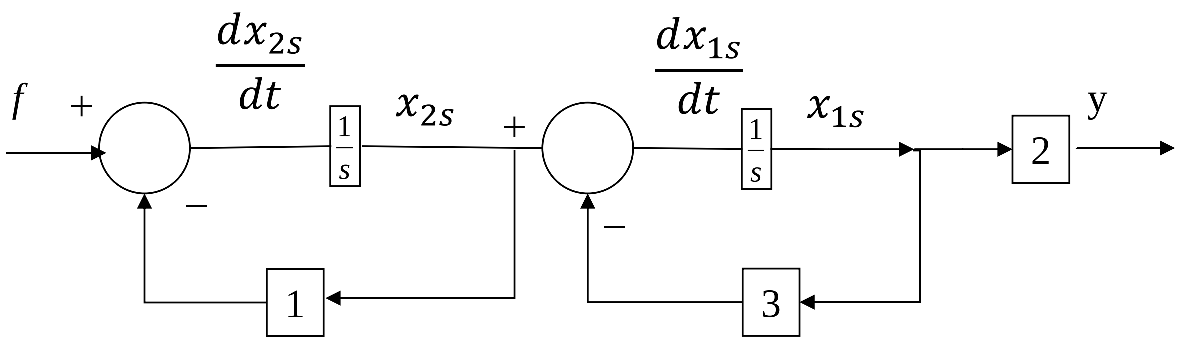
The block diagram for $\frac{1}{s+a}$ is



$$\frac{dr}{dt} = w - ar$$
$$sR(s) = W(s) - aR(s)$$
$$\frac{R(s)}{W(s)} = \frac{1}{s+a}$$

So $G(s)$ can be represented in block diagram form as follows

$$G(s) = \frac{2}{s^2 + 4s + 3} = 2\left(\frac{1}{s+1}\right)\left(\frac{1}{s+3}\right)$$



$$\frac{dx_{1s}}{dt} = -3x_{1s} + x_{2s}$$

$$\frac{dx_{2s}}{dt} = -x_{2s} + f$$
$$y = 2x_{1s}$$

Now write the state-space representation in matrix form

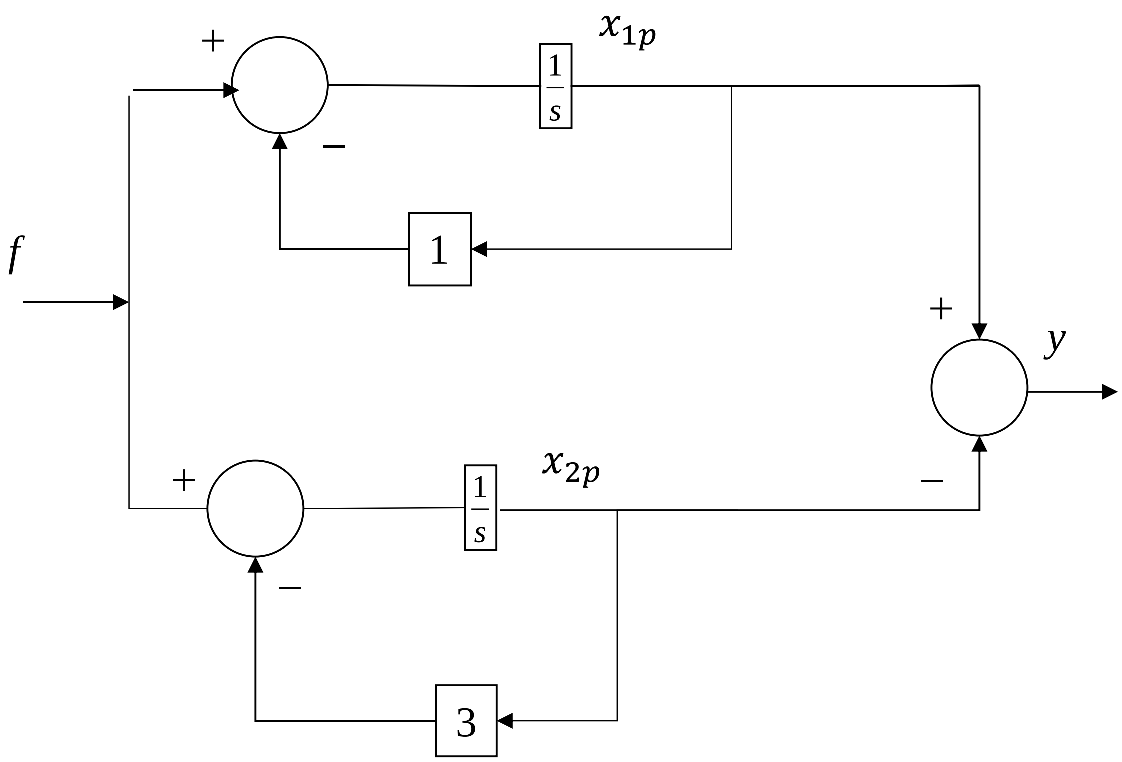
$$\frac{d}{dt}\begin{bmatrix} x_{1s}(t) \\ x_{2s}(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_{1s}(t) \\ x_{2s}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$
$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} x_{1s}(t) \\ x_{2s}(t) \end{bmatrix}$$

Parallel form state-space representation

Express $G(s)$ as the sum or difference of $\frac{1}{s+a}$ terms.

$$G(s) = \frac{2}{s^2 + 4s + 3} = \left(\frac{1}{s+1}\right) - \left(\frac{1}{s+3}\right)$$

This can be shown in block diagram format as follows



$$\frac{dx_{1p}}{dt} = -x_{1p} + f$$
$$\frac{dx_{2p}}{dt} = -3x_{2p} + f$$

$$\frac{d}{dt}\begin{bmatrix} x_{1p}(t) \\ x_{2p}(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_{1p}(t) \\ x_{2p}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} f(t)$$

$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_{1p}(t) \\ x_{2p}(t) \end{bmatrix}$$

Note that the state matrix is diagonal. This is important for computations as solution of the state equations (to be dealt with in a later lecture) involves exponentials of the state matrix and it is much easier to get the exponential of a diagonal matrix.

4.4 Transforming system models to state-space form

Derivation of the state-space model from the transfer function model of the system.

4.4.1 Transfer function → State space

Case 1: when the input does *not* involve derivatives

- Consider the following third-order transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- Converting this to the time domain gives:

$$\frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = ku$$

or

$$\frac{d^3 y}{dt^3} = -a_2 \frac{d^2 y}{dt^2} - a_1 \frac{dy}{dt} - a_0 y + ku$$

- We simply define the states as $x_1 = y, x_2 = \frac{dy}{dt}$ and $x_3 = \frac{d^2 y}{dt^2}$, leading to the following state model:

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{d^2 y}{dt^2} = x_3$$

$$\frac{dx_3}{dt} = \frac{d^3 y}{dt^3} = -a_2 x_3 - a_1 x_2 - a_0 x_1 + ku$$

$$y = x_1$$

giving:

$$
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

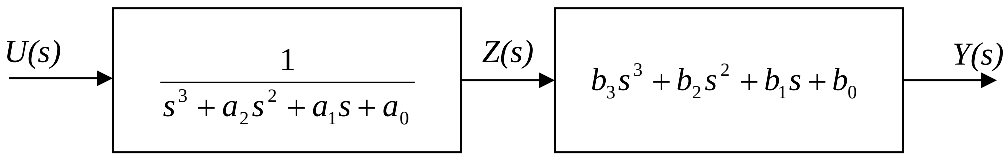
4.4.2 Transfer function → State space
Case 2: when the input does involve derivatives

- Consider the following third-order transfer function:

$$
G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}
$$

- Firstly, we split the transfer function into two parts by defining an intermediate variable *Z(s)* as follows:

$$
\frac{Z(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \qquad \text{and} \qquad \frac{Y(s)}{Z(s)} = b_3 s^3 + b_2 s^2 + b_1 s + b_0
$$



- Converting to the time domain gives:

$$
\frac{d^3 z}{dt^3} + a_2 \frac{d^2 z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z = u
$$

and

$$
b_3 \frac{d^3 z}{dt^3} + b_2 \frac{d^2 z}{dt^2} + b_1 \frac{dz}{dt} + b_0 z = y
$$

- Setting the states as $x_1 = z, x_2 = \frac{dz}{dt}$ and $x_3 = \frac{d^2z}{dt^2}$, we get the state equation as follows:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = -a_2x_3 - a_1x_2 - a_0x_1 + u$$

Once again, we have the **controllable canonical form**.

giving:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

- In terms of the output equation:

  - if $b_3 = 0$ then: $y = b_2\frac{d^2z}{dt^2} + b_1\frac{dz}{dt} + b_0z = b_2x_3 + b_1x_2 + b_0x_1$

    giving: $y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

    (*Note, the D matrix is 0 in this case*)

  - if $b_3 \neq 0$ then: $y = b_2x_3 + b_1x_2 + b_0x_1 + b_3\frac{d^3z}{dt^3}$

    but: $\frac{d^3z}{dt^3} = \frac{dx_3}{dt} = -a_2x_3 - a_1x_2 - a_0x_1 + u$

    giving:

    $$y = (b_2 - b_3a_2)x_3 + (b_1 - b_3a_1)x_2 + (b_0 - b_3a_0)x_1 + b_3u$$

therefore:

$$y = \big[(b_0 - b_3 a_0) \quad (b_1 - b_3 a_1) \quad (b_2 - b_3 a_2)\big] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_3]\, u$$

- Note that the state matrix (i.e. matrix $\boldsymbol{A}$) is exactly the same for both types of transfer function models.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- This implies that the dynamics depend on the transfer function denominator only.

4.5 Obtaining transfer functions from state-space models

- In section 4.4, we saw how to go from a transfer function model to a state-space model (known as the **controllable canonical form**).

- It is also possible to go from a state-space model to a transfer function model as follows:

4.5.1 Continuous-time state-space model → transfer function

- Consider the following single-input-single-output continuous-time state-space model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \boldsymbol{D}u(t) \end{aligned}$$

- Taking the Laplace transform gives:

$$\begin{aligned} s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \\ Y(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s) \end{aligned}$$

- Rearranging the state equation gives:

$$(sI - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{B}U(s)$$

$$\Rightarrow \boldsymbol{X}(s) = (sI - \boldsymbol{A})^{-1}\boldsymbol{B}U(s)$$

- Substituting this equation into the output equation gives:

$$Y(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}U(s) + \mathbf{D}U(s)$$

- Hence, the transfer function is defined in terms of the state-space equation matrices as:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- **Example 4.2:** Determine the transfer function for the following system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t)$$

Solution

$$G(s) = \mathbf{C}(s\boldsymbol{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -2 \\ 3 & s-5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-5 & 2 \\ -3 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{s^2 - 5s + 6}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-5 \\ -3 \end{bmatrix}}{s^2 - 5s + 6} = \frac{s-5}{s^2 - 5s + 6}$$

Hence: $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{s-5}{s^2 - 5s + 6}$

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5. Solving the state equations

5.1 State transformations

- State-space representations are not unique. There are many selections of state variables which can describe a system.
- State variables can be real or fictitious.
- Some state representations lead to computationally attractive forms.
- Thus, by using the appropriate state transformation, we can obtain a state-space representation that leads to significantly easier computation.
- Consider the following state transformation:

$$\boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{z}(t)$$

- Here, $\boldsymbol{T}$ is any constant non-singular (i.e. invertible) n x n matrix. Since $\boldsymbol{T}$ is constant, we can write:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{T}\frac{d\boldsymbol{z}}{dt}$$

- Substituting these expressions into the state equation

$$\boldsymbol{T}\frac{d\boldsymbol{z}}{dt} = \boldsymbol{A}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{B}u$$

- Finally, premultiplying by $\boldsymbol{T}^{-1}$ gives us the new state equation:

$$\frac{d\boldsymbol{z}}{dt} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{T}^{-1}\boldsymbol{B}u$$

$$\frac{d\boldsymbol{z}}{dt} = \bar{\boldsymbol{A}}\boldsymbol{z} + \boldsymbol{T}^{-1}\boldsymbol{B}u$$

- The new output equation is:

$$y = \boldsymbol{C}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{D}u$$
$$y = \bar{\boldsymbol{C}}\boldsymbol{z} + \boldsymbol{D}u$$

- Note – the only condition on $\boldsymbol{T}$ is that it must have an inverse. Hence, there are an infinite number of state representations for the system.

- We are interested in finding a representation that leads to a diagonal $\overline{\boldsymbol{A}}$ matrix.

- So the issue becomes one of finding a suitable $\boldsymbol{T}$ so that $\overline{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$ is diagonal.

- Consider a matrix $\boldsymbol{A}$ with distinct eigenvalues λ_1, λ_2, ... λ_n. Let $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, ... $\boldsymbol{m}_n$ be the corresponding eigenvectors.

- The ***modal matrix***, $\boldsymbol{M}$, is formed from these eigenvectors:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \cdots & \boldsymbol{m}_n \end{bmatrix}$$

- Note that by definition, $\boldsymbol{A}\boldsymbol{m}_i = \lambda_i \boldsymbol{m}_i$. Hence,

$$\begin{aligned} \boldsymbol{AM} &= \boldsymbol{A}\begin{bmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \cdots & \boldsymbol{m}_n \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}\boldsymbol{m}_1 & \boldsymbol{A}\boldsymbol{m}_2 & \cdots & \boldsymbol{A}\boldsymbol{m}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\boldsymbol{m}_1 & \lambda_2\boldsymbol{m}_2 & \cdots & \lambda_n\boldsymbol{m}_n \end{bmatrix} \\ &= \boldsymbol{M}\Lambda \end{aligned}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- Since $\quad \boldsymbol{AM} = \boldsymbol{M}\Lambda \;\Rightarrow\; \Lambda = \boldsymbol{M}^{-1}\boldsymbol{AM}$

Thus selecting $\boldsymbol{T}=\boldsymbol{M}$ results in a diagonal matrix.

Forming the Modal Matrix

- Recall that eigenvalues are calculated as the roots of the **matrix characteristic equation**

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = 0$$

- The eigenvectors are then determined either by
  - solving $\boldsymbol{Am}_i = \lambda_i\, \boldsymbol{m}_i$ or
  - evaluating the cofactors of a row of $(\lambda \boldsymbol{I} - \boldsymbol{A})$

We shall use the second method using cofactors.

- **Example 5.1:** Determine the modal matrix for $\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}$

Solution

Eigenvalues:

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = 0 \Rightarrow \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda+1 & -2 \\ 1 & \lambda+4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)(\lambda+4)+2 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda+2)(\lambda+3) = 0$$

$$\Rightarrow \lambda_1 = -2, \lambda_2 = -3$$

Eigenvectors – ***(using cofactors)*** – here, the eigenvectors are obtained by writing the cofactors of **any** row of $(\lambda \boldsymbol{I} - \boldsymbol{A})$ in column format (note – make sure to allow for the correct sign, as shown below):

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda+1 & -2 \\ 1 & \lambda+4 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

Take row 1, for example. Delete the first row and column.

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda+1 & -2 \\ 1 & \lambda+4 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

This leaves λ+4. Now delete the first row and second column.

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda+1 & -2 \\ 1 & \lambda+4 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

This leaves 1 but because of its position, we enter –1.

$$\boldsymbol{m}_V = \begin{bmatrix} \lambda+4 \\ -1 \end{bmatrix}$$

Now we enter the eigenvalues to this vector:

$$\lambda = -2 \Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \lambda = -3 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Combining these gives the modal matrix:

$$\boldsymbol{M} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

The entries in the diagonal matrix must correspond to the associated vector in $\boldsymbol{M}$

$$\boldsymbol{\Lambda} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

So

$$\boldsymbol{M}\Lambda\boldsymbol{M}^{-1} = \boldsymbol{A}$$

5.2 Continuous-time solution

- The general linear Single Input Single Output (SISO) continuous-time state-space model is given by:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

- We will consider the unforced and forced responses separately.

Unforced response

- This is when the input is set to zero for all time, i.e.:

$$u(t) = 0, \ \forall t$$

- The state equation then becomes:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t)$$

- Consider the scalar case $\frac{dx(t)}{dt} = ax(t)$. This is a first-order differential equation and has a solution:

$$x(t) = e^{at}x(0)$$

- The question is ... can we use the same solution when a is replaced by $\boldsymbol{A}$, i.e.:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

What is $e^{\boldsymbol{A}t}$?

- Expanding e^{at} gives:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \ldots$$

- Using the same expansion for $e^{\boldsymbol{A}t}$ gives:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \frac{(\boldsymbol{A}t)^3}{3!} + \ldots$$

- This can be rewritten as:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \boldsymbol{A}^3\frac{t^3}{3!} + \ldots = \sum_{k=0}^{\infty}\boldsymbol{A}^k\frac{t^k}{k!}$$

- Hence, we can conclude that $e^{\boldsymbol{A}t}$ is an $n \times n$ matrix defined as:

$$e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty}\boldsymbol{A}^k\frac{t^k}{k!}$$

- Now consider $\frac{d}{dt}\left(e^{\boldsymbol{A}t}\right)$:

$$\begin{aligned}
\frac{d}{dt}\left(e^{\boldsymbol{A}t}\right) &= \frac{d}{dt}\left(\boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \boldsymbol{A}^3\frac{t^3}{3!} + \ldots\right) \\
&= \boldsymbol{0} + \boldsymbol{A} + 2\boldsymbol{A}^2\frac{t^1}{2!} + 3\boldsymbol{A}^3\frac{t^2}{3!} + \ldots \\
&= \boldsymbol{A} + \boldsymbol{A}^2\frac{t^1}{1!} + \boldsymbol{A}^3\frac{t^2}{2!} + \ldots \\
&= \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \ldots\right) = \boldsymbol{A}\,e^{\boldsymbol{A}t}
\end{aligned}$$

- This is exactly the same as for the scalar case, i.e. $\frac{d}{dt}\left(e^{at}\right) = ae^{at}$, and hence $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$ is indeed a valid solution for the state equation $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t)$.

- Note – most of the properties of $e^{\boldsymbol{A}t}$ are the same as e^{at}. For example:

$$\int e^{at}dt = \frac{1}{a}e^{at} = a^{-1}e^{at} \quad \rightarrow \int e^{\boldsymbol{A}t}dt = \boldsymbol{A}^{-1}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}^{-1}$$

Forced response

- Now, consider the situation when the input is not zero:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

- This can be written as:

$$\frac{d\mathbf{x}(t)}{dt} - \mathbf{A}\mathbf{x}(t) = \mathbf{B}u(t)$$

- Premultiplying by $e^{-\mathbf{A}t}$ gives:

$$e^{-\mathbf{A}t}\left(\frac{d\mathbf{x}(t)}{dt} - \mathbf{A}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}u(t)$$

- This is equivalent to:

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}u(t)$$

- Integrating both sides with respect to t from 0 to t gives:

$$\int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right)d\tau = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

- Since $\int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right)d\tau = e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\Big|_0^t = e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0)$, we can write:

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

- Finally, bringing $\mathbf{x}(0)$ to the RHS of the equation and multiplying both sides by $e^{\mathbf{A}t}$ gives:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

- If we start from a time t_0 instead of zero, then:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

Output calculation

- Once the state has been determined, the output is easily computed as:

$$\begin{aligned} y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t) \\ &= \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau \quad + \boldsymbol{D}u(t) \end{aligned}$$

- For continuous systems, the state transition matrix is defined as:

$$\Phi(t) = e^{\boldsymbol{A}t}$$

- Hence, the output can be rewritten as:

$$y(t) = \boldsymbol{C}\Phi(t-t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{C}\Phi(t-\tau)\boldsymbol{B}u(\tau)d\tau \quad + \boldsymbol{D}u(t)$$

The Modal matrix method to compute the state-transition matrix

To form the state transition matrix, it is much easier from a computational viewpoint if one is getting the exponential of a diagonal matrix.

- Consider $e^{\boldsymbol{A}t}$ when $\boldsymbol{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

- Expanding $e^{\boldsymbol{A}t}$ as a power series gives:

$$\exp\left(\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t\right)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^2\frac{t^2}{2!}+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^3\frac{t^3}{3!}+\ldots$$

$$=\begin{bmatrix}1+\lambda_1 t+\lambda_1{}^2\dfrac{t^2}{2!}+\lambda_1{}^3\dfrac{t^3}{3!}+\ldots & 0\\ 0 & 1+\lambda_2 t+\lambda_2{}^2\dfrac{t^2}{2!}+\lambda_2{}^3\dfrac{t^3}{3!}+\ldots\end{bmatrix}$$

$$=\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$$

- Thus, when $\boldsymbol{A}$ is a diagonal matrix, calculating $e^{\boldsymbol{A}t}$ is straightforward.
- Consider: $\dfrac{d\mathbf{x}(t)}{dt}=\mathbf{A}\mathbf{x}(t)$
- Let $\boldsymbol{x}(t)=\boldsymbol{M}\boldsymbol{z}(t)$, where $\boldsymbol{M}$ is the modal matrix for $\boldsymbol{A}$. This gives:

$$\frac{d\mathbf{z}(t)}{dt}=\mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{z}(t)=\Omega\mathbf{z}(t)$$

- Solving this equation, we obtain: $\boldsymbol{z}(t)=e^{\Omega t}\boldsymbol{z}(0)$
- Note: $\Omega=\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$
- But $\boldsymbol{z}(t)=\boldsymbol{M}^{-1}\boldsymbol{x}(t)$, hence: $\boldsymbol{x}(t)=\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}\boldsymbol{x}(0)=e^{\boldsymbol{A}t}\boldsymbol{x}(0)$
- Thus, the state transition matrix $e^{\boldsymbol{A}t}$ can be computed as:

$$\Phi(t)=e^{\boldsymbol{A}t}=\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}$$

- **Example 5.2:** Determine the state transition matrix for the system:

$$\frac{d}{dt}[\boldsymbol{x}] = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{in}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}$$

Hence, determine the unit-impulse response if the initial values of the states are zero.

Solution

Eigenvalues: $\lambda = -2, \ -3$

Modal matrix: $\lambda I - \boldsymbol{A} = \begin{bmatrix} \lambda & -2 \\ 3 & \lambda + 5 \end{bmatrix}$

$$\boldsymbol{M}_V = \begin{bmatrix} \lambda + 5 \\ -3 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 3 & 2 \\ -3 & -3 \end{bmatrix}$$

Thus: $\mathbf{M}^{-1} = \begin{bmatrix} 1 & 2/3 \\ -1 & -1 \end{bmatrix}$

Hence:

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} 3 & 2 \\ -3 & -3 \end{bmatrix} e^{\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}t} \begin{bmatrix} 1 & 2/3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 2/3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

$$y(t) = \boldsymbol{C}e^{At}\boldsymbol{B}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(t) = 2e^{-2t} - 2e^{-3t}$$

- **Example 5.3:** Determine the output for the system:

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\mathbf{x}(\mathrm{t}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0]\boldsymbol{x}(t)$$

The input $u(t)$ is a step and the initial state $\boldsymbol{x}(0) = [1 \;\; -1]^{\mathrm{T}}$.

Solution

From example 5.2: $$e^{\boldsymbol{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

Need to determine: $$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

$$e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

$$\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau \rightarrow e^{\boldsymbol{A}t}\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}u(\tau)d\tau$$

$$= e^{\boldsymbol{A}t}\int_0^t \begin{bmatrix} 3e^{2\tau}-2e^{3\tau} & 2e^{2\tau}-2e^{3\tau} \\ -3e^{2\tau}+3e^{3\tau} & -2e^{2\tau}+3e^{3\tau} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(1)d\tau$$

$$= e^{\boldsymbol{A}t}\int_0^t \begin{bmatrix} 2e^{2\tau}-2e^{3\tau} \\ -2e^{2\tau}+3e^{3\tau} \end{bmatrix}d\tau$$

$$= e^{\boldsymbol{A}t}\left(\begin{bmatrix} e^{2\tau}-\tfrac{2}{3}e^{3\tau} \\ -e^{2\tau}+e^{3\tau} \end{bmatrix}\Bigg|_0^t\right)$$

$$= e^{\boldsymbol{A}t}\left(\begin{bmatrix} e^{2t}-\tfrac{2}{3}e^{3t}-1+\tfrac{2}{3} \\ -e^{2t}+e^{3t}+1-1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3e^{-2t}-2e^{-3t} & 2e^{-2t}-2e^{-3t} \\ -3e^{-2t}+3e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix}\begin{bmatrix} e^{2t}-\tfrac{2}{3}e^{3t}-\tfrac{1}{3} \\ -e^{2t}+e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} \left(3-2e^{t}-e^{-2t}-2e^{-t}+\tfrac{4}{3}+\tfrac{2}{3}e^{-3t}\right)+\left(-2+2e^{t}+2e^{-t}-2\right) \\ \left(-3+2e^{t}+e^{-2t}+3e^{-t}-2-e^{-3t}\right)+\left(2-2e^{t}-3e^{-t}+3\right) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}+\tfrac{2}{3}e^{-3t}+\tfrac{1}{3} \\ e^{-2t}-e^{-3t} \end{bmatrix}$$

Hence:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

$$\Rightarrow \boldsymbol{x}(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + \begin{bmatrix} -e^{-2t} + \frac{2}{3}e^{-3t} + \frac{1}{3} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

$$\Rightarrow \boldsymbol{x}(t) = \begin{bmatrix} \frac{2}{3}e^{-3t} + \frac{1}{3} \\ -e^{-3t} \end{bmatrix}$$

$$y(t) = [1 \quad 0]\boldsymbol{x}(t)$$

$$y(t) = \left[\frac{2}{3}e^{-3t} + \frac{1}{3}\right]u(t)$$

Continuous-time solution – Laplace Techniques

- The general linear Single Input Single Output (SISO) continuous-time state-space model is given by:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

An alternative approach to the time-domain methods is to use of the Laplace Transform.
Remember

$$L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0), \qquad L\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \frac{df}{dt}(0), \quad \text{etc.}$$

The solution to the state-space equation with the Laplace Transform is calculated as follows:

$$s\boldsymbol{I}\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}U(s)$$
$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{x}(0) + \boldsymbol{B}U(s)$$
$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s)$$

So

$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s) + \boldsymbol{D}U(s)$$

- **Example:** Determine the unit-step response for the system:

$$\frac{d\boldsymbol{x}(t)}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}(t)$$

if the input $u(t)$ is a step and the initial value of the states is

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

.

Solution
The input is a unit step so

$$U(s) = \frac{1}{s}$$

The initial state is given

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s) + \boldsymbol{D}U(s)$$

$$Y(s) = [1 \quad 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$+ [1 \quad 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}$$

$$Y(s) = [1 \quad 0]\left(\begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + [1 \quad 0]\left(\begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}$$

$$Y(s) = [1 \quad 0]\frac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$+ [1 \quad 0]\frac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}$$

$$Y(s) = \frac{s+3}{s^2+5s+6} + \frac{2}{s^2+5s+6}\frac{1}{s}$$
$$= \left(\frac{s^2+3s}{s^2+5s+6}\right)\frac{1}{s} + \left(\frac{2}{s^2+5s+6}\right)\frac{1}{s}$$
$$= \left(\frac{s^2+3s+2}{s^2+5s+6}\right)\frac{1}{s}$$
$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$
$$s^2 + 3s + 2 = A(s^2+5s+6) + Bs(s+3) + Cs(s+2)$$

Equate coefficients of powers of s

$$1 = A + B + C$$
$$3 = 5A + 3B + 2C$$
$$2 = 6A$$

$$A = \frac{1}{3}, B = 0, C = 2/3$$

$$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} = \frac{1}{3s} + \frac{2}{3(s+3)}$$

$$y(t) = \frac{1}{3}(1 + 2e^{-3t})u(t)$$

Note that as should be the case, this matches the result in example 5.3.

- **Example:** Determine the unit-impulse response for the system:

$$\frac{d\boldsymbol{x}(t)}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t)$$

  and the initial values of the states are zero.

$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{2}{s^2 + 5s + 6}$$

Now use partial fractions

$$\frac{2}{s^2 + 5s + 6} = \frac{A}{s+2} + \frac{B}{s+3}$$
$$2 = A(s+3) + B(s+2)$$
$$2 = 3A + 2B$$
$$3A + 2B = 0$$
$$A = 2, B = -2$$
$$\frac{2}{s^2 + 5s + 6} = \frac{2}{s+2} - \frac{2}{s+3}$$
$$y(t) = (2e^{-2t} - 2e^{-3t})u(t)$$

Note that as should be the case, this matches the result in example 5.2.