5. Solving the state equations

5.1 State transformations

- State-space representations are not unique. There are many selections of state variables which can describe a system.
- State variables can be real or fictitious.
- Some state representations lead to computationally attractive forms.
- Thus, by using the appropriate state transformation, we can obtain a state-space representation that leads to significantly easier computation.
- Consider the following state transformation:

$$\boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{z}(t)$$

- Here, $\boldsymbol{T}$ is any constant non-singular (i.e. invertible) n x n matrix. Since $\boldsymbol{T}$ is constant, we can write:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{T}\frac{d\boldsymbol{z}}{dt}$$

- Substituting these expressions into the state equation

$$\boldsymbol{T}\frac{d\boldsymbol{z}}{dt} = \boldsymbol{A}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{B}u$$

- Finally, premultiplying by $\boldsymbol{T}^{-1}$ gives us the new state equation:

$$\frac{d\boldsymbol{z}}{dt} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{T}^{-1}\boldsymbol{B}u$$

$$\frac{d\boldsymbol{z}}{dt} = \bar{\boldsymbol{A}}\boldsymbol{z} + \boldsymbol{T}^{-1}\boldsymbol{B}u$$

- The new output equation is:

$$y = \boldsymbol{C}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{D}u$$
$$y = \bar{\boldsymbol{C}}\boldsymbol{z} + \boldsymbol{D}u$$

- Note – the only condition on $\boldsymbol{T}$ is that it must have an inverse. Hence, there are an infinite number of state representations for the system.

- We are interested in finding a representation that leads to a diagonal $\overline{\boldsymbol{A}}$ matrix.

- So the issue becomes one of finding a suitable $\boldsymbol{T}$ so that $\overline{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{AT}$ is diagonal.

- Consider a matrix $\boldsymbol{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$. Let $\boldsymbol{m}_1, \boldsymbol{m}_2, \dots \boldsymbol{m}_n$ be the corresponding eigenvectors.

- The ***modal matrix***, $\boldsymbol{M}$, is formed from these eigenvectors:
  $$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \cdots & \boldsymbol{m}_n \end{bmatrix}$$
- Note that by definition, $\boldsymbol{Am}_i = \lambda_i \boldsymbol{m}_i$. Hence,

$$\begin{aligned} \boldsymbol{AM} &= \boldsymbol{A}\begin{bmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \cdots & \boldsymbol{m}_n \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}\boldsymbol{m}_1 & \boldsymbol{A}\boldsymbol{m}_2 & \cdots & \boldsymbol{A}\boldsymbol{m}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\boldsymbol{m}_1 & \lambda_2\boldsymbol{m}_2 & \cdots & \lambda_n\boldsymbol{m}_n \end{bmatrix} \\ &= \boldsymbol{M}\Lambda \end{aligned}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- Since $\quad \boldsymbol{AM} = \boldsymbol{M}\Lambda \;\Rightarrow\; \Lambda = \boldsymbol{M}^{-1}\boldsymbol{AM}$

  Thus selecting $\boldsymbol{T}$=$\boldsymbol{M}$ results in a diagonal matrix.

Forming the Modal Matrix

- Recall that eigenvalues are calculated as the roots of the **matrix characteristic equation**

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = 0$$

- The eigenvectors are then determined either by
  - solving $\boldsymbol{A m}_i = \lambda_i \, \boldsymbol{m}_i$ or
  - evaluating the cofactors of a row of $(\lambda \boldsymbol{I} - \boldsymbol{A})$

We shall use the second method using cofactors.

- **Example 5.1:** Determine the modal matrix for $\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}$

Solution
Eigenvalues:

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = 0 \Rightarrow \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda+1 & -2 \\ 1 & \lambda+4 \end{vmatrix} = 0$$
$$\Rightarrow (\lambda+1)(\lambda+4)+2 = 0$$
$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0$$
$$\Rightarrow (\lambda+2)(\lambda+3) = 0$$
$$\Rightarrow \lambda_1 = -2, \lambda_2 = -3$$

Eigenvectors – ***(using cofactors)*** – here, the eigenvectors are obtained by writing the cofactors of **any** row of $(\lambda \boldsymbol{I} - \boldsymbol{A})$ in column format (note – make sure to allow for the correct sign, as shown below):

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda+1 & -2 \\ 1 & \lambda+4 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

Take row 1, for example. Delete the first row and column.

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda + 1 & -2 \\ 1 & \lambda + 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

This leaves λ+4. Now delete the first row and second column.

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda + 1 & -2 \\ 1 & \lambda + 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

This leaves 1 but because of its position, we enter –1.

$$\boldsymbol{m}_V = \begin{bmatrix} \lambda + 4 \\ -1 \end{bmatrix}$$

Now we enter the eigenvalues to this vector:

$$\lambda = -2 \Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \lambda = -3 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Combining these gives the modal matrix:

$$\boldsymbol{M} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

The entries in the diagonal matrix must correspond to the associated vector in $\boldsymbol{M}$

$$\boldsymbol{\Lambda} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

So

$$\boldsymbol{M} \Lambda \boldsymbol{M}^{-1} = \boldsymbol{A}$$

5.2 Continuous-time solution

- The general linear Single Input Single Output (SISO) continuous-time state-space model is given by:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

- We will consider the unforced and forced responses separately.

Unforced response

- This is when the input is set to zero for all time, i.e.:

$$u(t) = 0, \ \forall t$$

- The state equation then becomes:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t)$$

- Consider the scalar case $\frac{dx(t)}{dt} = ax(t)$. This is a first-order differential equation and has a solution:

$$x(t) = e^{at}x(0)$$

- The question is ... can we use the same solution when a is replaced by $\boldsymbol{A}$, i.e.:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

What is $e^{\boldsymbol{A}t}$?

- Expanding e^{at} gives:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \ldots$$

- Using the same expansion for $e^{\boldsymbol{A}t}$ gives:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \frac{(\boldsymbol{A}t)^3}{3!} + \ldots$$

- This can be rewritten as:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \boldsymbol{A}^3\frac{t^3}{3!} + \ldots = \sum_{k=0}^{\infty}\boldsymbol{A}^k\frac{t^k}{k!}$$

- Hence, we can conclude that $e^{\boldsymbol{A}t}$ is an $n \times n$ matrix defined as:

$$e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty}\boldsymbol{A}^k\frac{t^k}{k!}$$

- Now consider $\frac{d}{dt}\left(e^{\boldsymbol{A}t}\right)$:

$$\begin{aligned}
\frac{d}{dt}\left(e^{\boldsymbol{A}t}\right) &= \frac{d}{dt}\left(\boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \boldsymbol{A}^3\frac{t^3}{3!} + \ldots\right) \\
&= \boldsymbol{0} + \boldsymbol{A} + 2\boldsymbol{A}^2\frac{t^1}{2!} + 3\boldsymbol{A}^3\frac{t^2}{3!} + \ldots \\
&= \boldsymbol{A} + \boldsymbol{A}^2\frac{t^1}{1!} + \boldsymbol{A}^3\frac{t^2}{2!} + \ldots \\
&= \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \ldots\right) = \boldsymbol{A}\,e^{\boldsymbol{A}t}
\end{aligned}$$

- This is exactly the same as for the scalar case, i.e. $\frac{d}{dt}\left(e^{at}\right) = ae^{at}$, and hence $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$ is indeed a valid solution for the state equation $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t)$.

- Note – most of the properties of $e^{\boldsymbol{A}t}$ are the same as e^{at}. For example:

$$\int e^{at}dt = \frac{1}{a}e^{at} = a^{-1}e^{at} \quad \rightarrow \int e^{\boldsymbol{A}t}dt = \boldsymbol{A}^{-1}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}^{-1}$$

Forced response

- Now, consider the situation when the input is not zero:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

- This can be written as:

$$\frac{d\mathbf{x}(t)}{dt} - \mathbf{A}\mathbf{x}(t) = \mathbf{B}u(t)$$

- Premultiplying by $e^{-\mathbf{A}t}$ gives:

$$e^{-\mathbf{A}t}\left(\frac{d\mathbf{x}(t)}{dt} - \mathbf{A}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}u(t)$$

- This is equivalent to:

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}u(t)$$

- Integrating both sides with respect to t from 0 to t gives:

$$\int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right)d\tau = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

- Since $\int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right)d\tau = e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\Big|_0^t = e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0)$, we can write:

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

- Finally, bringing $\mathbf{x}(0)$ to the RHS of the equation and multiplying both sides by $e^{\mathbf{A}t}$ gives:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

- If we start from a time t_0 instead of zero, then:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

Output calculation

- Once the state has been determined, the output is easily computed as:

$$\begin{aligned} y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t) \\ &= \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau \quad + \boldsymbol{D}u(t) \end{aligned}$$

- For continuous systems, the state transition matrix is defined as:

$$\Phi(t) = e^{\boldsymbol{A}t}$$

- Hence, the output can be rewritten as:

$$y(t) = \boldsymbol{C}\Phi(t-t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{C}\Phi(t-\tau)\boldsymbol{B}u(\tau)d\tau \quad + \boldsymbol{D}u(t)$$

The Modal matrix method to compute the state-transition matrix

To form the state transition matrix, it is much easier from a computational viewpoint if one is getting the exponential of a diagonal matrix.

- Consider $e^{\boldsymbol{A}t}$ when $\boldsymbol{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

- Expanding $e^{\boldsymbol{A}t}$ as a power series gives:

$$\exp\left(\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t\right)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^2\frac{t^2}{2!}+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^3\frac{t^3}{3!}+\ldots$$

$$=\begin{bmatrix}1+\lambda_1 t+\lambda_1{}^2\frac{t^2}{2!}+\lambda_1{}^3\frac{t^3}{3!}+\ldots & 0\\ 0 & 1+\lambda_2 t+\lambda_2{}^2\frac{t^2}{2!}+\lambda_2{}^3\frac{t^3}{3!}+\ldots\end{bmatrix}$$

$$=\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$$

- Thus, when $\boldsymbol{A}$ is a diagonal matrix, calculating $e^{\boldsymbol{A}t}$ is straightforward.
- Consider: $\dfrac{d\mathbf{x}(t)}{dt}=\mathbf{Ax}(t)$
- Let $\boldsymbol{x}(t)=\boldsymbol{Mz}(t)$, where $\boldsymbol{M}$ is the modal matrix for $\boldsymbol{A}$. This gives:

$$\frac{d\mathbf{z}(t)}{dt}=\mathbf{M}^{-1}\mathbf{AMz}(t)=\Omega\mathbf{z}(t)$$

- Solving this equation, we obtain: $\boldsymbol{z}(t)=e^{\Omega t}\boldsymbol{z}(0)$
- Note: $\Omega=\boldsymbol{M}^{-1}\boldsymbol{AM}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$
- But $\boldsymbol{z}(t)=\boldsymbol{M}^{-1}\boldsymbol{x}(t)$, hence: $\boldsymbol{x}(t)=\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}\boldsymbol{x}(0)=e^{\boldsymbol{A}t}\boldsymbol{x}(0)$
- Thus, the state transition matrix $e^{\boldsymbol{A}t}$ can be computed as:

$$\Phi(t)=e^{\boldsymbol{A}t}=\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}$$

- **Example 5.2:** Determine the state transition matrix for the system:

$$\frac{d}{dt}[\boldsymbol{x}] = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{in}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}$$

Hence, determine the unit-impulse response if the initial values of the states are zero.

Solution

Eigenvalues: $\lambda = -2, \ -3$

Modal matrix: $\lambda I - \boldsymbol{A} = \begin{bmatrix} \lambda & -2 \\ 3 & \lambda + 5 \end{bmatrix}$

$$\boldsymbol{M}_V = \begin{bmatrix} \lambda + 5 \\ -3 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 3 & 2 \\ -3 & -3 \end{bmatrix}$$

Thus: $\mathbf{M}^{-1} = \begin{bmatrix} 1 & 2/3 \\ -1 & -1 \end{bmatrix}$

Hence:

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} 3 & 2 \\ -3 & -3 \end{bmatrix} e^{\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}t} \begin{bmatrix} 1 & 2/3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 2/3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

$$y(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{B}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(t) = 2e^{-2t} - 2e^{-3t}$$

- **Example 5.3:** Determine the output for the system:

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \mathbf{x}(\mathrm{t}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0]\boldsymbol{x}(t)$$

The input $u(t)$ is a step and the initial state $\boldsymbol{x}(0) = [1 \;\; -1]^{\mathrm{T}}$.

Solution

From example 5.2: $$e^{\boldsymbol{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

Need to determine: $$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

$$e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

$$\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau \rightarrow e^{\boldsymbol{A}t}\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}u(\tau)d\tau$$

$$= e^{\boldsymbol{A}t}\int_0^t \begin{bmatrix} 3e^{2\tau}-2e^{3\tau} & 2e^{2\tau}-2e^{3\tau} \\ -3e^{2\tau}+3e^{3\tau} & -2e^{2\tau}+3e^{3\tau} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(1)d\tau$$

$$= e^{\boldsymbol{A}t}\int_0^t \begin{bmatrix} 2e^{2\tau}-2e^{3\tau} \\ -2e^{2\tau}+3e^{3\tau} \end{bmatrix}d\tau$$

$$= e^{\boldsymbol{A}t}\left(\begin{bmatrix} e^{2\tau}-\tfrac{2}{3}e^{3\tau} \\ -e^{2\tau}+e^{3\tau} \end{bmatrix}\Bigg|_0^t\right)$$

$$= e^{\boldsymbol{A}t}\left(\begin{bmatrix} e^{2t}-\tfrac{2}{3}e^{3t}-1+\tfrac{2}{3} \\ -e^{2t}+e^{3t}+1-1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3e^{-2t}-2e^{-3t} & 2e^{-2t}-2e^{-3t} \\ -3e^{-2t}+3e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix}\begin{bmatrix} e^{2t}-\tfrac{2}{3}e^{3t}-\tfrac{1}{3} \\ -e^{2t}+e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} \left(3-2e^{t}-e^{-2t}-2e^{-t}+\tfrac{4}{3}+\tfrac{2}{3}e^{-3t}\right)+\left(-2+2e^{t}+2e^{-t}-2\right) \\ \left(-3+2e^{t}+e^{-2t}+3e^{-t}-2-e^{-3t}\right)+\left(2-2e^{t}-3e^{-t}+3\right) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}+\tfrac{2}{3}e^{-3t}+\tfrac{1}{3} \\ e^{-2t}-e^{-3t} \end{bmatrix}$$

Hence:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

$$\Rightarrow \boldsymbol{x}(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + \begin{bmatrix} -e^{-2t} + \frac{2}{3}e^{-3t} + \frac{1}{3} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

$$\Rightarrow \boldsymbol{x}(t) = \begin{bmatrix} \frac{2}{3}e^{-3t} + \frac{1}{3} \\ -e^{-3t} \end{bmatrix}$$

$$y(t) = [1 \quad 0]\boldsymbol{x}(t)$$

$$y(t) = \left[\frac{2}{3}e^{-3t} + \frac{1}{3}\right]u(t)$$