General instructions: *Put your name and RIN on* ***every page*** *of the exam. Write as neatly as possible – we must be able to read your responses. Complete all problems to the best of your ability. Be sure to clearly explain your answers, especially in your proofs. Keep your eyes on* ***your own*** *paper. When you are done, raise your hand and wait for your exam and crib sheet to be collected before leaving the room.*

Note on calculations: *It is acceptable (and expected) that you will leave answers as expressions rather than actual numbers in many cases. Please do not spend time performing manual calculations unless it will help you understand the problem better. Answers like* $\binom{10}{3} / 2^{10}$ *are perfectly fine as is.*

"Random chance plays a huge part in everybody's life." –Gary Gygax

1. Dungeons and Dragons uses six different kinds of dice, each with a different number of sides: 4, 6, 8, 10, 12, and 20. Each die is fairly weighted, and numbered from 1 up to the number of sides. (e.g. The 8-sided die has values from one through eight.) The notation "dN" indicates an N-sided die, and "KdN" means to roll K N-sided dice and add them up. (e.g. 3d6 means roll three 6-sided dice and add them.)

(a) There are several easy ways to produce a roll with a maximum value of 12: 3d4, 2d6, 1d12
Rank these (from low to high) in order of how likely they are to actually roll a 12. Show your work.

(b) Calculate the variance for each of these three rolls. How does this correlate with the probability of rolling a 12 (the highest possible value)?

"Last night I stayed up late playing poker with Tarot cards. I got a full house and four people died." –Stephen Wright

2. Tarot cards are a type of playing cards used for a variety of games throughout parts of Europe. In the late 19th century, they also became used for fortune-telling and other mystical activities.

There are many different variations of Tarot deck, but the most common one (and the one we will use for this problem) has 78 total cards, all unique. 22 of them are *Major Arcana* or *Trumps*, depicting various symbolic figures and natural phenomena (e.g. "The Magician" or "The Moon"). The other 56 cards are *Minor Arcana*, which are divided into four suits: cups, wands, swords, and pentacles. Each suit has cards in 14 ranks: ten numeric (1(ace) through 10) and four *court cards* (page, knight, queen, and king).

(a) One simple fortune-telling layout involves revealing three cards: past, present, and future. How many different layouts of three cards are possible?

(b) The Major Arcana are considered to have more importance when they appear in a reading. What is the probability of drawing three Major Arcana in the above scenario?

(c) Tarot card faces have a distinguishable top and bottom. When reading Tarot, different meanings are attributed to cards when they are right-side-up versus upside-down. How many different three-card layouts are possible if cards drawn upside-down are distinct from right-side-up?

(d) Finally, let's address Mr. Wright. Recall that a full house involves three cards of one rank and two cards of a different rank. If five cards are drawn from the top of a shuffled Tarot deck, what is the probability of drawing a full house? (*Note: Major Arcana have no rank, and thus cannot be part of a full house.*)

"Take nothing on its looks; take everything on evidence. There's no better rule." –Charles Dickens, *Great Expectations*

3a. A game involves flipping six fair coins. If you get more heads than tails, you win $20; otherwise, you lose $10. What is the expected value of this game?

3b. Assume all birthdays are equally likely. You ask random people passing by on the street what their birthday is. What is the expected number of people you need to ask before finding someone with the same birthday as you?

3c. An exam has 40 true/false questions worth 1 point each and 20 five-answer multiple choice questions worth 3 points each. Zaphod has not gone to class nor studied for the exam at all, and so guesses randomly on each question. What is Zaphod's expected score on this exam?

3d. During a typical year in troy, it is cloudy on 1/2 of the days and there is precipitation (rain, snow, sleet, what have you) on 3/8 of the days. Assume that it must be cloudy in order for there to be precipitation. In a year with exactly 200 cloudy days, what is the expected number of days with precipitation?

“Probability theory is nothing but common sense reduced to calculation.” –Pierre-Simon Laplace

4. A taxi was involved in a hit-and-run collision. There are two colors of taxis in the city: 75% of them are black, and 25% are blue. The collision occurred late at night in dim light, and in those conditions witnesses correctly identify the color of a taxi only 80% of the time. An eyewitness says the taxi involved in the collision was blue. Calculate the probability that the taxi involved was actually blue.

5. The table at right breaks down the underclass students at a high school by the average number of text messages they receive in a given week.

	Number of Texts		
Grade	*0-20*	*21-50*	*51+*
9th	25	55	30
10th	10	60	45
11th	5	40	70

a. What is the probability that an underclass student selected at random receives an average of 50 or fewer text messages in a week?

b. What is the probability that a 10th grader selected at random receives an average of 50 or fewer text messages in a week?

c. A randomly selected student receives over 50 text messages in a week. What is the probability that student is in 10th grade?

"I graduated from Douglass College without distinction. I was in the top 98% of my class and damn glad to be there. I slept in the library and daydreamed my way through history lecture. I failed math twice, never fully grasping probability theory. I mean, first off, who cares if you pick a black ball or a white ball out of the bag? And second, if you're bent over about the color, don't leave it to chance. Look in the damn bag and pick the color you want." –Stephanie Plum, *Hard Eight*

6. A class has 20 students. They are put into four lab groups of five students each.

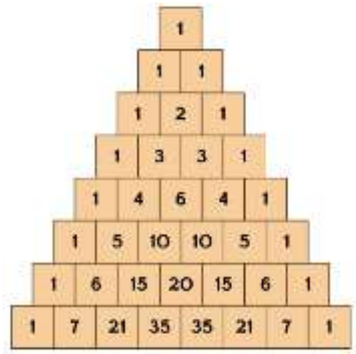
(a) Each lab group will be assigned to a distinct lab station. In how many unique ways can all 20 students be assigned to lab groups?

(b) What if the lab stations are indistinguishable from each other?

(c) Alex, Barbara, and Cori are three students in the class. How many ways are there to form a single lab group that includes at least one of these three students?

"Chance favors the prepared mind." –Louis Pasteur

7. Recall Pascal's Triangle, which gives us all of the values for $\binom{n}{k}$: rows 0 through 7 are shown in the picture at right. It is defined by the fact that all of the boxes except the 1s along the outside are the sum of the two boxes directly above. These facts are captured by Pascal's Identities:



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, for\ 1 \le k < n$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

Also observe that the sum of row n is 2^n. (For example, row 2 = 1 + 2 + 1 = 4 = 2^2.)

Prove via induction that, for n ≥ 2, that $\sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2$. *(Note that this is just the sum excluding the 1s on each end of the row; it's equivalent to proving the full sum = 2^n, but the mechanics of the proof are a bit easier.)* You may use Pascal's Identities as axioms in your proof.