Discretization of the Filtered Derivative Term

1. Continuous-Time Representation

The continuous-time filtered derivative in the Laplace domain is given by:

$$D_f(s) = \frac{T_d s}{1 + \frac{T_d}{N} s} E(s)$$

Rearranging:

$$(1 + \frac{T_d}{N} s) D_f(s) = T_d s E(s)$$

then,

$$D_f(s) + \frac{T_d}{N} D_f(s) s = T_d s E(s)$$

Taking the inverse Laplace transform:

$$D_f(t) + \frac{T_d}{N} \frac{dD_f(t)}{dt} = T_d \frac{de(t)}{dt}$$

This is a first-order differential equation for the filtered derivative term.

2. Discretization Using Backward Euler

To discretize the equation, we approximate the derivatives using the Backward Euler method:

$$\frac{dD_f}{dt} \approx \frac{D_k - D_{k-1}}{T_s}, \quad \frac{de}{dt} \approx \frac{e_k - e_{k-1}}{T_s}$$

where:

- D_k and D_{k-1} are the filtered derivative values at time steps k and $k-1$,
- e_k and e_{k-1} are the error values at time steps k and $k-1$,
- T_s is the sampling period.

Substituting these into the continuous time equation:

$$D_k + \frac{T_d}{N} \frac{D_k - D_{k-1}}{T_s} = T_d \frac{e_k - e_{k-1}}{T_s}$$

3. Solving for D_k

Rearranging the equation:

$$D_k + \frac{T_d}{NT_s} D_k = \frac{T_d}{NT_s} D_{k-1} + \frac{T_d}{T_s}(e_k - e_{k-1})$$

Factor out D_k:

$$D_k \left(1 + \frac{T_d}{NT_s}\right) = \frac{T_d}{NT_s} D_{k-1} + \frac{T_d}{T_s}(e_k - e_{k-1})$$

Dividing both sides by $1 + \frac{T_d}{NT_s}$:

$$D_k = \frac{\frac{T_d}{NT_s} D_{k-1}}{1 + \frac{T_d}{NT_s}} + \frac{\frac{T_d}{T_s}(e_k - e_{k-1})}{1 + \frac{T_d}{NT_s}}$$

Multiplying the numerator and the denominator by T_s,

$$D_k = \frac{\frac{T_d}{N} D_{k-1}}{Ts + \frac{T_d}{N}} + \frac{T_d(e_k - e_{k-1})}{Ts + \frac{T_d}{N}}$$

5. Interpretation

- The first term $\frac{D_{k-1} \cdot \frac{T_d}{N}}{Ts + \frac{T_d}{N}}$ is the leaky integrator term, allowing past derivative values to influence the current update, preventing sudden jumps.
- The second term $\frac{T_d \cdot (e_k - e_{k-1})}{Ts + \frac{T_d}{N}}$ is the filtered discrete derivative, smoothing the derivative estimate.
- The parameter N controls the amount of filtering, with larger values providing stronger smoothing.