Sample Problems - Solutions

Please note that $\arcsin x$ is the same as $\sin^{-1} x$ and $\arctan x$ is the same as $\tan^{-1} x$.

1. $\displaystyle\int xe^x \, dx$

Solution: We will integrate this by parts, using the formula

$$\int f'g = fg - \int fg'$$

Let $g(x) = x$ and $f'(x) = e^x$ Then we obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = x$
$f'(x) = e^x$	$g'(x) = 1$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int xe^x \, dx &= xe^x - \int e^x dx = \boxed{xe^x - e^x + C} \end{aligned}$$

We should check our result by differentiating the answer. Indeed,

$$\left(xe^x - e^x + C\right)' = e^x + xe^x - e^x = xe^x$$

and so our answer is correct.

2. $\displaystyle\int x \cos x \, dx$

Solution: Let $g(x) = x$ and $f'(x) = \cos x$ Then we obtain g' and f by differentiation and integration.

$f(x) = \sin x$	$g(x) = x$
$f'(x) = \cos x$	$g'(x) = 1$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int x \cos x \, dx &= x \sin x - \int \sin x \, dx = x \sin x - (-\cos x) = \boxed{x \sin x + \cos x + C} \end{aligned}$$

We should check our result by differentiating the answer. Indeed,

$$\left(x \sin x + \cos x + C\right)' = \sin x + x \cos x - \sin x = x \cos x$$

and so our answer is correct.

3. $\displaystyle\int xe^{-4x}\,dx$

Solution: Let $g(x) = x$ and $f'(x) = e^{-4x}$ Then we obtain g' and f by differentiation and integration. To compute $f(x)$, we will use substitution. Let $u = -4x$ then $du = -4dx$ and so $dx = \dfrac{du}{-4}$.

$$f(x) = \int e^{-4x}\,dx = \int e^{u}\,\frac{du}{-4} = -\frac{1}{4}\int e^{u}\,du = -\frac{1}{4}e^{u} + C = -\frac{1}{4}e^{-4x} + C$$

We will choose $C = 0$ and so $f(x) = -\dfrac{1}{4}e^{-4x}$.

$f(x) = -\frac{1}{4}e^{-4x}$	$g(x) = x$
$f'(x) = e^{-4x}$	$g'(x) = 1$

$$\begin{aligned}
\int f'g &= fg - \int fg' \quad \text{becomes} \\
\int xe^{-4x}\,dx &= -\frac{1}{4}xe^{-4x} - \int -\frac{1}{4}e^{-4x}\,dx = -\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x}\,dx = -\frac{1}{4}xe^{-4x} + \frac{1}{4}\left(-\frac{1}{4}e^{-4x}\right) + C \\
&= \boxed{-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C}
\end{aligned}$$

We check our result by differentiating the answer.

$$\left(-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C\right)' =$$

$$\begin{aligned}
&= -\frac{1}{4}\left(xe^{-4x}\right)' - \frac{1}{16}\left(e^{-4x}\right)' = -\frac{1}{4}\left(e^{-4x} + x\left(-4e^{-4x}\right)\right) - \frac{1}{16}\left(-4e^{-4x}\right) \\
&= -\frac{1}{4}e^{-4x} + xe^{-4x} + \frac{1}{4}e^{-4x} = xe^{-4x}
\end{aligned}$$

and so our answer is correct.

4. $\displaystyle\int \ln x\,dx$

Solution: Let $g(x) = \ln x$ and $f'(x) = 1$ Then we obtain g' and f by differentiation and integration.

$f(x) = x$	$g(x) = \ln x$
$f'(x) = 1$	$g'(x) = \frac{1}{x}$

$$\begin{aligned}
\int f'g &= fg - \int fg' \quad \text{becomes} \\
\int \ln x\,dx &= x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int 1\,dx = \boxed{x\ln x - x + C}
\end{aligned}$$

We check our result by differentiating the answer.

$$(x\ln x - x + C)' = \ln x + x \cdot \frac{1}{x} - 1 = \ln x$$

and so our answer is correct.

5. $\displaystyle\int \arcsin x \, dx$

Solution: Let $g(x) = \arcsin x$ and $f'(x) = 1$ Then we obtain g' and f by differentiation and integration.

$f(x) = x$	$g(x) = \arcsin x$
$f'(x) = 1$	$g'(x) = \dfrac{1}{\sqrt{1-x^2}}$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int \arcsin x \, dx &= x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \end{aligned}$$

We compute the integral $\displaystyle\int \frac{x}{\sqrt{1-x^2}} \, dx$ by substitution. Let $u = 1 - x^2$. Then $du = -2x dx$ and so $dx = \dfrac{du}{-2x}$.

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} \, dx &= \int \frac{x}{\sqrt{u}} \frac{du}{-2x} = -\frac{1}{2} \int \frac{1}{\sqrt{u}} \, du = -\frac{1}{2} \int u^{-1/2} \, du \\ &= -\frac{1}{2} \frac{u^{1/2}}{\frac{1}{2}} + C = -\sqrt{u} + C = -\sqrt{1-x^2} + C \end{aligned}$$

Thus the entire integral is

$$\int \arcsin x \, dx = x \arcsin x - \left(-\sqrt{1-x^2}\right) + C = \boxed{x \arcsin x + \sqrt{1-x^2} + C}$$

We check our result by differentiating the answer.

$$\left(x \arcsin x + \sqrt{1-x^2} + C\right)' =$$

$$\begin{aligned} &= (x \arcsin x)' + \left(\left(1-x^2\right)^{1/2}\right)' = \arcsin x + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) \\ &= \arcsin x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \arcsin x \end{aligned}$$

and so our answer is correct.

6. $\displaystyle\int \arctan x \, dx$

Solution: Let $g(x) = \arctan x$ and $f'(x) = 1$ Then we obtain g' and f by differentiation and integration.

$f(x) = x$	$g(x) = \arctan x$
$f'(x) = 1$	$g'(x) = \dfrac{1}{x^2+1}$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int \arctan x \, dx &= x \arctan x - \int x \cdot \frac{1}{x^2+1} \, dx = x \arctan x - \int \frac{x}{x^2+1} \, dx \end{aligned}$$

We compute the integral $\displaystyle\int \frac{x}{x^2+1}\,dx$ by substitution. Let $u = x^2+1$. Then $du = 2x dx$ and so $dx = \dfrac{du}{2x}$.

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\,\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(x^2+1\right) + C$$

Thus the entire integral is

$$\int \arctan x\,dx = \boxed{x\arctan x - \frac{1}{2}\ln\left(x^2+1\right) + C}$$

We check our result by differentiating the answer.

$$\left(x\arctan x - \frac{1}{2}\ln\left(x^2+1\right) + C\right)' =$$

$$\begin{aligned}
&= (x\arctan x)' - \frac{1}{2}\left(\ln\left(x^2+1\right)\right)' = \arctan x + x\cdot\frac{1}{x^2+1} - \frac{1}{2}\frac{1}{x^2+1}(2x) \\
&= \arctan x + \frac{x}{x^2+1} - \frac{x}{x^2+1} = \arctan x
\end{aligned}$$

so our answer is correct.

7. $\displaystyle\int e^x \sin x\,dx$

Solution: This is an interesting application of integration by parts. Let M denote the integral $\displaystyle\int e^x \sin x\,dx$.
Solution: Let $g(x) = \sin x$ and $f'(x) = e^x$ (Notice that because of the symmetry, $g(x) = e^x$ and $f'(x) = \sin x$ would also work.) We obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = \sin x$
$f'(x) = e^x$	$g'(x) = \cos x$

$$\begin{aligned}
\int f'g &= fg - \int fg' \quad \text{becomes} \\
\int e^x \sin x\,dx &= e^x \sin x - \int e^x \cos x\,dx
\end{aligned}$$

It looks like our method produced a new integral, $\displaystyle\int e^x\cos x\,dx$ that also requires integration by parts. We proceed: let $g(x) = \cos x$ and $f'(x) = e^x$. We obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = \cos x$
$f'(x) = e^x$	$g'(x) = -\sin x$

$$\begin{aligned}
\int f'g &= fg - \int fg' \quad \text{becomes} \\
\int e^x \cos x\,dx &= e^x \cos x - \int e^x(-\sin x)\,dx = e^x\cos x + \int e^x \sin x\,dx
\end{aligned}$$

$$\text{Thus } \int e^x\cos x\,dx = e^x\cos x + \int e^x \sin x\,dx$$

Now the result contains the original integral, $\int e^x \sin x$. At this point, it looks like we are getting nowhere because we are going in circles. However, this is not the case. Recall that we denote $\int e^x \sin x$ by M. Let us review the computation again:

$$\begin{aligned}
\int e^x \sin x \, dx &= e^x \sin x - \int e^x \cos x \, dx \\
\int e^x \sin x \, dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\
\int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx
\end{aligned}$$

This is the same as

$$M = e^x \sin x - e^x \cos x - M$$

This is an equation that we can solve for M.

$$\begin{aligned}
2M &= e^x \sin x - e^x \cos x \\
M &= \frac{1}{2} e^x (\sin x - \cos x)
\end{aligned}$$

Thus the answer is $\boxed{\frac{1}{2} e^x (\sin x - \cos x) + C}$. We check our result by differentiation.

$$\left(\frac{1}{2} e^x (\sin x - \cos x) \right)' =$$

$$\begin{aligned}
&= \frac{1}{2} (e^x)' (\sin x - \cos x) + \frac{1}{2} e^x (\sin x - \cos x)' = \frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2} e^x (\cos x + \sin x) \\
&= \frac{1}{2} e^x (\sin x - \cos x + \sin x + \cos x) = \frac{1}{2} e^x (2 \sin x) = e^x \sin x
\end{aligned}$$

so our answer is correct.

8. $\int \sin^2 x \, dx$

Solution: Note that this integral can be easily solved using substitution. This is because of the double angle formula for cosine, $\cos 2x = 1 - 2\sin^2 x \quad \Longrightarrow \quad \sin^2 x = \frac{1 - \cos 2x}{2}$. This solution can be found on our substitution handout. But at the moment, we will use this interesting application of integration by parts as seen in the previous problem.

Let M denote the integral $\int \sin^2 x \, dx$. Let $g(x) = \sin x$ and $f'(x) = \sin x$ Then we obtain g' and f by differentiation and integration.

$f(x) = -\cos x$	$g(x) = \sin x$
$f'(x) = \sin x$	$g'(x) = \cos x$

$$\begin{aligned}
\int f'g &= fg - \int fg' \quad \text{becomes} \\
\int \sin^2 x \, dx &= -\sin x \cos x - \int (-\cos x) \cos x \, dx = -\sin x \cos x + \int \cos^2 x \, dx \\
&= -\sin x \cos x + \int 1 - \sin^2 x \, dx = -\sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx \\
&= -\sin x \cos x + x - \int \sin^2 x \, dx
\end{aligned}$$

We obtained

$$\begin{aligned}
\int \sin^2 x \, dx &= -\sin x \cos x + x - \int \sin^2 x \, dx \quad \text{or} \\
M &= -\sin x \cos x + x - M \qquad \text{we solve for } M \\
2M &= -\sin x \cos x + x \\
M &= \frac{1}{2}\left(-\sin x \cos x + x\right) + C
\end{aligned}$$

So our answer is $\boxed{\frac{1}{2}\left(-\sin x \cos x + x\right) + C}$. We check our result by differentiating the answer.

$$\left(\frac{1}{2}\left(-\sin x \cos x + x\right) + C\right)' =$$

$$\begin{aligned}
&= \left(\frac{1}{2}\left(-\sin x \cos x + x\right) + C\right)' = \frac{1}{2}\left(-\sin x\left(-\sin x\right) + \left(-\cos x\right)\left(\cos x\right) + 1\right) \\
&= \frac{1}{2}\left(\sin^2 x - \cos^2 x + 1\right) = \frac{1}{2}\left(\sin^2 x + \underbrace{1 - \cos^2 x}_{\sin^2 x}\right) = \frac{1}{2}\left(2\sin^2 x\right) = \sin^2 x
\end{aligned}$$

so our answer is correct.

9. $\int \cos^2 x \, dx$

Solution: We do not need to integrate by parts (although it is good practice)

$$\begin{aligned}
\int \cos^2 x \, dx &= \int 1 - \sin^2 x \, dx = \int 1 \, dx - \int \sin^2 x \, dx = x - \frac{1}{2}\left(-\sin x \cos x + x\right) + C \\
&= \frac{1}{2}\sin x \cos x + \frac{1}{2}x + C = \boxed{\frac{1}{2}\left(x + \sin x \cos x\right) + C}
\end{aligned}$$

We check our result by differentiating the answer.

$$\left(\frac{1}{2}\left(x + \sin x \cos x\right) + C\right)' = \frac{1}{2}\left(1 + \cos^2 x - \sin^2 x\right) = \frac{1}{2}\left(\underbrace{1 - \sin^2 x}_{\cos^2 x} + \cos^2 x\right) = \frac{1}{2}\left(2\cos^2 x\right) = \cos^2 x$$

so our answer is correct.

10. $\displaystyle\int x^2 e^{-3x}\,dx$

Solution: We will need to integrate by parts twice. First, let $f'(x) = e^{-3x}$ and $g(x) = x^2$. Then

$f(x) = -\frac{1}{3}e^{-3x}$	$g(x) = x^2$
$f'(x) = e^{-3x}$	$g'(x) = 2x$

$$\begin{aligned}\int f'g &= fg - \int fg' \quad \text{becomes}\\ \int x^2 e^{-3x}\,dx &= -\frac{1}{3}e^{-3x}\left(x^2\right) - \int\left(-\frac{1}{3}e^{-3x}\right)2x\,dx = -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}dx\end{aligned}$$

and we can compute $\int xe^{-3x}dx$ by integrating by parts. Let $f'(x) = e^{-3x}$ and $g(x) = x$. Then

$f(x) = -\frac{1}{3}e^{-3x}$	$g(x) = x$
$f'(x) = e^{-3x}$	$g'(x) = 1$

$$\begin{aligned}\int f'g &= fg - \int fg' \quad \text{becomes}\\ \int xe^{-3x}\,dx &= -\frac{1}{3}e^{-3x}(x) - \int\left(-\frac{1}{3}e^{-3x}\right)dx = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\\ &= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{-3x}\right) + C = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C\end{aligned}$$

This is the result we need to compute the integral $\int x^2e^{-3x}\,dx$. So far we had this much:

$$\int x^2e^{-3x}\,dx = -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$$

to this we substitute our result $\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$:

$$\begin{aligned}\int x^2e^{-3x}\,dx &= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}dx = -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C_1\right)\\ &= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C\end{aligned}$$

Our result might look nicer if we factor out $-e^{-3x}$ or $-\frac{1}{27}e^{-3x}$. Then the final answer is

$\boxed{-e^{-3x}\left(\frac{1}{3}x^2 + \frac{2}{9}x + \frac{2}{27}\right) + C}$ or $\boxed{-\frac{1}{27}e^{-3x}\left(9x^2 + 6x + 2\right) + C}$.

We check via differentiation:

$$\begin{aligned}f'(x) &= \left(-\frac{1}{27}e^{-3x}\left(9x^2+6x+2\right)\right)' = -\frac{1}{27}\left(-3e^{-3x}\left(9x^2+6x+2\right) + e^{-3x}(18x+6)\right)\\ &= -\frac{1}{27}\left(e^{-3x}\left(-27x^2-18x-6\right) + e^{-3x}(18x+6)\right)\\ &= -\frac{1}{27}e^{-3x}\left(-27x^2-18x-6+18x+6\right) = x^2e^{-3x}\end{aligned}$$

and so our solution is correct.

11. $\displaystyle\int \frac{x^3}{\left(x^2+2\right)^2}\,dx$

Solution: this integral can be computed using at least three different methods: substitution (try $u = x^2+2$) or partial fractions or integration by parts. We will present integration by parts here.

First, let $f'(x) = \dfrac{x}{\left(x^2+2\right)^2}$ and $g(x) = x^2$. To compute f, we need to integrate $\dfrac{x}{\left(x^2+2\right)^2}$. We can do that by using substitution: Let $u = x^2+2$. Then $du = 2x\,dx$ and so $dx = \dfrac{du}{2x}$. So

$$\int \frac{x}{\left(x^2+2\right)^2}\,dx = \int \frac{x}{u^2}\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{1}{2}\left(-\frac{1}{u}\right) + C = -\frac{1}{2\left(x^2+2\right)} + C$$

Thus if $f'(x) = \dfrac{x}{\left(x^2+2\right)^2}$, then $f(x) = -\dfrac{1}{2\left(x^2+2\right)}$. Proceeding with the integration by parts, we write

$f(x) = -\dfrac{1}{2\left(x^2+2\right)}$	$g(x) = x^2$
$f'(x) = \dfrac{x}{\left(x^2+2\right)^2}$	$g'(x) = 2x$

$$\begin{aligned}
\int f'g &= fg - \int fg' \quad \text{becomes} \\
\int \frac{x}{\left(x^2+2\right)^2}\left(x^2\right)\,dx &= -\frac{1}{2\left(x^2+2\right)}\left(x^2\right) - \int -\frac{1}{2\left(x^2+2\right)}(2x)\,dx \\
&= -\frac{x^2}{2\left(x^2+2\right)} + \int \frac{x}{x^2+2}dx
\end{aligned}$$

and this second integral can be computed using the same substitution:

Let $w = x^2+2$. Then $dw = 2x\,dx$ and so $dx = \dfrac{dw}{2x}$

$$\int \frac{x}{x^2+2}dx = \int \frac{x}{w}\frac{dw}{2dx} = \frac{1}{2}\int \frac{1}{w}dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln\left(x^2+2\right) + C$$

and so the entire integral is then

$$\int \frac{x^3}{\left(x^2+2\right)^2}\,dx = -\frac{x^2}{2\left(x^2+2\right)} + \int \frac{x}{x^2+2}dx = \boxed{-\frac{x^2}{2\left(x^2+2\right)} + \frac{1}{2}\ln\left(x^2+2\right) + C}$$

We check via differentiation:

$$\begin{aligned}
f'(x) &= \left(-\frac{x^2}{2\left(x^2+2\right)} + \frac{1}{2}\ln\left(x^2+2\right) + C\right)' \\
&= -\frac{1}{2}\left(\frac{2x\left(x^2+2\right) - x^2(2x)}{\left(x^2+2\right)^2}\right) + \frac{1}{2}\frac{1}{x^2+2}(2x) \\
&= -\left(\frac{x\left(x^2+2\right) - x^3}{\left(x^2+2\right)^2}\right) + \frac{x}{x^2+2} = -\left(\frac{x^3+2x-x^3}{\left(x^2+2\right)^2}\right) + \frac{x}{x^2+2} \\
&= -\frac{2x}{\left(x^2+2\right)^2} + \frac{x\left(x^2+2\right)}{\left(x^2+2\right)^2} = \frac{-2x+x^3+2x}{\left(x^2+2\right)^2} = \frac{x^3}{\left(x^2+2\right)^2}
\end{aligned}$$

and so our solution is correct.

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