Discretizing the Dynamic Response of Current Carrying MEMS Devices

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[1] Base on the paper: Dynamic pull-in for micro-electromechanical device with a current-carrying conductor. Ji-Huan He, Daulet Nurakhmetov et al. (2019)

Review

In the previous note we derived the first and second order approximation of the system and compared the model with the original lumped elastic current carrying MEMS model.

In this note we proceed as follow:

i. Choose a proper sampling time T_s to discretize the system
ii. Use Euler's forward derivative approximation and linearize the state equations
iii. Finding the ZOH discrete equivalent of the first note's state-space equations
iv. Comparing the discrete and non-discrete responses for different sampling rates
v. Presenting the linear system's transfer function
vi. Comparing the Bode's diagrams of the last two systems for different sampling rates

Sampling Time Selection

In order to find a proper sampling time, we need to determine the pole locations of the system. This task is done by computing the eigenvalues of the matrix A.

$$Av - \lambda v = 0 \tag{12}$$

This leads to the following pole locations:

$$p_1 = -\frac{1}{2} + i\frac{\sqrt{2}}{2}$$

$$p_2 = -\frac{1}{2} - i\frac{\sqrt{2}}{2}$$

We select the sampling time to be around 30 to 60 times greater than the imaginary part of the poles. This sampling time is $T_s = 0.15$ seconds.

Euler's Forward Approximation

Euler's forward approximation of the derivative is defined as:

$$\dot{x}(kT_s) = \frac{x(kT_s + T_s)}{T_s} \tag{13}$$

In which k is the kth sample of the system.

We insert this equation into the state equation (5) to achieve the following discrete system.

$$x_1[k+1] = x_1[k] + T_s x_2[k] \tag{14}$$

$$x_2[k+1] = (1 - T_s)x_2[k] - T_s x_1[k] + \frac{T_s u[k]}{1 - x_1[k]}$$

This system describes the non-linear discrete model of the lumped elastic current carrying MEMS devices.

Now we need to linearize this model around the previous operating point to achieve the linear discrete model of the system. By taking the Jacobian, the discrete dynamical equation is:

$$x[k+1] = Jx[k] + Ku[k] \tag{15}$$

In which J and K are the following matrices:

$$J = \begin{bmatrix} 1 & 0.15 \\ -0.1125 & 0.85 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 \\ 0.1875 \end{bmatrix}$$

ZOH Equivalent of The Continuous System

First, we write $x_1(s)$ in terms of u(s) and then apply the ZOH transformation. From the continuous state-space equation, we have:

$$x_1(s) = \frac{1.25}{(s - p_1)(s - p_2)} u(s) \tag{16}$$

And the ZOH equivalent is derived from:

$$x_1(z) = \left(\frac{z-1}{z}\right) Z\left\{\frac{1.25}{s(s - p_1)(s - p_2)}\right\} u(z) \tag{17}$$

Taking Z-transform as stated leads to:

$$x_1(z) = \frac{(1.6334 - 1.6667z)(z-1)}{z^2 - 1.9799z + 1.0151} u(z) + \frac{5}{3} u(z) \tag{18}$$

Now we turn it back to the time domain and take $x_1[k+1] = x_2[k]$, which leads to these state space matrices:

$$J' = \begin{bmatrix} 0 & 1 \\ 1.0151 & 1 \cdot 9799 \end{bmatrix}$$

$$K' = \begin{bmatrix} 0 \\ 0.0584 \end{bmatrix}$$

We note that different choices for $x_2[k]$ may lead to different state-space matrices. Note that these calculations were done for Ts = 0.02. Which provides a better accuracy.

Comparing Discrete Vs Continuous Pulse response

First, we compare the non-linear discrete and continuous pulse responses for different amplitudes. Fig. (11-13) shows both of these responses.

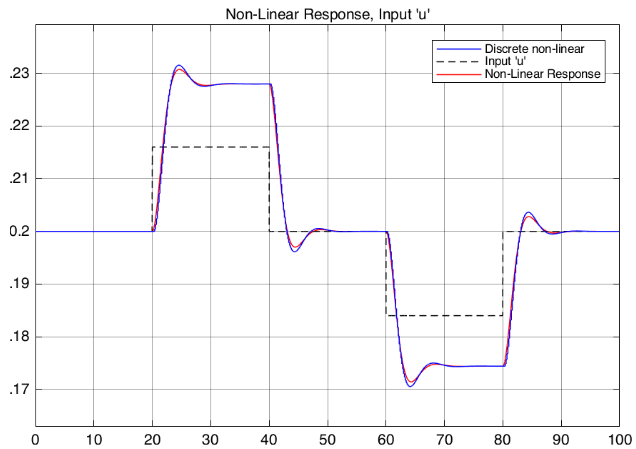

Figure 11 non-linear continuous vs discrete, 10%

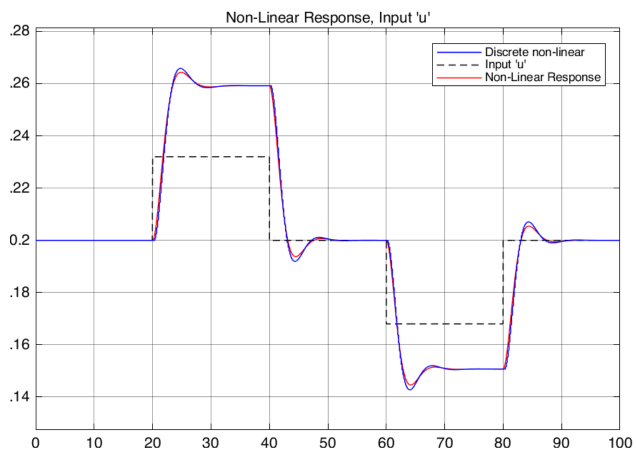

Figure 12 non-linear continuous vs discrete, 20%

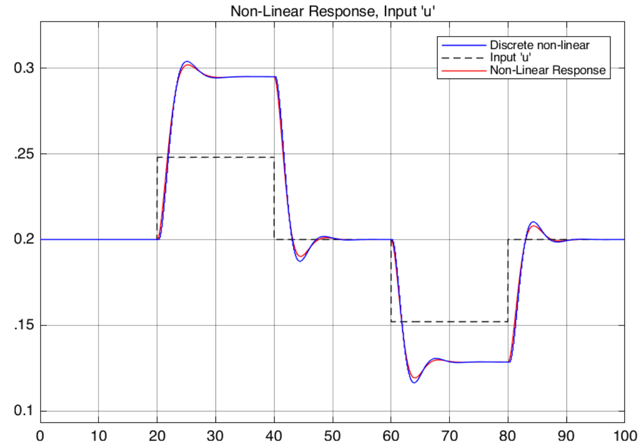

Figure 13 non-linear continuous vs discrete, 30%

Discrete and continuous non-linear responses match perfectly for the selected sampling time.

Second, we compare linearized continuous and discrete-time responses. Fig. (14-16) shows the responses.

We also want to compare the discrete-time responses for different sampling times. In order to achieve this, we keep the pulse amplitude constant and vary the sampling time.

Fig. (17-24) shows the linear system's response for an input amplitude of %30, with respect to operating point and different sampling times.

We observe that the discrete-time response overshoot increases as we increase the sampling time. It's also less accurate and possibly unstable for large sampling times like $10T_s$.

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The figures are generated by changing the 'amp' and 'Ts' parameters in the 'main.m' file.

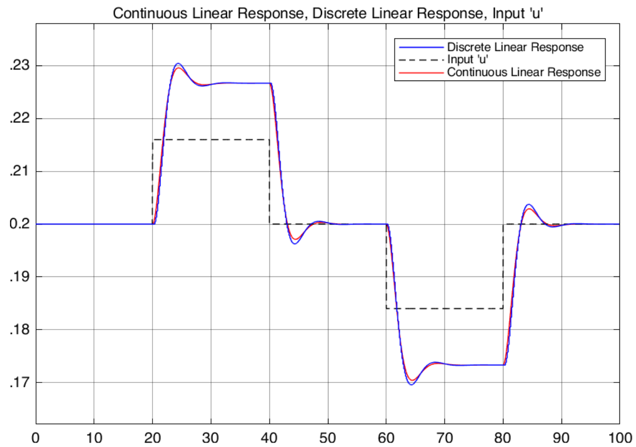

Figure 14 linear continuous vs discrete, 10%

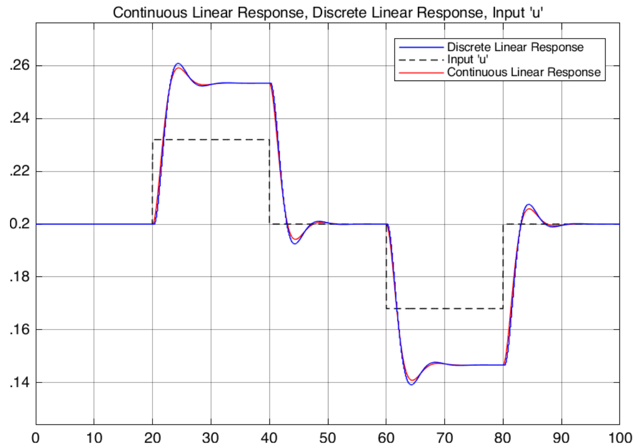

Figure 15 linear continuous vs discrete, 20%

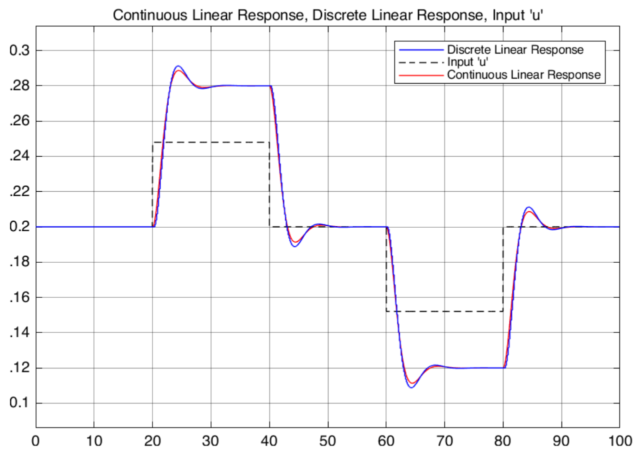

Figure 16 linear continuous vs discrete, 30%

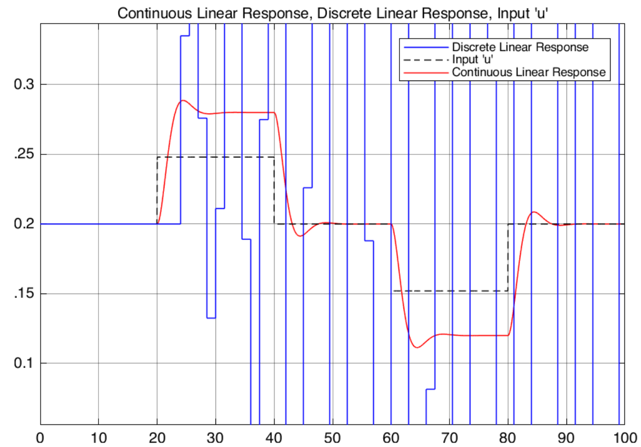

Figure 17 Linear, 10Ts, +30%

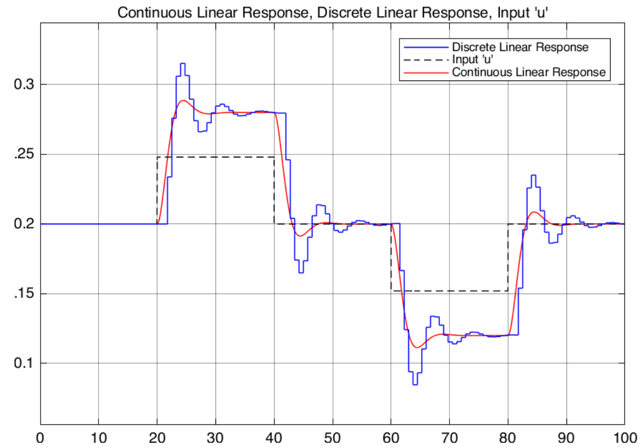

Figure 28 Linear, 5Ts, +30%

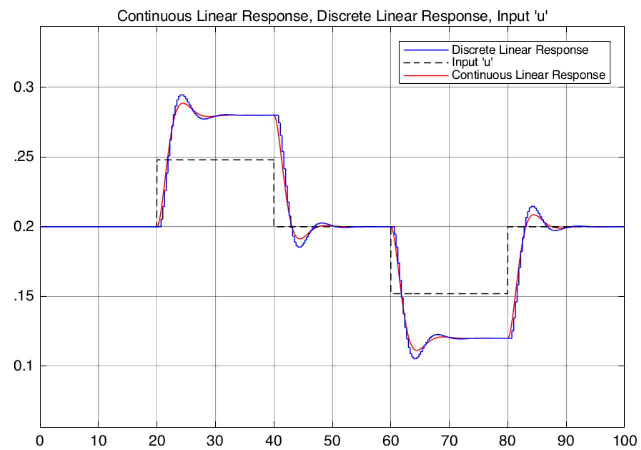

Figure 19 Linear, 2Ts, +30%

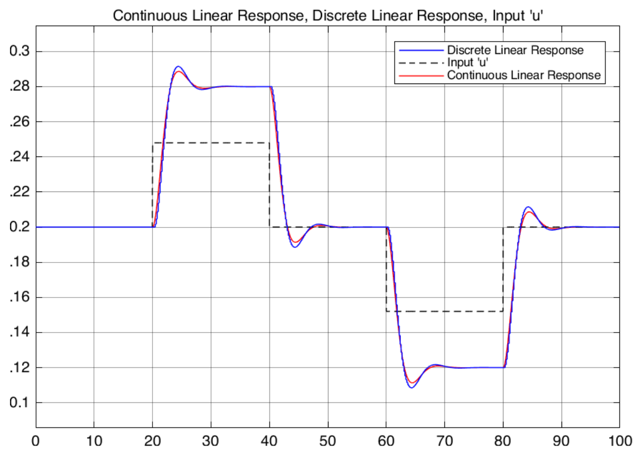

Figure 20 Linear, 1.1Ts, +30%

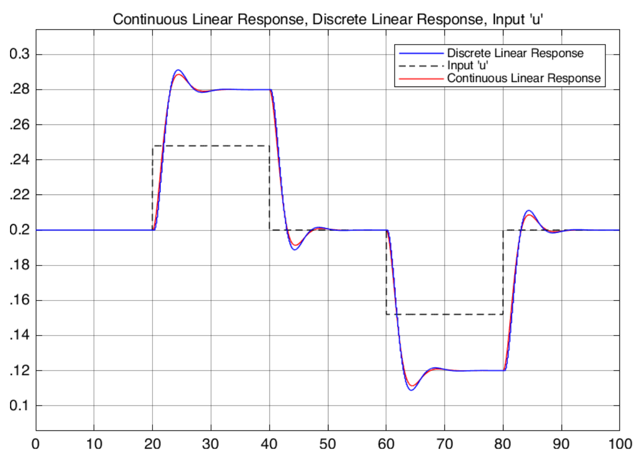

Figure 21 Linear, Ts, +30%

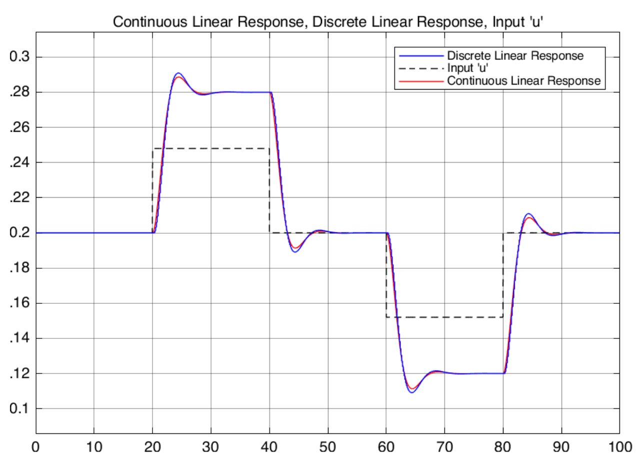

Figure 22 Linear, 0.9Ts, +30%

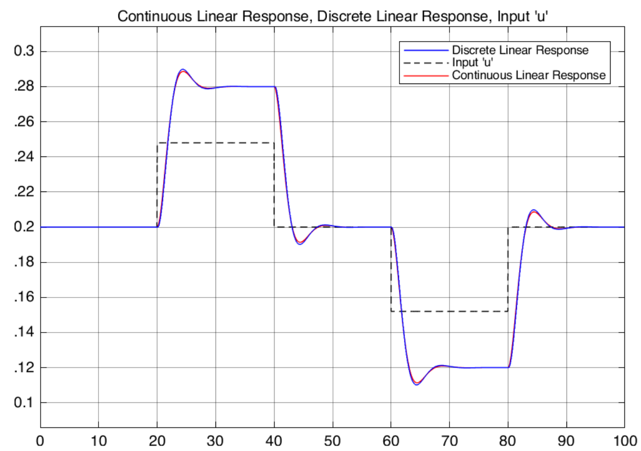

Figure 23 Non-linear, 0.5Ts, +30%

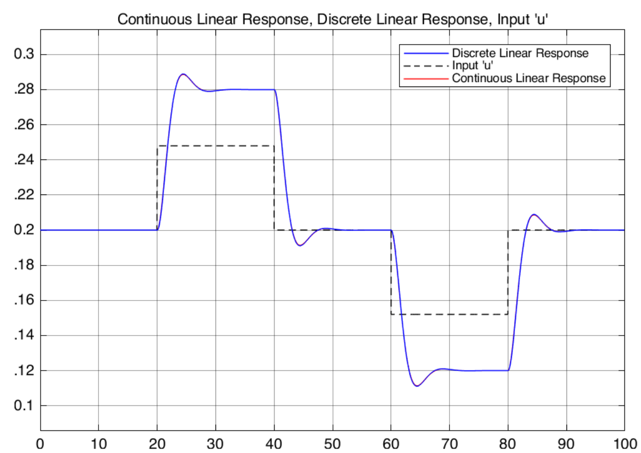

Figure 24 Non-linear, 0.1Ts, +30%

As we can observe, the gain in accuracy falls rapidly below our selected sampling rate.

Linear System's Transfer Functions

Since we're only interested in x_1, we want the transfer function of this variable with respect to u. Both discrete-time and continuous transfer functions are as follow:

$$H_1(s) = \frac{1.25}{(s^2 + s + 0.75)} \tag{16}$$

Which is base on taking Laplace transform of the state-space equation and solving for $x_1(s)$.

Similarly, discrete-time transfer function is:

$$H_1(z) = \frac{0.025}{(50z^2 - 99z + 49.015)} \tag{17}$$

Bode's Diagrams

To compare the affect of sampling on Bode's diagrams, we need to compute the transfer function for all the sampling times. The transfer function is as follows:

$$H_1(z) = \frac{1.25\, {T_s}^2}{\left(z^2 - (2 - T_s)z + 1 - T_s + \frac{3}{4}{T_s}^2\right)}$$

Now we use MATLAB to plot the Bode's diagrams in the following range. T = [10Ts, 2Ts, 1.1Ts, Ts, 0.9Ts, 0.5Ts, 0.1Ts]

The results are as follows:

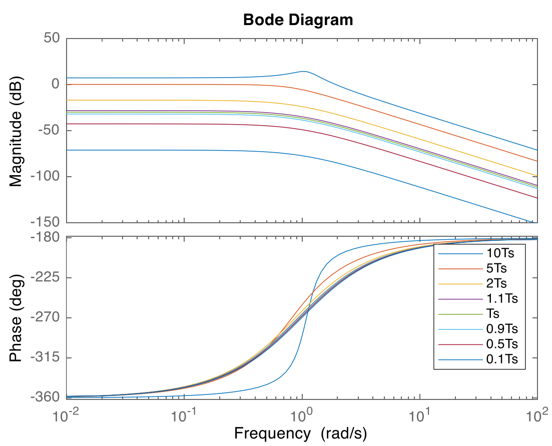

Figure 25 Bode's diagrams

The gain on average increases with sampling time as expected. This is due to the observation that in each step we have to update the states much more as sampling time increases. We can also notice that increasing the sampling time may lead to sharp phase changes and large gains at the cut-off frequency. This is not desired since it decreases the phase margin and eventually instability.