Dynamic Programming: Placing Parentheses

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Data Structures and Algorithms
Algorithmic Toolbox

Outline

1. Problem Overview
2. Subproblems
3. Algorithm
4. Reconstructing a Solution

How to place parentheses in an expression

$$1 + 2 - 3 \times 4 - 5$$

to maximize its value?

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Example

- $((((1 + 2) - 3) \times 4) - 5) = -5$

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- $((((1 + 2) - 3) \times 4) - 5) = -5$
- $((1 + 2) - ((3 \times 4) - 5)) = -4$

Answer

$$((1 + 2) - (3 \times (4 - 5))) = 6$$

Another example

What about

$$5 - 8 + 7 \times 4 - 8 + 9\,?$$

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$$5 - 8 + 7 \times 4 - 8 + 9\,?$$

Soon

We'll design an efficient dynamic programming algorithm to find the answer.

Outline

Placing parentheses

Input: A sequence of digits $d_1, \ldots, d_n$ and a sequence of operations $op_1, \ldots, op_{n-1} \in \{+, -, \times\}$.

Output: An order of applying these operations that maximizes the value of the expression

$$d_1 \, op_1 \, d_2 \, op_2 \cdots op_{n-1} \, d_n \, .$$

Intuition

- Assume that the last operation in an optimal parenthesizing of $5 - 8 + 7 \times 4 - 8 + 9$ is $\times$:

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- It would help to know optimal values for subexpressions $5 - 8 + 7$ and $4 - 8 + 9$.

However

We need to keep track for both the minimal and the maximal values of subexpressions!

Example: $(5 - 8 + 7) \times (4 - 8 + 9)$

$$\begin{aligned}
\min(5 - 8 + 7) &= (5 - (8 + 7)) = & -10 \\
\max(5 - 8 + 7) &= ((5 - 8) + 7) = & 4 \\
\min(4 - 8 + 9) &= (4 - (8 + 9)) = & -13 \\
\max(4 - 8 + 9) &= ((4 - 8) + 9) = & 5
\end{aligned}$$

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\end{aligned}$$

$$\max((5 - 8 + 7) \times (4 - 8 + 9)) = 130$$

Subproblems

- Let $E_{i,j}$ be the subexpression

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- Subproblems:

$$M(i,j) = \text{maximum value of } E_{i,j}$$
$$m(i,j) = \text{minimum value of } E_{i,j}$$

Recurrence Relation

$$M(i,j) = \max_{i \le k \le j-1} \begin{cases} M(i,k) & op_k & M(k+1,j) \\ M(i,k) & op_k & m(k+1,j) \\ m(i,k) & op_k & M(k+1,j) \\ m(i,k) & op_k & m(k+1,j) \end{cases}$$

$$m(i,j) = \min_{i \le k \le j-1} \begin{cases} M(i,k) & op_k & M(k+1,j) \\ M(i,k) & op_k & m(k+1,j) \\ m(i,k) & op_k & M(k+1,j) \\ m(i,k) & op_k & m(k+1,j) \end{cases}$$

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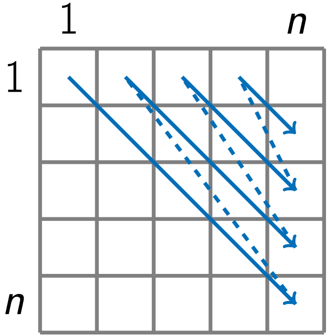
MinAndMax(i, j)

```
min ← +∞
max ← −∞
for k from i to j − 1:
    a ← M(i, k)  op_k  M(k + 1, j)
    b ← M(i, k)  op_k  m(k + 1, j)
    c ← m(i, k)  op_k  M(k + 1, j)
    d ← m(i, k)  op_k  m(k + 1, j)
    min ← min(min, a, b, c, d)
    max ← max(max, a, b, c, d)
return (min, max)
```

Order of Subproblems

- When computing $M(i, j)$, the values of $M(i, k)$ and $M(k + 1, j)$ should be already computed.
- Solve all subproblems in order of increasing $(j - i)$.

Possible Order



Parentheses($d_1\ op_1\ d_2\ op_2 \ldots d_n$)

```
for i from 1 to n:
   m(i,i) ← d_i ,  M(i,i) ← d_i
for s from 1 to n − 1:
   for i from 1 to n − s:
      j ← i + s
      m(i,j), M(i,j) ← MinAndMax(i,j)
return M(1,n)
```

Example: $5 - 8 + 7 \times 4 - 8 + 9$

5	-3	-10	-55	-63	-94
	8	15	36	-60	-195
		7	28	-28	-91
			4	-4	-13
				8	17
					9

m

5	-3	4	25	65	200
	8	15	60	52	75
		7	28	20	35
			4	-4	5
				8	17
					9

M

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