Problem 2

To show that the solution to

$$\min_{\beta_0,\alpha} \sum_{i=1}^{N} (1 - y_i f(x_i))_+ + \frac{\lambda}{2}\alpha^T K\alpha \tag{1}$$

is the same as the solution to

$$\min_{\beta_0,\theta} \sum_{i=1}^{N} \left[1 - y_i \left(\beta_0 + \sum_{m=1}^{\infty} \theta_m \phi_m(x_i)\right)\right]_+ + \frac{\lambda}{2}\sum_{m=1}^{\infty}\frac{\theta_m^2}{\delta_m} \tag{2}$$

We need first to find this particular Kernel, I will be choosing:

$$K(x,y) := h(x)^T h(y) \quad \text{for any } x, y \in \mathbb{R}^p$$

Now we will Substitute $K(x,y)$ with $h(x)^T h(y)$ in the f(x):

$$f(x) = \beta_0 + \sum_{i=1}^{N} \alpha_i K(x, x_i)$$

$$f(x) = \beta_0 + \sum_{i=1}^{N} \alpha_i h(x)^T h(x_i)$$

With:

$$\beta = \sum_{i=1}^{N} \alpha_i h(x_i)$$

f(x) becomes:

$$f(x) = \beta_0 + h(x)^T \beta$$

We know that the norm of β is:

$$\|\beta\|^2 = \beta^T \beta$$

And so with the β, the norm becomes:

$$\|\beta\|^2 = \left(\sum_{i=1}^{N} \alpha_i h(x_i)\right)^T \left(\sum_{i=1}^{N} \alpha_i h(x_i)\right)$$

$$\|\beta\|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j h(x_i)^T h(x_j)$$

Notice that we now see the Kernel definition in this equation:

$$\|\beta\|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j K(x_i, x_j)$$

In matrix notation:

$$\|\beta\|^2 = \alpha^T K \alpha$$

6. **Compare the Optimization Problems:**
The optimization problem (12.25) is given by:

$$\min_{\beta_0,\theta} \sum_{i=1}^{N} \left[1 - y_i \left(\beta_0 + \sum_{m=1}^{\infty} \theta_m \phi_m(x_i) \right) \right]_+ + \frac{\lambda}{2} \sum_{m=1}^{\infty} \frac{\theta_m^2}{\delta_m}$$

The optimization problem (12.29) is:

$$\min_{\beta_0,\alpha} \sum_{i=1}^{N} (1 - y_i f(x_i))_+ + \frac{\lambda}{2} \alpha^T K \alpha$$

Since:

$$f(x) = \beta_0 + h(x)^T \beta = \beta_0 + \sum_{i=1}^{N} \alpha_i K(x, x_i)$$

And,

$$\|\beta\|^2 = \alpha^T K \alpha$$

Thus, the solution to (12.29) is the same as the solution to (12.25) for this particular kernel that we have chosen!!