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1 Problem 1

The way I will organize my work is as follows for each error I will do both parts a and b

L2-Loss

a) Obtaining f^*, the population minimizer of the corresponding population risk with L2-Loss:

$$f^* = \arg\min_f E_{X,Y}[L(Y, f(X))]$$

Notice that since $\mathbf{X}$ is fixed at $\mathbf{X} = x$, the expected value is with respect to Y only. And since $Y \sim \text{Ber}(p)$ with values {-1,1}, and working locally i.e. on a particular f at a certain point, which when we minimize each one of these f we will get the overall f^* the minimizer

$$f^* = \arg\min_f [L(1, f).p + L(-1, f).(1 - p)]$$

Now since this is an L2-loss, thus,

$$f^* = \arg\min_f [(f - 1)^2.p + (f + 1)^2.(1 - p)] = \arg\min_f (f^2 - 4fp + 2f - 1)$$

Now take the derivative and set it equal to 0 to get

$$f^* = 2p - 1$$

and notice that it is not the case where f^* maps X to $\{-1, 1\}$, since it doesn't put a label as a prediction yet! Now we only care about minimizing its value, we can then put say for example if f at a certain point x_i $f(x_i) > 1/2$ then 1 else -1 or any other criterion depending on the situation

Notice that even now f^* is still unknown, since p_i for each Y_i is unknown, and so we will estimate it $p_i \approx \hat{p}_i$, in order to get $\hat{f} \approx f^*$ And so, $\hat{f} = 2\hat{p} - 1$ And so, with

$$\hat{p} = \frac{1 + ay}{2}$$

and on a particular point (x_i, y_i)

$$\hat{f}(x_i) = ay_i$$

Obviously we see that the larger the a is the more weight we add to the training set , if we set a = 0, then we put 0 weights on the training set and similar to the notes, in this particular case we can predict y by a flip of a fair coin! And that is what is meant by when we say a is a parameter that controls the degree of fit to the training data.

b.1) Computing $\hat{R}_{tr}$, the training error
This means that we are approximating Err(T) by $\hat{R}_{tr}$, at X=x

$$\hat{R}_{tr} = \frac{1}{n}\sum_{i=1}^{n} L(Y_i, \hat{f}(x_i)),$$

Now, at a particular $Y_i = y_i$, we get

$$\hat{r}_{tr} = \frac{1}{n}\sum_{i=1}^{n} L(y_i, \hat{f}(x_i)),$$

$$= \frac{1}{n}\sum_{i=1}^{n} (\hat{f}(x_i) - y_i)^2,$$

$$= \frac{1}{n}\sum_{i=1}^{n} (a^2 y_i^2 - 2ay_i^2 + y_i^2),$$

$$= \frac{(a-1)^2}{n}\sum_{i=1}^{n} y_i^2,$$

Notice that

$$\sum_{i=1}^{n} y_i^2 = n$$

Therefore,

$$\hat{r}_{tr} = (a-1)^2$$

Therefore, we can see that as a gets closer to one, $\hat{r}_{tr}$ becomes smaller and smaller (i.e. the training error) which is logical since a is the degree of fit to the training set. So for example when a=1, $\hat{r}_{tr}$ becomes equal to 0 which is consistent with our findings since $\hat{f}(x_i)$ would be exactly equal to y_i, (i.e. 0 Loss)

b.2) Computing $R = Err$, the average test error
Now let's derive $R = Err$, the average test error. Note that usually we would go to get an estimation for Err(T) but sometimes that is hard so

we try estimating Err over the rule $\hat{F}$.

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*,X^*}[L(y_i^*, \hat{f}(x_i^*))]$$

But since $X^* = X$ and X is fixed at a particular value $X_i = x_i$, and the loss is an L2-Loss, then

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*}[(y_i^* - \hat{f}(x_i))^2]$$

with $\hat{f}(x_i) = ay_i$,

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*}[(y_i^*)^2 - 2ay_i y_i^* + a^2 y_i^2]$$

Now since $Y_i^* \sim \text{Ber}(p_i)$, then the expected value with respect to Y^* becomes,

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_T[p_i(1 - 2ay_i + a^2 y_i^2) + (1 - p_i)(1 + 2ay_i + a^2 y_i^2)]$$

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{Y_i}[p_i(1 - 2ay_i + a^2) + (1 - p_i)(1 + 2ay_i + a^2)]$$

And similarly, $Y_i \sim \text{Ber}(p_i)$, then the expected value with respect to Y_i becomes with some basic mathematical formulas and using the fact that $E(h(Y)) = h(-1)P(Y = -1) + h(1)P(Y = 1)$, if $Y \sim$ Ber with values $\{-1, 1\}$,

$$\begin{aligned} Err &= 4a\frac{2}{n_{ts}} \sum_{i=1}^{n_{ts}} [p_i(1 - p_i)] + (a - 1)^2 \\ &= 4a\bar{e} + (a - 1)^2 \end{aligned}$$

Now in order to get the $a^* = a$ that minimizes the Err, we need to do the derivative with respect to a, and thus we get,

$$a = 1 - 2\bar{e}$$

and since $0 \leq \bar{e} \leq \frac{1}{2}$, we get $0 \leq a \leq 1$, which is consistent with the condition of a from the question !
We can easily observe that as a gets closer to 1, the Err derived from the test error gets larger and larger which is overfitting the model on training set (since we are making a bigger this means that we are giving more weights to the training set)

Now in order to get, the mean $= E_T\hat{f}(x_i, T)$, we will use to our advantage that T and Y_i have the same distribution since $X_i = x_i$ is fixed !

$$E_T\hat{f}(x_i, T) = E_{Y_i}\hat{f}(x_i, T)$$
$$E_{Y_i}\hat{f}(x_i, T) = aE_{Y_i}[y_i]$$
$$= 2ap_i - a$$
$$= a(2p_i - 1)$$

Now as for the variance we will use similar analysis, $E_T[(\hat{f}(x_i, T) - E_T\hat{f}(x_i, T))^2]$

$$Var = E_T[(ay_i - 2ap_i + a)^2]$$
$$= a^2E_T[(y_i - 2p_i + 1)^2]$$

And now using similar analysis as before we get,

$$Var = 4a^2p_i(1 - p_i)$$

L1-Loss

a) Obtaining f^*, the population minimizer of the corresponding population risk with L1-Loss:

$$f^* = \arg\min_f E_{X,Y}[L(Y, f(X))]$$

Notice that since $\mathbf{X}$ is fixed at $\mathbf{X} = x$, the expected value is with respect to Y only. And since $Y \sim \text{Ber}(p)$ with values {-1,1}, and working locally i.e. on a particular f at a certain point x_i, which when we minimize each one of these f we will get the overall f^* the minimizer

$$f^* = \arg\min_f[L(1, f).p + L(-1, f).(1 - p)]$$

Now since this is an L1-Loss $L(y, f) = |y - f|$, thus,

$$f^* = \arg\min_f[|1 - f|p + |1 + f|(1 - p)]$$

We have seen in class that in this case ,

$$f^*(x) = median(Y|X = x)$$

, And thus, one classifier that works is:

$$f^*(x) = \begin{cases} 1 & \text{if } p \geq \frac{1}{2}, \\ -1 & \text{otherwise.} \end{cases}$$

and notice that in this case f^* immediatly maps to $\{-1, 1\}$

Notice that even now f^* is still unknown, since p_i for each Y_i is unknown, and so we will estimate it $p_i \approx \hat{p}_i$, in order to get $\hat{f} \approx f^*$ And so, with

$$\hat{p} = \frac{1 + ay}{2}$$

and on a particular point (x_i, y_i),

$$\hat{f}(x_i) = \begin{cases} 1 & \text{if } \hat{p}_i > \frac{1}{2}, \\ -1 & \text{otherwise.} \end{cases}$$

$$\hat{f}(x_i) = \begin{cases} 1 & \text{if } ay_i > 0, \\ -1 & \text{otherwise.} \end{cases}$$

Which now we can say that

$$\hat{f}(x_i) = sign(ay_i)$$

But since $a > 0$, thus we get:

$$\hat{f}(x_i) = sign(y_i)$$

In the case where $a = 0$ we can decide with a fair flip of a coin!
Notice that in this particular loss we are not really paying attention to the value of a a only in the case where a is 0, but in all other cases we are fitting the model to exactly the training data (like in the previous loses when $a = 1$), so we can expect that the training error to be exactly 0.

b.1) Computing $\hat{R}_{tr}$, the training error. And now to test our hypothesis (Hope it is correct)

$$\hat{R}_{tr} = \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \hat{f}(x_i)),$$

Now, at a particular $Y_i = y_i$, we get

$$\hat{r}_{tr} = \frac{1}{n} \sum_{i=1}^{n} L(y_i, \hat{f}(x_i)),$$

$$= \frac{1}{n} \sum_{i=1}^{n} (|y_i - sign(y_i)|) = 0$$

b.2) Computing $R = Err$, the average test error

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*,X^*}[L(y_i^*, \hat{f}(x_i^*))]$$

But since $X^* = X$ and X is fixed at a particular value $X_i = x_i$, and the loss is an Exp-Loss, then

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*}[|y_i^* - sign(y_i)|],$$

with some similar computations as before we get

$$Err = 2\bar{e},$$

Here there is no min value of $a = a^*$available since Err is independent of a!

Now in order to get, the mean $= E_T \hat{f}(x_i, T)$, we will use to our advantage that T and Y_i have the same distribution since $X_i = x_i$ is fixed !

$$E_T \hat{f}(x_i, T) = E_{Y_i} \hat{f}(x_i, T)$$
$$E_{Y_i} \hat{f}(x_i, T) = 2p_i - 1$$

Now as for the variance we will use similar analysis, $E_T[(\hat{f}(x_i, T) - E_T \hat{f}(x_i, T))^2]$

$$Var = E_T[(ay_i - 2ap_i + a)^2]$$
$$= a^2 E_T[(y_i - 2p_i + 1)^2]$$

And now using similar analysis as before we get,

$$Var = 4 + 4p_i(1 - p_i)$$

Exponential

a) Obtaining f^*, the population minimizer of the corresponding population risk with Exp-Loss:

$$f^* = \arg\min_f E_{X,Y}[L(Y, f(X))]$$

Notice that since $\mathbf{X}$ is fixed at $\mathbf{X} = x$, the expected value is with respect to Y only. And since $Y \sim \text{Ber}(p)$ with values {-1,1}, and working locally i.e. on a particular f at a certain point x_i, which when we minimize each one of these f we will get the overall f^* the minimizer

$$f^* = \arg\min_f [L(1, f).p + L(-1, f).(1 - p)]$$

Now since this is an Exponential Loss $L(y, f) = e^{-yf}$, thus,

$$f^* = \arg\min_f [e^{-f}p + e^f(1 - p)] = \arg\min_f [e^{-f}p + e^f - e^f p]$$

Now take the derivative and set it equal to 0 to get

$$f^* = \frac{1}{2} \log(\frac{p}{1 - p})$$

and notice that it is not the case where f^* maps X to $\{-1, 1\}$, since it doesn't put a label as a prediction yet! Now we only care about minimizing its value, we can then put say for example if f at a certain point x_i , $f(x_i) > 1/2$ then 1 else -1 or any other criterion depending on the situation

Notice that even now f^* is still unknown, since p_i for each Y_i is unknown, and so we will estimate it $p_i \approx \hat{p}_i$, in order to get $\hat{f} \approx f^*$

And so, with

$$\hat{p} = \frac{1 + ay}{2}$$

and on a particular point (x_i, y_i),

$$\hat{f}(x_i) = \frac{1}{2} \log(\frac{1 + ay_i}{1 - ay_i})$$

Obviously we see that the larger the a is the more weight we add to the training set , if we set $a = 0$, then we put 0 weights on the training set and similar to the notes $\hat{f}$ will become 0, so in this particular case we can predict y by a flip of a fair coin! And that is what is meant by when we say a is a parameter that controls the degree of fit to the training data.

b.1) Computing $\hat{R}_{tr}$, the training error
This means that we are approximating Err(T) by $\hat{R}_{tr}$, at X=x

$$\hat{R}_{tr} = \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \hat{f}(x_i)),$$

Now, at a particular $Y_i = y_i$, we get

$$\hat{r}_{tr} = \frac{1}{n} \sum_{i=1}^{n} L(y_i, \hat{f}(x_i)),$$

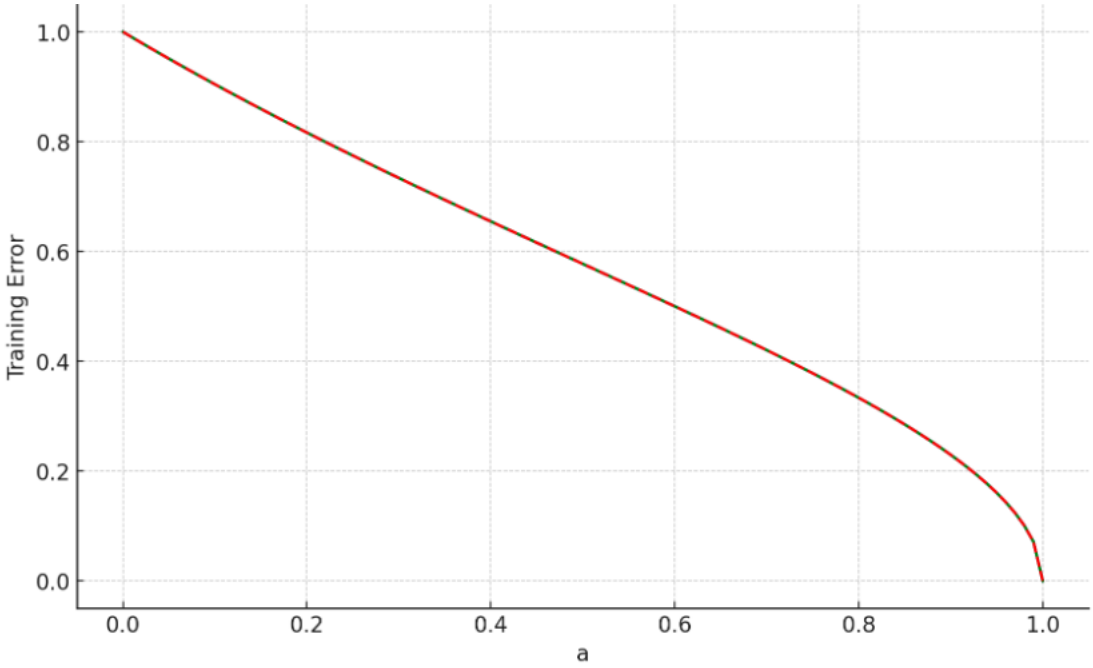
$$= \frac{1}{n} \sum_{i=1}^{n} (e^{-y_i \hat{f}(x_i)}),$$

$$= \frac{1}{n} \sum_{i=1}^{n} (e^{-\frac{1}{2} y_i \log(\frac{1+ay_i}{1-ay_i})}),$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\frac{1 - ay_i}{1 + ay_i})^{\frac{y_i}{2}},$$

And Now in order to understand how the $\hat{r}_{tr}$ changes with a, let's draw a plot, (notice that for both $y_i = 1$ or -1 the $\hat{r}_{tr}$ has the same formula and thus the same graph as we will see now), in this plot I will fix $y_i = 1$ (it

works with -1 as explained since same formula), after fixing it, I will vary a and see how the training error changes.



So as we see from the graph as a increase as the $\hat{r}_{tr}$ decreases which is logical since a is the degree of fit to the training data, and thus at $a = 1$ we can see that the training error becomes 0, because we are fitting exactly like the training data (this tends to have a negative effect on the test error because of concept of overfitting) !

b.2) Computing $R = Err$, the average test error
Now let's derive $R = Err$, the average test error. Note that usually we would go to get an estimation for Err(T) but sometimes that is hard so we try estimating Err over the rule $\hat{F}$.

$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*,X^*}[L(y_i^*, \hat{f}(x_i^*))]$$

But since $X^* = X$ and X is fixed at a particular value $X_i = x_i$, and the loss is an Exp-Loss, then

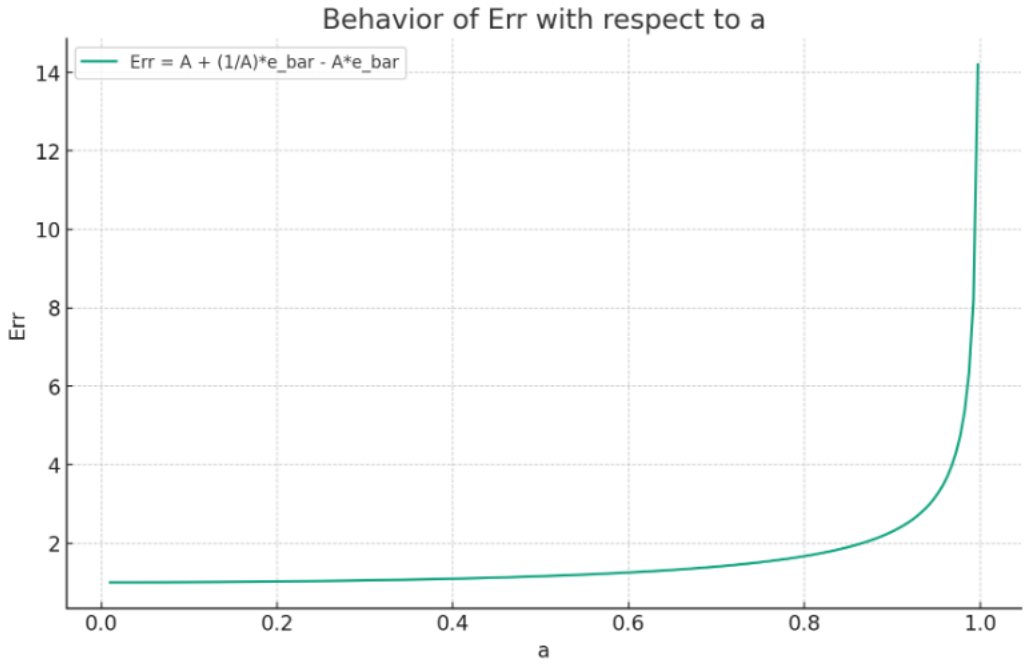
$$Err = \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*}[e^{-\frac{1}{2} y_i^* \log(\frac{1+ay_i}{1-ay_i})}]$$

$$= \frac{1}{n_{ts}} \sum_{i=1}^{n_{ts}} E_{T,Y^*}[(\frac{1-ay_i}{1+ay_i})^{\frac{y_i^*}{2}}]$$

After applying similar steps as before, we get finally

$$Err = A + \frac{1}{A}\bar{e} - A\bar{e}$$

with $A = (\frac{1-a}{1+a})^{\frac{1}{2}}$ Now We will explore how does the Err varies with a, with $\bar{e} = \frac{1}{2}$



Now in order to get, the mean $= E_T\hat{f}(x_i, T)$, we will use to our advantage that T and Y_i have the same distribution since $X_i = x_i$ is fixed !

$$E_T\hat{f}(x_i, T) = E_{Y_i}\hat{f}(x_i, T)$$

$$E_{Y_i}\hat{f}(x_i, T) = \frac{1}{2}(p_i \log(\frac{1}{A}) + \log(A) - p_i \log(A))$$

with $A = \frac{1-a}{1+a}$

Now as for the variance we will use similar analysis, $E_T[(\hat{f}(x_i, T) - E_T\hat{f}(x_i, T))^2]$

$$Var = E_T[(\frac{1}{2}\log(\frac{1+ay_i}{1-ay_i}) + \frac{1}{2}(p_i \log(\frac{1}{A}) + \log(A) - p_i \log(A)))^2]$$

And now using similar analysis as before we get,

$$Var = p_i(\log(\frac{1}{A}))^2 + \frac{1}{4}(1-p_i)(-(\log(A))^2 + p_i^2(\log(\frac{1}{A}))^2)$$