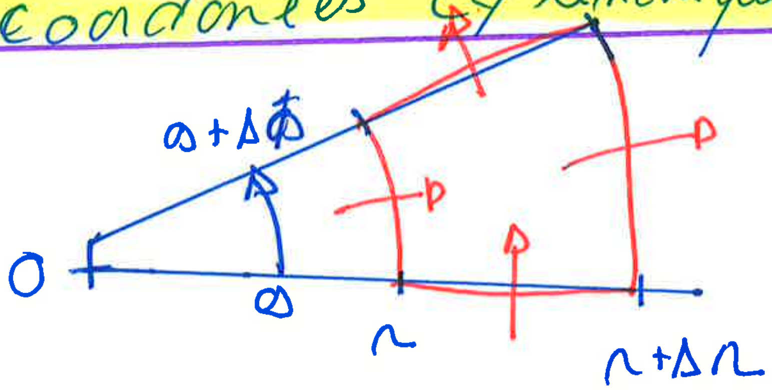
Q. 1

calculer l'équation de la chaleur en coordonées cylindriques.

Résolution.



⚠ piège il faut calculer le gradient en coordonnés cylindriques.

Re $\begin{cases} x = r\cos\theta \\ y = r\sin\theta . \end{cases}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \theta}.$$

$\boxed{x'_r = \partial_r x}$

$$\begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix} = \begin{pmatrix} x'_r & y'_r \\ x'_\theta & y'_\theta \end{pmatrix} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$$

$$\begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} \begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix}$$

(3)

$$\begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix} = \underbrace{\begin{pmatrix} x'_r & y'_r \\ x'_\theta & y'_\theta \end{pmatrix} \begin{pmatrix} r'_x & r'_y \\ \theta'_x & \theta'_y \end{pmatrix}}_{Id. =: P P^{-1}} \begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix}$$

or $\cancel{r_x}$ $x_r = \cos\theta \qquad y_r = \sin\theta.$

$x_\theta = -r\cos\theta \qquad y_\theta = r\cos\theta.$

$$P = \begin{pmatrix} \cos\theta & \sin\theta. \\ -r\cos\theta & r\cos\theta \end{pmatrix} \quad P^{-1} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta. \\ +r\cos\theta & \cos\theta \end{pmatrix}$$

il vient

$$\vec{\nabla} u = \partial_x u \, \vec{e}_x + \partial_y u \, \vec{e}_y.$$

$$= \frac{1}{r}\left(r\cos\theta \, \partial_r u \cancel{+} - \sin\theta \, \partial_\theta u\right) \vec{e}_x.$$

$$\frac{1}{r}\cdot\left(r\cos\theta \, \partial_r u + \cos\theta \, \partial_\theta u\right) \vec{e}_y.$$

$$= \cancel{r} \underbrace{\left(\cos\theta \, \vec{e}_x + \cos\theta \, \vec{e}_y\right)}_{\vec{e}_r} \partial_r u$$

$$+\frac{1}{r}\underbrace{\left(-\sin\theta \, \vec{e}_x + \cos\theta \, \vec{e}_y\right)}_{\vec{e}_\theta .} \partial_\theta u.$$

il vient donc au final :

$$\vec{\nabla}^{p} u = \partial_r u \, \vec{e}^{\,p}_r + \frac{1}{r} \partial_\theta u \, \vec{e}^{\,p}_\theta .$$

⚠ et non ~~$\vec{\nabla}^{p} u = \partial_r u \, \vec{e}^{\,p}_r + \partial_\theta u \, \vec{e}^{\,p}_\theta$~~

c'est pas intuitif !

En ~~...~~ revenant à notre bilan sur un élément de volume annulaire :

*

$$\partial_t \rho c_p T \, dV = - S_{r+\Delta r} \, q^r_{r+\Delta r, \theta} + S_r \, q^r_{r, \theta}$$

$$- S_{\theta+\Delta\theta} \, q^\theta_{r, \theta+\Delta\theta} + S_\theta \, q^\theta_{r, \theta} .$$

$$dV = \frac{1}{2}\left((r+\Delta r)^2 - r^2 \right) . \Delta\theta .$$

$$S_{r+\Delta r} = (r + \Delta r) \Delta\theta \qquad S_\theta = \Delta r .$$

$$S_r = r \, \Delta\theta . \qquad S_{\theta+\Delta\theta} = \Delta r .$$

$$\partial_t \rho c_p T \times \frac{1}{2}\left((r+\Delta r)^2 - r^2\right) \Delta\theta =$$

$$-(r+\Delta r)\Delta\theta \, q^r_{r+\Delta r,\theta} + r\Delta\theta \, q^r_{r,\theta}$$

$$- \Delta r \, q^\theta_{r,\theta+\Delta\theta} + \Delta r \, q^\theta_{r,\theta}.$$

$$(\partial_t \rho c_p T) \frac{1}{2} \frac{(r+\Delta r)^2 - r^2}{\Delta r} =$$

$$- \frac{(r+\Delta r) q^r_{r+\Delta r,\theta} - r \, q^r_{r,\theta}}{\Delta r}.$$

$$- \frac{q^\theta_{r,\theta+\Delta\theta} - q^\theta_{r\theta}}{\Delta\theta}.$$

Par passage à la limite il vient :

$$\partial_t \rho c_p T + \frac{1}{2} \times 2r =$$

$$- \frac{\partial}{\partial r} r q^r - \frac{\partial}{\partial\theta} q^\theta$$

Or on a vu que :

$$\vec{\nabla} T = \partial_r T \, \vec{e}_r + \frac{1}{r} \partial_\theta T \, \vec{e}_\theta .$$

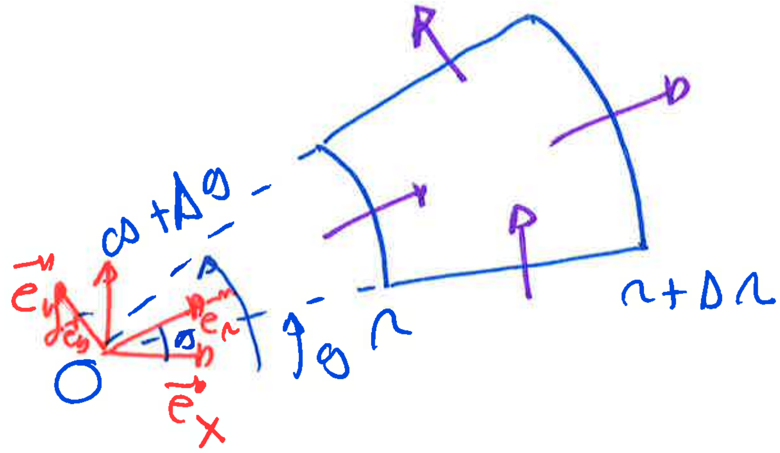
en base cylindrique.

d'où au final :

$$\frac{\partial}{\partial t} T = \frac{\alpha}{r} \left(\frac{\partial}{\partial r} r \frac{\partial}{\partial r} T + \frac{1}{r} \frac{\partial^2}{\partial \theta^2} T \right)$$

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} T + \frac{\alpha}{r^2} \frac{\partial^2}{\partial \theta^2} T .$$

avec $\boxed{\alpha = \frac{\lambda}{\rho c_p}}$.



$\theta+\Delta\theta$
$\vec{e}_y$
$\vec{e}_\theta$
$\vec{e}_r$
θ
O
$\vec{e}_x$
θ
r
$r+\Delta r$

Equation de diffusion cylindrique. ①

Implémentation

$$\Omega = [r_i, r_0] \times [0, 2\pi]$$

$$\frac{1}{r} \partial_r r \partial_r u + \frac{\alpha}{r^2} \partial_\theta^2 u = S$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} u + \frac{\alpha}{r^2} \frac{\partial}{\partial \theta} u = S$$

$$\frac{1}{r} \left(\frac{\partial}{\partial r} u + r \frac{\partial^2}{\partial r^2} u \right) + \frac{\alpha}{r^2} \frac{\partial}{\partial \theta} u = S.$$

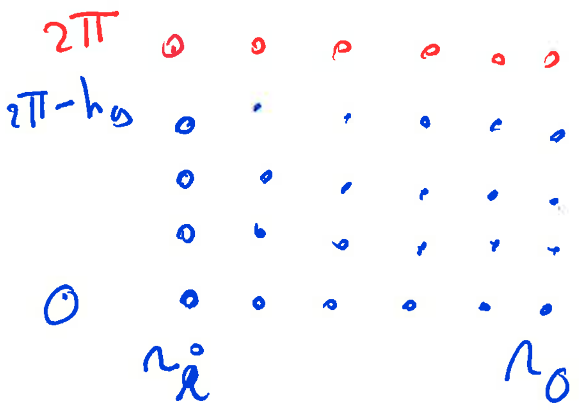
$$\boxed{\frac{\partial^2}{\partial r^2} u + \frac{1}{r} \frac{\partial}{\partial r} u + \frac{\alpha}{r^2} \frac{\partial}{\partial \theta} u = S}$$

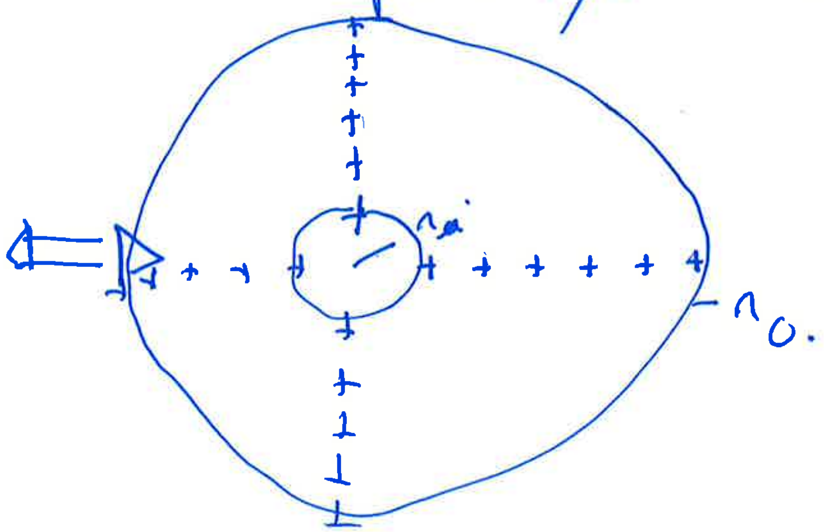
Les conditions aux limites

$$u(r_i, \theta) = u_i \qquad \text{et} \quad u(r, \theta + 2\pi) = u(r, \theta)$$

$$u(r_0, \theta) = u_0$$

périodique en θ





Pour les noeuds intérieurs.

$$\frac{1}{\Delta r^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{1}{r_i}\cdot\frac{1}{\Delta r}\left(u_{i+1,j} - u_{i-1,j}\right)$$

$$+ \frac{\alpha}{r_i}\,\frac{1}{\Delta\theta^2}\left(u_{i,j+1} - u_{i,j-1}\right) = S_{ij}$$

Pour les CL en r

$$\boxed{u_{-1,j} = u(r_i,\theta) = u_{in}} \text{ et } \boxed{u_{N_r+1,j} = u(r_0,\theta) = u_{out}}$$

(i) (ii)

et $\boxed{\begin{array}{l} u_{i,-1} = u_{i,n_\theta-1} \\ u_{i,n_\theta} = u_{i,0}. \end{array}}$ (iii) c'est la traduction de la périodicité.