Optimal Inputs for Linear System Identification

RAMAN K. MEHRA, MEMBER, IEEE

Abstract—This paper considers the design of optimal inputs for identifying parameters in linear dynamic systems. The criterion used for optimization is the sensitivity of the system output to the unknown parameters as expressed by the weighted trace of the Fisher information matrix. It is shown that the optimal energy constrained input is an eigenfunction of a positive self-adjoint operator corresponding to its largest eigenvalue. Several different representations of the optimal input and several methods for its numerical computation are considered. The results are extended to systems with process noise, and the relationship to other criteria for input design are brought out. Three analytical examples are solved in closed form which show that the optimal input is a sum of sine and cosine functions at appropriate frequencies. Optimal elevator deflections for identifying the short period parameters of C-8 aircraft are computed numerically and compared with the doublet input currently in use.

Introduction

THE importance of input selection for system identification has been recognized for a long time, although a unified mathematical treatment has emerged only recently. Some of the earlier attempts at input design were based on frequency domain methods and engineering judgment. An interesting discussion of the relative advantages and disadvantages of oscillatory versus pulse type inputs for aircraft flight testing is contained in [1]. A large amount of literature exists on pseudorandom binary sequence (PRBS) inputs which have been found to provide improved identification for a large number of systems [2]–[4]. However, PRBS inputs use very little information about the known properties of the system. Since in a number of physical systems some *a priori* information is available about the modes of the system (e.g., short period mode, phugoid mode, etc., of an aircraft's longitudinal motion), one can use this information to design inputs for identifying these modes more precisely. This is the basis of the approach considered in this paper.

The present work is related to that of Aoki and Staley [5], Levadi [6], Nahi and Wallis [7], and Levin [8] on input signal design for system identification, and to that of McAulay [9] and Esposito [10] for signal synthesis. Aoki and Staley [5] consider single-input single-output discrete-time systems. The results presented here for multiinput multioutput continuous-time linear systems are conceptually similar to theirs, but the computational methods used are entirely different. Levadi's results [6] are only applicable to the case in which the unknown parameters enter linearly in the system impulse response. Levine's results [8] are applicable when linear regression is used to estimate the unknown parameters.

The optimal inputs in the present work are derived under the assumption that the input is either energy or power constrained, and an efficient estimator is used to estimate the unknown parameters. In practice, a maximum likelihood estimator may be used. The optimal input which maximizes the simple or the weighted trace of information matrix is shown to be an eigenfunction of a positive self-adjoint operator. This input maximizes the sensitivity of the system output to the unknown parameters. The resulting eigenvalue problem is solved in a number of different ways, some of which are 1) transition matrix method, 2) Riccati equation method, 3) resolvent method, and 4) Ritz–Galerkin method. Several analytical examples are given to illustrate the procedure for determining optimal inputs.

The arrangement of the material is as follows. The problem is stated in Section I and is solved using the maximum principle for a single parameter in the gain matrix in Section II. Stronger results are derived in Section III using functional analysis, which also gives an operator-theoretic representation for optimal inputs. Two first-order examples are solved in closed form in Section IV. The extensions of these results to the identification of parameters in the system dynamic matrix are given in Section V. The multiparameter case and the process noise case are discussed in Sections VI and VII. Some numerical results are presented in Section VIII.

I. Problem Statement

Consider a time-invariant linear dynamic system

$$\dot{x} = Fx + Gu \tag{1}$$

where x is an $n \times 1$ state vector and u is an $m \times 1$ control vector. F and G are $n \times n$ and $n \times m$ matrices of unknown parameters. The output of the system is denoted by a $p \times 1$ vector $y(t)$ which is contaminated with white noise $v(t)$:

$$y(t) = Hx(t) + v(t). \tag{2}$$

H is a $p \times n$ matrix and $v(t)$ is a zero-mean Gaussian white noise process.

$$E\{v\} = 0,\ E\{v(t)v^T(\tau)\} = R\delta(t - \tau). \tag{3}$$

Let θ denote an $N \times 1$ vector of unknown but identifiable parameters in the above system. It is required to select the input $\{u(t), 0 \leq t \leq T\}$ to maximize a suitable

Manuscript received June 1, 1973. Paper recommended by H. Sorenson, Chairman of the IEEE S-CS Estimation and Identification Committee. This work was supported in part by the AFOSR under Contract F 44620-71-C-0077, and in part by the Office of Naval Research under Contract N00014-67-A-0298-0006.

The author is with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. 02138.

norm of the information matrix M subject to suitable constraints. The information matrix M for the unknown parameter set θ can be shown to be [7]

$$M = \int_0^T (\nabla_\theta x)^T H^T R^{-1} H (\nabla_\theta x)\, dt \tag{4}$$

where

$$\nabla_\theta = \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \cdots, \frac{\partial}{\theta_N}\right). \tag{5}$$

Notice that M^{-1} represents the Crámer–Rao lower bound for the covariance of an unbiased estimator of θ. The direct minimization of $\mathrm{tr}(M^{-1})$ leads to a highly nonlinear problem that can be solved by gradient methods. On the other hand, the maximization of $\mathrm{tr}(M)$ or $\mathrm{tr}(WM)$ where W is a suitably chosen weighting matrix leads to a much more tractable problem that can be solved by extensions of the existing linear-quadratic optimization methods. It can be seen from (4) that the maximization of $\mathrm{tr}(M)$ would lead to increased sensitivity of the outputs with respect to the unknown parameters. This is exactly what is needed for solving the "inverse problem", viz., determining the unknown parameters from the input–output data. In the next section, we introduce most of the important concepts for solving the above optimization problem by considering first the case of a scalar unknown parameter in G.

II. Optimal Energy-Constrained Input for One Parameter Using Maximum Principle

Let the input be energy constrained as follows:

$$\int_0^T u^T u\, dt = E. \tag{6}$$

Consider first the case where θ *is an unknown parameter in* G. The information M is scalar and the sensitivity function $\nabla_\theta x$ is obtained as follows:

$$\nabla_{\dot\theta} x = F\, \nabla_\theta x + (\nabla_\theta G) u. \tag{7}$$

$\nabla_\theta G$ is a matrix with all zero elements except a single one.

The maximization of M subject to constraint (6) is equivalent to the minimization of the performance index

$$J = \tfrac{1}{2} \int_0^T \left[-(\nabla_\theta x)^T H^T R^{-1} H (\nabla_\theta x) + \mu \left(u^T u - \frac{E}{T} \right) \right] dt \tag{8}$$

where μ is a constant multiplier chosen to keep (6) satisfied. This is a linear-quadratic problem for which the Euler–Lagrange equations are easily written down. For a somewhat different approach, see Anderson [22].

Hamiltonian:

$$\mathcal{H} = \tfrac{1}{2} \left[-(\nabla_\theta x)^T H^T R^{-1} H (\nabla_\theta x) + \mu \left(u^T u - \frac{E}{T} \right) \right] + \lambda^T [F \nabla_\theta x + \nabla_\theta G u] \tag{9}$$

where λ is an $n \times 1$ costate vector.

$$\dot\lambda = -\left(\frac{\partial \mathcal{H}}{\partial \nabla_\theta x} \right)^T$$

or

$$\dot\lambda = -F^T \lambda + H^T R^{-1} H\, \nabla_\theta x. \tag{10}$$

Stationarity Condition:

$$\mathcal{H}_u = 0$$

or

$$u^* = -\frac{1}{\mu} (\nabla_\theta G)^T \lambda. \tag{11}$$

The boundary conditions are homogeneous.

$$\nabla_\theta x(0) = 0, \quad \lambda(T) = 0.$$

Substituting for u^* in (7), we obtain the two-point boundary value problem

$$\frac{d}{dt} \begin{bmatrix} \nabla_\theta x \\ \lambda \end{bmatrix} = \begin{bmatrix} F, & -\frac{1}{\mu} (\nabla_\theta G)(\nabla_\theta G)^T \\ H^T R^{-1} H, & -F^T \end{bmatrix} \begin{bmatrix} \nabla_\theta x \\ \lambda \end{bmatrix}. \tag{12, 13}$$

Since the boundary conditions are homogeneous, the solution is trivial, viz., $\nabla_\theta x \equiv 0$, $\lambda \equiv 0$, $\mu \equiv 0$, except for certain values of μ which are the eigenvalues of the two-point boundary value problem. In other words, the problem is of the Sturm–Liouville type [12]. The eigenvalues and the optimal input can be determined in a number of ways. Two possible methods are 1) the transition matrix method, and 2) the Riccati equation method.

A. Transition Matrix Method

Let $\Phi(t,0;\mu)$ denote the transition matrix of (12), (13) for a particular μ.

Then

$$\begin{bmatrix} \nabla_\theta x(T) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} \Phi_{xx}(T,0;\mu) & \Phi_{x\lambda}(T,0;\mu) \\ \Phi_{\lambda x}(T,0;\mu) & \Phi_{\lambda\lambda}(T,0;\mu) \end{bmatrix} \begin{bmatrix} \nabla_\theta x(0) \\ \lambda(0) \end{bmatrix}. \tag{14, 15}$$

The equation in (15) along with the boundary conditions gives

$$\lambda(T) = \Phi_{\lambda\lambda}(T,0;\mu)\lambda(0) = 0. \tag{16}$$

For a nontrivial solution

$$\left|\Phi_{\lambda\lambda}(T,0;\mu)\right| = 0. \tag{17}$$

Equation (17) is the eigenvalue equation for the Hamiltonian system (12). It is a nonlinear algebraic equation in μ and can be solved by a Newton–Raphson iteration. In general, there is an infinite set of eigenvalues, but we will be only interested in the largest value of μ which will be shown to minimize J (Section III).

B. Riccati Equation Method

The eigenvalues μ are functions of the interval length T. Therefore, one can fix μ and determine T for which $\Phi_{\lambda\lambda}(T,0;\mu)$ becomes singular. Another way is to use the

Riccati matrix $P(t)$ defined by the relationship

$$\nabla_\theta x(t) = P(t)\lambda(t). \tag{18}$$

An equation for $P(t)$ is obtained by differentiating both sides of (18) and substituting from (12).

$$\nabla_\theta \dot{x} = \dot{P}\lambda + P\dot{\lambda}$$

or

$$\left[FP - \frac{1}{\mu}(\nabla_\theta G)(\nabla_\theta G)^T\right]\lambda = [\dot{P} + PH^TR^{-1}H\,P - P\,F^T]\lambda$$

or

$$\dot{P} = FP + PF^T - PH^TR^{-1}HP - \frac{1}{\mu}(\nabla_\theta G)(\nabla_\theta G)^T \tag{19}$$

$$P(0) = 0. \tag{20}$$

The Riccati equation (19) differs from the usual Riccati equation of the linear-quadratic problem since the forcing term (last term) in (19) enters negatively. Equation (18) can also be written as

$$\lambda(t) = P^{-1}(t)\nabla_\theta x(t)$$

whenever P^{-1} exists. At final time $t = T$, since $\lambda(T) = 0$,

$$P^{-1}(T) = 0, \tag{21}$$

which means that a conjugate point exists at $t = T$.

Equation (21) provides us with a method to obtain critical interval length T corresponding to an eigenvalue μ. The Riccati equation (19) is integrated forward in time for a particular μ, using initial conditions (20). When the elements of $P(t)$ become very large, the critical length T corresponding to an eigenvalue is being reached. Now $P^{-1}(t)$ is integrated using the equation

$$\frac{d}{dt}(P^{-1}) = -P^{-1}\dot{P}\,P^{-1}$$

or

$$\frac{d}{dt}(P^{-1}) = -P^{-1}F - F^TP^{-1} + H^TR^{-1}H + \frac{1}{\mu}P^{-1}(\nabla_\theta G)(\nabla_\theta G)^TP^{-1}. \tag{22}$$

At the critical interval length T, all the elements of P^{-1} go to zero. It follows from the Sturmian property [12] that the smallest T corresponds to the largest eigenvalue μ.

The Riccati equation method is similar to the invariant imbedding method of Alspaugh *et al.* [11] for determining eigenvalues in the problem of buckling of beams. By this method, one obtains a curve relating μ and T. (See Fig. 2.)

After the critical length T corresponding to the largest value of μ has been determined, (12), (13) are solved forward in time using $\lambda(0)$ obtained from (16) and (17) as an eigevnector of $\Phi_{\lambda\lambda}(T,0;\mu)$ corresponding to the zero eigenvalue. Thereby the boundary condition $\lambda(T) = 0$ is automatically satisfied. A unique value of $\lambda(0)$ is found by using the normalization condition of (6). A flow chart of the method is shown in Fig. 1.

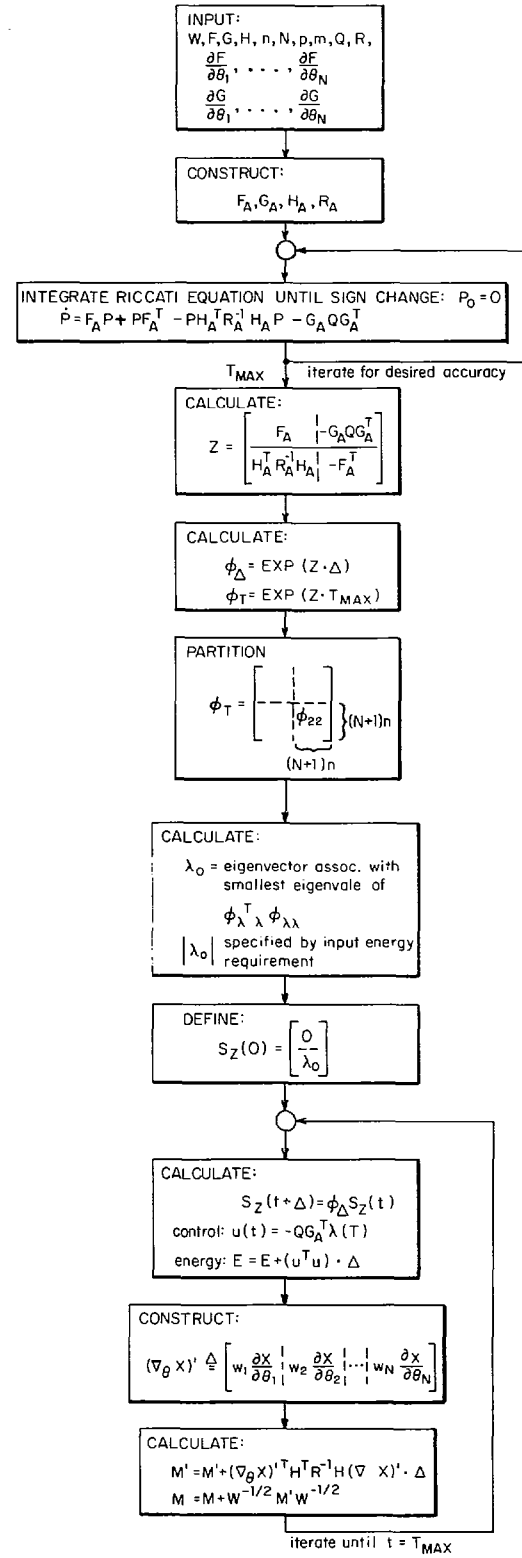

Fig. 1. Flow chart of optimal input computer program.

III. Application of Functional Analysis

In the last section, the optimal input u was characterized in terms of the solution to a two-point boundary value problem. In this section, we show that optimal u is an eigenfunction of a positive self-adjoint operator corresponding to the largest eigenvalue μ.

Let A denote the operator corresponding to (7), viz.,

$$A[u] = \int_0^T e^{F(t-\tau)} (\nabla_\theta G) u(\tau) d\tau. \tag{23}$$

Let $A^*[\cdot]$ denote the adjoint operator to $A[\cdot]$:

$$A^*[w] = (\nabla_\theta G)^T \int_t^T e^{F^T(s-\tau)} w(s) ds. \tag{24}$$

Let $\langle u,w \rangle$ denote the inner product:

$$\langle u,w \rangle = \int_0^T u^T(t) w(t) dt. \tag{25}$$

The information matrix M can be written as

$$M = \langle u, A^* H^T R^{-1} H\, Au \rangle. \tag{26}$$

The energy constraint of (6) is written as

$$\langle u,u \rangle = E.$$

It is well known [12] that M is maximized subject to the above constraint by u^*, which is an eigenfunction corresponding to the largest eigenvalue of the operator $A^*H^T R^{-1}HA$. Furthermore, since $A^*H^TR^{-1}HA$ is a positive self-adjoint operator, all its eigenvalues are real and positive [12]. For finite T, the operator is also bounded and has a finite maximum eigenvalue. The optimal input u^* is the eigenfunction of $A^*H^TR^{-1}HA$ corresponding to this eigenvalue and normalized according to $\langle u,u \rangle = E$.

$$A^*H^TR^{-1}HA\, u = \mu\, u. \tag{27}$$

Also,

$$\max_u M = \mu E. \tag{28}$$

To show the relationship of the above eigenvalue problem with the two-point boundary value of (12), define

$$z = Au \tag{29}$$

and

$$\eta = \int_t^T e^{F^T(s-t)} H^T R^{-1} Hz\,(s) ds. \tag{30}$$

Then it is easily shown that [23]

$$z = \nabla_\theta x$$

$$\eta = \lambda.$$

In order to relate these results to those of Levadi [6], (27) is written as a Fredholm integral equation of the second kind. After some manipulations, one gets

$$\int_0^T d\tau \wedge (t,\tau) u(\tau) = \mu\, u\,(t) \tag{31}$$

where

$$\wedge (t,\tau) = \int_0^T K(t,\tau,s) ds. \tag{32}$$

$$K(t,\tau,s) = \begin{cases} (\nabla_\theta G)^T e^{F^T(s-t)} H^T R^{-1} H\, e^{F(s-\tau)} \nabla_\theta G, & s \geq t \geq \tau \\ (\nabla_\theta G)^T e^{F^T(s-\tau)} H^T R^{-1} H\, e^{F(s-t)} \nabla_\theta G, & s \geq \tau \geq t \end{cases} \tag{33}$$

over $0 \leq t, s, \tau \leq T$.

The above representation suggests two other methods for obtaining μ and $u(t)$. These are the Ritz–Galerkin method [12] and the resolvent method [19].

A. Ritz–Galerkin Method

In this method, $u(t)$ is expanded in a series of orthogonal functions.

$$u(t) = \sum_{k=1}^{L} a_k \phi_k(t) \tag{34}$$

where L, the number of terms in the expansion is chosen large enough to give an adequate representation for $u(t)$ on $[0,T]$. One possible choice is

$$\phi_k(t) = \cos\left(k - \frac{1}{2}\right)\frac{\pi t}{T}, \qquad k = 1,\cdots,L \tag{35}$$

which automatically satisfies the boundary condition

$$u(T) = 0.$$

The coefficients a_k, $k = 1,\cdots,L$ are chosen to minimize the squared error in satisfying the integral equation. This is achieved by making the error orthogonal to ϕ_j, $j = 1,\cdots,L$.

$$\int_0^T dt \left[\int_0^T \wedge (t,\tau) \sum_{k=1}^{L} a_k \phi_k(\tau) d\tau - \mu \sum_{k=1}^{L} a_k \phi_k(t)\right] \phi_j(t) = 0 \tag{36}$$

or

$$\sum_{k=1}^{L} a_k \left\{\int_0^T dt \int_0^T d\tau \wedge (t,\tau)\, \phi_k(\tau)\, \phi_j(t) - \mu \delta_{j,k}\right\} = 0.$$

For a nontrivial solution,

$$\left|\int_0^T dt \int_0^T d\tau \wedge (t,\tau) \phi_k(\tau)\, \phi_j(t) - \mu \delta_{j,k}\right| = 0 \tag{37}$$

where $|\cdot|$ denotes the determinant of the $L \times L$ matrix with rows $j = 1,\cdots,L$ and columns $k = 1,\cdots,L$. The optimal input is given by the eigenvector corresponding to the largest eigenvalue of this matrix.

B. Resolvent Method

The resolvent method is based on the resolvent identity [19].

$$R_\mu - R_{\mu'} = (\mu - \mu')\, R_\mu\, R_{\mu'} \tag{38}$$

where

$$R_\mu = (\mu I - A^*H^TR^{-1}HA)^{-1}.$$

Using this identity, initial value problems are set up in μ

for the resolvent kernel and the input u. These equations are integrated forward starting from $\mu = 0$ until the first pole of R_μ is encountered. The advantage of this method over the Riccati equation method of Section II-B is that μ is determined for a fixed T and not vice versa. (For details see [19].)

IV. Examples

We now apply the above results to an example considered by Levadi [6].

A. First-Order System with Unknown Gain

Consider the system

$$\dot{x} = -x + bu \tag{39}$$

where x and u are scalars and b is the unknown gain.

$$y = x + v \tag{40}$$

where v is a correlated noise process with autocorrelation function

$$E\{v(t)v(\tau)\} = c\, e^{-a|t-\tau|}. \tag{41}$$

It is required to estimate b only.

Levadi's [6] results can be easily derived as follows.

$v(t)$ can be represented as

$$\dot{v} = -av + \epsilon \tag{42}$$

where ϵ is a white noise process and

$$E\{\epsilon\} = 0, \quad E\{\epsilon(t)\epsilon(\tau)\} = 2ac\,\delta(t-\tau). \tag{43}$$

A new measurement can be generated by differentiating (40). (This procedure is similar to that of Bryson and Johansen [13].)

$$\begin{aligned}\dot{y} &= \dot{x} + \dot{v} \\ &= (a-1)x + bu - ay + \epsilon.\end{aligned} \tag{44}$$

Now $\dot{y}$ can be regarded as a new measurement which has white noise ϵ in it. The new information matrix is

$$M = \int_0^T \frac{1}{2ac}\,[(a-1)\nabla_b x + u]^2 dt \tag{45}$$

where

$$\dot{\nabla_b x} = -\nabla_b x + u. \tag{46}$$

The following equations of optimality are easily derived.

$$u = -\frac{1}{2\left(\mu - \frac{1}{2ac}\right)}\left[\lambda - \frac{a-1}{ac}\nabla_b x\right] \tag{47}$$

$$\dot{\lambda} = \frac{(a-1)^2}{ac}\nabla_b x + \frac{a-1}{ac}u + \lambda. \tag{48}$$

An equation only in terms of u can also be obtained from (46)–(48):

$$\ddot{u} - \left[1 - \frac{a^2-1}{2\mu ac - 1}\right]u = 0. \tag{49}$$

The eigenvalue equation is

$$\tan(wT + \phi) = \frac{w}{a} \tag{50}$$

where

$$w = \left[-1 + \frac{a^2-1}{2\mu ac - 1}\right]^{\frac{1}{2}} \tag{51}$$

$$\phi = \tan^{-1} w. \tag{52}$$

The optimal input $u^* = A \sin(wt + \phi)$.

The optimum value of w is chosen to maximize μ. From (51),

$$\mu = \frac{1}{2ac}\left[1 + \frac{a^2-1}{1+w^2}\right]. \tag{53}$$

It is seen from (53) that when $a^2 > 1$, the maximum of μ is attained for the smallest value of w. This corresponds to the case when the noise is wide band. For the narrow-band noise case, viz., $a^2 < 1$, the second term in (53), is negative and the maximum of μ, viz., $1/2ac$ is reached at $w = \infty$. The practical implication of this result is that the input should be of as high frequency as possible. Since the noise is narrow band, this increases the high-frequency signal-to-noise ratio at the output.

In [23], a second-order example corresponding to the short-period mode of an aircraft is solved in closed form. The optimal input is the sum of two sine waves at frequencies closely bracketing the natural frequency of the system.

V. Extension to Unknown Parameter in F

In Sections II–IV, we only considered the case in which θ is an unknown in G. Consider the case in which θ is an unknown in F. The sensitivity equation is

$$\dot{\nabla_\theta x} = F\,\nabla_\theta x + (\nabla_\theta F)x. \tag{54}$$

Since (60) involves x, it is necessary to consider (1). Let x_A denote the augmented state vector

$$x_A = \begin{bmatrix} x \\ \nabla_\theta x \end{bmatrix}; \tag{55}$$

then

$$\dot{x}_A = F_A x_A + G_A u \tag{56}$$

where

$$F_A = \begin{bmatrix} F & 0 \\ \nabla_\theta F & F \end{bmatrix} \tag{57}$$

$$G_A = \begin{bmatrix} G \\ 0 \end{bmatrix}. \tag{58}$$

The performance index J can be written as

$$J = \tfrac{1}{2}\int_0^T \left[-x_A{}^T H_A{}^T R^{-1} H_A x_A + \mu \left(u^T u - \frac{E}{T} \right) \right] dt \quad (59)$$

where

$$H_A = \begin{bmatrix} & \cdot & \\ 0 & \cdot & I \\ & \cdot & \end{bmatrix}. \quad (60)$$

The two-point boundary value problem becomes

$$\begin{bmatrix} \dot{x}_A \\ \dot{\lambda}_A \end{bmatrix} = \begin{bmatrix} F_A & -\frac{1}{\mu} G_A G_A{}^T \\ H_A{}^T R^{-1} H_A & -F_A{}^T \end{bmatrix} \begin{bmatrix} x_A \\ \lambda_A \end{bmatrix} \quad (61)$$

$$-u^* = -\frac{1}{\mu} G_A{}^T \lambda_A \quad (62)$$

$$x_A(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix} \quad (63)$$

$$\lambda_A(T) = 0. \quad (64)$$

This problem can be solved by methods similar to the ones discussed earlier in Section II. The dimension of the problem, however, is doubled.

Notice that θ appears explicitly in (61). Therefore, the optimal input u cannot be obtained in one iteration unless a certain probability distribution is given for θ (see Aoki and Staley [5]). Otherwise, one starts with an estimate of θ to obtain a suboptimal u. Then the estimate of θ is refined from actual data and u is recomputed. This approach is similar to the sequential design approach of Wald [21].

VI. Extension to the Multiparameter Case

In the multiparameter case, M is a matrix and the selection of a suitable norm for optimization becomes very important. This question has been discussed earlier by Aoki and Staley [5] and Nahi and Napjus [18] who have considered the following norms:

1) $\mathrm{tr}(M)$
2) $\mathrm{tr}(M^{-1})$
3) $|M|$ or $|M^{-1}|$
4) $\mathrm{tr}(QM)$—weighted trace.

Two sets of useful inequalities for these norms are [5], [18]

$$\left(\frac{1}{N}\,\mathrm{tr}\,M\right)^N \geq \prod_{i=1}^{N} m_{ii} \geq \det M = (\det \bar{P})^{-1} \geq \prod_{i=1}^{N} \frac{1}{\bar{p}_{ii}} \geq \left(\frac{1}{N}\,\mathrm{tr}\,\bar{P}\right)^{-N} \quad (65)$$

and

$$N^2 \geq \mathrm{tr}\,\bar{P} \cdot \mathrm{tr}\,M \geq \frac{N^2}{2}\left(1 + \frac{\alpha_{\max}}{\alpha_{\min}}\right) \quad (66)$$

where $\bar{P} = M^{-1}$, m_{ii} and $\bar{P}_{ii}$ are the diagonal elements of M and $\bar{P}$, and $\alpha_{\max}$ and $\alpha_{\min}$ are the maximum and minimum eigenvalues of M. In the second inequality, $\mathrm{tr}\,\bar{P} \cdot \mathrm{tr}M$ is a figure of merit which provides a measure of the tightness of the inequality (65).

The approach taken in this paper applies directly to $\mathrm{tr}(M)$ and $\mathrm{tr}(QM)$ since these are linear norms of M. The other norms require iterative hill-climbing methods for solution. The presence of local minima and singular arcs make the resulting optimization problem difficult to solve.

A straightforward method to maximize $\mathrm{tr}(M)$ is to define an augmented state vector x_A as follows:

$$x_A = \begin{bmatrix} x \\ \nabla_{\theta_1} x \\ \cdot \\ \cdot \\ \cdot \\ \nabla_{\theta_N} x \end{bmatrix}. \quad (67)$$

$(N+1)n \times 1$

The state equation for x_A can be written as

$$\dot{x}_A = F_A x_A + G_A u \quad (68)$$

where

$$F_A = \left[\begin{array}{c|c|c|c} F & 0 & \cdots & 0 \\ \hline \nabla_{\theta_1} F & F & & \\ \hline \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \hline \nabla_{\theta_N} F & & & F \end{array}\right],$$

$(N+1)n \times (N+1)n$

$$G_A = \begin{bmatrix} G \\ \nabla_{\theta_1} G \\ \cdot \\ \cdot \\ \cdot \\ \nabla_{\theta_N} G \end{bmatrix}. \quad (69)$$

Define

$$H_A \begin{bmatrix} 0 & H & & 0 \\ & & \cdot & \cdot \\ & & & \cdot \\ & & & 0 \\ 0 & 0 & 0 & H \end{bmatrix}$$

$(N+1)p \times (N+1)n.$

Then

$$\mathrm{tr}(M) = \int_0^T \mathrm{tr}[(\nabla_\theta x)^T H^T R^{-1} H \nabla_\theta x]\, dt = \int_0^T x_A{}^T H_A{}^T R_A{}^{-1} H_A x_A\, dt \quad (70)$$

where

$$R_A{}^{-1} = \begin{bmatrix} R^{-1} & & & \bigcirc \\ & R^{-1} & & \\ & & \cdot & \\ & & & \cdot \\ \bigcirc & & & R^{-1} \end{bmatrix}.$$

A comparison of (68) and (70) with (1) and (4) shows that the optimal input u for the multiparameter case can be obtained by solving the same kind of linear two-point boundary value problem as for the scalar case. This method, however, leads to a high-dimensional problem. Wilkie and Perkins [14] and Neuman and Sood [15] have shown that the sensitivity functions for all the parameters can be obtained by linear transformations of the sensitivity functions for a few of the parameters. Using these transformations, the dimension of the problem can be kept low.

An explicit transformation can be constructed from n independent columns of the controllability matrix $[G, FG, \cdots, F^{n-1}G]$. It is of the form $T^{-1} = [g_1, Fg_1, \cdots, F^{n_1-1}g_1, g_2, \cdots, F^{n_2-1}g_2, \cdots, F^{n_m-1}g_m]$ where $g_1 \cdots g_m$ are columns of G and $n_1, \cdots, n_m$ are minimal indices of the system. The matrices TFT^{-1} and TG of the transformed system are in a special canonical form with parameter set α. The sensitivity equations for α can be obtained by solving no more than $(m+1)n$ differential equations consisting of n system equations and n sensitivity equations for each input [15]. These equations define a new state-space representation which is both completely controllable and observable.[1] The performance index $\mathrm{tr}(M)$ can be expressed as a quadratic form in terms of this new state representation, and the optimal trajectories can be obtained by solving a Riccati equation of dimension at most $(m+1)n \times (m+1)n$.

A more direct approach is to construct a minimal realization for the augmented system (F_A, G_A, H_A) by considering only the completely controllable and observable parts of the system. Since the initial state is null, only this part of the system gets excited and contributes to the performance index. Thus the quadratic optimization problem needs to be solved only for the minimal system which is of dimension $(m+1)n$ or less [15]. Further details on this approach will be presented in a later paper.

VII. Extension to Systems with Process Noise

Consider a linear dynamic system

$$\dot{x} = Fx + Gu + \Gamma\eta \tag{71}$$

$$y = Hx + v \tag{72}$$

where $\eta(t)$ is a Gaussian white noise forcing function

$$E\{\eta(t)\} = 0, \quad E\{\eta(t)\eta^T(\tau)\} = Q\delta(t-\tau).$$

The information matrix M in this case is given in terms of the Kalman filter for the above system [16].

$$\dot{\hat{x}} = F\hat{x} + Gu + K(y - H\hat{x}) \tag{73}$$

$$K = \Sigma H^T R^{-1} \tag{74}$$

$$\dot{\Sigma} = F\Sigma + \Sigma F^T + \Gamma Q \Gamma - \Sigma H^T R^{-1} H \Sigma \tag{75}$$

where $\hat{x}$ denotes the best filtered estimate of x and Σ denotes the covariance of $\hat{x}$. The Kalman filter provides a linear causally invertible whitening transformation for the process y since the innovation sequence $(y - H\hat{x})$ is a Gaussian white noise sequence. The likelihood function is easily written in terms of the innovation sequence [20], [17]. The information matrix M is given as

$$M = \int_0^T E\{(\nabla_\theta \hat{x})^T H^T R^{-1} H (\nabla_\theta \hat{x})\}\, dt \tag{76}$$

where $E\{\cdot\}$ denotes the expectation and $\nabla_\theta \hat{x}$ denotes the sensitivity function of the filtered estimate $\hat{x}$ with respect to the unknown parameter vector θ. Note that both K and Σ are functions of θ so that the sensitivity equations are much more complicated than for the no process noise case. Moreover, M, in general, depends on the random quantities η and v so that its expected value needs to be maximized.

A special case arises when θ contains parameters from G only. Since K and P do not depend upon G, the sensitivity equation has a simple form:

$$\nabla_\theta \dot{\hat{x}} = (F - KH)\nabla_\theta \hat{x} + \nabla_\theta G u. \tag{77}$$

K is, in general, time varying, but if the system is completely controllable and observable, K reaches a constant steady-state value [16]. Then (77) is essentially similar to (7) except that F is replaced by $(F - KH)$. Thus most of the theory developed in Sections II–IV carries over to this case.

Example: Let us consider the system

$$\dot{x} = -x + \theta u + \eta$$

$$y = x + v \tag{78}$$

where

$$E[v] = 0, \; E[v(t)v(\tau)] = r\delta(t-\tau)$$

$$E\{\eta\} = 0, \; E\{\eta(t)\eta(\tau)\} = q\,\delta(t-\tau).$$

The filter sensitivity equation for θ under steady-state filter gain $k > 0$ is

$$\nabla_\theta \dot{\hat{x}} = -(1+k)\nabla_\theta \hat{x} + u. \tag{79}$$

Proceeding as in Section IV-A and defining

$$\omega = \left[\frac{1}{\mu r} - (1+k)^2\right]^{\frac{1}{2}}$$

or

$$\mu = \frac{1}{r[\omega^2 + (1+k)^2]}, \tag{80}$$

it is seen that the optimal input u^* obeys equations of type (50)–(52).

Notice that by increasing process noise q, the gain increases and μ decreases. Thus the information $M = \mu E$ for the same input energy E decreases. The frequencies ω, however, remain unchanged.

[1] The system (F_A, G_A, H_A) is not completely controllable and observable.

VIII. Numerical Results

In this section, numerical results obtained by using the Riccati equation method are presented. Optimal elevator deflection inputs for identifying parameters in the longitudinal short period equations of C-8 aircraft are computed and compared with the inputs currently in use. The state variables are the angle of attack α and the pitch rate q, and the input is the elevator command δ_e. The equations for the short period dynamics of the C-8 aircraft are

$$\begin{bmatrix} \dot{q} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -1.588 & -0.562 \\ 1 & 0.737 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} -1.66 \\ 0.005 \end{bmatrix} \delta_e$$

and the measurement equations are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} n_q \\ n_\alpha \end{bmatrix}.$$

In determining the power spectral densities of n_q and n_α, the values given in [15] (1 °/s error in q, and 2° in α) are multiplied by two times the correlation time of the noise sources, which is assumed to be 0.01 s. The measurement noise spectral density matrix is therefore given as

$$R = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.04 \end{bmatrix}.$$

In this experiment the data length T is fixed at 4 s. The appropriate $\mu_{\max}$ is found from a $\mu_{\max} - T$ curve shown in Fig. 2. The value of $\mu_{\max}$ associated with a T of 4 s is 0.015. The shape of the $\mu_{\max} - T$ curve in Fig. 2 is characteristic of the general relationship between these two variables.

For the $\mu_{\max}$ and R values indicated above, the optimal input with respect to the three parameters in the F matrix and the two parameters in G is given in Fig. 3. The energy of the input is 311 and $\mathrm{tr}\{M\} = 20\,460$. The check value of $\mu_{\max}E$ is approximately 20 200, indicating a numerical error of 0.1 percent. The determinant of M is computed to be 1×10^{15}, with the ratio of the largest to smallest eigenvalue of M being almost three orders of magnitude.

The eigenvalues of M are 11881, 7616, 758, 613, and 25. The standard deviations of the parameter estimates are

$$\text{Standard Deviation for } F = \begin{bmatrix} 0.167, & 0.0639 \\ & 0.035 \end{bmatrix}$$

$$\text{Standard Deviation for } G = \begin{bmatrix} 0.095 \\ 0.025 \end{bmatrix}, \quad \mathrm{tr}(M^{-1}) = 0.0484.$$

The results obtained by using a doublet input of equal energy and time duration (see Fig. 4) are

$$|M| = 7 \times 10^8, \quad \mathrm{tr}(M) = 1585$$

$$\text{Standard Deviation for } F = \begin{bmatrix} 0.147, & 0.247 \\ & 0.164 \end{bmatrix}$$

$$\text{Standard Deviation for } G = \begin{bmatrix} 0.0631 \\ 0.0346 \end{bmatrix}, \quad \mathrm{tr}(M^{-1}), = 0.12128.$$

The eigenvalues of M are 396, 232, 58, 13, and 10. It is seen that all the eigenvalues are smaller than those for the optimal input, even though some of the parameter standard deviations are lower with the doublet input.

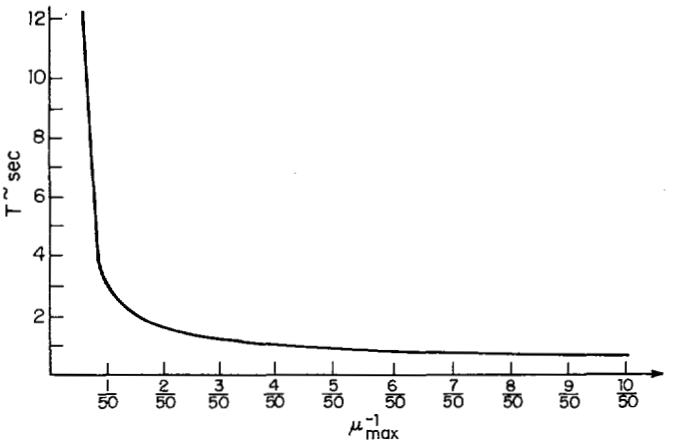

Fig. 2. $\mu_{\max}^{-1}$ versus T curve for a two-state five-parameter model.

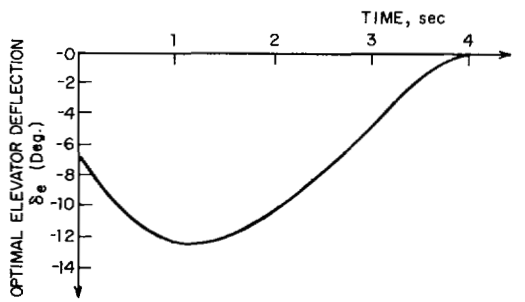

Fig. 3. Optimal input for short period longitudinal dynamics.

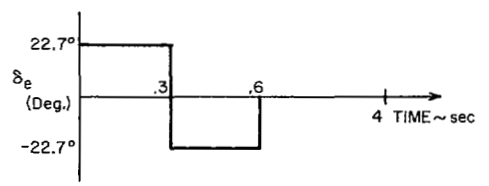

Fig. 4. Doublet input of same energy and time duration as optimal input.

IX. Conclusions

The problem of optimal input design for linear system identification is formulated as a linear-quadratic optimal control problem. It is shown that nontrivial solutions exist for certain values of the multiplier μ which correspond to the eigenvalues of a linear two-point boundary value problem. The optimal input is shown to be an eigenfunction of a positive self-adjoint operator corresponding to the largest eigenvalue μ. Numerical methods for the determination of eigenvalues and eigenfunctions of the linear two-point boundary value problem are discussed. Analytical results are derived for several first-order examples. Extensions of the results to systems with process noise are presented. Optimal elevator deflections for the short period mode of C-8 aircraft are computed and compared with the doublet inputs currently in use.

Acknowledgment

The author wishes to thank Dr. J. S. Tyler, Dr. D. E. Stepner, and Dr. J. Casti of Systems Control, Inc. for many stimulating discussions and suggestions.

References

[1] C. L. Muzzy and E. A. Kidd, "Measurement and interpretation of flight test data for dynamic stability and control," ch. 2 in *AGARD Flight Test Manual*, vol. 2. New York: Pergamon, 1959.
[2] P. A. N. Briggs, K. R. Godfrey, and P. H. Hammond, "Estimation of process dynamics characteristics by correlation methods using pseudorandom signals," presented at the IFAC Symp. Identification and Process Parameter Estimation, Prague, Czechoslovakia, June 1967.
[3] K. R. Godfrey, "The application of pseudorandom sequences to industrial processes and nuclear power plants," presented at the IFAC Symp. Identification and Process Parameter Estimation, Prague, Czechoslovakia, 1970.
[4] I. G. Cumming, "Frequency of input signal in identification," presented at the IFAC Symp. Identification and Process Parameter Estimation, Prague, Czechoslovakia, 1970.
[5] M. Aoki and R. M. Staley, "On input signal synthesis in parameter identification," *Automatica*, vol. 6, 1970.
[6] V. S. Levadi, "Design of input signals for parameter estimation," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 205–211, Apr. 1966.
[7] N. E. Nahi and D. E. Wallis, Jr., "Optimal inputs for parameter estimation in dynamic systems with white noise observation noise," *Preprints, Joint Automat. Contr. Conf.*, Boulder, Colo., Aug. 1969.
[8] M. J. Levin, "Estimation of a system pulse transfer function in the presence of noise," *IEEE Trans. Automat. Contr.*, vol. AC-9, pp. 229–235, Apr. 1964.
[9] R. J. McAulay, "Optimal control techniques applied to PPM signal design," *Inform. Contr.*, vol. 12, pp. 221–235, 1968.
[10] R. Esposito and M. A. Schumer, "Probing linear filters—Signal design for the detection problem," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 167–171, Mar. 1970.
[11] D. W. Alspaugh, H. H. Kagiwada, and R. E. Kalaba, "Application of invariant imbedding to the eigenvalue problems for buckling of columns," Rand Corp., Santa Monica, Calif., Rep. RM-5954-PR, Aug. 1969.
[12] K. Rektorys, Ed., *Survey of Applicable Mathematics*. Cambridge, Mass.: M.I.T. Press, 1969.
[13] A. E. Bryson, Jr. and D. E. Johansen, "Linear filtering for time-varying systems using measurements containing colored noise," *IEEE Trans. Automat. Contr.*, vol. AC-10, pp. 4–10, Jan. 1965.
[14] D. F. Wilkie and W. R. Perkins, "Generation of sensitivity functions for linear systems using low-order models," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 123–130, Apr. 1969.
[15] C. P. Neumann and A. K. Sood, "Sensitivity functions for multi-input linear time-invariant systems—II. Minimal order models," *Int. J. Contr.*, vol. 15, no. 3, pp. 451–463, 1972.
[16] R. E. Kalman, "New methods and results in linear prediction and filtering theory," in *Proc. Symp. Eng. Applications of Random Function Theory and Probability*. New York: Wiley, 1961.
[17] R. K. Mehra, "Identification of stochastic linear dynamic systems," *AIAA J.*, vol. 9, pp. 28–31, Jan. 1, 1971.
[18] N. E. Nahi and G. A. Napjus, "Design of optimal probing signals for vector parameter estimation," *Preprints, IEEE Conf. Decision and Contr.*, Miami, Fla., Dec. 1971.
[19] J. Casti and R. Kalaba, "On the equivalence between a Cauchy system and Fredholm integral equations," Dep. Elec. Eng., Univ. Southern California, Los Angeles, Rep. TR 70-21, Mar. 1970.
[20] T. Kailath, "A general likelihood-ratio formula for random signals in Gaussian noise," *IEEE Trans. Inform. Theory*, vol. IT-15, pp. 350–361, May 1969.
[21] A. Wald, *Sequential Analysis*. New York: Wiley, 1948.
[22] B. D. O. Anderson, "A quadratic performance index maximization problem," *Int. J. Contr.*, vol. 12, no. 5, pp. 897–907, 1970.
[23] R. K. Mehra *et al.*, "Dual control and identification methods for avionic systems—Part II, Optimal inputs for linear system identification," Final Rep., SCI Project 5971, AFOSR Contract F44620-71-C-0077, May 1972.

Raman K. Mehra (S'67–M'68) was born in Lahore, Pakistan, on February 10, 1943. He received the B.S. degree in electrical engineering from Punjab Engineering College, Chandigarh, India, in 1964, and the M.S. and Ph.D. degrees in engineering from Harvard University, Cambridge, Mass., in 1965 and 1968, respectively.

During the summer of 1965 he worked at Bell Laboratories, Andover, Mass., and from 1966 to 1967 he was a Research Assistant at Harvard University. From September 1967 to September 1969 he was employed by the Analytic Sciences Corporation, Reading, Mass., where he applied modern estimation and control theory to problems in inertial navigation. From October 1969 until September 1972 he was, respectively, a Research Engineer, a Senior Research Engineer, and Manager of System Identification and Estimation Division at Systems Control, Inc., Palo Alto, Calif. He is currently an Associate Professor of Engineering and Applied Mathematics on the Gordon McKay Endowment at Harvard University. His interests lie in the areas of trajectory optimization; linear, nonlinear, and adaptive filtering; smoothing; system identification; experimental design; and stochastic control. He has applied system identification techniques to aircraft parameter identification, human operator modeling, process control, and forecasting.

Dr. Mehra was the recipient of the 1971 Donald P. Eckman Award for outstanding contributions to automatic control.