Optimal Input System Identification for Homogeneous and Nonhomogeneous Boundary Conditions

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Abstract. A recent paper by Mehra has considered the design of optimal inputs for linear system identification. The method proposed involves the solution of homogeneous linear differential equations with homogeneous boundary conditions. In this paper, a method of solution is considered for similar-type problems with nonhomogeneous boundary conditions. The methods of solution are compared for the homogeneous and nonhomogeneous cases, and it is shown that, for a simple numerical example, the optimal input for the nonhomogeneous case is almost identical to the homogeneous optimal input when the former has a small initial condition, terminal time near the critical length, and energy input the same as for the homogeneous case. Thus tentatively, solving the nonhomogeneous problem appears to offer an attractive alternative to solving Mehra's homogeneous problem.

Key Words. System identification, optimal inputs, parameter estimation, two-point boundary-value problems, optimization.

1. Introduction

Mehra (Ref. 1) shows in a recent paper that the design of optimal inputs for linear system identification involves the solution of two-point boundary-value problems with homogeneous boundary conditions. Nontrivial solutions exist for certain values of the multiplier q, which correspond to the boundary-value problem eigenvalues. The optimal input is obtained for the largest eigenvalue. Similar-type problems for nonhomogeneous boundary-value problems, i.e., for nonzero initial

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conditions, are considered in Ref. 2. In this paper, the methods of solution for the homogeneous (Ref. 1) and nonhomogeneous (Ref. 2) cases are compared and numerical results for a simple example are given.

Utilizing Mehra's approach, the problem may be stated as follows. Consider the time-invariant linear system

$$\dot{x}(t) = Fx(t) + Gy(t), \tag{1}$$

$$z(t) = Hx(t) + v(t), \tag{2}$$

where x is an $n \times 1$ state vector, y is an $m \times 1$ control vector, and z is a $p \times 1$ measurement vector. F, G, and H are respectively $n \times n$, $n \times m$, and $p \times n$ matrices. The vector v is a zero mean Gaussian white noise process:

$$E[v(t)] = 0, \tag{3}$$

$$E[v(t)\, v^T(\tau)] = R\, \delta(t - \tau). \tag{4}$$

Let a denote an unknown parameter in the above system. Then, it is desired to determine the optimal input such that the weighted sensitivity (Refs. 1 and 3)

$$M = \int_0^{T_f} x_a{}^T H^T R^{-1} H x_a \, dt \tag{5}$$

is maximized, subject to the input energy constraint

$$\int_0^{T_f} y^T y \, dt = E, \tag{6}$$

where

$$x_a = \partial x / \partial a. \tag{7}$$

Consider a to be an unknown parameter in the matrix F. Then,

$$\dot{x}_a = Fx_a + F_a x, \tag{8}$$

where

$$F_a = \partial F / \partial a. \tag{9}$$

Let x_A equal the augmented state vector

$$x_A = \begin{bmatrix} x \\ x_a \end{bmatrix}. \tag{10}$$

Then,

$$\dot{x}_A = F_A x_A + G_A y, \tag{11}$$

where

$$F_A = \begin{bmatrix} F & 0 \\ F_a & F \end{bmatrix}, \tag{12}$$

$$G_A = \begin{bmatrix} G \\ 0 \end{bmatrix}. \tag{13}$$

The performance index can be expressed as the return

$$J = \max_y \left\{ \tfrac{1}{2} \int_0^{T_f} [x_A{}^T H_A{}^T R^{-1} H_A x_A - q y^T y] \, dt \right\}, \tag{14}$$

where H_A is the $p \times 2n$ matrix

$$H_A = [0, H]. \tag{15}$$

Utilizing Pontryagin's maximum principle, the Hamiltonian function is

$$\mathscr{H} = \tfrac{1}{2}[-x_A{}^T H_A{}^T R^{-1} H_A x_A + q y^T y] + \lambda^T [F_A x_A + G_A y]. \tag{16}$$

The costate vector $\lambda(t)$ is the solution of the vector differential equation

$$\dot{\lambda} = -\left(\frac{\partial \mathscr{H}}{\partial x_A}\right)^T = H_A{}^T R^{-1} H_A x_A - F_A{}^T \lambda. \tag{17}$$

The vector $y(t)$ that maximizes $\mathscr{H}$ is

$$\partial \mathscr{H} / \partial y = q y + G_A{}^T \lambda = 0, \qquad y = -(1/q)\, G_A{}^T \lambda. \tag{18}$$

The two-point boundary-value problem is then given by

$$\begin{bmatrix} \dot{x}_A \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} F_A & -(1/q)\, G_A G_A{}^T \\ H_A{}^T R^{-1} H_A & -F_A{}^T \end{bmatrix} \begin{bmatrix} x_A \\ \lambda \end{bmatrix}, \tag{19}$$

with the boundary conditions

$$x_A = \begin{bmatrix} x(0) \\ 0 \end{bmatrix}, \qquad \lambda(T_f) = 0. \tag{20}$$

Equation (19) can also be expressed in the form

$$\dot{x}_B = A x_B, \tag{21}$$

where

$$x_B = \begin{bmatrix} x_A \\ \lambda \end{bmatrix}, \tag{22}$$

$$A = \begin{bmatrix} F_A & -(1/q)\, G_A G_A{}^T \\ H_A{}^T R^{-1} H_A & -F_A{}^T \end{bmatrix}. \tag{23}$$

The derivations are given above for only a single unknown parameter a, but can easily be extended to the case where several parameters are unknown (Refs. 1 and 4).

Since a is unknown, an initial estimate of a must be utilized in order to obtain the optimal input. Then, utilizing the parameter estimation techniques as described in Ref. 5, a new estimate of a is found and the process repeated until a satisfactory estimate of a is obtained.

2. Solution for Homogeneous Boundary Conditions

Mehra's method of solution (Ref. 1) is described briefly in the following paragraphs for the case where the boundary conditions in Eqs. (20) are homogeneous, i.e., $x(0) = 0$. The solution is trivial, except for certain values of q which are the eigenvalues of the two-point boundary-value problem. To obtain the optimal input, the Riccati matrix and the transition matrix are defined as follows.

The Riccati matrix $P(t)$ is defined by the relation

$$x_A(t) = P(t)\, \lambda(t). \tag{24}$$

Differentiating Eq. (24), substituting from Eq. (19), and rearranging, we have

$$\dot{P} = F_A P + P F_A{}^T - P H_A{}^T R^{-1} H_A P - (1/q)\, G_A G_A{}^T, \tag{25}$$

$$P(0) = 0. \tag{26}$$

Let $\Phi(t; q)$ denote the transition matrix of Eq. (19) for a particular value of q. Then,

$$\begin{bmatrix} x_A(T_f) \\ \lambda(T_f) \end{bmatrix} = \begin{bmatrix} \Phi_{xx}(T_f\,; q) & \Phi_{x\lambda}(T_f\,; q) \\ \Phi_{\lambda x}(T_f\,; q) & \Phi_{\lambda\lambda}(T_f\,; q) \end{bmatrix} \begin{bmatrix} x_A(0) \\ \lambda(0) \end{bmatrix}. \tag{27}$$

The second equation in (27), along with the boundary conditions, gives

$$\lambda(T_f) = \Phi_{\lambda\lambda}(T_f\,; q)\, \lambda(0) = 0. \tag{28}$$

For a nontrivial solution, the determinant of the matrix must equal zero:

$$|\Phi_{\lambda\lambda}(T_f\,;q)| = 0. \tag{29}$$

The optimal input is then obtained as follows.

(a) The matrix Riccati equation (25) with initial condition (26) is integrated forward in time for a particular value of q. When the elements of $P(t)$ become very large, the critical length T_{crit} has been reached. The terminal time T_f is set equal to T_{crit}.

(b) The transition matrix is obtained by integrating

$$\dot{\Phi}(t; q) = A\Phi(t; q), \tag{30}$$

$$\Phi(0; q) = I, \tag{31}$$

from $t = 0$ to $t = T_f = T_{\text{crit}}$. The matrix A is defined by Eq. (23).

(c) The initial costate vector $\lambda(0)$ is obtained from Eqs. (28) and (29) as an eigenvector of $\Phi_{\lambda\lambda}(T_f; q)$ corresponding to the zero eigenvalue. A unique value of $\lambda(0)$ is found by using the normalization condition of the input energy constraint (6).

(d) Equation (19) is integrated forward in time using $\lambda(0)$ obtained above. The optimal input is obtained utilizing Eq. (18).

If a particular terminal time T_f is desired, then the matrix Riccati equation must be integrated several times with different values of q, in order to determine the value of q corresponding to the desired T_f.

Some numerical errors are introduced in the solution because the critical length T_{crit}, and the eigenvector of $\Phi_{\lambda\lambda}(T_f; q)$ associated with the zero eigenvalue, cannot be determined exactly. The accuracy of the solution can be improved (Refs. 1 and 6) by integrating P^{-1}, when appropriate, and finding the eigenvector associated with the smallest eigenvalue of $\Phi_{\lambda\lambda}^T\Phi_{\lambda\lambda}$.

3. Solution for Nonhomogeneous Boundary Conditions

When the boundary conditions are nonhomogeneous, the two-point boundary-value problem given by Eqs. (19) and (20) can be solved using the method of complementary functions (Ref. 2). This method makes use of the homogeneous differential equations

$$\dot{H} = AH, \qquad H(0) = I, \tag{32}$$

where H is a $4n \times 4n$ matrix (not the measurement matrix defined previously) and A is the matrix defined by Eq. (23). The solution to equation (21) is given by

$$x_B = Hc_B, \tag{33}$$

where the vector of constants

$$c_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{34}$$

is determined from the equation

$$\begin{bmatrix} x_A(0) \\ \lambda(T_f) \end{bmatrix} = \begin{bmatrix} H_{11}(0) & H_{12}(0) \\ H_{21}(T_f) & H_{22}(T_f) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{35}$$

The optimal trajectory is obtained by integrating Eq. (32) from time $t = 0$ to time $t = T_f$ and storing the values of $H(t)$ at the boundaries. The vector c_B is evaluated and utilized to obtain the full set of initial conditions of x_B from Eq. (33). Equation (21) is then integrated from time $t = 0$ to $t = T_f$ and the optimal input is obtained utilizing Eq. (18).

For a given value of q, the optimal return J [given by Eq. (14)] will increase as the critical length is approached. The input energy also increases and becomes infinite at the critical length. Thus, the desired input energy can be obtained for a given value of q by evaluating Eqs. (33) and (21) for several different values of the terminal time T_f, where $T_f < T_{\text{crit}}$. The desired input energy is then obtained by plotting a curve of input energy versus T_f and finding the terminal time corresponding to the desired input energy. Alternately, a different procedure may be utilized by finding the value of q corresponding to the desired input energy for a given value of T_f.

4. Example

Consider the scalar linear differential equation,

$$\dot{x} = -ax + y, \tag{36}$$

$$z = x + v, \tag{37}$$

where v is a zero mean Gaussian white noise process with variance r. The initial condition is given by

$$x(0) = c. \tag{38}$$

The optimal input is to be determined such that the return is maximized. The two-point boundary-value problem is given by Eq. (19), where

$$F_A = \begin{bmatrix} -a & 0 \\ -1 & -a \end{bmatrix}, \tag{39}$$

$$G_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{40}$$

$$H_A = [0, 1]. \tag{41}$$

Equation (19) then becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -a & 0 & -1/q & 0 \\ -1 & -a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 1 & 0 & a \end{bmatrix} \begin{bmatrix} x \\ x_a \\ \lambda_1 \\ \lambda_2 \end{bmatrix}, \tag{42}$$

with boundary conditions

$$x(0) = c, \qquad x_a(0) = 0, \tag{43}$$

$$\lambda_1(T_f) = 0, \qquad \lambda_2(T_f) = 0. \tag{44}$$

5. Numerical Results

To obtain numerical results, a fourth-order Runge–Kutta method with grid intervals of 0.01 sec was utilized for the numerical integrations. All results were obtained on the GE-265 time-sharing computer.

The following parameters are assumed to be given:

$$a = 0.1, \qquad q = 0.075, \qquad r = 1.$$

Then, for the homogeneous boundary conditions, the matrix Riccati equation (25) is integrated until overflow occurs. This occurs at the critical length $T_{\text{crit}} = 1.02$ sec. The transition matrix is obtained by integrating Eq. (30) from $t = 0$ to $t = T_{\text{crit}}$. The normalized eigenvector of $\Phi_{\lambda\lambda}(T_f; q)$ corresponding to the zero eigenvalue is then determined and is given by

$$\lambda(0) = \begin{bmatrix} 0 \\ -1.373 \end{bmatrix}. \tag{45}$$

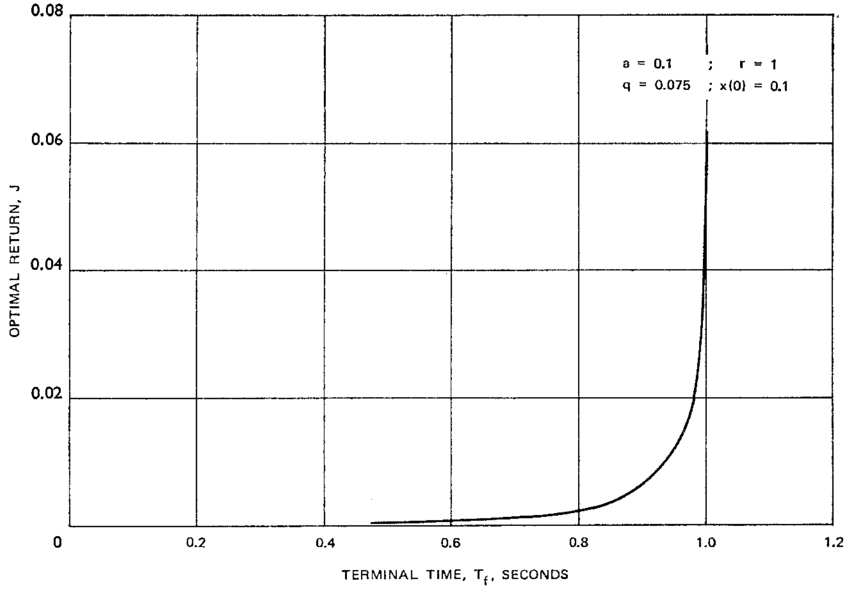

Fig. 1. Optimal return versus terminal time for nonhomogeneous boundary conditions.

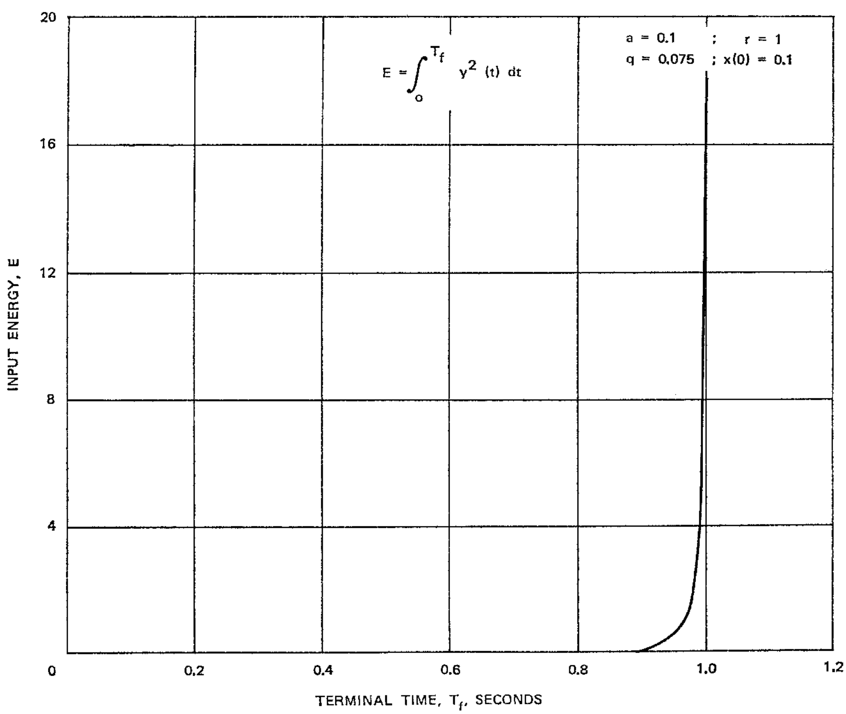

Fig. 2. Input energy versus terminal time for nonhomogeneous boundary conditions.

To obtain the desired input energy, the above equation is multiplied by an appropriate scalar constant. Equation (19) is then integrated forward in time to obtain the optimal input and state variables.

For the nonhomogeneous boundary conditions, it is assumed that

$$x(0) = c = 0.1. \tag{46}$$

Then, the optimal input can be found using either the method of complementary functions or the analytical solution (Ref. 5). The optimal return as a function of the terminal time T_f is shown in Fig. 1. The optimal return increases as the terminal time approaches the critical length T_{crit}. The energy input also increases as shown in Fig. 2. Figure 3 shows the optimal input as a function of time for $T_f = 1$ sec. Also shown is the optimal input for the homogeneous case with $T_f = 1.02$ sec and the same input energy as for the nonhomogeneous case. It is seen that the solutions are almost identical.

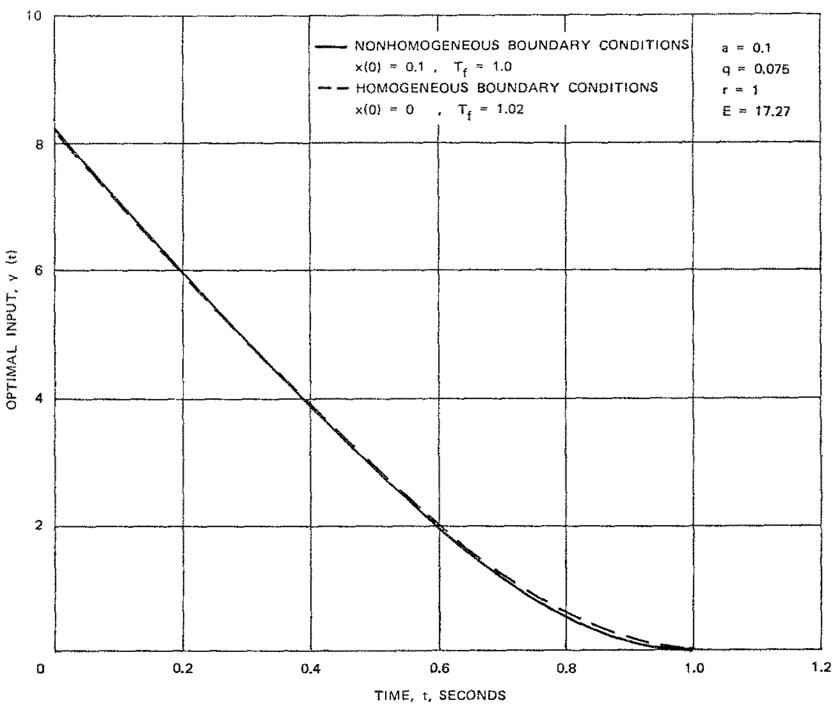

Fig. 3. Optimal input versus time.

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