1D Mahera

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1 Introduction

In this article, we will discuss an example involving one parameter.

2 Example with One Parameter

Consider the following system:

$$\dot{x} = -x + \theta \cdot u \tag{1}$$

$$y = x + v \tag{2}$$

Where:

- $\dot{x}$ represents the time derivative of x,
- θ is a constant parameter,
- u is an input, and
- v is a gaussian white noise with $E(v) = 0$ and $E(v(t)v(\tau)) = \sigma^2\delta(t - \tau)$.

The Information matrix can be shown to be:

$$M = \int_0^T \frac{1}{\sigma^2} x_\theta^2 \, d\theta \tag{3}$$

Where:

- x_θ is the derivative of x with respect to θ,
- T is the time horizon,
- σ^2 is the variance of the noise process v.

Let the input be energy constrained as follows:

$$\int_0^T u^2 dt = E \tag{4}$$

The sensitivity function $\dot{x_\theta}$ is obtained as follows:

$$\dot{x_\theta} = -x_\theta + u \tag{5}$$

The maximization of M is subject to a constraint (**??**) is equivalent to the minimization of the performance index:

$$J = -\frac{1}{2} \int_0^T \frac{-x_\theta^2}{\sigma^2} + (u^2 - \frac{E}{T}) dt \tag{6}$$

The Pontryagin Maximum Principle (PMP) gives us the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\left[\frac{-x_\theta^2}{\sigma^2} + (u^2 - \frac{E}{T})\right] + \lambda[-x_\theta + u] \tag{7}$$

Where λ is the costate scalar

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x_\theta}$$

$$\dot{\lambda} = -\lambda + \frac{-x_\theta}{\sigma^2} \tag{8}$$

Maximization Condition:

$$\mathcal{H}_u = 0$$

or:

$$u^\star = -\frac{1}{\mu}\lambda \tag{9}$$

The boundary conditions are homogeneous.

$$x_\theta(0) = 0, \quad \lambda(T) = 0$$

Substituting for u in (**??**), we obtain the two-point boundary value problem:

$$\frac{d}{dt}\begin{bmatrix} x_\theta \\ \lambda \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{\mu} \\ \frac{1}{\sigma^2} & 1 \end{bmatrix}\begin{bmatrix} x_\theta \\ \lambda \end{bmatrix} \tag{10}$$

Let $\Phi(T, 0; \mu)$ be the transition matrix of (**??**) for a particular μ

$$\begin{bmatrix} x_\theta(T) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} \Phi_{xx}(T,0;\mu) & \Phi_{x\lambda}(T,0;\mu) \\ \Phi_{\lambda x}(T,0;\mu) & \Phi_{\lambda\lambda}(T,0;\mu) \end{bmatrix}\begin{bmatrix} x_\theta(0) \\ \lambda(0) \end{bmatrix} \tag{11}$$

The second equation in (**??**) and the boundary conditions gives

$$\lambda(T) = \Phi_{\lambda\lambda}(T,0;\mu)\lambda(0) = 0 \tag{12}$$

For a non trivial solution

$$|\Phi_{\lambda\lambda}(T,0;\mu)| = 0 \tag{13}$$

Equation (**??**) is the eigenvalue equation for the Hamiltonian system (**??**). It can be solved by a Newton-Raphson iteration.
The Ricatti