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  - Current state & operator => properties of next state

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- Nothing else is a sentence

Precedence

$\neg$	highest
$\wedge$	
$\vee$	
$\rightarrow$	
$\leftrightarrow$	lowest

$A \vee B \wedge C$	$A \vee (B \wedge C)$
$A \wedge B \rightarrow C \vee D$	$(A \wedge B) \rightarrow (C \vee D)$
$A \rightarrow B \vee C \leftrightarrow D$	$(A \rightarrow (B \vee C)) \leftrightarrow D$

- Precedence rules enable "shorthand" form of sentences, but formally only the fully parenthesized form is legal.
- Syntactically ambiguous forms allowed in shorthand only when semantically equivalent: $A \wedge B \wedge C$ is equivalent to $(A \wedge B) \wedge C$ and $A \wedge (B \wedge C)$

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- *holds*(P, i) iff i(P) = **t**
- *fails*(P, i) iff i(P) = **f**

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Truth Tables

P	Q	$\neg$ P	P $\wedge$ Q	P $\vee$ Q	P $\rightarrow$ Q	Q $\rightarrow$ P	P $\leftrightarrow$ Q
f	**f**	**t**	**f**	**f**	**t**	**t**	**t**
f	**t**	**t**	**f**	**t**	**t**	**f**	**f**
t	**f**	**f**	**f**	**t**	**f**	**t**	**f**
t	**t**	**f**	**t**	**t**	**t**	**t**	**t**

Some important shorthand

- $\phi \to \psi \equiv \neg \phi \vee \psi$ (conditional, implication)

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t	**f**	**f**	**f**	**t**	**f**	**t**	**f**
t	**t**	**f**	**t**	**t**	**t**	**t**	**t**

Note that implication is not "causality", if P is **f** then P $\to$ Q is **t**

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All are finitely decidable.

Examples

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$b \vee d \vee (b \rightarrow d)$ $b \vee d \vee \neg b \vee d$	valid	

Satisfiability

- Related to constraint satisfaction
- Given a sentence S, try to find an interpretation i such that *holds*(S,i)
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- Better methods:
    - heuristic search
    - constraint propagation
    - stochastic search

Satisfiability problems

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- Scheduling nurses to work in a hospital
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  - constraints on the schedule are represented using logical expressions over the variables
- Finding bugs in software
  - propositional variables represent state of the program
  - use logic to describe how the program works and to assert there is a bug
  - if the sentence is satisfiable, you've found a bug!

A Good Lecture?

A Good Lecture?

Imagine we knew that:

- If today is sunny, then Tomas will be happy ($S \rightarrow H$)
- If Tomas is happy, the lecture will be good ($H \rightarrow G$)
- Today is sunny (S)

Should we conclude that the lecture will be good?

Checking Interpretations

Checking Interpretations

S	H	G
t	t	t
t	t	f
t	f	t
t	f	f
f	t	t
f	t	f
f	f	t
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Checking Interpretations

S	H	G	S→ H	H→G	S
t	t	t	t	t	t
t	t	f	t	f	t
t	f	t	f	t	t
t	f	f	f	t	t
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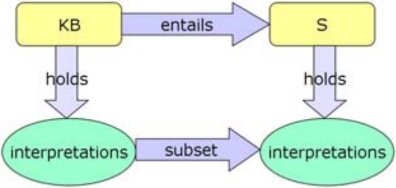
f	t	f	t	f	f	f
f	f	t	t	t	f	t
f	f	f	t	t	f	f

Adding a Variable

L	S	H	G	S→H	H→G	S	G
t	t	t	t	t	t	t	t
t	t	t	f	t	f	t	f
t	t	f	t	f	t	t	t
t	t	f	f	f	t	t	f
t	f	t	t	t	t	f	t
t	f	t	f	t	f	f	f
t	f	f	t	t	t	f	t
t	f	f	f	t	t	f	f
f	t	t	t	t	t	t	t
f	t	t	f	t	f	t	f
…	…	…					

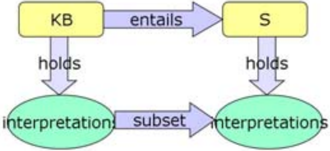
Entailment

A knowledge base (KB) *entails* a sentence S iff every interpretation that makes KB true also makes S true



Computing Entailment

- enumerate all interpretations
- select those in which all elements of KB are true
- check to see if S is true in all of those interpretations



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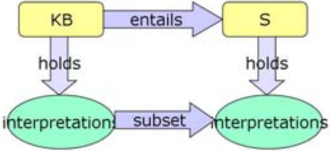
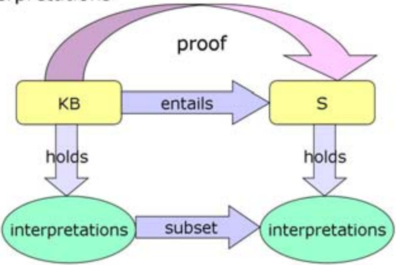


Way too many interpretations, in general!!

Entailment and Proof

A proof is a way to test whether a KB entails a sentence, without enumerating all possible interpretations



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- If inference rules are *sound*, then any S you can prove from KB is entailed by KB

- If inference rules are *complete*, then any S that is entailed by KB can be proved from KB

Natural Deduction

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$$\frac{\begin{array}{c}\alpha \rightarrow \beta \\ \alpha\end{array}}{\beta}$$

Modus ponens

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$$\frac{\begin{array}{c}\alpha \rightarrow \beta \\ \neg \beta\end{array}}{\neg \alpha}$$

Modus tolens

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Some inference rules:

Modus ponens	Modus tolens	And-introduction
$\dfrac{\alpha \to \beta \quad \alpha}{\beta}$	$\dfrac{\alpha \to \beta \quad \neg \beta}{\neg \alpha}$	$\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$

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And-introduction

$$\frac{\alpha \wedge \beta}{\alpha}$$

And-elimination

Natural deduction example

Prove S

Step	Formula	Derivation

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- Natural deduction uses lots of inference rules which introduces a large branching factor in the search for a proof.
- In general, you need to do "proof by cases" which introduces even more branching.

Prove R

1	$P \vee Q$
2	$Q \rightarrow R$
3	$P \rightarrow R$

Propositional Resolution

- Resolution rule:

$$\frac{\begin{array}{l}\alpha \vee \beta \\ \neg\beta \vee \gamma\end{array}}{\alpha \vee \gamma}$$

- Single inference rule is a sound and complete proof system

- Requires all sentences to be converted to conjunctive normal form

Conjunctive Normal Form

- Conjunctive normal form (CNF) formulas:

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  - A, B, and ¬ C are literals, each of which is a variable or the negation of a variable.
  - Each clause is a requirement that must be satisfied and can be satisfied in multiple ways
  - Every sentence in propositional logic can be written in CNF

Converting to CNF

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4. Every sentence can be converted to CNF, but it may grow exponentially in size

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2. Drive in negations

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3. Distribute

$$(\neg A \vee \neg C \vee D) \wedge (\neg B \vee \neg C \vee D)$$

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- Resolution refutation:
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Propositional Resolution

- Resolution rule:

$$\frac{\begin{array}{l}\alpha \vee \beta \\ \neg\beta \vee \gamma\end{array}}{\alpha \vee \gamma}$$

- Resolution refutation:
  - Convert all sentences to CNF
  - Negate the desired conclusion (converted to CNF)

Propositional Resolution

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- Resolution refutation:
  - Convert all sentences to CNF
  - Negate the desired conclusion (converted to CNF)
  - Apply resolution rule until either
    - Derive false (a contradiction)
    - Can't apply any more

Propositional Resolution

- Resolution rule:

$$\frac{\begin{array}{l}\alpha \vee \beta \\ \neg\beta \vee \gamma\end{array}}{\alpha \vee \gamma}$$

- Resolution refutation:
  - Convert all sentences to CNF
  - Negate the desired conclusion (converted to CNF)
  - Apply resolution rule until either
    - Derive false (a contradiction)
    - Can't apply any more
- Resolution refutation is sound and complete
  - If we derive a contradiction, then the conclusion follows from the axioms
  - If we can't apply any more, then the conclusion cannot be proved from the axioms.

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given

Propositional Resolution Example

Prove R

1	P v Q
2	$P \to R$
3	$Q \to R$

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \to R$
3	$Q \to R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated conclusion

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated conclusion

Propositional Resolution Example

Prove R

1	P v Q
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	P v Q	Given
2	$\neg$ P v R	Given
3	$\neg$ Q v R	Given
4	$\neg$ R	Negated conclusion
5	Q v R	1,2

Propositional Resolution Example

Prove R

1	P v Q
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion
5	Q v R	1,2
6	¬ P	2,4

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated conclusion
5	$Q \vee R$	1,2
6	$\neg P$	2,4
7	$\neg Q$	3,4

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated conclusion
5	$Q \vee R$	1,2
6	$\neg P$	2,4
7	$\neg Q$	3,4
8	R	5,7

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated conclusion
5	$Q \vee R$	1,2
6	$\neg P$	2,4
7	$\neg Q$	3,4
8	R	5,7
9	•	4,8

Propositional Resolution Example

Prove R

1	P v Q
2	P → R
3	Q → R

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion
5	Q v R	1,2
6	¬ P	2,4
7	¬ Q	3,4
8	R	5,7
9	•	4,8

false v R

¬ R v false

false v false

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated conclusion
5	$Q \vee R$	1,2
6	$\neg P$	2,4
7	$\neg Q$	3,4
8	R	5,7
9	•	4,8

$$\frac{\begin{array}{l}\text{false} \vee R \\ \neg R \vee \text{false}\end{array}}{\text{false} \vee \text{false}}$$

The Power of False

Prove Z

1	P
2	$\neg$ P

Step	Formula	Derivation

The Power of False

Prove Z

1	P
2	¬ P

Step	Formula	Derivation
1	P	Given
2	¬ P	Given
3	¬ Z	Negated conclusion

The Power of False

Prove Z

1	P
2	¬ P

Step	Formula	Derivation
1	P	Given
2	¬ P	Given
3	¬ Z	Negated conclusion
4	•	1,2

The Power of False

Prove Z

1	P
2	$\neg$ P

Step	Formula	Derivation
1	P	Given
2	$\neg$ P	Given
3	$\neg$ Z	Negated conclusion
4	•	1,2

Note that $(P \wedge \neg P) \rightarrow Z$ is valid

The Power of False

Prove Z

1	P
2	¬ P

Step	Formula	Derivation
1	P	Given
2	¬ P	Given
3	¬ Z	Negated conclusion
4	•	1,2

Note that $(P \wedge \neg P) \rightarrow Z$ is valid

Any conclusion follows from a contradiction – and so strict logic systems are very brittle.

Example Problem

Convert to CNF

Prove R

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Example Problem

Prove R

1	$(P \to Q) \to Q$
2	$(P \to P) \to R$
3	$(R \to S) \to \neg(S \to Q)$

Convert to CNF

- $\neg(\neg P \vee Q) \vee Q$
- $(P \wedge \neg Q) \vee Q$
- $(P \vee Q) \wedge (\neg Q \vee Q)$
- $(P \vee Q)$

Example Problem

Prove R

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Convert to CNF

- $\neg(\neg P \vee Q) \vee Q$
- $(P \wedge \neg Q) \vee Q$
- $(P \vee Q) \wedge (\neg Q \vee Q)$
- $(P \vee Q)$

- $\neg(\neg P \vee P) \vee R$
- $(P \wedge \neg P) \vee R$
- $(P \vee R) \wedge (\neg P \vee R)$

Example Problem

Prove R

1	$(P \to Q) \to Q$
2	$(P \to P) \to R$
3	$(R \to S) \to \neg(S \to Q)$

Convert to CNF

- $\neg(\neg P \vee Q) \vee Q$
- $(P \wedge \neg Q) \vee Q$
- $(P \vee Q) \wedge (\neg Q \vee Q)$
- $(P \vee Q)$

- $\neg(\neg P \vee P) \vee R$
- $(P \wedge \neg P) \vee R$
- $(P \vee R) \wedge (\neg P \vee R)$

- $\neg(\neg R \vee S) \vee \neg(\neg S \vee Q)$
- $(R \wedge \neg S) \vee (S \wedge \neg Q)$
- $(R \vee S) \wedge (\neg S \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee \neg Q)$
- $(R \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee \neg Q)$

Resolution Proof Example

Prove R

1	$(P \to Q) \to Q$
2	$(P \to P) \to R$
3	$(R \to S)$ $\to \neg (S \to Q)$

1	$P \vee Q$	
2	$P \vee R$	
3	$\neg P \vee R$	
4	$R \vee S$	
5	$R \vee \neg Q$	
6	$\neg S \vee \neg Q$	
7	$\neg R$	Neg

Resolution Proof Example

Prove R

1	$(P \to Q) \to Q$
2	$(P \to P) \to R$
3	$(R \to S) \to \neg (S \to Q)$

1	$P \vee Q$	
2	$P \vee R$	
3	$\neg P \vee R$	
4	$R \vee S$	
5	$R \vee \neg Q$	
6	$\neg S \vee \neg Q$	
7	$\neg R$	Neg
8	S	4,7
9	$\neg Q$	6,8
10	P	1,9
11	R	3,10
12	•	7,11

Proof Strategies

Proof Strategies

- Unit preference: prefer a resolution step involving an unit clause (clause with one literal).
  - Produces a shorter clause – which is good since we are trying to produce a zero-length clause, that is, a contradiction.
- Set of support: Choose a resolution involving the negated goal or any clause derived from the negated goal.
  - We're trying to produce a contradiction that follows from the negated goal, so these are "relevant" clauses.
  - If a contradiction exists, one can find one using the set-of-support strategy.