First-Order Logic

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- In first-order logic, variables refer to things in the world and, furthermore, you can quantify over them: talk about all of them or some of them without having to name them explicitly.

FOL motivation

- Statements that cannot be made in propositional logic but can be made in FOL

FOL motivation

Statements that cannot be made in propositional logic but can be made in FOL

- When you paint a block with green paint, it becomes green.
  - In propositional logic, one would need a statement about every single block, one cannot make the general statement about all blocks.

FOL motivation

Statements that cannot be made in propositional logic but can be made in FOL

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- When you sterilize a jar, all the bacteria are dead.
  - In FOL, we can talk about all the bacteria without naming them explicitly.

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  - In propositional logic, one would need a statement about every single block, one cannot make the general statement about all blocks.
- When you sterilize a jar, all the bacteria are dead.
  - In FOL, we can talk about all the bacteria without naming them explicitly.
- A person is allowed access to this Web site if they have been formally authorized or they are known to someone who has access.

FOL syntax

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- Term
  - Constant symbols: Fred, Japan, Bacterium39

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  - Variables: x, y, a

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  - A predicate symbol applied to zero or more terms: On(a,b), Sister(Jane, Joan), Sister(mother-of(John), Jane)

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  - If v is a variable and Φ is a sentence, then $\forall v.\Phi$ and $\exists v.\Phi$ are sentences.

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- Sentence
  - A predicate symbol applied to zero or more terms: On(a,b), Sister(Jane, Joan), Sister(mother-of(John), Jane)
  - $t_1 = t_2$
  - If v is a variable and Φ is a sentence, then $\forall v.\Phi$ and $\exists v.\Phi$ are sentences.
  - Closure under sentential operators: $\wedge \vee \rightarrow \neg$ ( )

FOL Interpretations

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    (binary relation is a set of pairs)

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    (called "domain of discourse" or "universe")
  - Maps constant symbols to elements of U
  - Maps predicate symbols to relations on U
    (binary relation is a set of pairs)
  - Maps function symbols to functions on U
    (function is a binary relation with a single pair for each element in U, whose first item is that element)

Denotation of Terms

Terms name objects in U

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- I(Fred)   if Fred is constant, then given

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Denotation of Terms

Terms name objects in U

- I(Fred) if Fred is constant, then given
- I(x) if x is a variable, then undefined
- $I(f(t_1, \ldots, t_n))$ $I(f)(I(t_1), \ldots, I(t_n))$

Holds

When does a sentence hold in an interpretation?

Holds

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- P is a relation symbol
- $t_1, \ldots, t_n$ are terms

holds($P(t_1, \ldots, t_n)$, I) iff $\langle I(t_1), \ldots, I(t_n)\rangle \in I(P)$

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Brother(Jon, Joe)??

- I(Jon) =  [an element of U]

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Brother(Jon, Joe)??

- I(Jon) =  [an element of U]
- I(Joe) =  [an element of U]

Holds

When does a sentence hold in an interpretation?

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holds($P(t_1, \ldots, t_n)$, I) iff $<I(t_1), \ldots, I(t_n)> \in I(P)$

Brother(Jon, Joe)??

- I(Jon) =  [an element of U]
- I(Joe) =  [an element of U]
- I(Brother) = {<>,<>, <...,...>, ... }

Holds

When does a sentence hold in an interpretation?

- P is a relation symbol
- $t_1, \ldots, t_n$ are terms

holds($P(t_1, \ldots, t_n)$, I) iff $<I(t_1), \ldots, I(t_n)> \in I(P)$

Brother(Jon, Joe)??

- I(Jon) =  [an element of U]
- I(Joe) =  [an element of U]
- I(Brother) = {< ,  >, <   >, <…,…>, … }

- holds(Brother(Jon, Joe), I)

Equality

holds($t_1 = t_2$, I) iff I(t_1) is the same object as I(t_2)

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Jon = Jack ?

- I(Jon) =  [an element of U]
- I(Jack) =  [an element of U]

- holds(Jon = Jack, I)

Semantics of Quantifiers

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Extend an interpretation I to bind variable x to element $a \in U$: $I_{x/a}$

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Extend an interpretation I to bind variable x to element $a \in U$: $I_{x/a}$

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Extend an interpretation I to bind variable x to element $a \in U$: $I_{x/a}$

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Quantifier applies to formula to right until an enclosing right parenthesis:

Semantics of Quantifiers

Extend an interpretation I to bind variable x to element $a \in U$: $I_{x/a}$

- holds($\forall x.\Phi$, I) iff holds(Φ, $I_{x/a}$) for all $a \in U$
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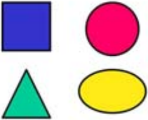
Quantifier applies to formula to right until an enclosing right parenthesis:

$$(\forall x.P(x) \vee Q(x)) \wedge \exists x.R(x) \rightarrow Q(x)$$

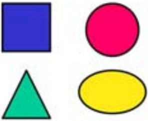
FOL Example Domain



The Real World

FOL Example Domain

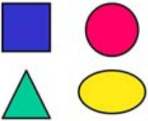
- U = {}



The Real World

FOL Example Domain

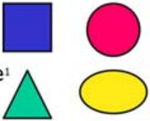
- U = {}
- Constants: Fred



The Real World

FOL Example Domain

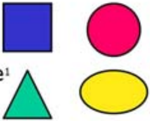
- U = {  }
- Constants: Fred
- Preds: Above^2, Circle^1, Oval^1, Square^1



The Real World

FOL Example Domain

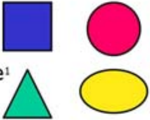
- U = {}
- Constants: Fred
- Preds: Above^2, Circle^1, Oval^1, Square^1
- Function: hat



The Real World

FOL Example Domain

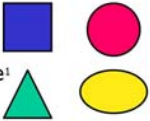
- U = {■, ▲, ●, ⬬}
- Constants: Fred
- Preds: Above^2, Circle^1, Oval^1, Square^1
- Function: hat
- I(Fred) = ▲



The Real World

FOL Example Domain

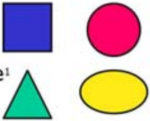
- U = {■, ▲, ●, ⬬}
- Constants: Fred
- Preds: Above^2, Circle^1, Oval^1, Square^1
- Function: hat
- I(Fred) = ▲
- I(Above) = {<■ ,▲ >,<● ,⬬>}



The Real World

FOL Example Domain

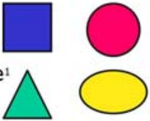
- U = {■, ▲, ●, ⬬}
- Constants: Fred
- Preds: Above^2, Circle^1, Oval^1, Square^1
- Function: hat
- I(Fred) = ▲
- I(Above) = {<■, ▲>, <●, ⬬>}
- I(Circle) = {<●>}



The Real World

FOL Example Domain

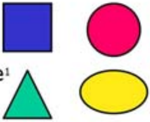
- U = {■, ▲, ●, ⬭}
- Constants: Fred
- Preds: Above[2], Circle[1], Oval[1], Square[1]
- Function: hat
- I(Fred) = ▲
- I(Above) = {<■, ▲>, <●, ⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>, <⬭>}



The Real World

FOL Example Domain

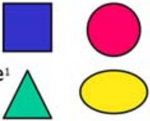
- U = {■, ▲, ●, ⬭}
- Constants: Fred
- Preds: Above², Circle¹, Oval¹, Square¹
- Function: hat
- I(Fred) = ▲
- I(Above) = {<■ ,▲ >,<● ,⬭>}
- I(Circle) = {<● >}
- I(Oval) = {<● >,<⬭>}
- I(hat) = {<▲ ,■ >,<⬭,● >,< ■,■ >,<● ,● >}



The Real
World

FOL Example Domain

- U = {■, ▲, ●, ⬭}
- Constants: Fred
- Preds: Above^2, Circle^1, Oval^1, Square^1
- Function: hat
- I(Fred) = ▲
- I(Above) = {<■, ▲>, <●, ⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>, <⬭>}
- I(hat) = {<▲, ■>, <⬭, ●>, <■, ■>, <●, ●>}
- I(Square) = {<▲>}



The Real World

FOL Example

- holds(Square(Fred), I) ?

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬭>}
- I(hat) = {<▲,■>,<⬭,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

FOL Example

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬭>}
- I(hat) = {<▲,■>,<⬭,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(Square(Fred), I) ?
  yes
- holds(Above(Fred, hat(Fred)), I) ?

FOL Example

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●> <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(Square(Fred), I) ?

  yes
- holds(Above(Fred, hat(Fred)), I) ?
  - I(hat(Fred)) = ■

FOL Example

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(Square(Fred), I) ?
  yes
- holds(Above(Fred, hat(Fred)), I) ?
  - I(hat(Fred)) = ■
  - holds(Above(▲, ■), I) ?

FOL Example

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬭>}
- I(hat) = {<▲,■>,<⬭,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(Square(Fred), I) ?
  yes
- holds(Above(Fred, hat(Fred)), I) ? no
  - I(hat(Fred)) = ■
  - holds(Above(▲, ■), I) ? no

FOL Example

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(Square(Fred), I) ?
  yes
- holds(Above(Fred, hat(Fred)), I) ? no
  - I(hat(Fred)) = ■
  - holds(Above(▲, ■), I) ? no
- holds(∃x. Oval(x), I) ?

FOL Example

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(Square(Fred), I) ?
  yes
- holds(Above(Fred, hat(Fred)), I) ? no
  - I(hat(Fred)) = ■
  - holds(Above(▲, ■), I) ? no
- holds(∃x. Oval(x), I) ? yes
  - holds(Oval(x), $I_{x/●}$) ? yes

FOL Example: Continued

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ?

FOL Example: Continued

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
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- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ?
  - holds(∃y. Above(x,y) v Above(y,x), $I_{x/▲}$) ?

FOL Example: Continued

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬭>}
- I(hat) = {<▲,■>,<⬭,●>
<■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ?
  - holds(∃y. Above(x,y) v Above(y,x), $I_{x/▲}$) ? yes

    holds(Above(x,y) v Above(y,x), $I_{x/▲,y/■}$) ? yes

FOL Example: Continued

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ? yes
  - holds(∃y. Above(x,y) v Above(y,x), $I_{x/▲}$) ? yes
    holds(Above(x,y) v Above(y,x), $I_{x/▲,y/■}$) ? yes
  - verify for all other values of x

FOL Example: Continued

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- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ? yes
  - holds(∃y. Above(x,y) v Above(y,x), Ix/▲) ? yes
    holds(Above(x,y) v Above(y,x), Ix/▲,y/■) ? yes
  - verify for all other values of x
- holds(∀ x. ∀ y. Above(x,y) v Above(y,x), I) ?

FOL Example: Continued

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ? yes
  - holds(∃y. Above(x,y) v Above(y,x), $I_{x/\blacktriangle}$) ? yes
    holds(Above(x,y) v Above(y,x), $I_{x/\blacktriangle, y/\blacksquare}$) ? yes
  - verify for all other values of x

- holds(∀ x. ∀ y. Above(x,y) v Above(y,x), I) ?
  - holds(Above(x,y) v Above(y,x), $I_{x/\blacksquare, y/\bullet}$) ?

FOL Example: Continued

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
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- I(hat) = {<▲,■>,<⬬,●> <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ? yes
  - holds(∃y. Above(x,y) v Above(y,x), $I_{x/▲}$) ? yes
    holds(Above(x,y) v Above(y,x), $I_{x/▲,y/■}$) ? yes
  - verify for all other values of x

- holds(∀ x. ∀ y. Above(x,y) v Above(y,x), I) ? no
  - holds(Above(x,y) v Above(y,x), $I_{x/■,y/●}$) ? no

Writing FOL

Writing FOL

- Cats are mammals [Cat^1, Mammal^1]

Writing FOL

- Cats are mammals   [Cat^1, Mammal^1]
  - $\forall$ x. Cat(x) $\rightarrow$ Mammal(x)

Writing FOL

- Cats are mammals [Cat^1, Mammal^1]
  - $\forall$ x. Cat(x) $\rightarrow$ Mammal(x)
- Jane is a tall surveyor [Tall^1, Surveyor^1, Jane]

Writing FOL

- Cats are mammals [Cat^1, $Mammal^1$]
  - $\forall$ x. Cat(x) $\rightarrow$ Mammal(x)
- Jane is a tall surveyor [$Tall^1$, $Surveyor^1$, Jane]
  - Tall(Jane) $\wedge$ Surveyor(Jane)

Writing FOL

- Cats are mammals [Cat^1, $Mammal^1$]
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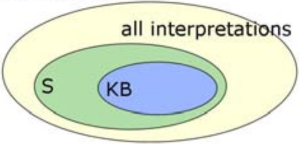

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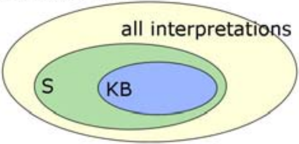

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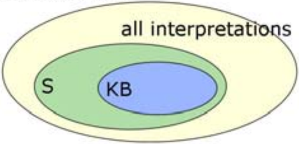

- Computing entailment is impossible in general, because there are infinitely many possible interpretations
- Even computing holds is impossible for interpretations with infinite universes

Intended Interpretations

$KB : (\forall x.\, \text{Circle}(x) \rightarrow \text{Oval}(x)) \wedge (\forall x.\, \text{Square}(x) \rightarrow \neg \text{Oval}(x))$

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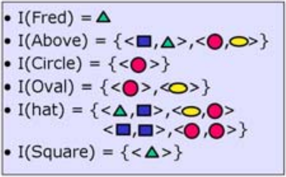

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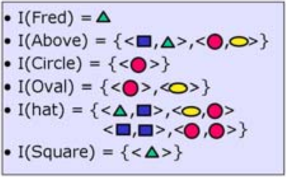

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- We know holds(KB, I)
- We wonder whether holds(S, I)
- We could ask: Does KB entail S?
- Or we could just try to check whether holds(S, I)

- I(Fred) = ▲
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- Does KB hold in I_1?

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- Does KB hold in I_1?
- Yes, but can't answer via enumerating U
- S also holds in I_1
- No way to verify mechanically

U_1 = {1, 2, 3, …}
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An Argument for Entailment

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KB doesn't entail S_1!

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- Can't evaluate it directly by enumerating interpretations
- So, we'll do proofs
- In FOL, if S is entailed by KB, then there is a finite proof of S from KB

First-Order Resolution

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$$\frac{\begin{array}{c}\forall x.\ P(x) \rightarrow Q(x) \\ P(A)\end{array}}{Q(A)}$$

uppercase letters: constants

lowercase letters: variables

First-Order Resolution

$$\frac{\forall x.\ P(x) \rightarrow Q(x) \qquad P(A)}{Q(A)}$$

Syllogism:

All men are mortal

Socrates is a man

Socrates is mortal

uppercase letters: constants

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———
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Equivalent by definition of implication

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Substitute A for x, still true

then

Propositional resolution

First-Order Resolution

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Syllogism:
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$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

Equivalent by definition of implication

Two new things:

- converting FOL to clausal form
- resolution with variable substitution

$$\frac{\neg P(A) \vee Q(A) \qquad P(A)}{Q(A)}$$

Substitute A for x, still true

then

Propositional resolution

Clausal Form

- like CNF in outer structure
- no quantifiers

$$\forall x.\ \exists y.\ P(x) \rightarrow R(x, y)$$

$$\Downarrow$$

$$\neg P(x) \vee R(x, F(x))$$

Converting to Clausal Form

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1. Eliminate arrows

$$\alpha \leftrightarrow \beta \Rightarrow (\alpha \to \beta) \wedge (\beta \to \alpha)$$

$$\alpha \to \beta \Rightarrow \neg \alpha \vee \beta$$

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$$\alpha \rightarrow \beta \Rightarrow \neg\alpha \vee \beta$$

2. Drive in negation

$$\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \wedge \neg\beta$$

$$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$$

$$\neg\neg\alpha \Rightarrow \alpha$$

$$\neg\forall x.\ \alpha \Rightarrow \exists x.\ \neg\alpha$$

$$\neg\exists x.\ \alpha \Rightarrow \forall x.\ \neg\alpha$$

Converting to Clausal Form

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$$\alpha \leftrightarrow \beta \Rightarrow (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$$

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$$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$$

$$\neg\neg\alpha \Rightarrow \alpha$$

$$\neg\forall x.\ \alpha \Rightarrow \exists x.\ \neg\alpha$$

$$\neg\exists x.\ \alpha \Rightarrow \forall x.\ \neg\alpha$$

3. Rename variables apart

$$\forall x.\ \exists y.\ (\neg P(x) \vee \exists x.\ Q(x, y)) \Rightarrow$$

$$\forall x_1.\ \exists y_2.\ (\neg P(x_1) \vee \exists x_3.\ Q(x_3, y_2))$$

Skolemization

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   - substitute new name for each existential var

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$$\exists x.\, \mathrm{P}(x) \Rightarrow \mathrm{P}(\mathrm{Fred})$$

$$\exists x, y.\, \mathrm{R}(x, y) \Rightarrow \mathrm{R}(\mathrm{Thing1}, \mathrm{Thing2})$$

$$\exists x.\, \mathrm{P}(x) \wedge \mathrm{Q}(x) \Rightarrow \mathrm{P}(\mathrm{Fleep}) \wedge \mathrm{Q}(\mathrm{Fleep})$$

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$$\exists x.\ P(x) \wedge \exists x.\ Q(x) \Rightarrow P(\text{Frog}) \wedge Q(\text{Grog})$$

$$\exists y.\ \forall x.\ \text{Loves}(x, y) \Rightarrow \forall y.\ \text{Loves}(x, \text{Englebert})$$

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$$\forall x.\ \exists y.\ \text{Loves}(x, y) \Rightarrow \forall x.\ \text{Loves}(x, \text{Beloved}(x))$$

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   - substitute new function of all universal vars in outer scopes

$$\forall x.\ \exists y.\ \text{Loves}(x,y) \Rightarrow \forall x.\ \text{Loves}(x, \text{Beloved}(x))$$
$$\forall x.\ \exists y.\ \forall z.\ \exists w.\ P(x,y,z) \wedge R(y,z,w) \Rightarrow$$
$$P(x,F(x),z) \wedge R(F(x),z,G(x,z))$$

Convert to Clausal Form: Last Steps

5. Drop universal quantifiers

   $\forall x.\ \text{Loves}(x, \text{Beloved}(x)) \Rightarrow \text{Loves}(x, \text{Beloved}(x))$

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   $$\mathrm{P}(z) \vee (\mathrm{Q}(z,w) \wedge \mathrm{R}(w,z)) \Rightarrow \{\{\mathrm{P}(z), \mathrm{Q}(z,w)\}, \{\mathrm{P}(z), \mathrm{R}(w,z)\}\}$$

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$$P(z) \vee (Q(z,w) \wedge R(w,z)) \Rightarrow \{\{P(z), Q(z,w)\}, \{P(z), R(w,z)\}\}$$

7. Rename the variables in each clause

$$\{\{P(z), Q(z,w)\}, \{P(z), R(w,z)\}\} \Rightarrow \{\{P(z_1), Q(z_1,w_1)\}, \{P(z_2), R(w_2,z_2)\}\}$$

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$\exists x.\ D(x) \wedge O(J,x)$

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$\forall$ x. ($\exists$ y. D(y) $\wedge$ O(x,y)) $\rightarrow$ L(x)

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$\forall$ x. $\forall$ y. $\neg$(D(y) $\wedge$ O(x,y)) v L(x)
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More converting to clausal form

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e. Tuna is a cat
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f. All cats are animals
¬ C(x) v A(x)

A Proof Using Resolution

We are given:

$$\forall x\ [P(x) \Rightarrow [\forall y\ [P(y) \Rightarrow P(f(x,y))] \wedge \\ \neg\forall y\ [Q(x,y) \Rightarrow P(y)]]]$$

where: P, Q are predicates
x, y are variables
f is a function

Step 1:

Eliminate Implication

use $x \Rightarrow y \equiv \neg x \vee y$

We obtain:

$$\forall x\ [\neg P(x) \vee [\forall y\ [\neg P(y) \vee P(f(x,y))] \wedge \\ \neg\forall y\ [\neg Q(x,y) \vee P(y)]]]$$

Step 2

Reduce Scope of Negation Symbols

Use:

- DeMorgan's Laws
  $\neg(X \wedge Y) \equiv \neg X \vee \neg Y$
  $\neg(X \vee Y) \equiv \neg X \wedge \neg Y$
- $\neg(\neg X) \equiv X$
- And from Quantifier Meaning Rules:
  $\neg\forall x \;\; P(x) \equiv \exists x[\neg P(x)]$
  $\neg\exists x \;\; P(x) \equiv \forall x[\neg P(x)]$

Step 2 Continued:

Reduce Scope of Negation Symbols

Actual Reduction:

$$\forall x\ [\neg P(x) \vee [\forall y\ [\neg P(y) \vee P(f(x,y))] \wedge$$
$$\underbrace{\neg \forall y\ [\neg Q(x,y) \vee P(y)]}_{\textit{only underlined part}}]]$$

$$\Downarrow \neg \forall x\ \ P(x) \equiv \exists x\ \ [\neg P(x)]$$

$$\exists y\ \ [\neg[\neg Q(x,y) \vee P(y)]]$$

$$\Downarrow \text{DeMorgan's Law}$$

$$\exists y\ \ [\neg[\neg Q(x,y)] \wedge \neg P(y)]$$

$$\Downarrow \text{Double Negation}$$

$$\exists y\ \ [Q(x,y) \wedge \neg P(y)]$$

Step 3

Standardize Variables

Rename variable associated with each quantifier so it is unique. Can do this without altering truth values (in quantifiers only).
For example:

$$\forall x \ [P(x) \Rightarrow \exists x \ Q(x)]$$

is rewritten

$$\forall x \ [P(x) \Rightarrow \exists y \ Q(y)]$$

Result in Example:

$$\forall x \ [\neg P(x) \vee [\forall y \ [\neg P(y) \vee P(f(x,y))] \wedge \underline{\exists y \ [Q(x,y) \wedge \neg P(y)]}]]$$

$$\Downarrow \forall x \ [\neg P(x) \vee [\forall y \ [\neg P(y) \vee P(f(x,y))] \wedge \underline{\exists w \ [Q(x,w) \wedge \neg P(w)]}]]$$

Step 4

Get Rid of Existential Quantifiers:
"Skolemize"

How to use Skolem Functions:

$$\forall y \ [\exists x \ \ P(x, y)]$$

For all y, there exists an x, <u>perhaps depending on y</u>, such that $P(x, y)$.
Skolem function $g(y)$ maps y onto the necessary x.

$$\forall y \ \ P[g(y), y)]$$

<u>Rule:</u>
Replace each occurrence of existential quantifier <u>variable</u> by a Skolem function whose arguments are universally quantified <u>variables</u> whose scopes <u>include</u> the existential quantifier being eliminated.

Step 4 Continued

Example of Skolemization

$$\underline{\forall x\ \ P(x)} \wedge \forall y \forall z\ \ [\exists w\ \ [P(y,z,w) \wedge R(y)]]$$

not in scope

$$\Downarrow$$

$$\forall x\ \ P(x) \wedge \forall y \forall z\ \ [P(y,z,\underline{g(y,z)}) \wedge R(y)]]$$

Result in Example Proof:

$$\forall x\ \ [\neg P(x) \vee [\forall y\ \ [\neg P(y) \vee P(f(x,y))] \wedge$$
$$\underline{\exists w}\ \ [Q(x,\underline{w}) \wedge \neg P(\underline{w})]]]$$

$\Downarrow$ only x in scope

$$\forall x\ \ [\neg P(x) \vee [\forall y\ \ [\neg P(y) \vee P(f(x,y))] \wedge$$
$$[Q(x,g(x)) \wedge \neg P(g(x))]]]$$

Step 5

Move all Universal Quantifiers to Left
<u>Prenex</u> Form

This can be done because all variables are unique to each universal quantifier.

Result in Example Proof:

$$\forall x\ [\neg P(x) \vee [\underline{\forall y}\ [\neg P(y) \vee P(f(x,y))] \wedge$$
$$[Q(x,g(x)) \wedge \neg P(g(x))]]]$$
$$\Downarrow$$
$$\forall x \underline{\forall y}\ [\neg P(x) \vee [\neg P(y) \vee P(f(x,y))] \wedge$$
$$[Q(x,g(x)) \wedge \neg P(g(x))]]$$

Step 6

Rewrite in Conjunctive Normal Form

Resulting in a conjunction of a set of disjunctions of literals.
(i.e., move disjunctions down to the literals.)

Use the Distributive laws repeatedly.

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

Result in Example Proof:

$$\begin{array}{l} \forall x \forall y\ [\neg P(x) \vee [\neg P(y) \vee P(f(x,y))] \wedge \\ \quad [Q(x,g(x)) \wedge \neg P(g(x))]] \\ \qquad\qquad \Downarrow \text{apply distributive law twice} \\ \forall x \forall y\ [[\neg P(x) \vee \neg P(y) \vee P(f(x,y))] \wedge \\ \quad [\neg P(x) \vee Q(x,g(x))] \wedge \\ \quad [\neg P(x) \vee \neg P(g(x))]] \end{array}$$

Step 7

Eliminate Universal Quantifiers

All variables are assumed to be universally quantified.

Result in Example Proof:

$$[\neg P(x) \vee \neg P(y) \vee P(f(x,y))] \wedge$$
$$[\neg P(x) \vee Q(x, g(x))] \wedge$$
$$[\neg P(x) \vee \neg P(g(x))]$$

Step 8

Eliminate $\wedge$ Symbols

Separate into clauses

Result in Example Proof:

$$\neg P(x) \vee \neg P(y) \vee P(f(x,y))$$
$$\neg P(x) \vee Q(x, g(x))$$
$$\neg P(x) \vee \neg P(g(x))$$

Step 9

Rename Variables

No variable symbol should appear in more than one clause.

(Standardizing the variables apart).

Result in Example Proof:

$$\neg P(x1) \vee \neg P(y) \vee P(f(x1, y))$$
$$\neg P(x2) \vee Q(x2, g(x2))$$
$$\neg P(x3) \vee \neg P(g(x3))$$

Done! (Whew!!)

Now we can **start** actual proving by resolution from these ground clauses.

Resolution Theorem Proving

- Proof by contradiction

- Unification is the major process:
  matching while propagating variable bindings

Procedure

1. Negate theorem trying to prove
2. Add to axiom set
3. Convert all axioms to clause form
4. Resolve clauses until:
   a) Empty clause produced -> T
   b) No resolvable clauses -> F

Empty clause denotes contradiction

Completeness and Decidability

- Complete: If KB entails S, then we can prove S from KB

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Completeness and Decidability

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- Gödel's Completeness Theorem: *There exists a complete proof system for FOL*
- Robinson's Completeness Theorem: *Resolution refutation is a complete proof system for FOL*

- *FOL is semi-decidable:* if the desired conclusion follows from the premises then eventually resolution refutation will find a contradiction.
  - If there's a proof, we'll halt with it
  - If not, maybe we'll halt, maybe not

Adding Arithmetic

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- Gödel's Incompleteness Theorem: *There is no consistent, complete proof system for FOL + Arithmetic.*
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  P = "P is not provable."
  - If P is true: it's not provable (incomplete)
  - If P is false: it's provable (inconsistent)