Assignment 8

Development of optimal smoothing to increase the estimation accuracy

By

Group 19

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1) Generateing a true trajectory X_i of an object motion disturbed by normally distributed random acceleration. and mesurment Z_i for the trajectory

$$X_i = X_{i-1} + V_{i-1}T + a_{i-1}\frac{T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$
$$Z_i = X_i + \eta$$

Size of trajectory is 300 points.

Initial conditions: $X_1 = 5;\ V_1 = 0;\ T = 0.1;$

Variance of noise a_i : $\sigma_a^2 = 0.2^2$

Variance of noise Z_i : $\sigma_\eta^2 = 20^2$

```
clc; clear

N=200;                % size of trajectory
x = zeros(N,1);       % real data for position
V = zeros(N,1);       % real data for velocity
a = zeros(N,1);       % noise
z = zeros(N,1);       % measurements
s_alpha = 0.2^2;      % sigma_w^2
s_eta = 20^2;

x(1) = 5;             % initial position
V(1) = 1;             % initial velocity
```

```
a(1) = normrnd(0,sqrt(s_eta));
z(1) = x(1) + a(1);
T = 1;              % time step

for i = 2:N
    a(i) = normrnd(0,sqrt(s_alpha));
    x(i) = x(i-1) + V(i-1)*T + a(i)*T^2 /2;
    V(i) = V(i-1) + a(i)*T;
    z(i) = x(i) + normrnd(0,sqrt(s_eta));
end
```

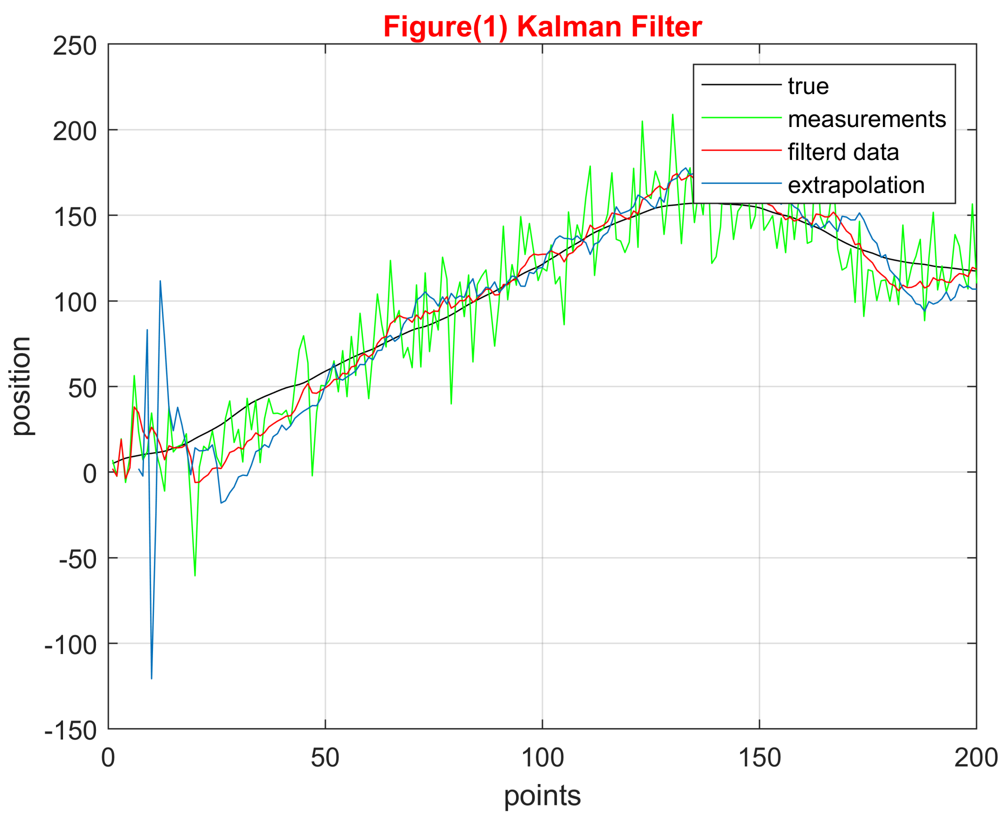
2) Present the system at state space taking into account that only measurements of coordinate X_i are available.

$$X_i = \Phi X_{i-1} + \mathrm{G}a_{i-1}$$
$$Z_i = H_i X_i + \eta_i$$

3) Develop Kalman filter algorithm to estimate state vector X_i (extrapolation and filtration)

```
phi = [1 T;0 1];    % transition matrix that relates Xi to Xi-1
G = [T^2/2;T];      % input matrix, that determines how random acceleration affects state vecto
H = [1 0];          % observation matrix
Q = G*G'*s_alpha;   % covariance matrix of state noise
K = zeros(N,2);     %  array to save filter gain


I = eye(2);
R = s_eta;          % covariance matrix of measurements noise

X = zeros(N,2);     % state vector, that describes full state of the system
x_extrapolated = zeros(N-6,1);
X(1,:)=[2,0];                   % Initial filtered estimate
P_filteration = [10000 0; 0 10000];         % Initial filtration error covariance matrix
P_value = zeros(N,1);           % array to save the caculation error
K(1,:) = (P_filteration*H'*(H*P_filteration*H'+R)^-1)';  % initial value of filter gain

m = 7;                  % extrapolation steps
P_filteration_cell = cell(N,1);     % cell to save the P_filteration matrices
P_prediction_cell = cell(N,1);     % cell to save the P_prediction matrices
P_filteration_cell{1} = P_filteration;

% kalman filter
for i = 2:N
    % prediction(extrapolation)
        X_predicted = phi * X(i-1,:)';      % Prediction of state vector at time i using i - 1
        P_prediction = phi*P_filteration*phi' + Q;                 % Prediction error covarian
    % filteration
        X(i,:) = X_predicted + K(i-1,:)'*(z(i)- H*X_predicted);     % Improved estimate by inco
        K_updated = P_prediction*H'*(H*P_prediction*H'+R)^-1;     % Filter gain, weight of resi
        P_filteration = (I - K_updated*H)*P_prediction;            % Filtration error covarian
    % extrapolation
        if i >= m
```

```
            x1 = phi^(m-1) * X(i-m+1,:)';
            x_extrapolated(i-m+1)=x1(1);
        end
    % save values to be used in the smoothing step
        P_filteration_cell{i} = P_filteration;
        P_prediction_cell{i} = P_prediction;

P_value(i)=sqrt(P_filteration(1,1));
K(i,:) = K_updated;

end

figure()
plot(x,'k'); hold on
plot(z,'g'); hold on
plot(X(:,1),'r'); hold on
plot(7:200,x_extrapolated)
legend('true','measurements','filterd data','extrapolation')
title ('Figure(1) Kalman Filter','color','r'); xlabel('points'); ylabel('position')
grid on
```

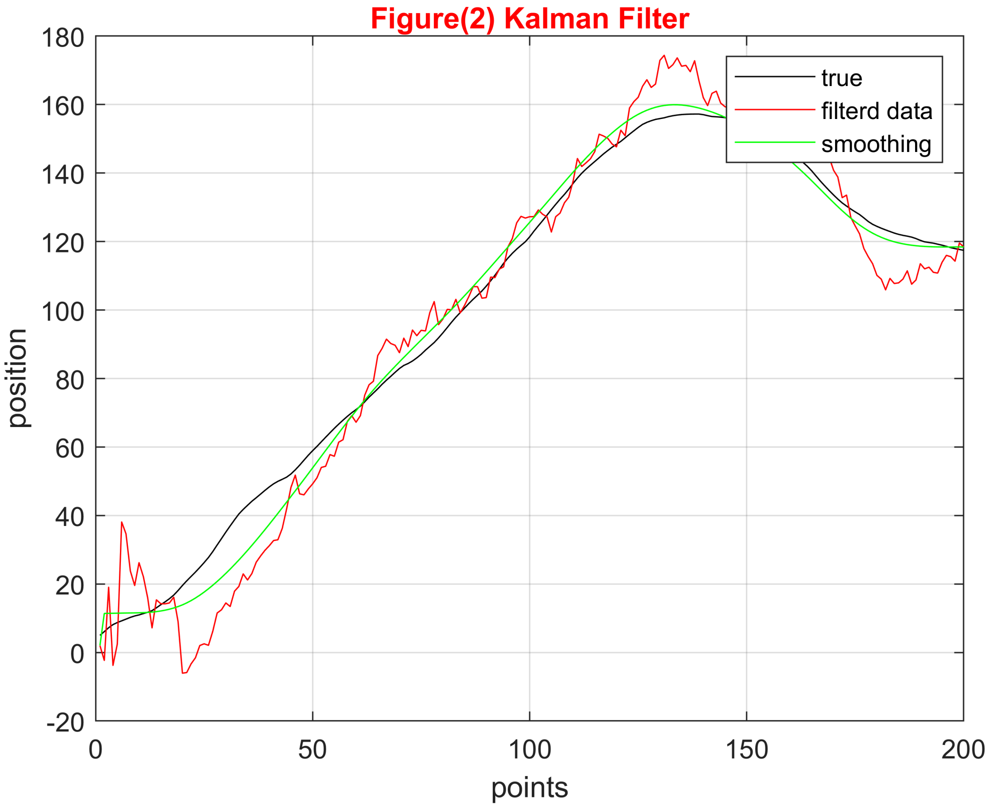


Comment:

In figure(1) above, the true trajectory, the measurements, the Kalman filter estimation and extapolation with 7 steps ahead are graphed against each other. It can be seen that the estimation values are kind of following the trajectory with some disturbances. And plotting the filter with different random trajectories leads to the same results. Also, the extrapolation follows the behavior of the estimation more, which makes sense as it's based on it. Moreover, it's more noisy at the beginning of the trajectory so it cannot be trusted at this region as an estimation of the true model, and lastly we can predict that both the error of filter estimation and the extrapolation will have the same behavior with close values.

4) Develop backward smoothing algorithm to get improved estimates of state vector X_i

```
 % backward sommothing
        X_smoothed = zeros(N,2);
        P_smoothing_value_X = zeros(N-1,1);
        P_smoothing_value_V = zeros(N-1,1);
        P_filteration_cell{1} = P_filteration;
        X_smoothed(N,:) = X(N,:);
        P_smoothing = P_filteration_cell{N};

        for i =  N-1:-1:1
            A = P_filteration_cell{i}*phi'/P_prediction_cell{i+1};
            X_smoothed(i,:) = X(i,:)' + A*(X_smoothed(i+1,:)' - phi*X(i,:)');
            P_smoothing = P_filteration_cell{i}+A*(P_smoothing -P_prediction_cell{i+1})*A';
            P_smoothing_value_X(i) = sqrt(P_smoothing(1,1));
            P_smoothing_value_V(i) = sqrt(P_smoothing(2,2));
        end

figure()
plot(x,'k'); hold on
%plot(z,'y'); hold on
plot(X(:,1),'r'); hold on
%plot(7:200,x_extrapolated); hold on
plot(X_smoothed(:,1),'g')
hold off
%legend('true','measurements','filterd data','extrapolation','smoothing')
legend('true','filterd data','smoothing')
title ('Figure(2) Kalman Filter','color','r'); xlabel('points'); ylabel('position')
grid on
```

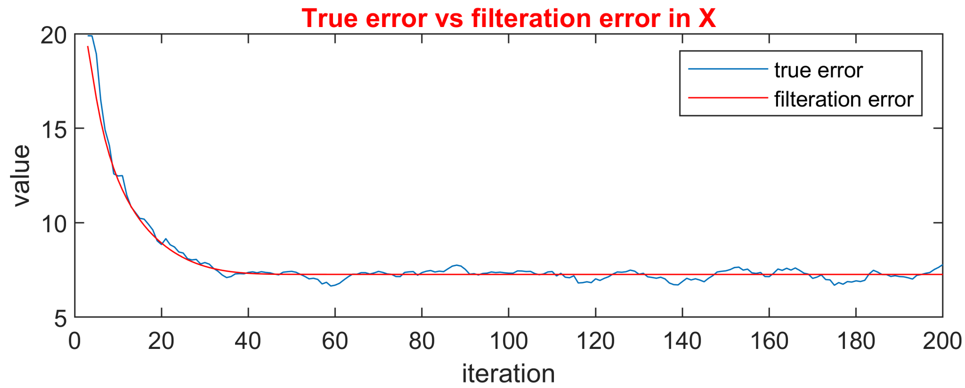


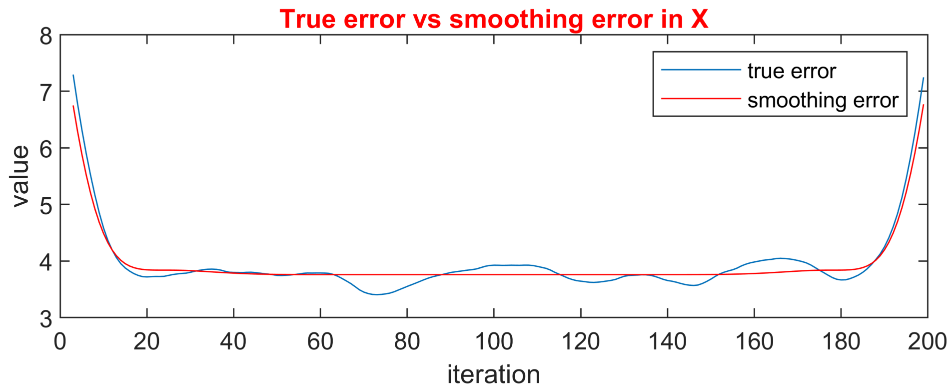
comment:

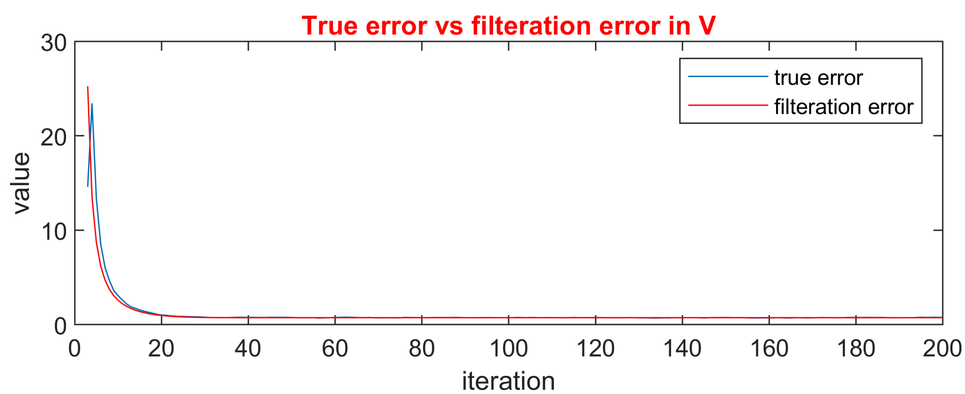
Compairng the backward smoothed results to the kalman fileter output. the error after smoothing is very small and the filter can capture the dynamics of the system to a great extent. Although the kalman filter results still good approximations the using of the Kalman somoothing clearly enhanced the rsults.

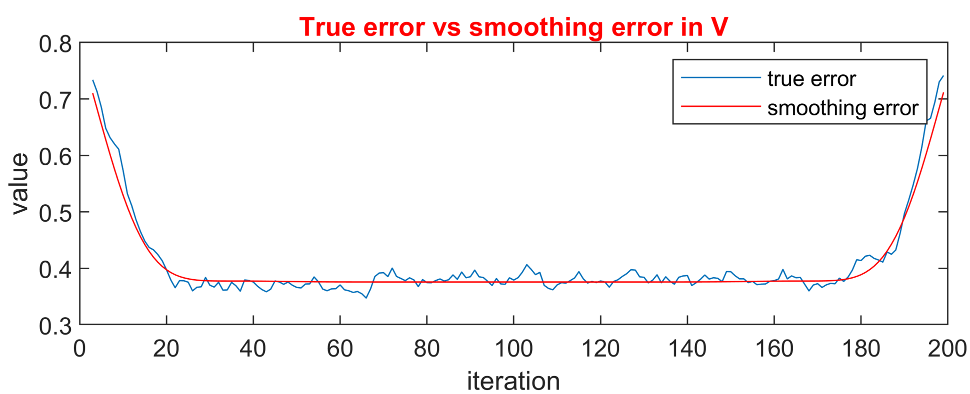
5) Make $M = 500$ **runs of smoothing and compare true estimation error with errors of smoothing** $P_{i,N}$ **provided by smoothing algorithm.**

```
clc; clear; close all;
% code for M=500 runs
M_runs_smoothing(500)
```


True error vs filteration error in X
true error
filteration error
value
iteration


True error vs smoothing error in X
true error
smoothing error
value
iteration


True error vs filteration error in V
true error
filteration error
value
iteration


True error vs smoothing error in V
true error
smoothing error
value
iteration

Comment:

From the above for both the X,V errors we an get two things:

1. Comparing the true smoothing error to the true filtration error, the smoothed results gives better approximation of the true trajectory eliminating the greatest portion of noise and providing smaller error values.
2. Comparing the true smoothing error to the estimated smoothing error calculated by the smoothing P matrix, the filter (if tuned correctly) can provide a very good error approximation.

We also can notice unlike the filtered results the error values increased on the right part of the curve, that is due to the backward smoothing. The filter takes some iterations to reach a steady state regarding the smoothing error. For the left part in the curve the error decreases as the filtered data itself reaches the steady state so when the smoothing filter is already tuned the error source is the previous step in the Kalman filtration.

Learning Log:

In this assignment we further enhanced the results of Kalman filter estimations by smoothing them using optimum smoothing and if we conclude in bullet points:

1. Kalman filter’s parameters (if tuned correctly) can be used to further enhance the filtered results and provide optimum smoothing and good error estimation
2. As always, the key element in applying any processing to data is understanding the dynamics of your system to avoid any over smoothing or wrong noise filtration. for example choosing the right model, right parameters and applying the suitable method.