Assignment 7
Tracking in conditions of correlated biased state and measurement noise

Performance -Monday, September 21, 2020
Due to submit a performance report – Friday, September 25, 2020

The objectives of this laboratory work is first to analyze the sensitivity of estimation results obtained by a Kalman filter that does not take into account correlation of state and measurement noise. Second is to develop optimal Kalman filter that takes into account correlated random acceleration to track a moving object. This problem is typical for many practical control and forecasting problems. This will bring about a deeper understanding of main difficulties of practical Kalman filter implementation and skills to overcome these difficulties to get optimal assimilation output. Additional important outcome of this exercise is experience in developing algorithms to improve Kalman filter estimates.

This laboratory work is performed in the class by students as in teams of 4 and the team will submit one document reporting about the performance. Within your group, you may discuss all issues openly, and discuss and debate until you reach a consensus.

This laboratory work consists of two parts:

I. Sensitivity of estimation results obtained by a Kalman filter that does not take into account the correlation of state noise (acceleration) and measurement noise.
II. Development of optimal Kalman filter in conditions of correlated state noise.

Here is the recommended procedure for part I:

Sensitivity of estimation results obtained by a Kalman filter that doesn't take into account the correlation of state noise (acceleration) and measurement noise.

1. Generate a true trajectory X_i of an object motion disturbed by a correlated in time random acceleration. Let's assume that this random acceleration is first-order Gauss-Markov process. It means that

$$a_i = e^{-\lambda T} a_{i-1} + \zeta_i \; (1)$$

ζ_i - uncorrelated random noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;
σ_a^2 – variance of correlated noise σ_a^2

λ – value that is inverse to correlation interval.
For example,
(a) if $\lambda = 1000$, then $a_i = \sigma_a \xi_i$ and is uncorrelated noise (substitute 1000 to equation (1) for λ)
(b) if $\lambda = 0.1$, then a_i is correlated noise on interval over 10 steps. It means that inside every 10 steps correlation is significant.

Hint to generate correlated noise a_i
$a_i = e^{-\lambda T} a_{i-1} + \sigma_a \xi_i \sqrt{1 - e^{-2\lambda T}}$
ξ_i – random uncorrelated unbiased noise with variance $\sigma_\xi^2 = 1$.
$\sigma_a = 0.2$ –standard deviation of random acceleration.
$T = 1$ – time interval between measurements.

Then true trajectory is generated using random acceleration a_i obtained according to equation (1). First use $\lambda = 1000$ that means that a_i is uncorrelated noise,

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$

Size of trajectory is $N = 200$ points.
Initial conditions: $x_1 = 5; V_1 = 1;$

2. Generate measurements z_i of the coordinate x_i

$$z_i = x_i + \eta_i$$

η_i – is random correlated noise. This is also first-order Gauss-Markov process.

$$\eta_i = e^{-\lambda T}\eta_{i-1} + \varsigma_i \ (2)$$

ς_i - uncorrelated random noise with variance $\sigma_\varsigma^2 = \sigma_\eta^2(1 - e^{-2\lambda T})$;
$\sigma_\eta^2 = 20^2$ – variance of measurement noise.

First use $\lambda = 1000$ that means that η_i is uncorrelated noise.

3. Obtain estimates of state vector $X = \begin{vmatrix} x \\ V \end{vmatrix}$ by Kalman filter over $M = 500$ runs and compare true estimation errors with errors of estimation $P_{i.i}$ provided by Kalman filter algorithm. This is optimal Kalman filter as assumptions about uncorrelated noise are true.

4. Generate trajectory and measurements again (repeat items 1 and 2), but use $\lambda = 0.1$ to generate random acceleration a_i according to Equation (1). It means that a_i is correlated noise on interval over 10 steps.

5. Obtain estimates of state vector $X = \begin{vmatrix} x \\ V \end{vmatrix}$ by Kalman filter over $M = 500$ runs in these new conditions. Compare true estimation errors with errors of estimation $P_{i.i}$ provided by Kalman filter algorithm. This is non-optimal Kalman filter as assumptions about uncorrelated state noise are not true. Random acceleration is correlated noise.

6. Compare results for optimal and non-optimal filter.

7. Generate trajectory and measurements again (repeat items 1 and 2), but use $\lambda = 1000$ to generate random acceleration a_i according to Equation (1) (a_i is uncorrelated noise). But use $\lambda = 0.1$ to generate random measurement noise η_i according to Equation (2). It means that noise η_i is correlated on interval over 10 steps.

8. Obtain estimates of state vector $X = \begin{vmatrix} x \\ V \end{vmatrix}$ by Kalman filter over $M = 500$ runs in these new conditions. Compare true estimation errors with errors of estimation $P_{i.i}$ provided by Kalman filter algorithm. This is also non-optimal Kalman filter as assumptions about uncorrelated measurement noise are not true. η_i - is correlated noise. Estimate how useful this filter is by comparing the resulted true estimation error with the standard deviation of measurement noise.

9. Conclude neglecting of which noise leads to greater accuracy decrease?

10. Generate trajectory and measurements again (repeat items 1 and 2) in conditions of both correlated state noise a_i and correlated measurement noise η_i. Use $\lambda = 0.1$ to generate random acceleration a_i according to Equation (1) and use $\lambda = 0.1$ to generate random measurement noise η_i according to Equation (2).

11. Obtain estimates of state vector $X = \begin{vmatrix} x \\ V \end{vmatrix}$ by Kalman filter over $M = 500$ runs in these new conditions. Compare true estimation errors with errors of estimation $P_{i.i}$ provided by Kalman filter algorithm. Estimate how useful this filter is by comparing the resulted true estimation error with the standard deviation of measurement noise.

Here is the recommended procedure for part II:

Development of optimal Kalman filter in conditions of correlated state noise

1. Generate a true trajectory X_i of an object motion disturbed by a correlated in time random acceleration with variance σ_a^2. Let's assume that this random acceleration is first-order Gauss-Markov process. It means that

$$a_i = e^{-\lambda T} a_{i-1} + \zeta_i \ (1)$$

ζ_i - uncorrelated random noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;
σ_a^2 – variance of correlated noise $\sigma_a^2 = 0.2^2$;
$T = 1$ – time interval between measurements.

λ – value that is inverse to correlation interval.
For example,
(c) if $\lambda = 1000$, then $a_i = \sigma_a \xi_i$ and is uncorrelated noise
(substitute 1000 to equation (1) for λ)
(b) if $\lambda = 0.1$, then a_i is correlated noise on interval over 10 steps. It means that inside every 10 steps correlation is significant.

Then true trajectory is generated using random acceleration a_i obtained according to equation (1). Use $\lambda = 0.1$ that means that a_i is correlated noise on interval over 10 steps.

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$

Size of trajectory is $N = 200$ points.
Initial conditions: $x_1 = 5; V_1 = 1;$

1. Generate measurements z_i of the coordinate x_i

$$z_i = x_i + \eta_i$$

η_i –normally distributed uncorrelated random noise with zero mathematical expectation and variance $\sigma_\eta^2 = 20^2$.

2. Preparations to develop optimal Kalman filter algorithm that takes into account correlated acceleration.

   (a) In case of correlated random acceleration the motion of object is described by following equations

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$
$$a_i = e^{-\lambda T}a_{i-1} + \zeta_i$$

Here x_i – coordinate, V_i – velocity, a_i – correlated random acceleration.
Here $e^{-\lambda T}a_{i-1}$ depends on previous value of random acceleration a_{i-1}
ζ_i - uncorrelated random noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;
λ – value that is inverse to correlation interval;
T – time interval between measurements.

   (b) To apply Kalman filter let's present the system at state space.
   State vector in this case is presented as

$$X_i = \begin{vmatrix} x_i \\ V_i \\ a_i \end{vmatrix}$$

   It means that state vector X_i is extended by inclusion of correlated random acceleration a_i. Therefore, Kalman filter algorithm will provide estimates not only of coordinate x_i and velocity V_i, but also acceleration a_i. If random acceleration is correlated, then it is characterized by a certain dynamics that should be estimated in contrast to white noise.

   State equation is given by

$$X_i = \Phi X_{i-1} + G\zeta_i \ (2)$$

   Determine transition matrix Φ and input matrix G for the motion given in (a)
   $G\zeta_i$ - state noise;
   ζ_i - uncorrelated noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;

   Measurement equation is given by

$$z_i = \mathrm{H}X_i + \eta_i \ (3)$$

   Determine the observation matrix H if only coordinate x_i is measured.

   Check that determined matrices Φ, G, and H satisfy state-space model (2,3)

3. Initial conditions needed for Kalman filter algorithm
   Initial filtered estimate

$$X_0 = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$$

   Initial filtration error covariance matrix

$$P_{0,0} = \begin{vmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{vmatrix}$$

4. Obtain estimates of state vector $X_i = \begin{vmatrix} x_i \\ V_i \\ a_i \end{vmatrix}$ by Kalman filter over $M = 500$ runs and compare true estimation error with errors of estimation $P_{i.i}$ provided by Kalman filter algorithm.
   (a) error of filtered estimates of coordinate x_i;
   (b) error of filtered estimates of velocity V_i;
   (c) error of filtered estimates of acceleration a_i;

   Compare also errors of filtered estimates with errors of extrapolated estimates of x_i, V_i, a_i

5. Make conclusions to the Assignment.
   Conclusions should be done in a form of a learning log.
   A learning log is a journal which evidences your **own learning and skills development.**
   It is not just a diary or record of **"What you have done"**
   but a record of **what you have learnt, tried and critically reflected upon.**

6. Prepare performance report and submit to Canvas:
   Performance report should include 2 documents:
   1) A report (PDF) with performance of all the items listed above
   2) Code (PDF)

Notes:

- PDF report should contain the names of team members, number of the assignment
- All questions of the assignment should be addressed
- All figures should have a caption, all axes should have labels, a legend to curves should be given, and short conclusions/discussions/results related to figures should be provided.
- The overall conclusion to the assignment should be provided in a form of a learning log.