Numeric Analysis

ex4

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ex 3.1

- firstly we prove $\|X\|_2 \le \|X\|_1$, by induction on n.

Base case $n=1$: $\|X\|_2 = \sqrt{x_1^2} = |x_1| = \|X\|_1$.

step: assume that the inequality holds on $n-1$, we'll prove that it holds on n. (instead of proving $\|X_2\| \le \|X\|_1$ we'll prove $\|X\|_2^2 \le \|X\|_1^2$)

$$(\|X\|_1)^2 = (|X_1| + \dots + |X_n|)^2 = (|X_1| + \dots + |X_{n-1}|)^2 + 2(|X_1| + \dots + |X_{n-1}|)|X_n| + X_n^2 \underset{\text{step}}{\ge}$$

$$x_1^2 + \dots + X_{n-1}^2 + 2(|X_1| + \dots + |X_{n-1}|)|X_n| + (X_n)^2 \ge X_1^2 + \dots + X_n^2 = (\|X\|_2)^2.$$

Now we'll prove $\|X\|_1 \le \sqrt{n}\,\|X\|_2$ using cauchy-schwarz inequality $|\langle v,u\rangle| \le \|v\|\cdot\|u\|$ on $v = (|X_1|, \dots, |X_n|)$ $u = (1, \dots, 1)$ we get

$$\|X\|_1 = |X_1| + \dots + |X_n| \underset{C.S}{\le} \sqrt{x_1^2 + \dots + X_n^2} \cdot \sqrt{1^2 + \dots + 1^2} = \sqrt{n} \cdot \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{n} \cdot \|X\|_2$$

- $\|X\|_\infty \le \|X\|_2 \le \sqrt{n}\,\|X\|_2$: $\|X\|_\infty^2 = (\max_{i\in[n]} |X_i|)^2 \le X_1^2 + \dots + X_n^2 = (\|X\|_2)^2 \Rightarrow \|X\|_\infty \le \|X\|_2$

$$\|X\|_2 = \sqrt{X_1^2 + \dots + X_n^2} \le \sqrt{n \cdot \max_{i\in[n]} |X_i|^2} = \sqrt{n}\,\|X\|_\infty$$

- $\|X\|_\infty \underset{\text{definition of norm}}{\le} \|X\|_1 \le n\|X\|_1$: $\|X\|_\infty = \max_{i\in[n]} |X_i| \le |X_1| + \dots + |X_n| = \|X\|_1 \le n \cdot \max_{j\in[n]} |X_j| = n \cdot \|X\|_\infty$

ex 3.3

$$\|A\|_1 = \sup_{\|x\|_1=1} \|Ax\|_1 = \sup_{\|x\|_1=1} \left(\left\| \begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{matrix} \right\|_1 \right) = \sup_{\|x\|_1} \left(\sum_{i=1}^{n} |a_{i1}x_1 + \dots a_{in}x_n| \right) \overset{\text{triangle inequality}}{\leq}$$

$$\sup_{\|x\|_1=1} \left(\sum_{i=1}^{n} (|a_{i1}x_1| + \dots + |a_{in}x_n|) \right) = \sup_{\|x\|_1=1} \left(\sum_{j=1}^{n} |x_j| (|a_{1j}| + \dots + |a_{nj}|) \right) \leq \sup_{\|x\|_1=1} \sum_{j=1}^{n} |x_j| \max_{j\in[n]} (|a_{1j}| + \dots + |a_{nj}|)$$

$$= \sup_{\|x\|_1=1} \left(\max_{j\in[n]} (|a_{1j}| + \dots + |a_{nj}|) \cdot \underbrace{\sum_{j=1}^{n} |x_j|}_{=1} \right) = \max_{j\in[n]} (|a_{1j}| + \dots + |a_{nj}|) = \max_{j\in[n]} \sum_{i=1}^{n} |a_{ij}|$$

In the other direction we want to prove $\sup_{\|x\|_1=1} \|Ax\|_1 \geq \max_{j\in[n]} \sum_{i=1}^{n} |a_{ij}|$

let $\hat{j}$ be the index s.t $\sum_{i=1}^{n} |a_{i\hat{j}}| = \max_{j\in[n]} \sum_{j\in[n]} |a_{ij}|$.

If we take $\hat{x} = (0, \dots, 0, 1, 0, \dots, 0)$ then $\|\hat{x}\|_1 = 1$. $\|A\|_1 = \sup_{\|x\|_1=1} \|Ax\|_1 \geq \|A\hat{x}\|_1 =$

$= \sum_{i=1}^{n} |a_{i\hat{j}}| = \max_{j\in[n]} \sum_{i=1}^{n} |a_{ij}|$. $\boxtimes$

ex 3.4

(a) let $\|\cdot\|$ be a norm space on $\mathbb{C}^n$, $\|x\|' = \|Sx\|$.

- Triangle inequality, let $x_1, x_2 \in \mathbb{C}^n$: $\|x_1 + x_2\|' = \|(x_1 + x_2)S\| = \|Sx_1 + Sx_2\| \leq$ we know that $Sx_i \in \mathbb{C}^n$, since $\|\cdot\|$ is a norm we get $\overset{*}{\leq} \|Sx_1\| + \|S.x_2\| = \|x_1\|' + \|x_2\|'$
- Absolute homogeneity: let $x \in \mathbb{C}^n$, λ be a scalar: $\|\lambda x\|' = \|S \cdot \lambda x\| = \|\lambda \cdot Sx\| =$ $= |\lambda| \cdot \|Sx\| = |\lambda| \cdot \|x\|'$.
- Positivity: let $x \in \mathbb{C}^n$ then $\|x\|' = \|Sx\| \overset{\text{positivity of } \|\cdot\|}{\geq} 0$, $\|x\| = 0 \Leftrightarrow \|Sx\| = 0$, since S is invertible rank$(S) = n$. we get that $Sx = 0 \Leftrightarrow x = 0$

(b) let $\|\cdot\|$ be the norm operator induced by the above vector norm

$\|A\|' = \|SAS^{-1}\|$. $\|A\|' = \|SAS^{-1}\| = \sup_{\|x\|=1} \|(SAS^{-1})x\| = \sup_{\|SS^{-1}x\|=1} \|SA(S^{-1}x)\| \overset{*}{=}$

denote $v = S^{-1}x$ then we get $\overset{*}{=} \sup_{\|Sv\|=1} \|SAv\| = \sup_{\|v\|'=1} \|Av\|'$

$\boxtimes$