ELC 423 , ELCN323 & ELCn323

ELECTROMAGNETIC FIELD THEORY

Summer Semester 2021/2022

CHAPTER 3

Electrostatic Field In Material Space

Week 3, Lecture 3

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Lecture's Topics

- **Materials Classification**
- **Electric Field in Dielectrics and Conductors**
- **Resistance and Capacitance as Boundary Value Problems**
- **Boundary Condition in the case of Dielectrics**
- **Boundary Condition in the case of Conductors**
- **Solved Examples**

Properties of Materials

Materials Classification

- **If ($\sigma >> 1$)**

The material is referred to as a metal or a conductor

- **If ($\sigma << 1$)**

The material is referred to as an insulator or nonconductor or dielectric.

σ = material conductivity [mhos per meter (℧/m)] or [Siemens per meter (S/m)].

A material whose conductivity lies between those of metals and insulators is called a semiconductor.

Properties of Materials

When electric field is going through a medium, two phenomena can take place:

- **If the material is conductive** (has free electrons): Electric current can be created and related to the electric field by **Ohm's law**.

$$\mathbf{J} = \sigma\,\mathbf{E}, \quad \mathbf{J} = \Delta \mathbf{I} \,/\, \Delta \mathbf{s}, \quad \mathbf{I} = \frac{\mathbf{dQ}}{\mathbf{dt}}, \quad \mathbf{I} = \int_{s} \mathbf{J}\,.\,\mathbf{ds}$$

- **If the material is dielectric** (has no free electrons): Polarization of the material molecules can take place and polarization flux density is produced.

$$\mathbf{P} = \chi\,.\,(\varepsilon_0\,\mathbf{E})$$

Electric Quantities in Conductors

Uniform Cross Section Conductors

The Electric Resistance:

Assume the electric field is uniform, its magnitude is:

$$\mathbf{E} = \frac{\mathbf{V}}{\boldsymbol{l}}, \qquad \mathbf{J} = \frac{\mathbf{I}}{\mathbf{S}}, \qquad \mathbf{J} = \boldsymbol{\sigma}\,\mathbf{E}$$

$$\frac{\mathbf{I}}{\mathbf{S}} = \boldsymbol{\sigma}\,\frac{\mathbf{V}}{\boldsymbol{l}}$$

$$\therefore \quad \mathbf{R} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\boldsymbol{l}}{\boldsymbol{\sigma}\,\mathbf{S}} = \frac{\boldsymbol{\rho}_{\mathbf{c}}\,\boldsymbol{l}}{\mathbf{S}} = \frac{\mathbf{1}}{\mathbf{G}}$$

- G is the conductance of the material [mho]
- $\rho_c = 1/\sigma$ is the resistivity of the material [$\frac{m}{mho}$]

Electric Quantities in Conductors

The Electric Capacitance

$$C = \frac{Q}{V} = \frac{Q}{I\,R} = \frac{Q}{J\,S\,R} = \frac{D}{J\;R} = \frac{\varepsilon\,E}{\sigma\,E\;\,R} = \frac{1}{R}\,\frac{\varepsilon}{\sigma}$$

ε = The material dielectric constant [Farad/m]

The Electric Power

P = rate of change of energy W

= force times velocity

$$dP = I\,.\,dV = (\sigma\,E\;\,ds)\,.\,(E\,dl) = \sigma\,E^2(dl\,.\,ds) = \sigma\,E^2\,dv$$

$$P = \iiint_V \sigma\,E^2\,dv = V\,.\,I\quad [\text{watt}]$$

Electric Quantities in Conductors

Non Uniform Cross Section Conductors

In this case, Resistance and Capacitance can be calculated as Boundary Value Problems

Resistance Calculation:

$$\mathbf{R} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\int \mathbf{E} \,.\, \mathbf{d}\boldsymbol{l}}{\int \mathbf{J} \,.\, \mathbf{ds}} = \frac{\int \mathbf{E} \,.\, \mathbf{d}\boldsymbol{l}}{\int \boldsymbol{\sigma} \,\mathbf{E} \,.\, \mathbf{ds}}$$

$$= \frac{\int_0^{l_1} \mathbf{E} \,.\, \mathbf{d}\boldsymbol{l}_1}{\int_0^{l_2} \int_0^{l_3} \boldsymbol{\sigma}\, \mathbf{E}\, (\mathbf{d}\boldsymbol{l}_2 .\, \mathbf{d}\boldsymbol{l}_3)}$$

where $(l_1, l_{2,}, l_3)$ are the distances in the direction of coordinate axes.

Electric Quantities in Conductors

Capacitance Calculation

The capacitance C of the capacitor is defined as the ratio of the magnitude of the charge on one of the plates to the potential difference between them:

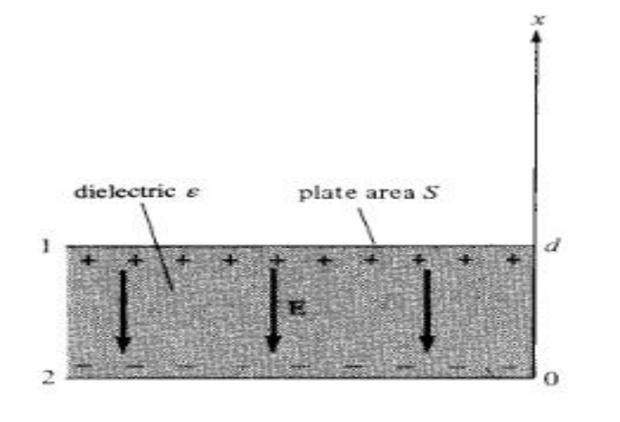
$$\mathbf{C} = \frac{\mathbf{Q}}{\mathbf{V}} = \frac{\varepsilon \oint \mathbf{E} \cdot \mathbf{ds}}{\int \mathbf{E} \cdot \mathbf{dl}} \qquad \mathbf{C} = \frac{\mathbf{1}}{\mathbf{R}} \frac{\varepsilon}{\sigma}$$

The (-) sign before V = $-\int \mathbf{E} \cdot \mathrm{dl}$ is dropped because we only interest with the absolute value of V.

Three cases for capacitance calculation as boundary value problem are introduced.

Capacitance Calculation As A Boundary Value Problem

Case 1: Parallel Plate Capacitor:



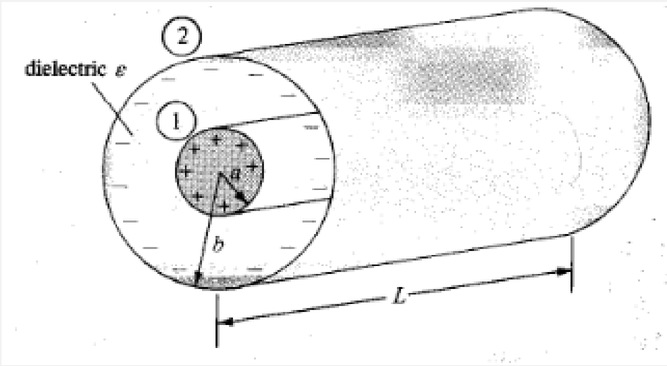
$$Q = \varepsilon \oint \mathbf{E} \,.\, ds = -\,\varepsilon\, E_x\, S, \qquad \text{Hence:} \qquad \mathbf{E} = -\frac{Q}{\varepsilon\, S}\,\mathbf{a}_x$$

$$V = -\int_2^1 \mathbf{E}\,.\,dl = -\int_0^d \left[-\frac{Q}{\varepsilon S}\right].\,dx = \frac{Qd}{\varepsilon S}$$

Therefore: $C = \dfrac{Q}{V} = \dfrac{\varepsilon S}{d}$, $\quad R = \dfrac{1}{C}\,\dfrac{\boldsymbol{\varepsilon}}{\boldsymbol{\sigma}}$

Capacitance Calculation as a Boundary Value Problem

Case 2: Coaxial Capacitor:

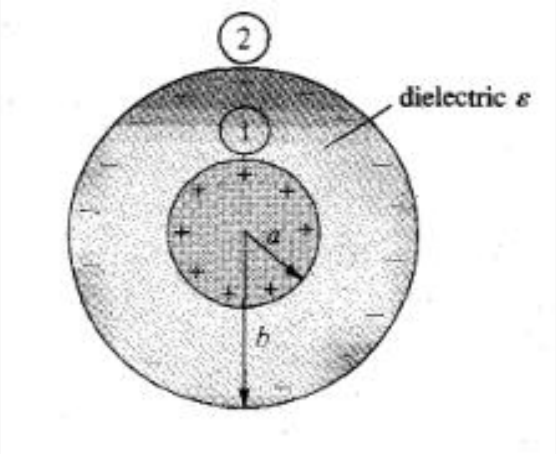


$$Q = \varepsilon \oint \mathbf{E} \,.\, ds = \varepsilon E_\rho \, 2\,\pi\rho l,$$ Hence: $$\mathbf{E} = \frac{Q}{\varepsilon\, 2\,\pi\rho l}\, \mathbf{a}_\rho$$

$$V = -\int_2^1 \mathbf{E} \,.\, dl = -\int_b^a \left[\frac{Q}{\varepsilon\, 2\,\pi\rho l}\right] .\, d\rho = \frac{Q}{2\,\pi\,\varepsilon\, l} \ln\frac{b}{a}$$

Therefore: $$C = \frac{Q}{V} = \frac{2\,\pi\,\varepsilon\, l}{\ln\frac{b}{a}},$$ $$R = \frac{\mathbf{1}}{C}\,\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\sigma}}$$

Case 3: Spherical Capacitor:



$$Q = \varepsilon \oint \mathbf{E}\,.\,ds = \varepsilon E_r\, 4\,\pi r^2,$$ Hence: $$\mathbf{E} = \frac{Q}{4\,\pi\varepsilon r^2}\,\mathbf{a}_r$$

$$V = -\int_2^1 \mathbf{E}\,.\,dl = -\int_b^a \left[\frac{Q}{4\,\pi\varepsilon r^2}\right].\,dr = \frac{Q}{4\,\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Therefore: $$C = \frac{Q}{V} = \frac{4\,\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]}, \qquad R = \frac{\mathbf{1}}{C}\,\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\sigma}}$$

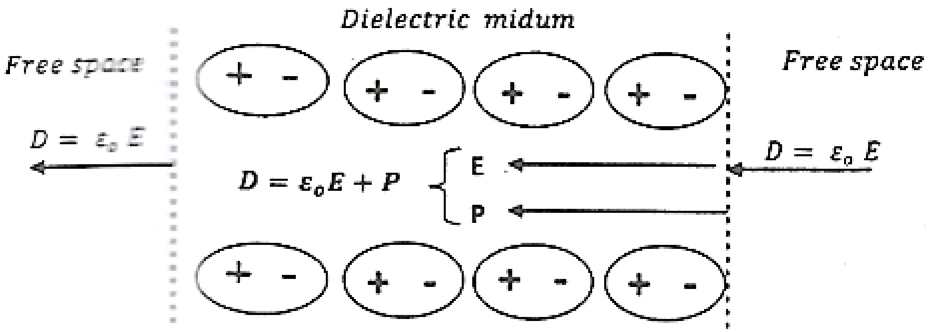
Polarization in Dielectrics

When the electric field go through a dielectric medium, the atoms of the medium get polarized (separated).

- The positive charges in the direction of **E**.
- The negative charges in the inverse direction of **E**.

The polarization flux density **P** is added to the applied electric flux density **D**, and the total flux density inside the medium will be:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$



Polarization in Dielectrics

P is defined by dipole moment per unit volume, (recall chapter 1, P =Q d)

Its value depend upon electric field intensity and material properties so it can defined by:

$$\mathbf{P} = \chi \,.\, (\varepsilon_0\, \mathbf{E})$$

Where χ is the electric susceptibility (sensitivity) of the material.

$$\mathbf{D} = \varepsilon_0\, \mathbf{E} + \mathrm{P} = \varepsilon_0\, \mathbf{E} + \varepsilon_0\, \chi\, \mathbf{E}$$

$$= \varepsilon_0\, \mathbf{E}\,(1 + \chi) = \varepsilon_0\, \varepsilon_r\, \mathrm{E} = \varepsilon\, \mathbf{E}$$

Polarization in Dielectrics

ε_r = relative permittivity (dimension less), always >1

ε_0 = the free space permittivity $\left[\frac{\text{Farad}}{\text{m}}\right]$.

In general, in homogenous isotropic media **D** and **E** are in the same direction.

Dielectric is linear: if ε dose not change with **E**

is homogenous: if ε dose not change from point to point.

is isotropic if ε dosn't change with direction

All formulas used in free space can be applied in material space by replacing ε_0 by ε

Polarization in Dielectrics

EXAMPLE 3.1

The electric field intensity in polystyrene (ε_r = 2.55) filling the space between the plates of a parallel-plate capacitor is 10 [kV/m]. The distance (d) between the plates is 1.5 [mm]. Calculate:

(a) D

(b) P

(c) The surface charge density of free charge on the plates

(d) The surface density of polarization charge

(e) The potential difference between the plates

Polarization in Dielectrics

SOLUTION

a) $\mathbf{D} = \varepsilon_0\, \varepsilon_r\, \mathbf{E} = \frac{10^{-9}}{36\pi}\,.\,(2.55).\,(10)^4 = 225.4\,\frac{\mathrm{nC}}{\mathrm{m}^2}$

b) $\mathbf{P} = \chi_e \varepsilon_0 \mathrm{E} = \frac{10^{-9}}{36\pi}\,.\,(1.55).\,(10)^4 = 137\ \mathrm{nC/m^2}$

c) $\rho_s = \mathbf{D}\,.\,\mathrm{a_n} = \mathrm{D_n} = 225.4\ \mathrm{nC/m^2}$

d) $\rho_{Ps} = \mathbf{P}\,.\,\mathrm{a_n} = \mathrm{P_n} = 137\ \mathrm{nC/m^2}$

e) $\mathrm{V} = \mathrm{E\,d} = 10^4\,(1.5\mathrm{x}10^{-3}) = 15\ \mathrm{V}$

Polarization in Dielectrics

EXAMPLE 3.2

A dielectric sphere ($\boldsymbol{\varepsilon}_{\mathbf{r}}$ = 5.7) of radius 10 [cm] has a point charge 2 [pC] placed at its center.

Calculate:

(a) The surface density of polarization charge on the surface of the sphere

(b) The force exerted by the charge on a -4 [pC] point charge placed on the sphere

Polarization in Dielectrics

SOLUTION

a) apply Coulomb's or Gauss's law to obtain Q

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \quad \mathbf{E} = \frac{Q}{4\pi\varepsilon_0\,\varepsilon_r\, r^2}\,\mathbf{a_r} \Longrightarrow \mathrm{P} = \frac{\chi_e Q}{4\pi\,\varepsilon_r\, r^2}\,\mathbf{a_r}$$

$$\rho_{Ps} = \mathbf{P}\,.\,\mathbf{a_r} = \frac{(\boldsymbol{\varepsilon}_\mathbf{r} - 1)Q}{4\pi\,\boldsymbol{\varepsilon}_\mathbf{r}\,\mathbf{r}^2} = \frac{(4.7)2 \text{ x } 10^{-12}}{4\pi\,(5.7)\,100 \text{ x } 10^4} = 13.12\,\left[\frac{\mathrm{pC}}{\mathrm{m}^2}\right]$$

b) Using Coulomb’s law, we have:

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\boldsymbol{\varepsilon}_0\,\boldsymbol{\varepsilon}_\mathbf{r}\,\mathbf{r}^2}\,\mathbf{a_r} = \frac{(-4)\,(2)\,10^{-24}}{4\pi\,\frac{10^{-9}}{36\pi}\,(\mathbf{5.7})\,(\mathbf{100})(\mathbf{10})^{-4}}\,\mathbf{a_r}$$

$$= -1.263\,\mathbf{a_r} \quad [\mathrm{pN}]$$

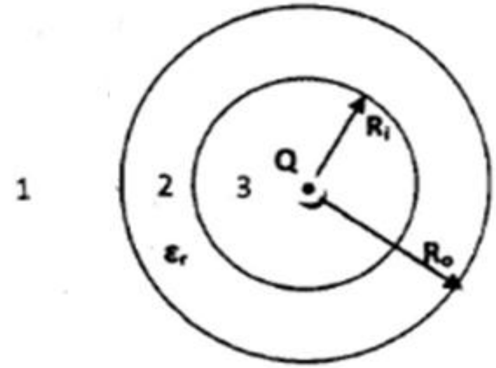
Polarization in Dielectrics

EXAMPLE 3.3

A dielectric spherical shell having a dielectric constant $\varepsilon_r(r, \theta, \varphi)$ where $R_i \leq r \leq R_0$, A positive charge Q_1 is located at the center of the shell.

Find **E**, **D**, **P**, and V, in the regions:

a) $R_0 \leq r < \infty$, b) $R_i \leq r < R_0$, c) $0 \leq r < R_i$



Polarization in Dielectrics

SOLUTION

Use Gauss theorem to find D at any distance r:

$$\oint \mathbf{D.dS} = D * 4\pi r^2 \rightarrow D(r) = \frac{Q}{4\pi r^2}, \quad 0 < r < \infty$$

Region 1	Region 2	Region 3
$\mathbf{R_0 \leq r < \infty}$	$\mathbf{R_i \leq r < R_0}$	$\mathbf{0 \leq r < R_i}$
$\mathbf{E_1 = \frac{D}{\varepsilon_0}}$	$\mathbf{E_2 = \frac{D}{\varepsilon_0 \varepsilon_r}}$	$\mathbf{E_3 = \frac{D}{\varepsilon_0}}$
$\mathbf{P_1 = D - \varepsilon_0 E_1 = 0}$	$\mathbf{P_2 = D - \varepsilon_0 E_2 = \varepsilon_0 E_2 (\varepsilon_r - 1)}$	$\mathbf{P_3 = D - \varepsilon_0 E_3 = 0}$
$v_1 = KQ \frac{1}{R_0}$	$v_2 = KQ\left[\frac{1}{R_0} + \frac{1}{\varepsilon_r}\left(\frac{1}{R_i} - \frac{1}{R_0}\right)\right]$	$V_3 = KQ\left[\frac{1}{R_0} + \frac{1}{\varepsilon_r}\left(\frac{1}{R_i} - \frac{1}{R_0}\right) + \left(\frac{1}{R} - \frac{1}{R_i}\right)\right]$

Polarization in Dielectrics

Note: Apply the voltage Law to get the potential :

$$v_1 = -\int_{\infty}^{R_0} \mathbf{E}.\,\boldsymbol{dl} = -\int_{\infty}^{R_0} KQ\frac{1}{r^2}dr = KQ\left(\frac{1}{R_0} - \frac{1}{\infty}\right) = KQ\frac{1}{R_0}$$

$$v_2 = -\int_{\infty}^{R_o} KQ\frac{1}{r^2}dr - \int_{R_0}^{R_i} KQ\frac{1}{\varepsilon_r r^2}dr = KQ\left(\frac{1}{R_0} + \frac{1}{\varepsilon_r}\left(\frac{1}{R_i} - \frac{1}{R_0}\right)\right)$$

$$v_3 = -\int_{\infty}^{R_o} KQ\frac{1}{r^2}dr - \int_{R_0}^{R_i} KQ\frac{1}{\varepsilon_r r^2}dr - \int_{R_i}^{R} KQ\frac{1}{r^2}dr$$

$$= KQ\left(\frac{1}{R_0} + \frac{1}{\varepsilon_r}\left(\frac{1}{R_i} - \frac{1}{R_0}\right) + \left(\frac{1}{R} - \frac{1}{R_i}\right)\right)$$

Boundary Conditions

So far, we have considered the existence of the electric field in a homogeneous medium.

If the field exists in a region consisting of two different media, boundary conditions can be used to determine the field on one side of the boundary if the field on the other side is known.

Two cases will be introduced:

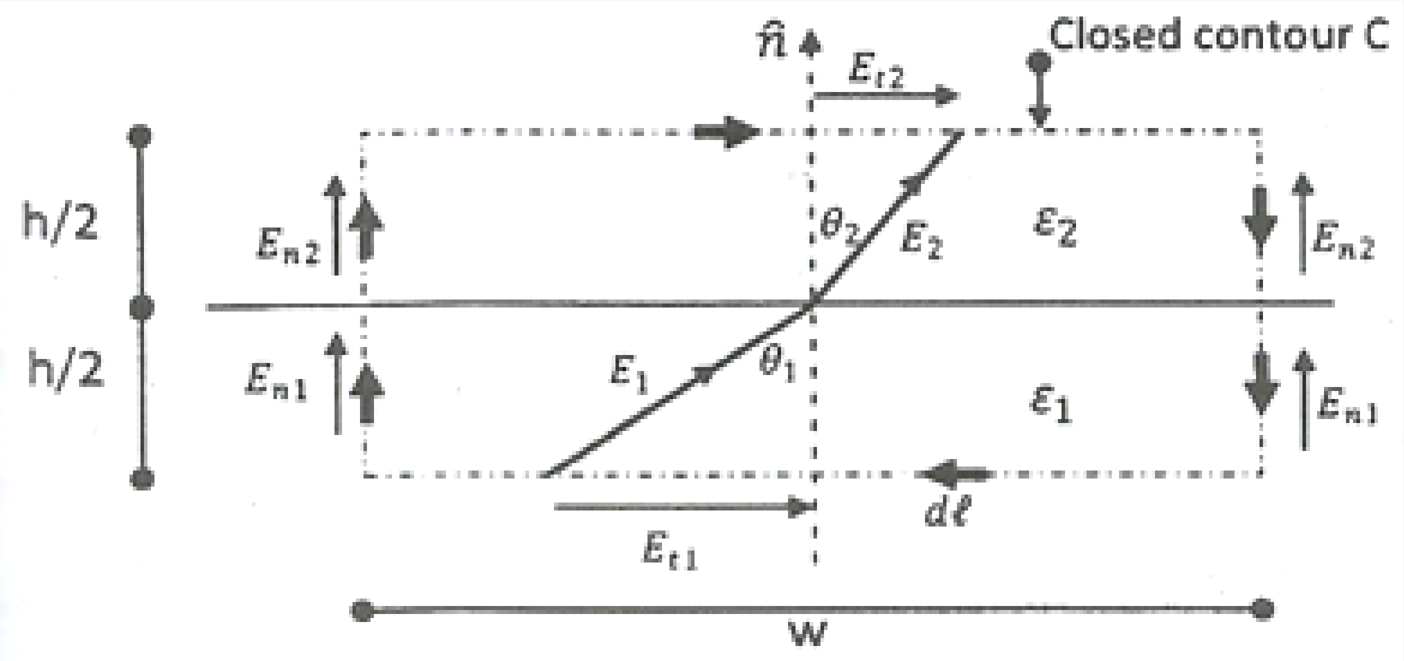
The boundary conditions at an interface separating a dielectric (ε_{r1}) and another dielectric (ε_{r2}).

The boundary conditions at an interface separating two conducting materials given by (ε_1, σ_1 and ε_2, σ_2) .

Boundary Conditions of 2 Dielectric Materials

Assuming two mediums ε_1 , ε_2 separated by the boundary surface S and having $\mathbf{E}_1$, $\mathbf{E}_2$ respectively .

Assuming one of them is known and the other need to be determined



Boundary Conditions of 2 Dielectric Materials

To find relation for tangential fields apply voltage's law over the shown rectangular path

$$\oint \mathbf{E}.\,\mathbf{d}\boldsymbol{l} = 0 = \mathrm{E}_{2t}\,\mathrm{W} - \mathrm{E}_{1t}\,\mathrm{W} = 0$$

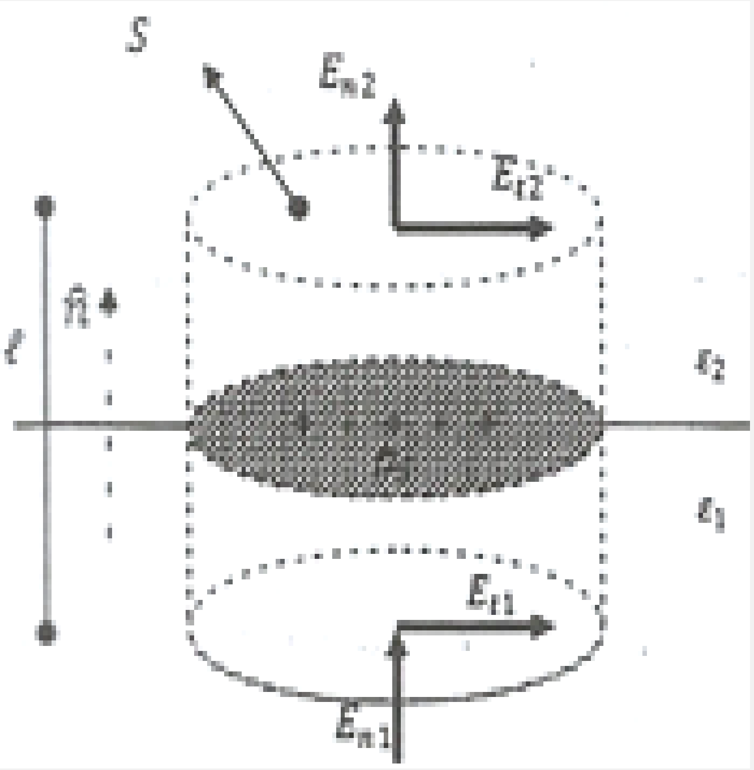
$$\mathbf{E_{2t}} = \mathbf{E_{1t}} \quad , \quad \frac{\mathbf{D_{2t}}}{\boldsymbol{\varepsilon}_2} = \frac{\mathbf{D_{1t}}}{\boldsymbol{\varepsilon}_1}$$

E is continuous across the surface, **D** is discontinuous across the surface.

Note the integration over the two sides normal to the boundary are zero since h is too small, h → o.

Boundary Conditions of 2 Dielectric Materials

To find relation for Normal fields apply Gauss's law over a cylindrical surface of axis Normal to the boundary as in Figure



Boundary Conditions of 2 Dielectric Materials

$$\oint_S \mathbf{D} \,.\, \mathbf{ds} = \mathrm{Q}_{\text{in}}$$

$$-\mathbf{D_{1n}\Delta s} + \mathbf{D_{2n}\Delta s} = \mathbf{Q}_{\mathbf{in}} = \boldsymbol{\rho_s}\, \mathbf{\Delta s}$$

$$\therefore \boldsymbol{\varepsilon_2}\, \mathbf{E_{2n}} = \boldsymbol{\varepsilon_1}\, \mathbf{E_{1n}} + \boldsymbol{\rho_s}$$

When ρ_s =0

$$\mathbf{D_{1n}} = \mathbf{D_{2n}}$$

To find angle of incidence θ_1 and transmit θ_2. Use the equations:

$$\boldsymbol{\theta_1} = \tan^{-1}\frac{E_{1t}}{E_{1n}}\left(\frac{D_{1t}}{D_{1n}}\right) \quad \theta_2 = \tan^{-1}\frac{E_{2t}}{E_{2n}}\left(\frac{D_{2t}}{D_{2n}}\right)$$

Note: the integration over the cylindrical side is zero because $\ell \rightarrow 0$

Boundary Conditions of 2 Conducting Materials

Assuming two mediums $\boldsymbol{\varepsilon_1}$, $\boldsymbol{\varepsilon_2}$ separated by the boundary surface S and having $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$ respectively .

Assuming The field of one of them is known and the other need to be determined.

For horizontal component of E:

Do closed loop integration along the boundary you get. Tangential components of electric field **E** at the boundary should be equal to:

$$\mathbf{E}_{1t} = \mathbf{E}_{2t}, \qquad \text{then,} \qquad \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$$

Boundary Conditions of 2 Conducting Materials

For vertical component of E:

Do a closed surface integration over a normal cylinder across the boundary, the vertical components of current and field density should be equal to:

$$J_{1n} = J_{2n}, \quad \text{then} \quad \sigma_1 E_{1n} = \sigma_2 E_{2n}$$

$$D_{1n} = D_{2n}, \quad \text{then} \quad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

Boundary Conditions of 2 Dielectric Materials

EXAMPLE 3.4

A homogenous dielectric (ε_r = 2.5) fills region 1 ($x < 0$) while region 2 ($x > 0$) is free space. **Find:**

- $\mathbf{D}_2$ if: $\mathbf{D}_1 = 12\,\mathbf{a}_x - 10\,\mathbf{a}_y + 4\,\mathbf{a}_z$ [nC/m²].
- The angle bet. $\mathbf{E}_2$ and the normal to the surface.

SOLUTION:

Since:

$$\mathbf{a}_n = \mathbf{a}_x$$

$$\mathbf{D}_{1n} = 12\,\mathbf{a}_x$$

$$\mathbf{D}_{1t} = -10\,\mathbf{a}_y + 4\,\mathbf{a}_z$$

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} = 12\,\mathbf{a}_x$$

Boundary Conditions of 2 Dielectric Materials

$$\mathbf{E}_{2t} = \mathbf{E}_{1t}$$

$$\frac{\mathbf{D}_{2t}}{\varepsilon_2} = \frac{\mathbf{D}_{1t}}{\varepsilon_1}$$

$$\mathbf{D}_{2t} = \frac{\varepsilon_2}{\varepsilon_1}\mathbf{D}_{1t}$$

$$= \frac{1}{2.5}(-10\,\mathbf{a}_y + 4\,\mathbf{a}_z) = -4\,\mathbf{a}_y + 1.6\,\mathbf{a}_z$$

$$\mathbf{D}_2 = \mathbf{D}_{2n} + \mathbf{D}_{2t}$$

$$= 12\,\mathbf{a}_x - 4\,\mathbf{a}_y + 1.6\,\mathbf{a}_z \quad \text{n}\,\mathbf{C}/\,\text{m}^2$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{4^2 + 1.6^2}}{12} = \frac{4.3081}{12} = 0.359$$

$$\theta_2 = 19.75^0$$