ELC 423 , ELCN323 & ELCn323 ELECTROMAGNETIC FIELD THEORY

Summer Semester 2021/2022

CHAPTER 2

Electrostatic Boundary Value Problems

Week 2, Lecture 2

Dr. Ibrahim Amin

GRADES

Course Codes: ELC 423, ELCN323, ELCn323

➢ **Mid Term Exam**	**10 %**	**20 %**
➢ **Semester work**	**20 %**	**40 %**
❖ **4 Exercises**	**2 % each**	**5 % each**
❖ **4 Quizzes**	**3 % each**	**5 % each**
➢ **Final Term Exam**	**70 %**	**40%**

COURSE PLAN

Weeks	Topics	Assessments	Grades
Week 1 Date 26/07/2022	CHAPTER 1 Electrostatic Field in Free Space Relevant Mathematics Coulomb's law and field intensity Electric potential Electric field due to continuous charge distribution Gauss's law and its applications Energy density in electrostatic fields Solved Examples	Quiz (1) Exercise (1)	3% or 5% 2% or 5%

COURSE PLAN

Week 2 Date **2/08/2022**	**CHAPTER 2** **Electrostatic Boundary Value Problems** **Image method in the case of:** **A charge in front of an Infinite grounded surface** **A charge in front of two orthogonal infinite grounded surfaces** **A Charge in front of grounded sphere**	**Quiz (2)**	**3 % or 5 %**
	Curl & Divergence of E **Laplace & Poisson's equations** **Solution of Laplace's equation in Cartesian , Cylindrical and Spherical Coordinates** **Solved Examples** **Revision & Mid Term Preparation**	**Exercise (2)**	**2 % or 5 %**

COURSR PLAN

Week 3 Date 09/08/2022	**Mid Term Exam** **CHAPTER 3** **Electrostatic Field in Material Space** Properties of materials Convection and conduction currents Electric field in conductors Resistance, capacitance calculation Boundary conditions of 2 dielectric materials and 2 conducting materials Solved Examples	Covering weeks (1,2)	10 % or 20 %

COURSE PLAN

Week 4 Date 16/08/2022	**CHAPTER 4** **Magneto Static Fields** Static fields analogy Ampere circuital's law Biot Savart's law Curl & Divergence Magnetic vector of potential $\mathbf{A}$ Boundary condition between two magnetic media The magnetic force Solved Examples	Quiz(3) Exercise (3)	3 % or 5 % 2 % or 5 %

COURSE PLAN

Week 5 Date 23/08/2022	CHAPTER 5 Time Varying Field Maxwell’s equations solution Plane wave equation Velocity of electromagnetic wave in free space Penetration depth in conductors Solved Examples	Quiz (4) Exercise (4)	3 % or 5 % 2 % or 5 %

COURSE PLAN

Week 6 Date 30/08/2022	Final Exam		70 % or 40 %

References

- 🕮 Sadiku, M. N. (1995), Elements of Electromagnetics, USA, Oxford University Press Inc.
- 🕮 Hayt, . W. H., Buck, J. A, (2000) Engineering Electromagnetics, USA, McGraw-Hill.
- 🕮 Nannapneni, R. N. (1997), Elements of Engineering Electromagnetic, USA, Prentice Hall, Inc.
- 🕮 Cheng, D. K. (1989), Field and Wave Electromagnetics, USA, Addison Wesley Publishing Company Inc.
- 🕮 El Wakil, M. M. (2012), Electromagnetic Field Theory Notes, Cairo, Modern Academy for Engineering and Technology Press.

Lecture's Topics

- **Chapter 2 Electrostatic Boundary Value Problems**
  - Boundary Value Problems Classification
  - Image Method
  - Case-Study
    - **Case 1** A charge against one conducting surface
    - **Case 2** A charge against two conducting surfaces
    - **Case 3** A charge against a conducting Sphere
- Curl and Divergence of Electric Field
- Poisson's Equation
- Laplace's Equation
- Finding E & V using Laplace and Poisson's Equations in Cartesian, cylindrical and Spherical Coordinates

Electrostatic Problems Classification

In Chapter 1, **E** was calculated as follows:

- For a given static charge distribution:

Coulomb's law or Gauss's law were used

$$\mathbf{E} = \frac{K\,Q}{|\mathbf{R}|^2}\,\mathbf{R} \qquad Q_{in} = \oint \mathbf{D}\,.\,ds$$

- For a given potential **in one direction** :

$$\mathbf{E} = -\,\nabla V$$

Electrostatic Boundary Value Problems

In this chapter, **PRACTICAL** electrostatic problems will be considered.

PRACTICAL means that, only electrostatic condition (**charge** or potential) at some BOUNDARIES are known.

Laplace's or Poisson's equations or **Image method** can be used to find **E** THROUGH OUT THE REGION.

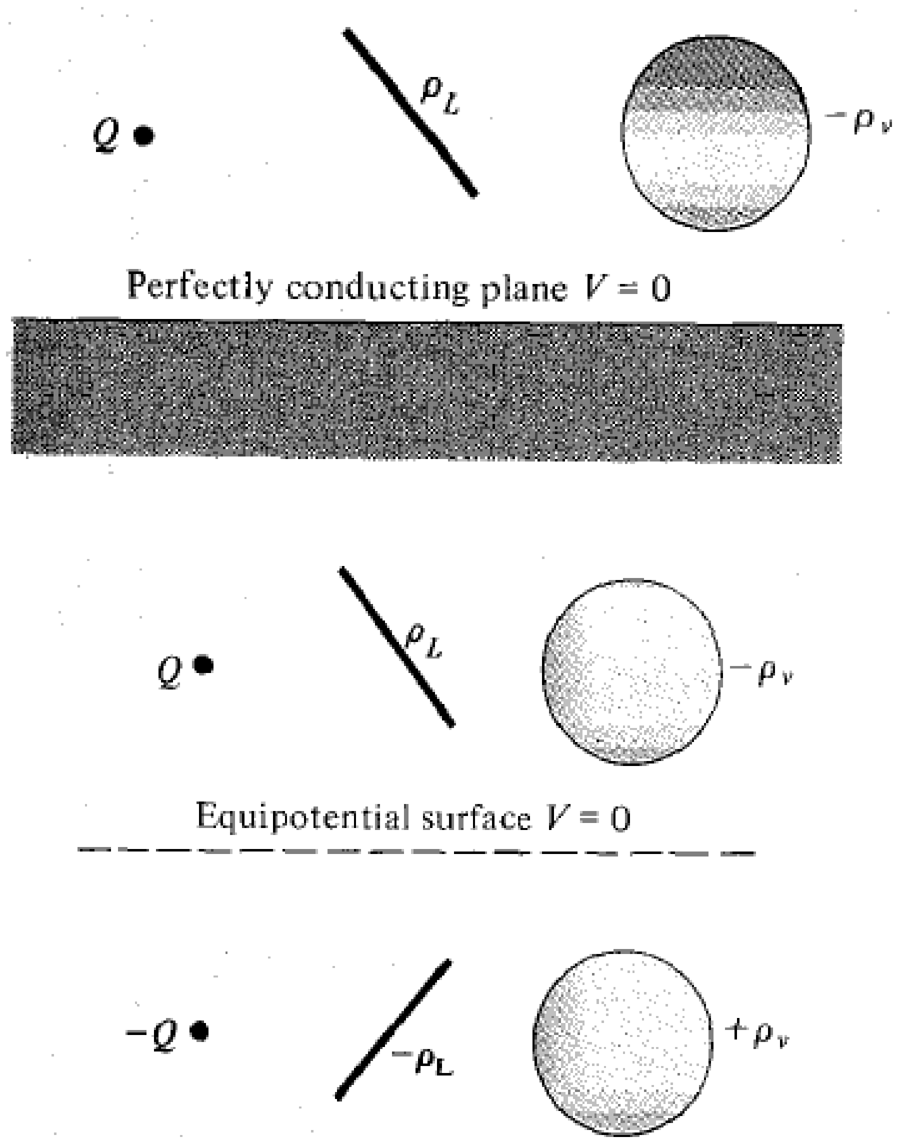
Electrostatic Boundary Value Problems Using Image Method

The image method is used to determine V, E, D, and ρ_s due to static charge distribution in the presence of conductors.

Using this method, we can find E using Coulomb's law by replacing the conducting surface by an equivalent image charge distribution.

By this method, we avoid solving Poisson's or Laplace's equations but rather utilize the fact that a conducting surface is an equipotential.

Electrostatic Boundary Value Problems Using Image Method



Electrostatic Boundary Value Problems Using Image Method

Three cases will be investigated:

A case of a charge in front of an infinite grounded conducting surface.

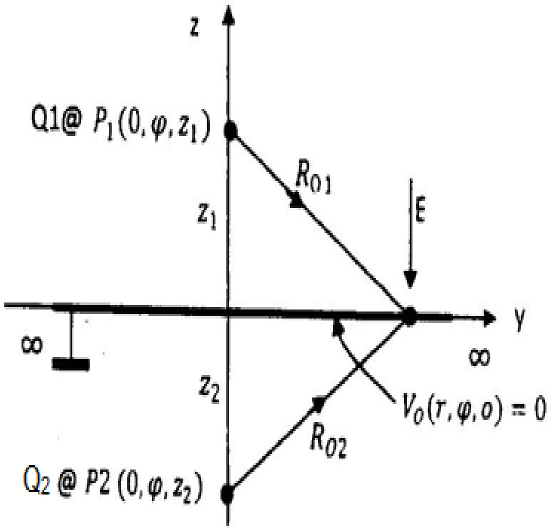
A case of a charge in front of two orthogonal infinite grounded conducting surfaces.

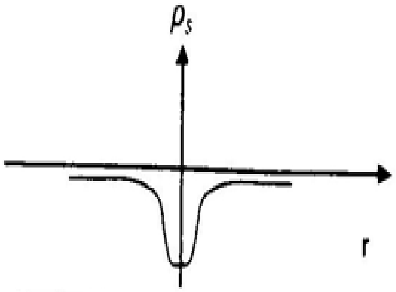
A case of a charge in front of grounded conducting sphere.

Electrostatic Boundary Value Problems Using Image Method

Image Method

A case of a static charge or a line charge or volume charge distribution in front of an infinite grounded surface.





The condition on the image charge should be:

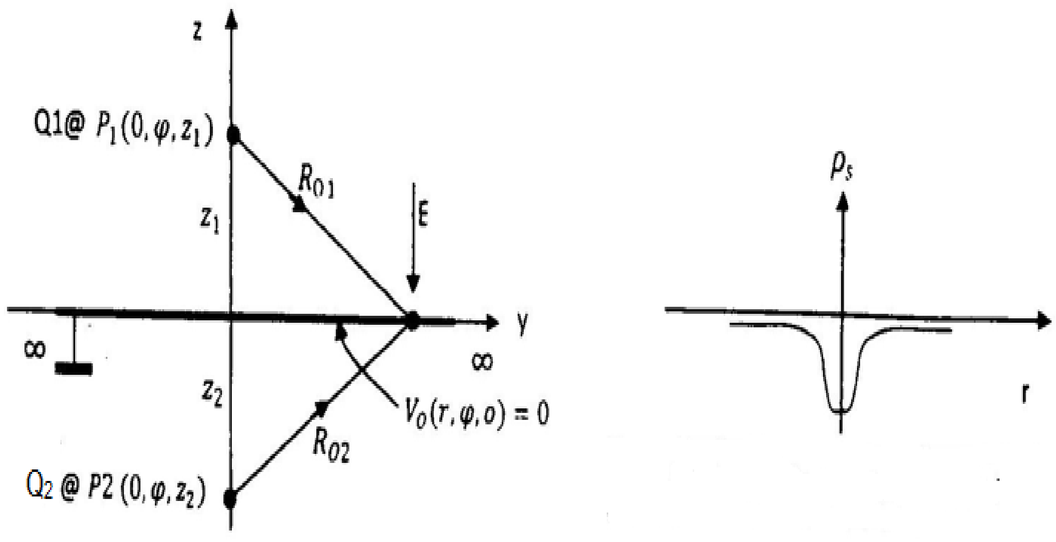
$$Q = Q_1 = -Q_2 \qquad \rho_l = -\rho_l \qquad \rho_v = -\rho_v$$

$$R = |R_{01}| = |R_{02}|,\ z = z_2 = -z_2$$

Electrostatic Boundary Value Problems Using Image Method

Case 1: A Charge in Front of an Infinite Grounded Surface

Consider a charge Q_1 at point $P_1 = (0, \varphi, z_1)$ over an infinite horizontal conducting surface. At the conducting surface points, potential $V(r, \varphi, 0) = 0$ where $0 \le r \le \infty$ and $0 \le \varphi \le 2\pi$.



Electrostatic Boundary Value Problems Using Image Method

At region of definition where $z \geq 0$, the conducting surface can be replaced by an equivalent charge to get the same field at the region of definition.

The image charge Q_2 $(0, \varphi, z_2)$ should satisfy the boundary conditions at the surface.

$$V(r, \varphi, 0) = 0 = V_{01} + V_{02} = \frac{kQ_1}{|\mathbf{R}_{01}|} + \frac{kQ_2}{|\mathbf{R}_{02}|} = 0$$

So the condition on image charge should be:

$$\therefore Q = Q_1 = -Q_2, \quad R = |R_{01}| = |R_{02}|, \quad z = z_2 = -z_2$$

Electrostatic Boundary Value Problems Using Image Method

So the field is given by the real charge Q and the image charge – Q in the region of definition $z \geq 0$.

Note that the field $\mathbf{E}(r, \varphi, z) = 0$ for $z \leq 0$.

Electric field intensity at the surface $P_0\,(r, \varphi, 0)$

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{KQ_1}{|\mathbf{R}_{01}|^3}(\mathbf{r} - \mathbf{z}) + \frac{-KQ_1}{|\mathbf{R}_{02}|^3}(\mathbf{r} + \mathbf{z})$$

$$= \frac{KQ}{R^3}(-2\mathbf{z}) = -\,2\,K\,Q\,\frac{z}{(z^2 + y^2)^{\frac{3}{2}}}\,.\,(\hat{z})$$

Note: **E** at any point $(r, \varphi, 0)$ on the conducting surface is normal to the surface.

Electrostatic Boundary Value Problems Using Image Method

Electric flux density

$$\mathbf{D} = \varepsilon_0 \, \mathbf{E} = \frac{-2\,\varepsilon_0\, K\, Q\, z}{(z^2 + y^2)^{\frac{3}{2}}}(\hat{z})$$

Surface charge density

$$\rho_s\,(r, \varphi, 0) = \mathbf{D} = \varepsilon_0 \, \mathbf{E} = \frac{-2\;\varepsilon_0\, K\, Q\, z}{(z^2 + y^2)^{\frac{3}{2}}}$$

Force induced on the charge due to the surface:

$$\mathbf{F} = \frac{K\, Q_1 Q_2}{R^3}\mathbf{R} = \frac{K\, Q^2}{(-2z)^3}\;(-2\mathbf{z})\quad \text{atractive force}$$

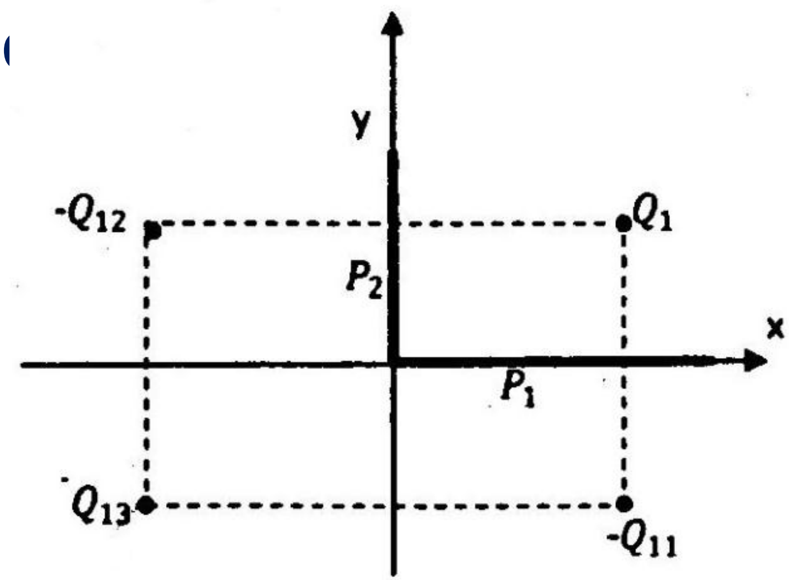
Total charge induced on the conducting surface.

$$Q = \int_{\varphi=0}^{2\pi}\int_{r=0}^{\infty}\rho_s\; r\, dr\, d\varphi = -Q_1$$

See solved problem SP 2.1, page 40.

Image Method

A case of a charge or a line charge distribution or volume charge distribtion in front of two orthogonal infinite ground



The condition on the image charges should be:

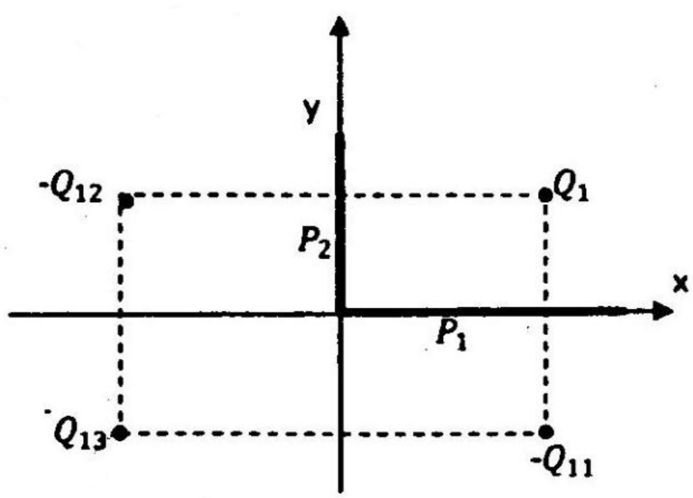
$$\mathbf{Q = Q_1\,(\rho_l) = -\,Q_{12}\,(-\rho_l) = Q_{13}\,(\rho_l) = -\,Q_{14}\,(\rho_l)}$$

Are equi-distance from the two surfaces

Electrostatic Boundary Value Problems Using Image Method

Case 2: A Charge in Front of two Orthogonal Infinite Grounded Surfaces

Consider a charge Q_1 (x_1, y_1, z_1) in the front of two orthogonal planes P_1 $(x, 0, z)$ and P_2 $(0, y, z)$. The image charges and its locations are chosen symmetrical around the planes and opposite sign to satisfy condition of zero voltage on the two surfaces.



Electrostatic Boundary Value Problems Using Image Method

Three charges are chosen to replace the effect of the two surfaces on the field such that:

$V_1(x, 0, z) = 0$ for the horizontal surface

$V_2(0, y, z) = 0$ for the vertical surface.

$$V_1(x, 0, z) = K\,Q\left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} + \frac{1}{R_4}\right)$$

$$V_2(0, y, z) = K\,Q\left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} + \frac{1}{R_4}\right)$$

Electrostatic Boundary Value Problems Using Image Method

Field at the surface:

The electric field at any point on the two surfaces is normal to each surface. E is always normal to the metal surfaces (Why). The tangent components = 0).

The induced force on the charge:

The force induced on the real charge due to the conducting plates is the resultant forces between image charges and the original charge. In that case three forces should be added.

$$\mathbf{F} = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14}$$

SOLVED EXAMPLES

EXAMPLE 2.1

A charge Q_1 = 2 [nC], placed at point P_1 (2, 1, 0) in front of two grounded infinite conducting planes: vertical P_2 (0, y, z) and horizontal P_3 (x, 0, z). Find the potential at observation point P_0 (5, 5, 0).

SOLUTION

Use the image method to replace the two planes:

The real charge at Q_1(2, 1, 0) = 2 [nC]. First image at Q_2 (2, -1, 0) = -2 [nC].
Second image at Q_3 (-2, 1, 0) = -2 [nC]. Third image at Q_4 (-2, -1, 0) = -2 [nC].

SOLVED EXAMPLE

The potential at any point:

$$V(x, y, z) = V_1 + V_2 + V_3 + V_4$$

$$= \frac{K}{R_1} Q_1 + \frac{K}{R_2} Q_2 + \frac{K}{R_3} Q_3 + \frac{K}{R_4} Q_4$$

$$= K\, Q \left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} + \frac{1}{R_4}\right)$$

$R_{01} = (5, 5, 0) – (2, 1, 0) = (3, 4, 0)$, thus $R_1 = \sqrt{9 + 16} = \sqrt{25}$

$R_{02} = (5, 5, 0) – (2, -1, 0) = (3, 6, 0)$, thus $R_1 = \sqrt{9 + 36} = \sqrt{45}$

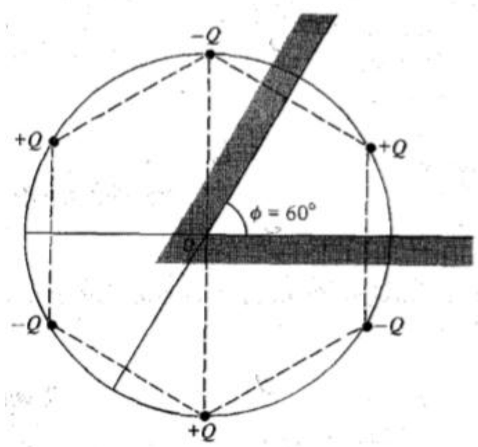
$R_{03} = (5, 5, 0) – (-2, 1, 0) = (7, 4, 0)$, thus $R_1 = \sqrt{49 + 16} = \sqrt{65}$

$R_{04} = (5, 5, 0) – (-2, -1, 0) = (3, 4, 0)$, thus$R_1 = \sqrt{49 + 36} = \sqrt{85}$

Electrostatic Boundary Value Problems Using Image Method

In general, when the method of images is used for a system consisting of a point charge between two semi-infinite conducting planes inclined at an angle (ϕ in degrees), the number of images is given by:

$$N = \left(\frac{360^0}{\phi} - 1\right)$$



The charge and its images all lie on a circle.

Electrostatic Boundary Value Problems Using Image Method

Electric Field in case of point charge:

$$\mathbf{E} = \frac{K\,Q}{R^2}\hat{\mathbf{R}} = \frac{K\,Q}{R^3}\,\mathbf{R} \qquad [N/C] \text{ or } [v/m] \qquad R = |\mathbf{R}|$$

Potential in case of point charge:

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{K\,Q}{r} = \frac{K\,Q}{|\mathbf{R}|} \qquad r = |\mathbf{R}|$$

Electric Field in case of infinite line charge dist.:

$$\mathbf{E} = \frac{K\,2\rho_l}{R}\hat{\mathbf{R}} = \frac{K\,2\rho_l}{R^2}\,\mathbf{R} \qquad [N/C] \text{ or } [v/m]$$

Potential in case of infinite line charge dist.:

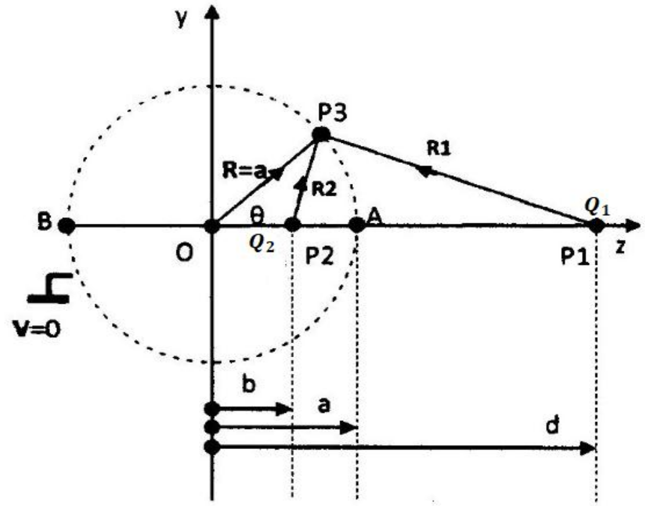
$$V = \frac{2\rho_l}{4\pi\varepsilon_0}\ln r = K\,2\rho_l \ln r$$

Electrostatic Boundary Value Problems Using Image Method

Case 3: A Charge in Front of Grounded Sphere

A charge Q_1 is placed at point P_1 (d, 0, φ) in front of metallic sphere of zero voltage. The field due to charge Q_1 and the spherical surface P_s (a, θ, φ) is given by image method.

Assume the image of the charge Q_1 on the surface is Q_2 located at point P_2 (b, 0, φ).



Electrostatic Boundary Value Problems Using Image Method

Applying the boundary condition at the surface of the sphere, V_s (a, θ, φ) = 0

$$V_s\,(a,\,\theta,\,\varphi) = \frac{K\,Q_1}{R_1} + \frac{K\,Q_2}{R_2} = 0$$

The boundary condition is satisfied when:

$$c = -\ \frac{Q_2}{Q_1} = \left|\frac{R_2}{R_{1)}}\right|$$

To solve these equations in two unknowns Q_2 and b, find R_1 and R_2 at two points on the sphere surface A (a, 0, φ), B (a, π, φ).

Electrostatic Boundary Value Problems Using Image Method

At point A $(a, 0, \varphi)$, $R_1 = (d - a)$, $R_2 = (a - b)$

At point B (a, π, φ), $R_1 = (d + a)$, $R_2 = (b + a)$

If two fractions $\frac{A}{B} = \frac{C}{D}$ then, $\frac{A}{B} = \frac{C}{D} = \frac{A+C}{B+D} = \frac{A-C}{B-D}$

$$c = -\frac{Q_2}{Q_1} = \left|\frac{R_2}{R_{1)}}\right| = \frac{a-b}{d-a} = \frac{a+b}{a+d} = \frac{a}{d} = \frac{b}{a}$$

$$Q_2 = -Q_1\frac{a}{d}, \quad b = \frac{a^2}{d}$$

The electric field can be proved that it is normal to the sphere surface, see solved problem SP2 .2, page 41.

Electrostatic Boundary Value Problems Using Image Method

The force acting on the charge due to the conducting sphere is the force between the real charge Q_1 and the image charge Q_2:

$$\mathbf{F} = \frac{K\, Q_1 Q_2}{|\mathbf{R}|^2}(\hat{z}) = \frac{-KQ^2}{|d-b|^2}(\hat{z})$$

Which is attractive force towards the center of the sphere.

SOLVED EXAMPLES

EXAMPLE 2.2

In the previous figure, if a = 0.5 [m], d= 1 [m], and Q_1 = - 50 [mC]. Find Q_2 and b.

SOLUTION:

$$c = -\frac{Q_2}{Q_1} = \frac{b}{a} = \frac{a}{d}$$

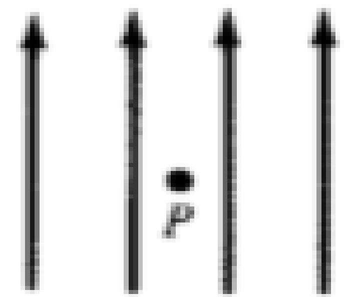
$$Q_2 = -\,Q_1\frac{a}{d} = -0.5\,(-50\,x10^{-3}) = 25\ [mC].$$

$$a^2 = bd, \qquad \text{thus: } b = 0.5^2/1 = 0.25\ [m]$$

Curl of the Electric Field

Curl of Any Vector

The curl of any vector **A** at a given point P is the rotational vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero.



(a) curl at P points out of the page; (b) curl at P is zero

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{max} \mathbf{a_n}$$

Its direction is normal to the area

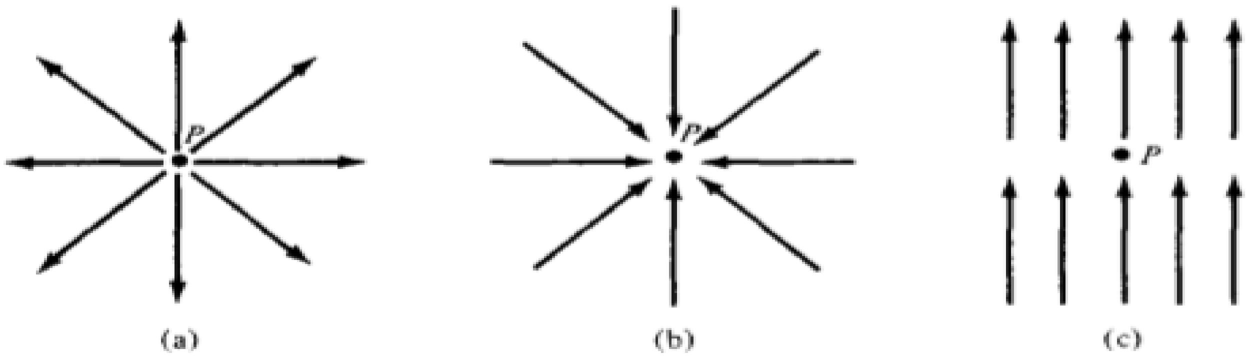
Curl of the Electric Field

Using the previous definition of the curl we get the very important result:

$$\nabla \times \mathbf{E} = \frac{\oint_c \mathbf{E} \,.\, \mathrm{dL}}{\Delta \mathrm{S}} = 0$$

Divergence of Any Vector

The divergence of any vector **A** at a given point P is the outward flux per unit volume as the volume shrinks about P.



positive divergence   negative divergence   zero divergence

$$\text{div}\,\mathbf{A} = \nabla.\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\,.d\mathbf{S}}{\Delta v}$$

where Δv is the volume enclosed by the closed surface S in which P is located.

Divergence of the Electric Field

Using the previous definition of the divergence we get the very important result:

$$\nabla \cdot \mathbf{D} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v_i} = \frac{Q}{\Delta v_i} = \rho_v$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

Δv_i is the incremental volume element enclosed by surface S

Laplace's and Poisson's Equations

$$\nabla . \mathbf{D} = \rho_v \quad , \quad \nabla . \mathbf{E} = \frac{\rho_v}{\varepsilon}$$

Poisson's Equation

Since: $\mathbf{E} = -\nabla V$ **T**herefore: $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$

Laplace's Equation

In free space, $\rho_v = 0$ **T**herefore: $\nabla^2 V = 0$

Laplace's and Poisson's Equations

When the electrostatic problem is described by potential distribution at some BOUNDARYS on a given surface:

Laplace equation $\nabla^2 \mathrm{V} = 0$

Poisson's equation $\nabla^2 \mathrm{V} = \rho_v$

should be:

solved to find potential V and field intensity **E** THROUGH OUT THE REGION.

Laplace Equation in Cartesian Coordinates

To solve any boundary value problem completely, you should have:

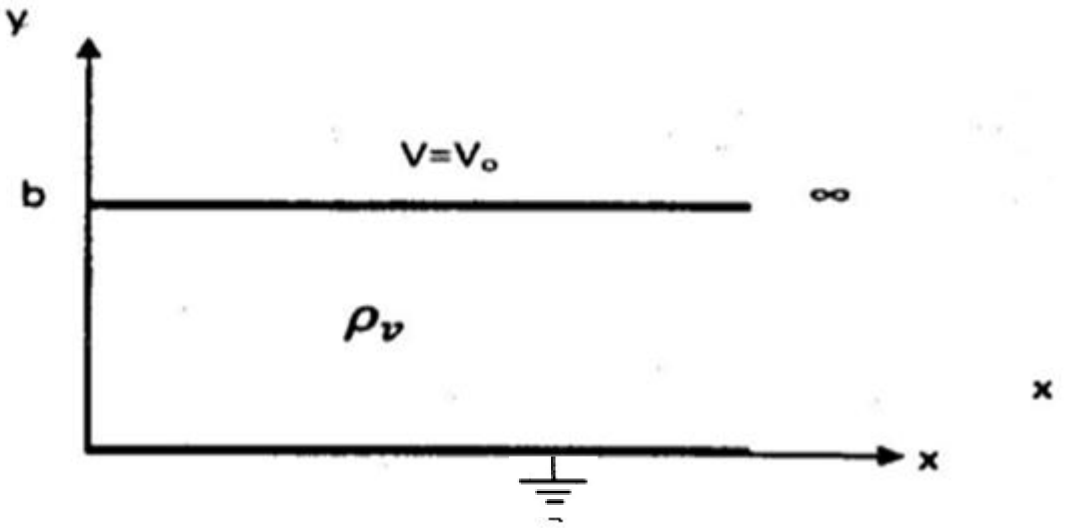
- The appropriate differential equation.
  - Laplace's Equation
  - Poisson's Equation
- The solution region.
- The prescribed boundary conditions.

Laplace Equation in Cartesian Coordinates

EXAMPLE 2.3

Application of Poisson equation

Consider the region confined between two conductors as shown in Figure. This region has constant volume charge density ρ_v. Determine the potential distribution in the space between the conductors.



Laplace Equation in Cartesian Coordinates

SOLUTION

At any point P(x, y, z) where $0 < y < b$ potential is given by Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

As V is independent of x and z, we have:

$$\frac{d^2 V}{dy^2} = -\frac{\rho_v}{\varepsilon}$$

Solution of this equation is:

$$\frac{dV}{dy} = -\frac{\rho_v}{\varepsilon} y + c_1, \quad V = -\frac{\rho_v}{2\,\varepsilon} y^2 + c_1 y + c_2$$

Laplace Equation in Cartesian Coordinates

Apply the 1st boundary condition: $V(0) = 0$

Thus: $c_2 = 0$ $\qquad V = -\dfrac{\rho_v}{2\,\varepsilon}\,y^2 + c_1 y$

Apply the 2nd boundary condition: $V(b) = V_0$

Thus: $V_0 = -\dfrac{\rho_v}{2\,\varepsilon}\,b^2 + c_1 b\,,\quad c_1 = \dfrac{1}{b}\left(V_0 + \dfrac{\rho_v}{2\,\varepsilon}b^2\right)$

Thus: $V = -\dfrac{\rho_v}{2\,\varepsilon}\,y^2 + \left(V_0 + \dfrac{\rho_v}{2\,\varepsilon}b^2\right)\dfrac{y}{b}$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dy} = \left(-\frac{\rho_v}{\varepsilon}\,y + \frac{\left(V_0 + \frac{\rho_v}{2\varepsilon}b^2\right)}{b}\right)\hat{Y}$$

Laplace Equation in Cylindrical Coordinates

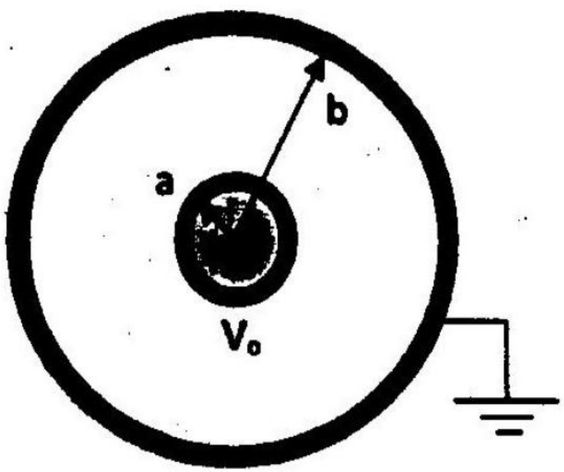
EXAMPLE 2.4

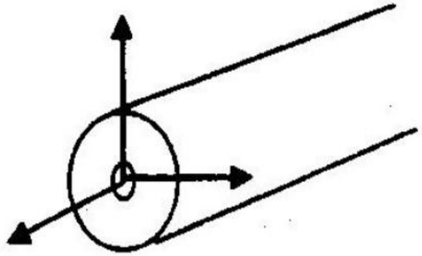
Consider a very long coaxial cable.

The inner conductor has a radius a and is maintained at a potential $V(a,\varphi,z) = V_0$.

The outer conductor has a potential $V(b,\varphi,z) = 0$.

Determine the potential distribution in the space between the two conductors.





Laplace Equation in Cylindrical Coordinates

SOLUTION

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial^2 V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

By symmetry the potential is independent of φ and z

V_φ and V_z are constants then γ_φ^2 and $\gamma_z^2 = 0$, apply constant condition we get;

$$\gamma_r{}^2 = 0$$

Thus:

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dV(r)}{\partial r}\right) = 0$$

$$r\,\frac{dV(r)}{dr} = c_1\,, \qquad \frac{dV(r)}{dr} = \frac{c_1}{r} \qquad V(r) = c_1 \ln r + c_2$$

Laplace Equation in Cylindrical Coordinates

Apply the first boundary condition:

$$V(\mathrm{b}) = 0,\ \mathrm{c}_2 = -\mathrm{c}_1 \ln \mathrm{b}$$

$$V(\mathrm{r}) = -\mathrm{c}_1 (\ln \mathrm{b} - \ln \mathrm{r}) = -\mathrm{c}_1 \ln \left(\frac{\mathrm{b}}{\mathrm{r}}\right)$$

Apply the second boundary condition:

$$V(\mathrm{a}) = \mathrm{V_o} = -\mathrm{c}_1 \ln \left(\frac{\mathrm{b}}{\mathrm{a}}\right), \quad \text{thus: } \mathrm{c}_1 = \frac{-\mathrm{V}_0}{\ln \left(\frac{\mathrm{b}}{\mathrm{a}}\right)}$$

$$V(\mathrm{r}) = \frac{\mathrm{V}_0}{\ln \left(\frac{\mathrm{b}}{\mathrm{a}}\right)} \ln \left(\frac{\mathrm{b}}{\mathrm{r}}\right)$$

Laplace Equation in Cylindrical Coordinates

$$V(\mathrm{r}) = \frac{V_0}{\ln\left(\frac{b}{a}\right)} \ln\left(\frac{b}{r}\right) = \frac{V_0}{\ln\left(\frac{b}{a}\right)} (\ln b - \ln r)$$

$$E_r = - \nabla V(\mathrm{r})$$

$$\frac{d}{dx} \ln U = \frac{1}{U} \cdot \frac{dU}{dx}$$

$$E_r = - \frac{dV}{dr} = - \frac{V_0}{\ln\left(\frac{b}{a}\right)} \left(- \frac{1}{r}\right)$$

$$= \frac{V_0}{r \ln\left(\frac{b}{a}\right)} \text{ v/m}$$

Laplace Equation in Spherical Coordinates

The Laplace operator in spherical coordinates is :

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\varphi}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

The constants equation should be:

$$\gamma_R^{\,2} + \gamma_\theta^{\,2} + \gamma_\varphi^{\,2} = 0$$

The separation of variables equations:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = \gamma_R^{\,2}$$

$$\frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = \gamma_\theta^{\,2}$$

$$\frac{1}{R^2\sin^2\varphi}\frac{\partial^2 V}{\partial\varphi^2} = \gamma_\varphi^{\,2}$$

Laplace Equation in Spherical Coordinates

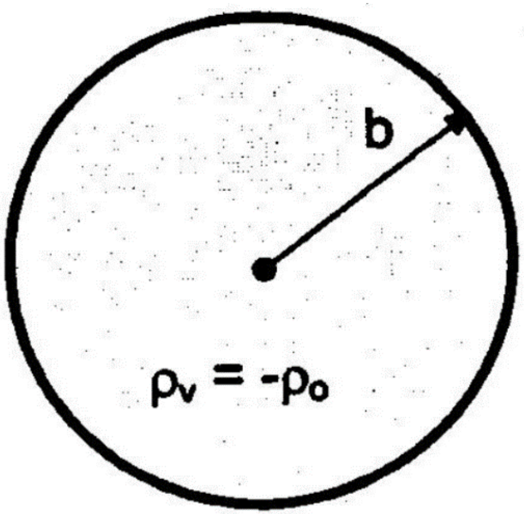
EXAMPLE 2.5

Application of Laplace and Poisson equations

Determine the Electric field both inside and outside a spherical cloud of electrons with a uniform volume charge density:

$\rho_v = -\rho_0$ for $0 \leq R \leq b$

$\rho_v = 0$ for $R > b$.



Laplace Equation in Spherical Coordinates

SOLUTION

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2 \sin^2\varphi}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

As the potential is independent of θ and φ

In region 1, where $R_1 \leq b$ i. e., inside the sphere:

$$\nabla^2 V(R_1) = \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dV}{dR}\right) = -\frac{\rho_v}{\varepsilon} = \frac{\rho_0}{\varepsilon}$$

$$R^2\frac{dV}{dR} = \int \frac{\rho_0}{\varepsilon} R^2\, dR + c_1 = \frac{\rho_0}{\varepsilon}\left(\frac{R^3}{3}\right) + c_1$$

Laplace Equation in Spherical Coordinates

Then the solution of this equation is:

$$\frac{dV}{dR} = \frac{\rho_0}{\varepsilon}\left(\frac{R}{3}\right) + c_1\frac{1}{R^2}$$

$$V = \frac{\rho_0}{\varepsilon}\left(\frac{R^2}{6}\right) - \frac{c_1}{R} + c_2$$

Apply boundary condition at R = 0, potential should not be infinite so $c_1 = 0$

Then:

$$V(R_1) = \frac{\rho_0}{\varepsilon}\left(\frac{R^2}{6}\right) + c_2$$

$$\mathbf{E}(R_1) = -\nabla V = -\frac{dV}{dR} = -\frac{\rho_0}{\varepsilon}\left(\frac{R}{3}\right)$$

Laplace Equation in Spherical Coordinates

In region 2, where $R_2 \geq b$ i. e., outside the sphere:

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dV}{dR}\right) = 0$$

Thus: $R^2\frac{dV}{dR} = c_3$, $V(R_2) = c_3\frac{-1}{R} + c_4$

Boundary condition at V(∞) = 0, make $c_4 = 0$,

Thus: $V(R_2) = \frac{-c_3}{R}$, $\mathbf{E}(R_2) = -\nabla V = \frac{c_3}{R^2}$

To find constants c_2 and c_3, apply the boundary conditions at the surface of sphere where R = b.

Thus: $\mathbf{E}_1(b) = -\frac{\rho_0}{\varepsilon}\left(\frac{b}{3}\right)$, $\mathbf{E}_2(b) = \frac{c_3}{b^2}$

Laplace Equation in Spherical Coordinates

Thus:

$$c_3 = -\frac{\rho_0}{\varepsilon}\left(\frac{b^3}{3}\right)$$

$$V_1(b) = \frac{\rho_0}{\varepsilon}\left(\frac{b^2}{6}\right) + c_2, \qquad V_2(b) = \frac{-c_3}{b} = \frac{\rho_0}{\varepsilon}\left(\frac{b^2}{3}\right)$$

Thus:

$$c_2 = \frac{\rho_0}{\varepsilon}\left(\frac{b^2}{3} - \frac{b^2}{6}\right) = \frac{\rho_0}{\varepsilon}\left(\frac{b^2}{3}\right)$$

Substitute for c_2 and c_3, we get the final solution:

$$V_1(R) = \frac{\rho_0}{\varepsilon}\left(\frac{2\,b^2 + R^2}{6}\right) \qquad V_2(R) = \frac{\rho_0}{\varepsilon}\left(\frac{b^3}{3\,R}\right)$$

$$\mathbf{E}_1(R) = -\frac{\rho_0}{\varepsilon}\left(\frac{R}{3}\right) \qquad \mathbf{E}_2(R) = -\frac{\rho_0}{\varepsilon}\left(\frac{b^3}{3\,R^2}\right)$$

The solution is similar to the simple solution using Gauss's law.

Quiz 2

1- Two point charges of 5 nC and -20 nC are located at (-3, 2, 4) and (1, 0, 5) above a grounded conducting plane z = 2. Calculate **D** at (3,4,8) and (1, 1, -1).

2- A negative point charge $Q_1 = -100\ [\mathrm{mC}]$ is placed at a distance d = 50 [cm] in front of a grounded metallic sphere of radius a = 25 [cm]. **Find**:

- The value of its image Q_2 and its place with respect to the sphere center.
- The force acting on the charge Q_1 due to the conducting sphere.