ELC 423 , ELCN323 & ELCn323

ELECTROMAGNETIC FIELD THEORY

Summer Semester 2021/2022

CHAPTER 1

ELECTROSTATIC FIELD IN FREE SPACE

Week 1, Lecture 1

Dr. Ibrahim Amin

GRADES

Course Codes: ELC 423, ELCN323, ELCn323

➢ **Mid Term Exam**	**10 %**	**20 %**
➢ **Semester work**	**20 %**	**40 %**
❖ **4 Exercises**	**2 % each**	**5 % each**
❖ **4 Quizzes**	**3 % each**	**5 % each**
➢ **Final Term Exam**	**70 %**	**40%**

COURSE PLAN

Weeks	Topics	Assessments	Grades
Week 1 Date 26/07/2022	**CHAPTER 1** **Electrostatic Field in Free Space** Relevant Mathematics Coulomb's law and field intensity Electric potential Electric field due to continuous charge distribution Gauss's law and its applications Energy density in electrostatic fields Solved Examples	Quiz (1) Exercise (1)	3% or 5% 2% or 5%

COURSE PLAN

Week 2 Date **2/08/2022**	**CHAPTER 2** **Electrostatic Boundary Value Problems** **Image method in the case of:** **A charge in front of an Infinite grounded surface** **A charge in front of two orthogonal infinite grounded surfaces** **A Charge in front of grounded sphere** **Curl & Divergence of E** **Laplace & Poisson’s equations** **Solution of Laplace’s equation in Cartesian , Cylindrical and Spherical Coordinates** **Solved Examples** **Revision & Mid Term Preparation**	**Quiz (2)** **Exercise (2)**	**3 % or 5 %** **2 % or 5 %**

COURSR PLAN

Week 3 Date 09/08/2022	**Mid Term Exam** **CHAPTER 3** **Electrostatic Field in Material Space** Properties of materials Convection and conduction currents Electric field in conductors Resistance, capacitance calculation Boundary conditions of 2 dielectric materials and 2 conducting materials Solved Examples	Covering weeks (1,2)	10 % or 20 %

COURSE PLAN

Week 4 Date 16/08/2022	CHAPTER 4 Magneto Static Fields Static fields analogy Ampere circuital's law Biot Savart's law Curl & Divergence Magnetic vector of potential A Boundary condition between two magnetic media The magnetic force Solved Examples	Quiz(3) Exercise (3)	3 % or 5 % 2 % or 5 %

COURSE PLAN

Week 5 Date 23/08/2022	**CHAPTER 5** **Time Varying Field** **Maxwell's equations solution** **Plane wave equation** **Velocity of electromagnetic wave in free space** **Penetration depth in conductors** **Solved Examples**	Quiz (4) Exercise (4)	3 % or 5 % 2 % or 5 %

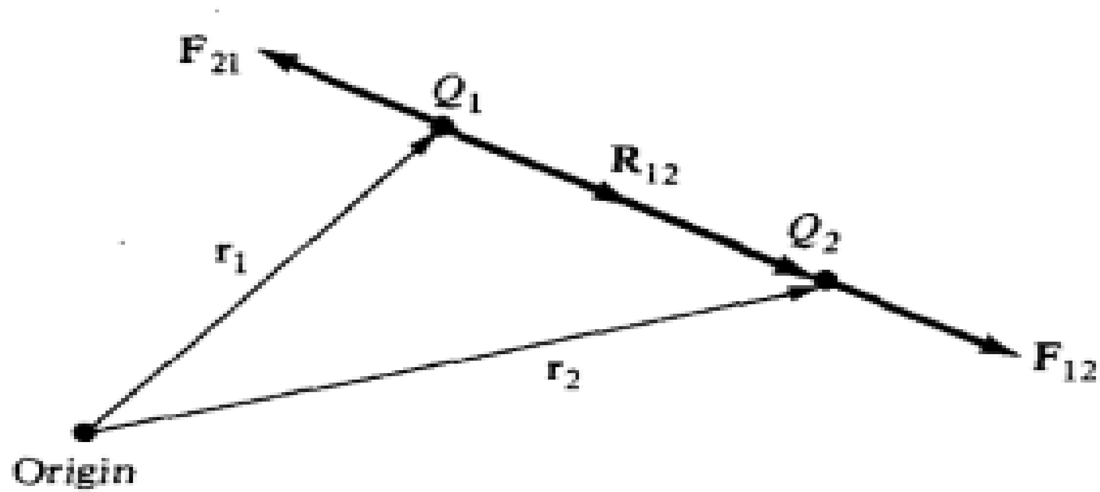
COURSE PLAN

Week 6 Date 30/08/2022	**Final Exam**		70 % or 40 %

References

- Sadiku, M. N. (1995), Elements of Electromagnetics, USA, Oxford University Press Inc.
- Hayt, . W. H., Buck, J. A, (2000) Engineering Electromagnetics, USA, McGraw-Hill.
- Nannapneni, R. N. (1997), Elements of Engineering Electromagnetic, USA, Prentice Hall, Inc.
- Cheng, D. K. (1989), Field and Wave Electromagnetics, USA, Addison Wesley Publishing Company Inc.
- El Wakil, M. M. (2012), Electromagnetic Field Theory Notes, Cairo, Modern Academy for Engineering and Technology Press.

COULOMB'S LAW OF FORCE



$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 . Q_2}{|\mathbf{R}_{12}|^2}\,\widehat{\mathbf{R}}_{12} \quad = K\,\frac{Q_1 . Q_2}{|\mathbf{R}_{12}|^3}\,\mathbf{R}_{12} \quad [\text{Newton}]$$

$$\mathbf{R}_{12} = (\,\mathbf{r}_2 - \,\mathbf{r}_1) = \,(\,P_2 - P_1)\ \text{(origin reference)}$$

COULOMB'S LAW OF ELCTRIC FIELD

$$\mathbf{E} = \frac{\mathbf{F}}{\mathrm{Q}}$$

$$\therefore\ \mathbf{E}_{12} = \frac{\mathrm{K}\,\mathrm{Q}_1}{|\mathbf{R}_{12}|^2}\mathbf{R}_{12} = \frac{\mathrm{K}\,\mathrm{Q}_1}{|\mathbf{R}_{12}|^3}\mathbf{R}_{12}$$ [N/C] or [v/m]

$$\mathbf{E}_2 = \frac{\mathrm{K}\,\mathrm{Q}}{|\mathbf{R}|^2}\mathbf{R} = \frac{\mathrm{K}\,\mathrm{Q}}{|\mathbf{R}|^3}\mathbf{R}$$ [N/C] or [v/m]

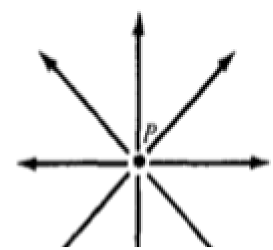
POTENIAL's LAW

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{K\,Q}{r} = \frac{K\,Q}{|\mathbf{R}|} \qquad r = |\mathbf{R}|$$

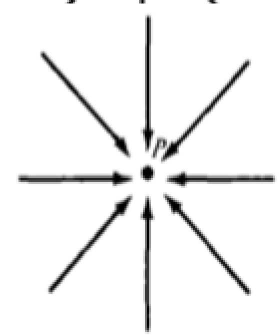
R … Position vector = observation point – source point (origin reference)

r … Position Vector length = $\sqrt{\sum[(\text{observation} - \text{source})\ \text{points componentes}]^2}$

Electric Field Lines Direction in the case of Point (Static) Charges



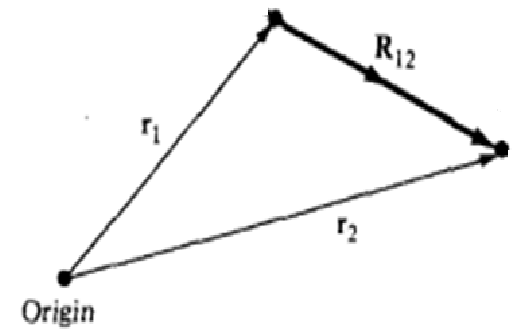
Outward in case of positive charge



Inward in case of negative charge

Lecture 1 Relevant Mathematics

The position Vector



The position vector $\mathbf{R} = (\mathbf{r_2} - \mathbf{r_1})$,

$\mathbf{r_2} = \mathrm{P_2}$ - (0, 0, 0) $\mathbf{r_1} = \mathrm{P_1}$ - (0, 0, 0)

$\widehat{\mathrm{R}}$= unit vector given by $= \frac{\mathbf{R}}{|\mathbf{R}|}$ dimensionless.

Differential Lengths

$$d\mathbf{l} = dx\,\mathbf{a_x} + dy\,\mathbf{a_y} + dz\,\mathbf{a_z}$$

$$d\mathbf{l} = d\rho\,\mathbf{a}_\rho + \rho d\varphi\, a_\varphi + dz\,\mathbf{a_z}$$

$$d\mathbf{l} = dr\,\mathbf{a_r} + r d\theta\,\mathbf{a}_\theta + r\,\sin_\theta d\varphi\, a_\varphi$$

Vector Multiplication

$A\,.\,B = AB\cos\theta_{AB}$ scaler

$A \times B = AB\sin\theta_{AB}\,\mathbf{an}$ vector

COULOMB'S LAW AND FIELD INTENSITY

one coulomb = 6 x 10^{18} electrons

electron charge e= -1.6019 x 10^{-19} C.

$\widehat{\mathbf{R}}_{12}$= unit vector given by $= \frac{\mathbf{R_{12}}}{|\mathbf{R_{12}}|}$ dimensionless.

The vector $\mathbf{R}_{12} = (\,\mathbf{r_2} - \mathbf{r_1})$, $[\text{meter}]$

$\mathbf{r_2} = \mathrm{P}_{\mathbf{2}}$ - (0, 0, 0) $\qquad$ $\mathbf{r_1} = \mathrm{P}_{\mathbf{1}}$ - (0, 0, 0)

The constant $\mathrm{K} = \frac{1}{4\pi\epsilon_0} = 9 \text{ x } 10^9 \;\; [\frac{\text{meter}}{\text{Farad}}]$

The notation F_{21} means that observation point at 2 and source point at 1.

$$\varepsilon_0 = \frac{10^{-4}}{36\,\pi} = 8.85\text{x}10^{-12} \quad \left[\frac{\text{farad}}{\text{meter}}\right]$$

= free space dielectric constant.

COULOMB'S LAW FOR N POINT CHARGE

If there are N charges Q_1, Q_2, …, Q_N located at points with position vectors r_1, r_2,. . ., r_N.

The resultant force **F** on the charge Q located at point r is the vector sum of the forces exerted on Q by each of these charges:

$$\mathbf{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{Q_k\,(\mathbf{r} - \mathbf{r_k})}{|\mathbf{r} - \mathbf{r_k}|^3}$$

The electric field intensity **E** on the charge Q located at that point is:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{Q_k\,(\mathbf{r} - \mathbf{r_k})}{|\mathbf{r} - \mathbf{r_k}|^3}$$

SOLVED EXAMPLES

EXAMPLE 1.1

Force between two discrete charges.

A point charge Q_1 (-3, 7, 4)= 2 x 10^{-3} [C], Q_2 (2, 4, -1) = - 5 x 10^{-3} [C].

Find the force on the charge Q_2.

SOLUTION

Using Coulomb's law: $\mathbf{F_{12}} = K \frac{Q_1.Q_2}{|\mathbf{R}_{12}|^3} \mathbf{R}_{12}$

$$\mathbf{R}_{12} = (\mathbf{P_2} - \mathbf{P_1}) = (2, 4, -1) - (-3, 7, 4) = (5, -3, -5)$$

$$\therefore |\mathbf{R}_{12}| = \sqrt{25 + 9 + 25} = \sqrt{59}$$

$$\mathbf{F}_{12} = \frac{9 \text{ x } 10^9 \text{x } (-10) \text{ x } 10^{-3} \text{ x } 10^{-3}}{\sqrt{59}^3} (5, -3, -5) \text{ [N]}$$

SOLVED EXAMPLES

EXAMPLE 1.2 Force due to three point charges.

Q_1 (-3, 7, -4) = 2 x10^{-3} [C], Q_2 (2, 4, -1) = -5 x10^{-3} [C], Q_3 (1, 3, 5) = 4 x10^{-3} [C].

Find the force acting on the charge Q_2 (2, 4, -1).

$$\mathbf{F_2} = \mathbf{F_{12}} + \mathbf{F_{32}} = K\,\frac{Q_2 Q_1}{|\mathbf{R}_{12}|^3}\,\mathbf{R}_{12} + K\,\frac{Q_2 Q_3}{|\mathbf{R}_{32}|^3}\,\mathbf{R}_{32}$$

$$\mathbf{R}_{12} = (\mathbf{P_2} - \mathbf{P_1}) = (2, 4, -1) - (-3, 7, -4) = (5, -3, 3)$$

$$\therefore |\mathbf{R}_{21}| = \sqrt{25 + 9 + 9} = \sqrt{43}$$

$$\mathbf{R}_{32} = (\mathbf{P_2} - \mathbf{P_3}) = (2, 4, -1) - (1, 3, 5) = (1, 1, -6)$$

$$\therefore |\mathbf{R}_{23}| = \sqrt{1 + 1 + 36} = \sqrt{38}$$

$$\mathbf{F_2} = (9 \text{ x } 10^9)(-5 \text{ x } 10^{-3})[\left(\frac{2 \text{ x } 10^{-3}}{\sqrt{43}^3}\right) \text{x } (5, -3,\ 3)$$

$$+ \left(\frac{4 \text{ x } 10^{-3}}{\sqrt{38}^3}\right) \text{x } (1,\ 1, -6)] \qquad [\text{N}]$$

SOLVED EXAMPLES

EXAMPLE 1.3

Two point charges -4 [μC] and 5 [μC] are located at (2,-1, 3) and (0, 4, -2), respectively. Find the potential at (1, 0, 1) assuming zero potential at infinity.

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_2|}$$

$$|\mathbf{r} - \mathbf{r}_1| = |(1,0,1) - (2,-1,3)| = (-1,1,-2) = \sqrt{6}$$

$$|\mathbf{r} - \mathbf{r}_2| = |(1,0,1) - (0,4,-2)| = (1,-4,3) = \sqrt{26}$$

$$V(1,0,1) = \frac{10^{-6}}{4\pi 10^{-9}} [\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}]$$

$$= 9 \text{ x } 10^{3} (-1.633 + 0.9806) = -5.872 \ \ [\text{kV}]$$

SOLVED EXAMPLES

EXAMPLE 1.4

Field and potential due to discrete charge:

Determine electric field **E** and potential V at the observation point P_0 (-0.2, 0.0, -1.3) due to charge Q_2 (0.2, 0.1, -1.5) = 5 [nC] in free space.

Apply the Coulomb's law for field and potential:

$$E_0 = K\frac{Q_2}{|\mathbf{R}|^3}\,\mathbf{R} \quad , \quad V_0 = K\frac{Q_2}{|\mathbf{R}|}$$

$$\mathbf{R} = (\mathbf{P}_0 - \mathbf{P}_2) = (-0.2, 0.0, -1.3) - (0.2, 0.1, -1.5)$$

$$= (-0.4, -0.1, 0.2)$$

$$\therefore |\mathbf{R}| = \sqrt{0.16 + 0.01 + 0.04} = \sqrt{0.21}$$

$$\mathbf{E}_0 = (9\text{x}10^9)\left(\frac{5\text{x}10^{-9}}{\sqrt{0.21}^3}\right)\text{x}(-0.4, -0.1, 0.2) \qquad [\frac{\text{V}}{\text{m}}]$$

$$V_0 = (9\text{x}10^9)\left(\frac{5\text{x}10^{-9}}{\sqrt{0.21}}\right) = \frac{45}{\sqrt{0.21}} \quad [\text{V}]$$

Lecture 1 Relevant Mathematics

Differential Areas

$$d\mathbf{s} = dx\,dy\,\mathbf{a}_z = dy\,dz\,\mathbf{a}_x = dz\,dx\,\mathbf{a}_y$$

$$d\mathbf{s} = \rho d\varphi\,dz\,\mathbf{a}_\rho = d\rho\,dz\,a_\varphi = \rho d\varphi\,d\rho\,\mathbf{a}_z$$

$$d\mathbf{s} = r^2 \sin\theta\,d\theta\,d\varphi\,\mathbf{a}_r = r\,dr\,d\theta\,a_\varphi = r^2 \sin\theta\,dr\,d\theta\,\mathbf{a}_\theta$$

Unit Vectors Relations

Cartesian vs. Cylindrical coordinates

$$\mathbf{a}_\rho = \cos\varphi\,\mathbf{a}_x + \sin\varphi\,\mathbf{a}_y$$

$$\mathbf{a}_\varphi = -\sin\varphi\,a_x + \cos\varphi\,\mathbf{a}_y$$

$$\mathbf{a}_z = \mathbf{a}_z\,.$$

Special Integrals

$$\int \frac{dx}{\sqrt{a^2+x^2}^{3}} = \frac{x}{a^2\sqrt{a^2+x^2}}$$

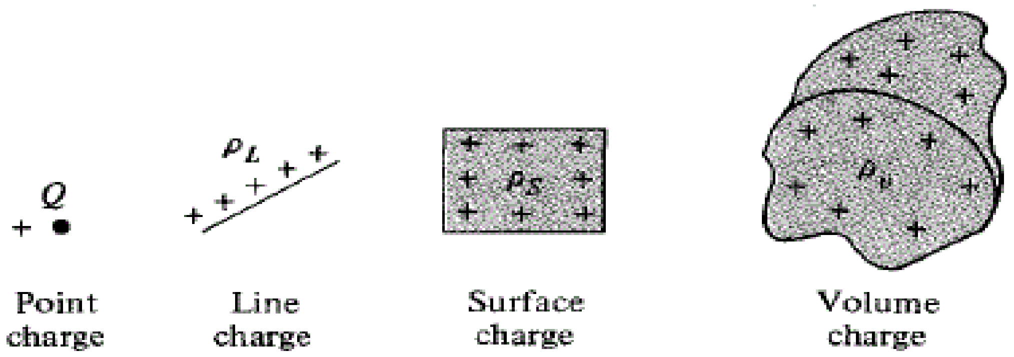
$$\int \frac{x\,dx}{\sqrt{a^2+x^2}^{3}} = \frac{-1}{\sqrt{a^2+x^2}}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln(x+\sqrt{x^2+a^2})$$

$$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}$$

ELECTRIC FIELD DUE TO CONTINUOUS CHARGE DISTRIBUTION

Various Charge Distributions



It is customary to denote:

The line charge density by: ρ_l [C/m]

The surface charge density by: ρ_s [C/m^2]

The volume charge density by: ρ_v [C/m^3]

ELECTRIC FIELD DUE TO CONTINUOUS CHARGE DISTRIBUTION

The electric field intensity due to each of charge distributions ρ_l, ρ_s, and ρ_v may be regarded as the summation of the field contributed by all point charges making up these distributions.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k}{|\mathbf{r} - \mathbf{r}_k|^2}\,\boldsymbol{a}_R$$

By replacing Q_k with charge element dQ = ρ_l dl or or ρ_s ds or ρ_v dv and integrating, you can get:

ELECTRIC FIELD DUE TO CONTINUOUS CHARGE DISTRIBUTION

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dQ}{\mathbf{R}^2}\,\mathbf{a}_R = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho_l\, dl}{\mathbf{R}^2}\,\mathbf{a}_R \\ \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho_s\, ds}{\mathbf{R}^2}\,\mathbf{a}_R \\ \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho_v\, dv}{\mathbf{R}^2}\,\mathbf{a}_R \end{cases}$$

Potential Due To Continuous Charge Distribution

For n point charges $Q_1, Q_2, .. Q_n$ located at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, .. \mathbf{r}_n$, then the potential at $\mathbf{r}$ is:

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\mathbf{r} - \mathbf{r}_k|}$$

For continuous charge distributions

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_l \, dl}{r}$$ for line charge

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_s \, ds}{r}$$ for surface charge

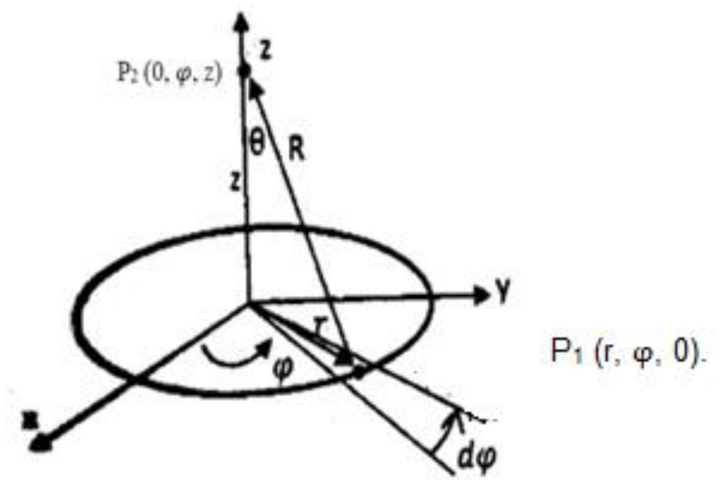
$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_v \, dv}{r}$$ for volume charge

Solved Examples

EXAMPLE 1.5

Field and potential due to charge distribution on a circular ring.

Find the electric field $\mathbf{E}_2$ and potential V_2 at the axis of a charged ring P_2 (0, φ, z). The ring located at z = 0, with radius = r, and charge density = ρ_l (r, φ, z) [C/m]



Solved Examples

SOLUTION

Observation point at P_2 (0, φ, z) on the z axis.

Line charge distribution on the ring is given by:

$$\rho_l\,(r, \varphi, 0)\left[\frac{C}{m}\right], \text{ where } 0 \le \varphi \le 2\pi$$

Apply Coulomb's law to find the electric field and potential at P_2:

$$\mathbf{E}_2 = K \int \frac{dQ}{|\mathbf{R}|^3}\,\mathbf{R} \qquad V_2 = K \int \frac{dQ}{|\mathbf{R}|}$$

$$dQ(r, \varphi, 0) = \rho_l\, dl = \rho_l\, r d\varphi$$

Solved Examples

$$\mathbf{R} = (\mathbf{P}_2 - \mathbf{P}_1) = (0, \varphi, z) - (r, \varphi, 0) = (-\mathbf{r}, 0, \mathbf{z}),$$

$$|\mathbf{R}| = \sqrt{r^2 + z^2}$$

$$\therefore\ \mathbf{E}_2 = K \int \frac{\rho_l\, r\, d\varphi}{\sqrt{r^2 + z^2}^{\,3}} (-\mathbf{r}, 0, \mathbf{z}) \qquad V_2 = K \int \frac{\rho_l\, r\, d\varphi}{\sqrt{r^2 + z^2}}$$

$$\mathbf{r} = r\,\hat{\mathbf{r}}$$

$$\mathbf{E}_2 = \cancel{-K \frac{\rho_l\,(r^2)}{\sqrt{r^2 + z^2}^{\,3}} \int_0^{2\pi} d\varphi\, \hat{r}} + K \frac{\rho_l\, r\, z}{(r^2 + z^2)^{3/2}} \int_0^{2\pi} d\varphi\, \hat{z}$$

Solved Examples

$$\mathbf{E}_2 = \cancel{-K\frac{\rho_l\,(r^2)}{\sqrt{r^2+z^2}^{\,3}}\int_0^{2\pi} d\varphi\,\hat{r}} + K\frac{\rho_l\; r\; z}{(r^2+z^2)^{3/2}}\int_0^{2\pi} d\varphi\;\hat{z}$$

r is independent of φ, but $\hat{r}$ is dependent on φ

$$\hat{r} = \cos\varphi\;\hat{x} + \sin\varphi\;\hat{y}$$

$$\hat{r} = \int_0^{2\pi}(\cos\varphi\;\hat{x} + \sin\varphi\;\hat{y})\,d\varphi = 0,$$

$\Rightarrow$ r component = 0

$$\mathbf{E}_2 = K\frac{\rho_l\,2\pi r}{(r^2+z^2)}\frac{z}{\sqrt{r^2+z^2}} = K\,.\,\frac{2\pi r\rho_l}{R^2}\,.\,\frac{z}{R} = K\,.\frac{Q}{R^2}\left(\frac{z}{R}\right)\hat{z}$$

Solved Examples

$$V_2 = \int_0^{2\pi} dV = K \int_0^{2\pi} \frac{\rho_l \; r \, d\varphi}{\sqrt{r^2 + z^2}} = K \, . \, \frac{2\pi r \rho_l}{\sqrt{r^2 + z^2}} = K \frac{Q}{R}$$

Note that:

the relation bet. unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$; $\mathbf{a}_\rho$, $\mathbf{a}_\varphi$, $\mathbf{a}_z$ are:

$$\mathbf{a}_\rho = \cos\varphi \, \mathbf{a}_x + \sin\varphi \, \mathbf{a}_y$$

$$\mathbf{a}_\varphi = -\sin\varphi \, a_x + \cos\varphi \, \mathbf{a}_y$$

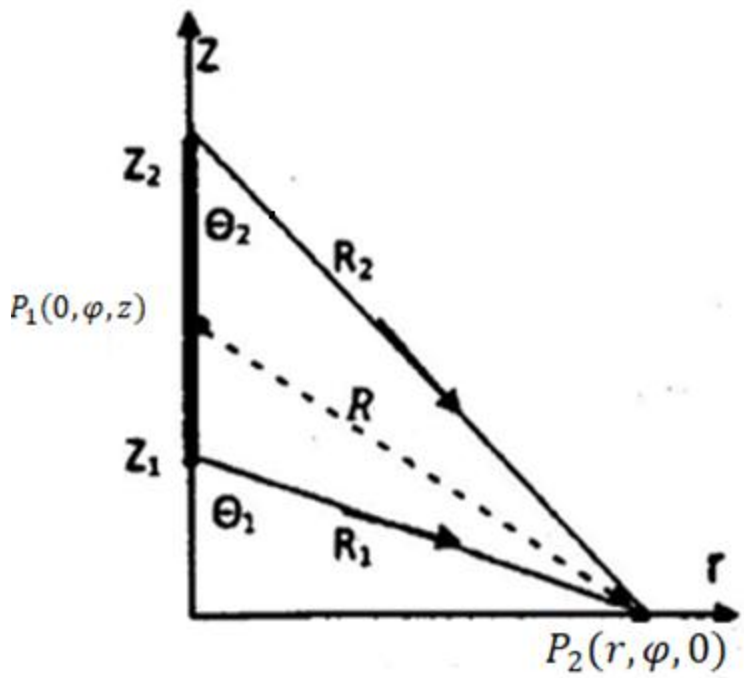
$$\mathbf{a}_z = \mathbf{a}_z \, .$$

Solved Examples

EXAMPLE 1.6

Field and potential due to line charge distribution

Find the electric field intensity $\mathbf{E}$ and the potential V at the point $P_2(r, \varphi, 0)$ due to a charged line segment of density: $\rho_l = \rho(0, \varphi, z)\ \left[\frac{c}{m}\right],\ z_1 \le z \le z_2,\ 0 \le \varphi \le 2\pi$.



Solved Examples

SOLUTION

The charge element on the line is $dQ = \rho\, dz$

Apply the coulomb's law for field and potential:

$$\mathbf{E}_2 = K \int \frac{dQ}{|\mathbf{R}|^3}\,\mathbf{R} \qquad V_2 = K \int \frac{dQ}{|\mathbf{R}|}$$

$$\mathbf{R} = (\mathbf{P}_2 - \mathbf{P}_1) = (r, \varphi, 0) - (0, \varphi, z) = (\mathbf{r}, 0, -\mathbf{z}) \ ,$$

$$|\mathbf{R}| = \sqrt{r^2 + z^2}$$

Solved Examples

$$\therefore\ \mathbf{E}_2 = K\int_{z_1}^{z_2}\frac{\rho\, dz}{\sqrt{(r^2+z^2)}^3}(\mathbf{r}, 0, -\mathbf{z}) \qquad V_2 = K\int_{z_1}^{z_2}\frac{\rho\, dz}{\sqrt{r^2+z^2}}$$

Do the integrals for the three axes:

$$\mathbf{E}_2 = K\,\rho\, r\int_{z_1}^{z_2}\frac{dz}{\sqrt{(r^2+z^2)}^3}\,\hat{r} - K\,\rho\int_{z_1}^{z_2}\frac{z\, dz}{\sqrt{(r^2+z^2)}^3}\,\hat{z}$$

$$= K\,\rho\, r\left[\frac{z}{r^2\sqrt{r^2+\ z^2}}\right]_{z_1}^{z_2}\hat{r} - K\,\rho\left[\frac{-1}{\sqrt{r^2+\ z^2}}\right]_{z_1}^{z_2}\hat{z}$$

$$= \frac{K\rho}{r}\left(\frac{z_2}{R_2} - \frac{z_1}{R_1}\right)\hat{r} - K\rho\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\hat{z}$$

Solved Examples

$$V_2 = K\int_{z_1}^{z_2} \frac{\rho\, dz}{\sqrt{r^2 + z^2}} = K\rho \int_{z_1}^{z_2} \frac{dz}{\sqrt{r^2 + z^2}}$$

$$= K\rho \left| \ln(z + \sqrt{r^2 + z^2}) \right|_{z_1}^{z_2}$$

$$= K\rho \ \ln\left(\frac{R_2 + z_2}{R_1 + z_1}\right)$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}^3} = \frac{x}{a^2\sqrt{a^2 + x^2}} ,$$

$$\int \frac{x\, dx}{\sqrt{a^2 + x^2}^3} = \frac{-1}{\sqrt{a^2 + x^2}} ,$$

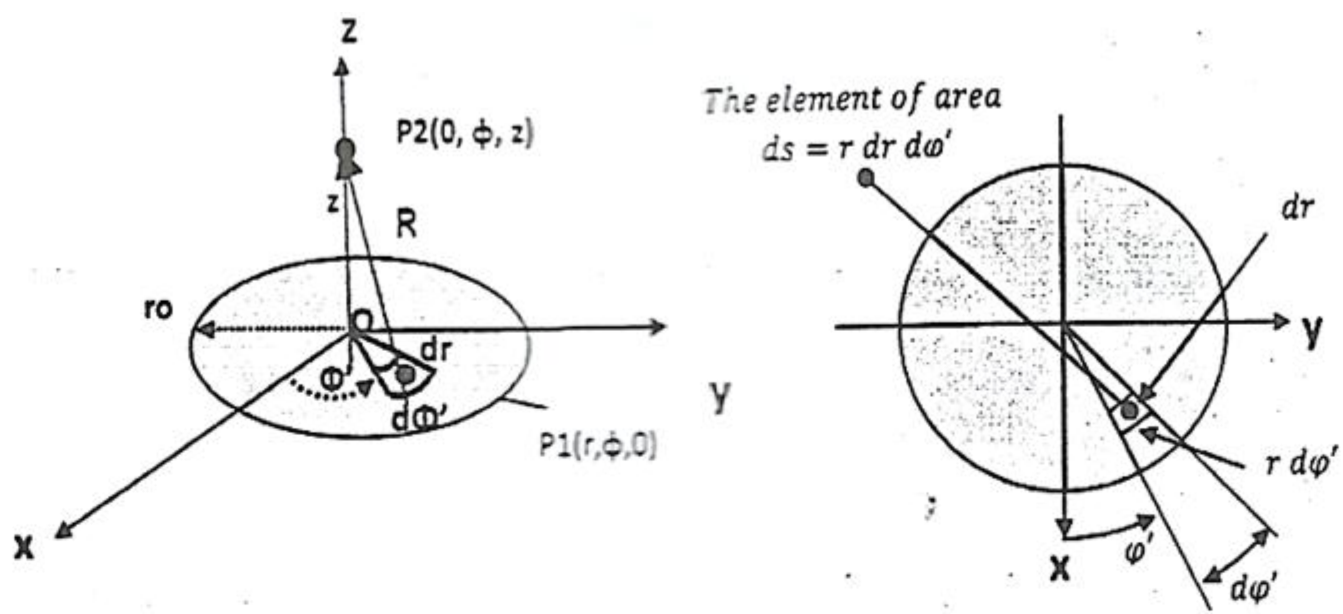
$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{x^2 + a^2})$$

Solved Examples

EXAMPLE 1.7

Field and Potential due to surface charge distribution

Find the electric field intensity E_2 , and the potential V_2 at $(0, \varphi, z)$ on the $\hat{z}$ axis due to a charged circular disk of uniform surface charge density $\rho_s = \rho\,(r, \varphi, 0)$ where of the disk radius $0 < r < r_0$ and $0 \leq \varphi \leq 2\pi$.



Solved Examples

SOLUTION

The charge element on the surface is:

$$dQ = \rho_s dS = \rho\ r d\varphi\ dr\ ,$$

Apply the coulomb's law for field and potential:

$$\mathbf{E}_2 = K \int \frac{dQ}{|\mathbf{R}|^3}\ \mathbf{R} \qquad V_2 = K \int \frac{dQ}{|\mathbf{R}|}$$

$$\mathbf{R} = (P_2 - P_1) = (0, \varphi, z) - (r, \varphi, 0) = (-\mathbf{r}, 0, \mathbf{z})\ ,$$

$$|\mathbf{R}| = \sqrt{r^2 + z^2}$$

Solved Examples

$$\therefore \mathbf{E}_2 = K \int \frac{dQ_1}{|R|^3}\,\mathbf{R} = K \int \rho \frac{r\,dr\,d\varphi}{(r^2+z^2)^{3/2}}\,(-\mathbf{r}, 0, \mathbf{z})$$

$$\cancel{= \int_0^{2\pi} d\varphi \int_{r=0}^{r_0} \frac{K\rho\,(-r^2)\,dr}{(r^2+z^2)^{\frac{3}{2}}}\,\hat{\mathbf{r}}} + \int_0^{2\pi} d\varphi \int_{r=0}^{r_0} \frac{K\,\rho\,z\,r\,dr}{(r^2+z^2)^{\frac{3}{2}}}\,\hat{\mathbf{z}}$$

$$E_{2z} = (2\pi) K\,\rho\,z \left[\frac{-1}{\sqrt{r^2+z^2}}\right]_0^{r_0} = 2\pi\,K\,\rho \left(1 - \frac{z}{\sqrt{r_0^2+z^2}}\right)$$

$$= \frac{\rho}{2\varepsilon_0}\left(1 - \frac{z}{R}\right)$$

Solved Examples

$$dV_2 = K\,\frac{dQ}{|\mathbf{R}|} = K\,\rho\,\frac{r\,dr\,d\varphi}{\sqrt{r^2+z^2}}$$

$$V_2 = K\,\rho\int_0^{2\pi} d\varphi \int_{r=0}^{r_0} \frac{r\,dr}{\sqrt{r^2+z^2}}$$

$$= 2\pi\;K\,\rho\left[\sqrt{r^2+z^2}\right]_0^{r_0}$$

$$= \frac{\rho}{2\varepsilon_0}\left(\sqrt{r_0^2+z^2} - z\right)$$

Electric Field from Electric Potential

Another way to obtain **E** is from the electric scalar potential V.

$$V = -\int \mathbf{E} \cdot \mathbf{d}\boldsymbol{l}$$

$$\mathbf{E} = -\ \nabla V$$

This way of finding **E** is easier because it is easier to handle scalars than vectors.

Vector Calculus

Gradient of a Scalar

The gradient of V is expressed in Cartesian, Cylindrical, and Spherical coordinates as:

$$\nabla V = \left[\frac{\partial V}{\partial x}\,\mathbf{a}_x + \frac{\partial V}{\partial y}\,\mathbf{a}_y + \frac{\partial V}{\partial z}\,\mathbf{a}_z\right]$$

$$\nabla V = \left[\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\,\mathbf{a}_z\right]$$

$$\nabla V = \left[\frac{\partial V}{\partial r}\,\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi\right]$$

Gauss's Law

In previous lectures, **E** due to any charge distribution was obtained from Coulomb's law.

In this lecture, when the charge distribution is symmetric, **E** can be calculated from **Gauss's law**.

Gauss's law states that the total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface. $\Psi = Q_{in}$

$$\Psi = \oint \mathbf{D} \,.\, ds = Q_{in} = \int \rho_v \, dv$$

D …electric flux density ρ_v …volume charge density

Gauss's Law

Symmetric charge distributions includes:

- **Rectangular symmetry**:
  if the charge distribution depends only on x or y or z

- **Cylindrical symmetry**:
  if the charge distribution depends only on ρ

- **Spherical symmetry**:
  if the charge distribution depends only on r

Vector Calculus

Differential Areas

$$d\mathbf{s} = dx\, dy\, \mathbf{a}_z = dy\, dz\, \mathbf{a}_x = dz\, dx\, \mathbf{a}_y$$

$$d\mathbf{s} = \rho d\varphi\, dz\, \mathbf{a}_\rho = d\rho\, dz\, a_{\varphi} = \rho d\varphi\, d\rho\, \mathbf{a}_z$$

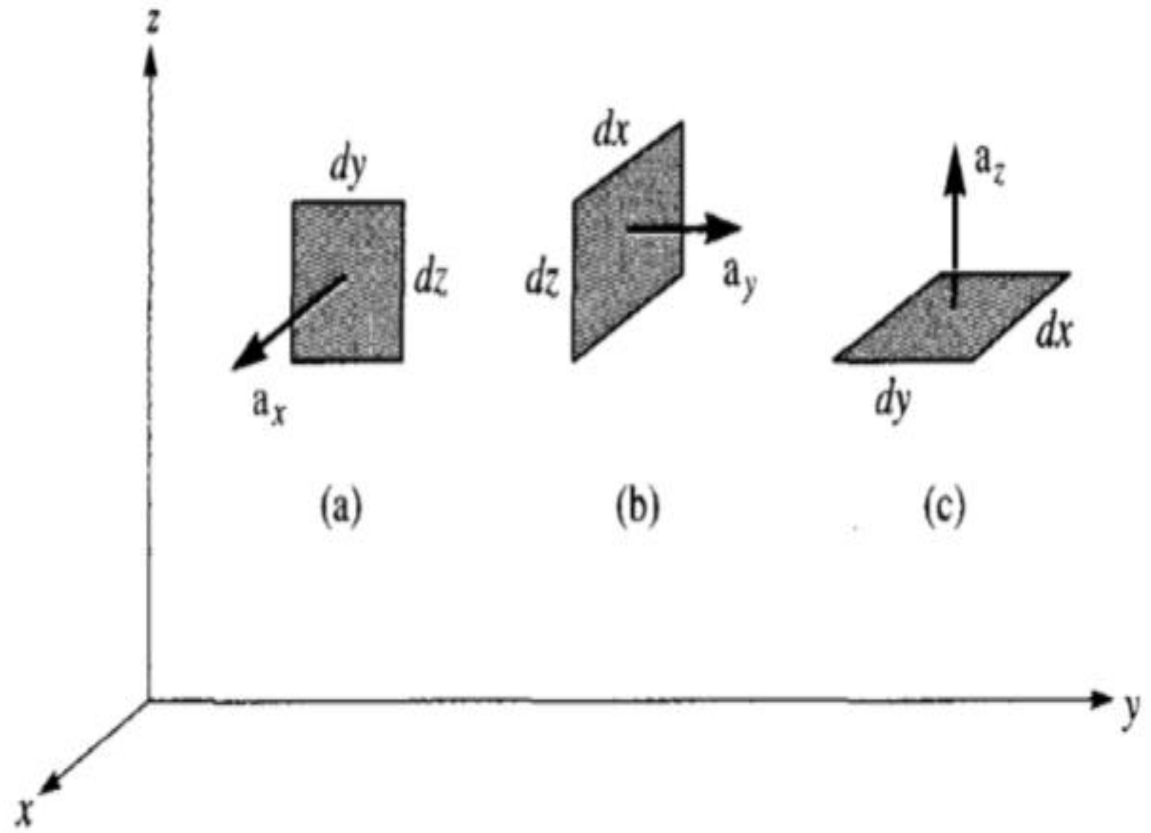
$$d\mathbf{s} = r^2 \sin\theta\, d\theta\, d\varphi\, \mathbf{a}_r = r\, dr\, d\theta\, a_\varphi = r^2 \sin\theta\, dr\, d\theta\, \mathbf{a}_\theta$$

Differential Volumes

$$dv = dx\, dy\, dz$$

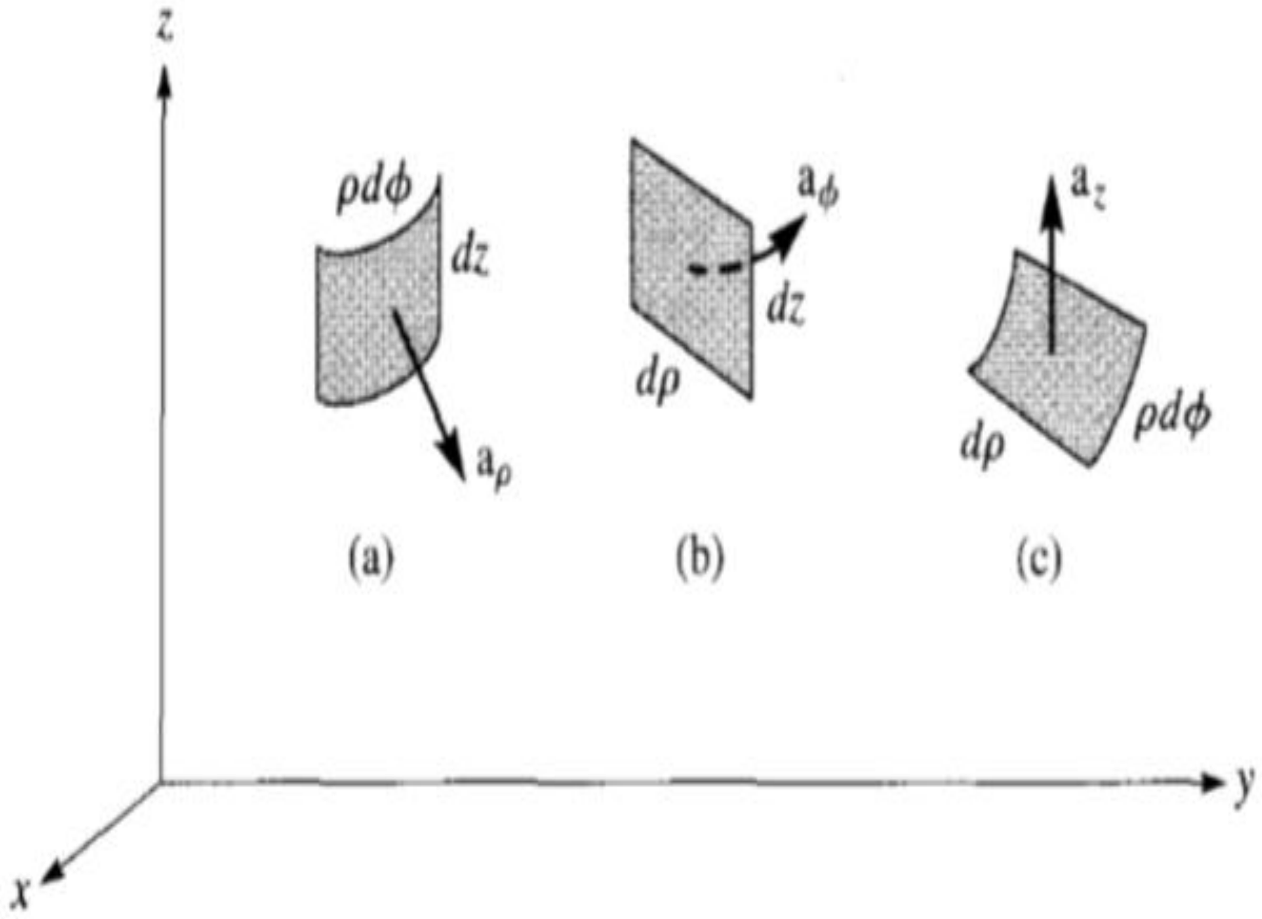
$$dv = \rho\, d\rho\, d\varphi\, dz$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$



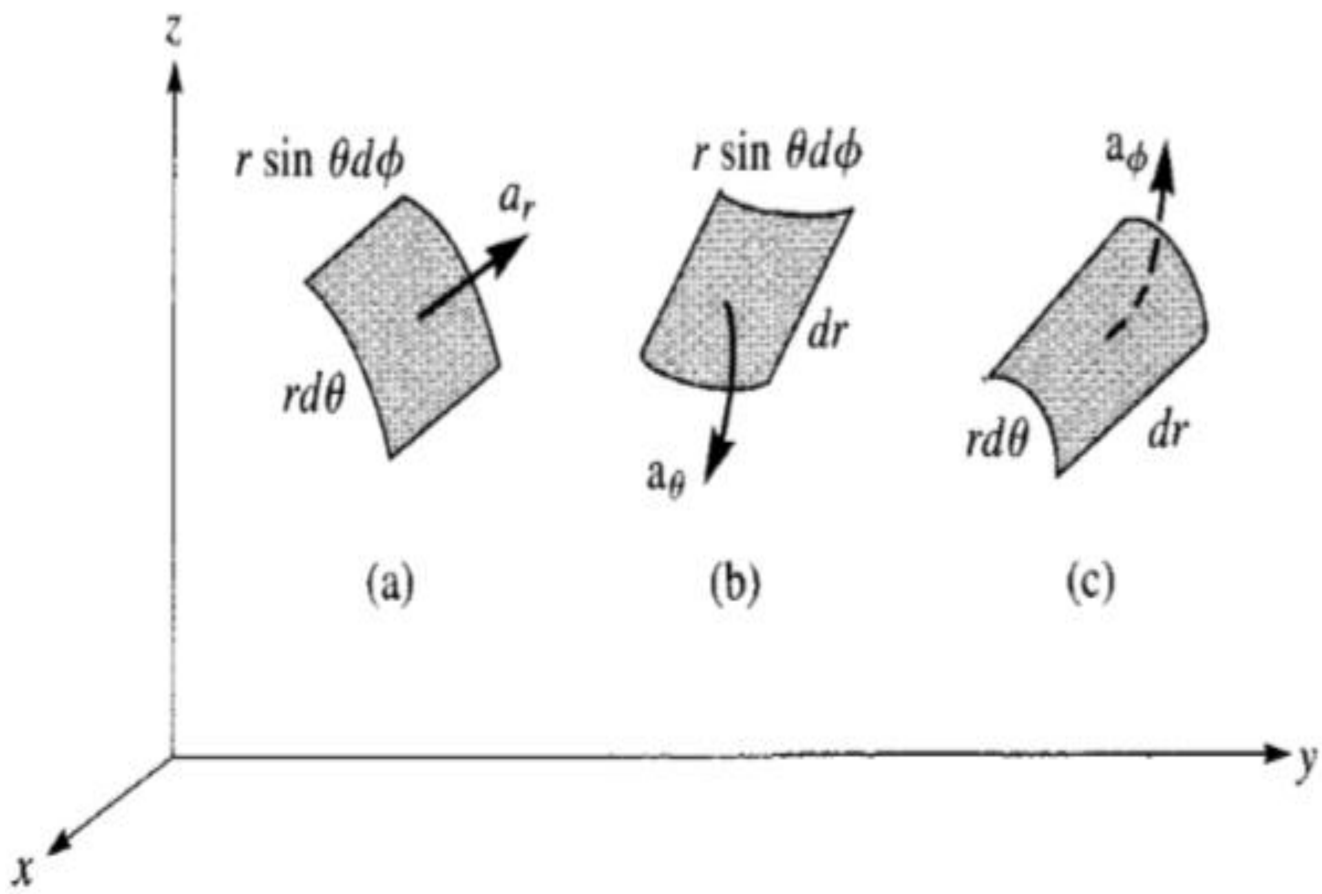
Differential normal areas in Cartesian coordinates

Lecture 1 Relevant Mathematics



Differential normal areas in Cylindrical coordinates

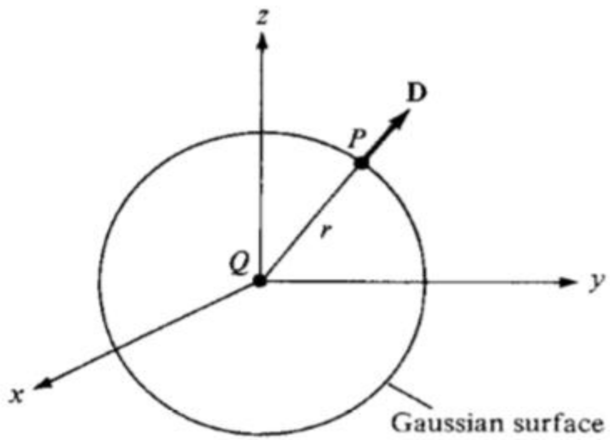
Lecture 1 Relevant Mathematics



Differential normal areas in Spherical coordinates

Field Density D and Intensity E Relation

Assuming a point charge Q located at the origin inside a sphere of radius r



The electric field at distance r by Coulomb Law is:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}$$

Field Density D and Intensity E Relation

The field at the same distance by Gauss's Law assuming **D** is uniform is:

$$Q = \oint \mathbf{D} \,.\, ds = D \,.\, \text{area of the sphere of raduis r}$$

$$Q = \mathbf{D} \,.\, 4\pi r^2$$

Therefore:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\,\hat{\mathbf{r}}$$

Comparing equations **E** and **D** equations:

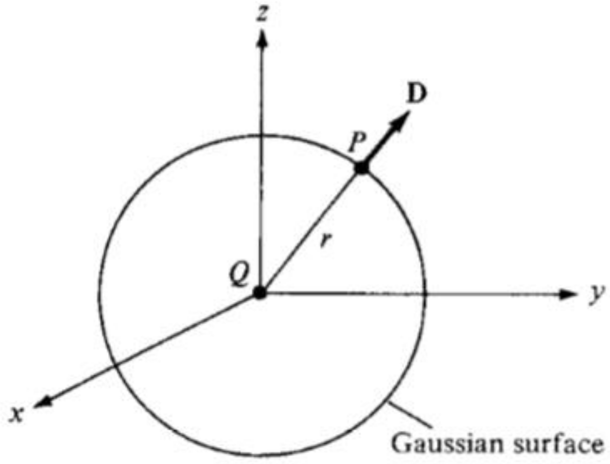
$$\mathbf{D} = \varepsilon_0 \mathbf{E} \qquad [C/m^2]$$

Applications of Gauss's Law

Case 1: Point Charge

Q is a point charge located at the origin

To determine **D** at a point P, the spherical surface centered at the origin is chosen as the Gaussian surface



Applications of Gauss's Law

As **D** is out warding from the point charge Q, so it is normal to the assumed spherical Gaussian surface,

$$\mathbf{D} = D_{\mathbf{r}}\, \mathbf{a_r}$$

Applying Gauss's law ($\Psi = Q_{in}$) gives:

$$Q = \oint \mathbf{D} \,.\, ds = D_{\mathbf{r}} \oint ds$$

$$= D_{\mathbf{r}} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\varphi = D_{\mathbf{r}}.\, 4\pi r^2$$

Therefore:

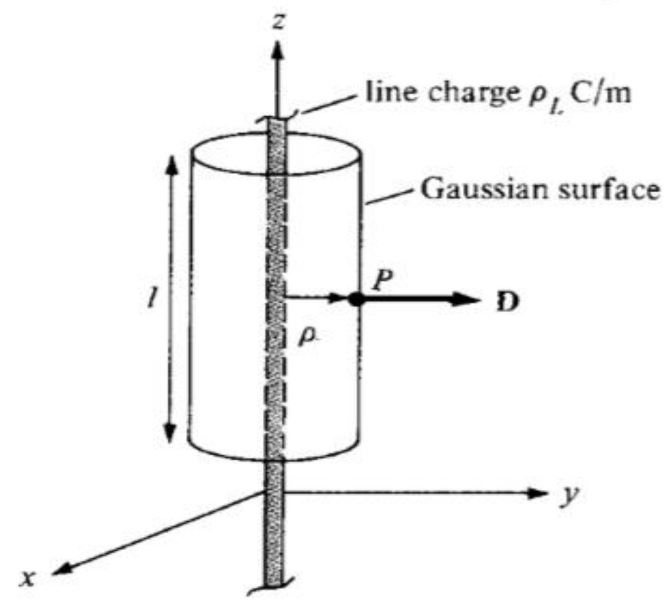
$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a_r} \qquad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \frac{Q}{4\pi r^2 \varepsilon_0}\mathbf{a_r}$$

Applications of Gauss's Law

Case 2: Infinite Line Charge

Assume an infinite line of uniform charge p_l (C/m) lies along the z-axis.

To determine **D** at a point P, choose a cylindrical surface containing P to satisfy the symmetry condition



Applications of Gauss's Law

As **D** is normal to the charged line segment, so it is normal to the assumed cylindrical Gaussian surface,

$$\mathbf{D} = D_r\,\mathbf{a_r}$$

Applying Gauss's law : $Q = \oint \mathbf{D}\,.\,ds$

$$Q = \int \rho_l\,dl = \rho_l\ l \qquad Q = \oint \mathbf{D}\,.\,ds = D_r \oint ds$$

$$= D_{\mathbf{r}}\ 2\pi r\, l$$

Therefore:

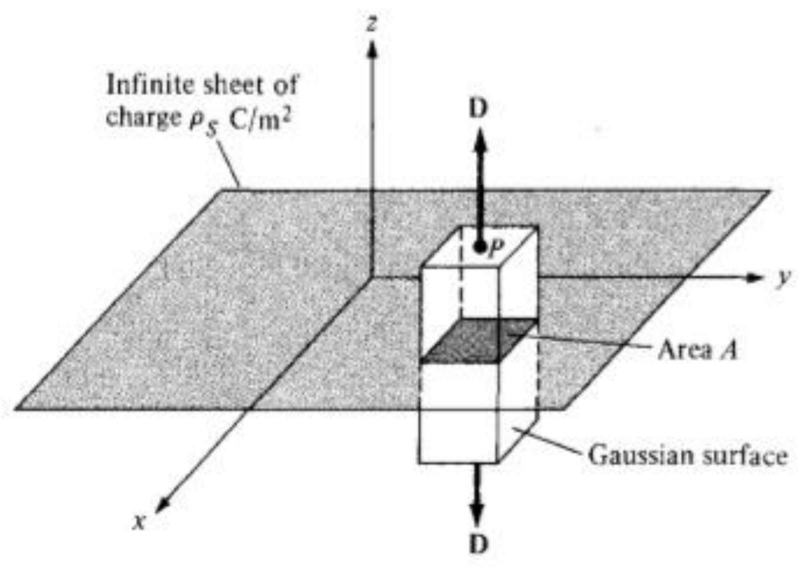
$$\mathbf{D} = \frac{\rho_l}{2\pi r}\mathbf{a_r} \qquad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \frac{\rho_l}{2\pi r\,\varepsilon_0}\mathbf{a_r}$$

Applications of Gauss's Law

Case 3: Infinite Sheet of Charge

Consider the infinite sheet of uniform charge ρ_s [C/m^2] lying on the z = 0 plane.

To determine **D** at point P, choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet



Applications of Gauss's Law

As **D** is normal to the sheet, so it is normal to the assumed Gaussian box surface,

$$\mathbf{D} = D_{\mathbf{z}}\, \mathbf{a}_{\mathbf{z}},$$

Applying Gauss's law gives:

$$Q = \rho_s A = \oint \mathbf{D} \,.\, ds = D_{\mathbf{z}} \oint ds = \mathbf{D}_{\mathbf{z}} \left[\int_{top} ds + \int_{bottom} ds\right]$$

Therefore:

$$\rho_s A = D_{\mathbf{z}} \,[A + A]$$

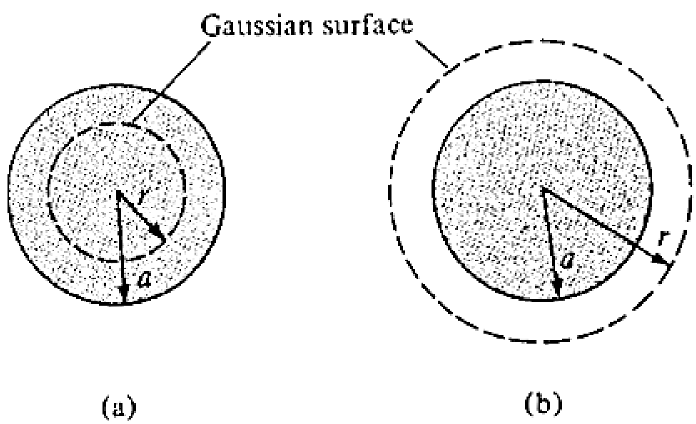
$$\mathbf{D} = \frac{\rho_s}{2} \mathbf{a}_{\mathbf{z}} \qquad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \frac{\rho_s}{2\varepsilon_0} \mathbf{a}_{\mathbf{z}}$$

Applications of Gauss's Law

Case 4: Uniformly Charged Sphere

Consider a sphere of radius a with a uniform charge p_v (C/m^3).

To determine **D** everywhere, we construct spherical Gaussian surfaces for cases r < a and r > a separately.



Applications of Gauss's Law

For r < a, the total charge enclosed by spherical surface of radius r is:

$$Q = \int \rho_v \, dv = \rho_v \int dv$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{4}{3} \pi r^3 \rho_v$$

$$= \oint \mathbf{D} \,.\, ds = D_r \oint ds = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\varphi$$

$$= \mathbf{D} \,.\, 4\pi r^2$$

Therefore:

$$\mathbf{D} = \frac{r}{3} \rho_v \, \mathbf{a_r} \qquad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0}$$

Applications of Gauss's Law

For r > a, the charge enclosed by the surface is the entire charge, that is:

$$Q = \int \rho_v \, dv = \rho_v \int \, dv$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{4}{3} \pi \, ra^3 \, \rho_v$$

$$= \oint \mathbf{D} \, . \, ds = D_r \oint ds = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\varphi$$

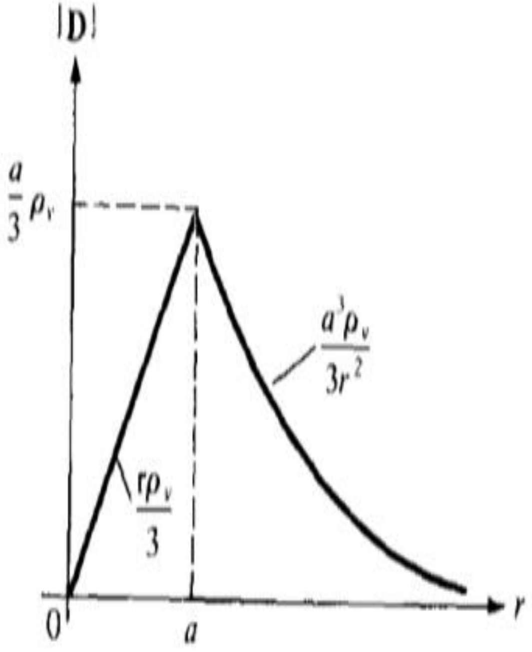
$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\varphi = \; D_r . \, 4\pi r^2$$

Therefore: $\mathbf{D} = \frac{a^3}{3r^2} \rho_v \, \mathbf{a_r}$ $\quad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0}$

Applications of Gauss's Law

For r < a $\mathbf{D} = \frac{r}{3}\rho_v\,\mathbf{a_r}$

For r > a $\mathbf{D} = \frac{a^3}{3r^2}\rho_v\,\mathbf{a_r}$



Energy Density In Electrostatic Fields

The energy required to bring a charge from a place to another place is:

$$W = Q\,V \qquad\qquad [\text{Joule}]$$

For N charges, the energy density in an electrostatic field is:

$$W = \frac{1}{2}\sum_{i=1}^{N} V_i Q_i$$

Where:

$$V_i = \sum_{j=1}^{N} \frac{K\,Q_j}{Ri_j} \text{ for } j \neq i$$

EXAMPLE 1.8

Two point charges $Q_1 = 2\ \mathrm{mc}$, and $Q_2 = -5\ [\mathrm{mC}]$, are located in free space at $P_1(-3, 7, -4)$ and $P_2(2, 4, -1)$ respectively.

Find the energy required to bring the charge $Q_3 = 4\ [\mathrm{mC}]$, from infinity to the point $P_3(1, 3, 5)$.

SOLUTION

The energy required to bring Q_3 to P_3 is given by:

$$W = [V_{31} + V_{32}] * Q_3 = Q_3\left(\frac{KQ_1}{R_{31}} + \frac{KQ_2}{R_{32}}\right),$$

SPLVED EXAMPLES

$$R_{31} = |(\mathbf{P}_3 - \mathbf{P}_1)| = |(1,3,\ 5) - (-3,7,-4)|$$
$$= \sqrt{16+16+81} = \sqrt{113}$$

$$R_{32} = |(\mathbf{P}_3 - \mathbf{P}_2)| = |(1,\ 3,\ 5) - (2,4,-1)|$$
$$= \sqrt{1+1+36} = \sqrt{38}$$

$$\therefore W = KQ_3\left(\frac{Q_1}{R_{31}} + \frac{Q_2}{R_{32}}\right)$$
$$= 9\,.\,10^9.\ 4\,.\,10^{-3}\left(\frac{1.10^{-3}}{\sqrt{113}} + \frac{(-5).\,10^{-3}}{\sqrt{38}}\right)$$

$$= -29.8 * 10^3 \quad [\text{Joul}]$$

SPLVED EXAMPLES

EXAMPLE 1.9

For the above example (1.8) find the electrostatic energy of the field constructed from the three Q_1, Q_2 and Q_3.

SOLUTION

$$W = \frac{1}{2}(Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32}))$$

$$W = \frac{1}{2}\left(Q_1\left(\frac{KQ_2}{R_{12}} + \frac{KQ_3}{R_{13}}\right) + Q_2\left(\frac{KQ_1}{R_{21}} + \frac{KQ_3}{R_{23}}\right) + Q_3\left(\frac{KQ_1}{R_{31}} + \frac{KQ_2}{R_{32}}\right)\right)$$

SPLVED EXAMPLES

$$R_{31} = R_{13} = \sqrt{113}\,, \qquad R_{32} = R_{23} = \sqrt{38}$$

$$R_{12} = R_{21} = |(P_1 - P_2)| = |(-3,7-4)-(2,4,-1)|$$

$$= \sqrt{25+9+9} = \sqrt{43}$$

$$W = \frac{1}{2} * 9.10^9 * 10^{-6}\left[2\left(\frac{-5}{\sqrt{113}} + \frac{4}{\sqrt{113}}\right) + (-5)\left(\frac{2}{\sqrt{43}} + \frac{4}{\sqrt{38}}\right) + 4\left(\frac{2}{\sqrt{113}} + \frac{-5}{\sqrt{38}}\right)\right]$$

$$= 36.16 * 10^{-6} \qquad [\mathrm{Joul}]$$

Vector Calculus

Divergence of a Vector

$$\nabla . \mathbf{A} = \frac{\partial \mathbf{A_x}}{\partial x} + \frac{\partial \mathbf{A_y}}{\partial y} + \frac{\partial \mathbf{A_z}}{\partial z}$$

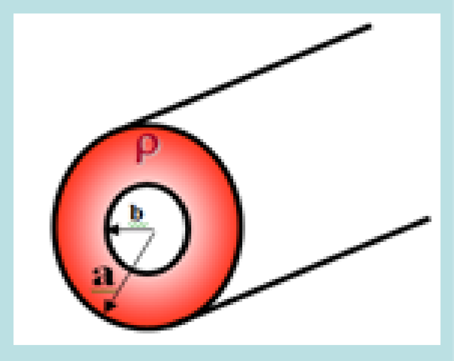
$$\nabla . \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, \mathbf{A_\rho}) + \frac{1}{\rho}\frac{\partial \mathbf{A_\phi}}{\partial \phi} + \frac{\partial \mathbf{A_z}}{\partial z}$$

$$\nabla . \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, \mathbf{A_r}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\mathbf{A_\theta} \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial \mathbf{A_\phi}}{\partial \phi}$$

Quiz 1

1- A point charge of 30 nC is located at the origin while plane y = 3 carries a surface charge density of $p_{s} = 10\ nC/m^{2}$. Find **D** at (0, 4 , 3).

2- Find the electric field inanity E anywhere inside and outside the hollow charged cylinder as shown in the Figure, with charge density $\rho = \rho_v$.



3- Three point charges $Q_1 = 1$ nC, $Q_2 = -2$ nC, and $Q_3 = 3$ nC are positioned one at a time and in that order at (0, 0, 0) , (1, 0, 0) and (0, 0, -1) respectively. Calculate the energy in the system after each charge is positioned.