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Project on:

Linear programming

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Contents

Introduction

A model, which is used for optimum allocation of scarce or limited resources to competing products or activities under such assumptions as certainty, linearity, fixed technology, and constant profit per unit, is linear programming.

Linear Programming is one of the most versatile,

powerful and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc. Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique that helps us is LINEAR PROGRAMMING. As a decision-making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more. In this chapter, let us discuss about various types of linear programming models.

1.1 What is a linear program?

Linear programming (LP) *or* ***Linear Optimization*** *may be defined as the problem of maximizing or minimizing a linear function that is subjected to linear constraints. The constraints may be equalities or inequalities. The optimization problems involve the calculation of profit and loss. Linear programming problems are an important class of optimization problems, that helps to find the feasible region and optimize the solution in order to have the highest or lowest value of the function.*

In other words, linear programming is considered as an optimization method to maximize or minimize the objective function of the given mathematical model with the set of some requirements which are represented in the linear relationship. The main aim of the linear programming problem is to find the optimal solution.

1.2 Linear Programming Applications

A real-time example would be considering the limitations of labors and materials and finding the best production levels for maximum profit in particular circumstances. It is part of a vital area of mathematics known as optimization techniques. The applications of LP in some other fields are

- *Engineering – It solves design and manufacturing problems as it is helpful for doing shape optimization*
- *Efficient Manufacturing – To maximize profit, companies use linear expressions*
- *Energy Industry – It provides methods to optimize the electric power system.*
- *Transportation Optimization – For cost and time efficiency.*

2. General Linear Programming Problem

A general mathematical way of representing a Linear Programming Problem (L.P.P.) is as given below:

Optimizing the objective function

$$Z = c_1\, x_1 + \; c_2\, x_2 \; + \ldots\ldots \; + c_n\, x_n$$

subjects to the conditions

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots + a_{1j}\, x_j + \ldots\ldots + a_{1n}\, x_n \; (\geq, =, \leq)\; b_1$$

$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots\ldots + a_{2j}\, x_j + \ldots\ldots + a_{2n}\, x_n \; (\geq, =, \leq)\; b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

....

....

$$a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots\ldots + a_{mj}\, x_j + \ldots\ldots + a_{mn}\, x_n \; (\geq, =, \leq)\; b_m$$

and all $x_j \geq 0$*. Where j = 1, 2, 3, … n*

Where all c_j *s,* b_i *s and* a_{ij} *s are constants and* x_j *s are decision variables. To show the relationship between left hand side and right-hand side the symbols ≤, =, ≥ are used. Any one of the signs may appear in real problems. Generally, ≤ sign is used for maximization*

problems and ≥ sign are used for minimization problems and in some problems, which are known as mixed problems we may have all the three signs. The word optimize in the above model indicates either maximize or minimize. The linear function, which is to be optimized, is the objective function.

The inequality conditions shown are constraints of the problem. Finally, all x_j *s should be positive, hence the non-negativity function.*

3. Properties of Linear Programming Model:

Any linear programming model (problem) must have the following properties:

(a) The relationship between variables and constraints must be linear.

(b) The model must have an objective function.

(c) The model must have structural constraints.

(d) The model must have non-negativity constraint.

4.The Steps for Formulating The Linear Programming are:

1. Identify the unknown decision variables to be determined and assign symbols to them.

2. Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.

3. Identify the objective or aim and represent it also as a linear function of decision variables.

Some Solved Problems:

Problem 1.1:

A company manufactures two products X and Y, which require, the following resources. The resources are the capacities machine M_1, M_2, *and* M_3. *The available capacities are 50,25, and 15 hours respectively in the planning period. Product X requires 1 hour of machine* M_2 *and 1 hour of machine* M_3. *Product Y requires 2 hours of machine* M_1, *2 hours of machine* M_2 *and 1 hour of* *machine M_3. The profit contribution of products X and Y are Rs.5/- and Rs.4/- respectively. Formulate a mathematical model of the problem.*

The contents of the statement of the problem can be summarized as follows:

Machines	*products*		*Availability in hours*
	X	*Y*	
M_1	*0*	*2*	*50*
M_2	*1*	*2*	*25*
M_3	*1*	*1*	*15*
Profit in RS per unit	*5*	*4*	

In the above problem, Products X and Y are competing candidates or variables. Machine capacities are available resources. Profit contribution of products X and Y are given. Now let us formulate the model.

Let the company manufactures x units of X and y units of Y. As the profit contributions of X and Y are Rs.5/- and Rs. 4/- respectively. The objective of the problem is to maximize the profit Z, hence objective function is:

Maximize

$Z = 5x + 4y$ *(OBJECTIVE FUNCTION)*

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

(LINEAR STRUCTURAL CONSTRAINTS)

For Machine M_1 $0x + 2y \leq 50$

For Machine M_2 $1x + 2y \leq 25$

For Machine M_3 $1x + 1y \leq 15$

But the company can stop production of x and y or can manufacture any amount of x and y. It cannot manufacture negative quantities of x and y. Hence, we have write,

Both x and y are ≥ 0

As the problem has got objective function, structural constraints, and non-negativity constraints and there exist a linear relationship between the variables and the constraints in the form of inequalities, the problem satisfies the properties of the Linear Programming Problem.

***Problem 1.2**:*

A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Rs. 2/- per unit and type B a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Solution:

Here shirts A and B are problem variables. Let the store stock 'a' units of A and 'b' units of B. As the profit contribution of A and B are Rs.2/- and Rs.5/- respectively, objective function is:

Maximize

$$Z = 2a + 5b$$

subjected to condition (s.t.)

Structural constraints are, stores can sell 400 units of shirt A and 300 units of shirt B and the storage capacity of both put together is 600 units. Hence the structural constraints are:

$1a + 0b \geq 400$ and $0a + 1b \leq 300$ for sales capacity and $1a + 1b \leq 600$ for storage capacity.

And non-negativity constraint is both a and b are ≥ 0. Hence the model is:

Maximize

$$Z = 2a + 5b$$

subjects to

$$1a + 0b \leq 400$$

$$0a + 1b \leq 300$$

$$1a + 1b \leq 600$$

And Both a and b are ≥ 0

Problem 1.3:

A patient consults a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin A and vitamin D. Doctor advises him to consume vitamin A and D regularly for a period of time so that he can regain his health. Doctor prescribes tonic X and tonic Y, which are having vitamin A, and D in certain proportion. Also advises the patient to consume at least 40 units of vitamin A and 50 units of vitamin Daily. The cost of tonics X and Y and the proportion of vitamin A and D that present in X and Y are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

Vitamins	tonics		Daily requirement in units
	X	Y	
A	2	4	40
D	3	2	50
Cost in Rs. Per unit	5	3	

Solution:

Let patient purchase x units of X and y units of Y.
Objective function:

Minimize

$$Z = 5x + 3y$$

Inequality for vitamin A is $2x + 4y \geq 40$ *(Here at least word indicates that the patient can consume more than 40 units but not less than 40 units of vitamin A daily).*

Similarly, the inequality for vitamin D is $3x + 2y \geq 50$*. For non–negativity constraint the patient cannot consume negative units. Hence both x and y must be* ≥ 0*.*

Now the l.p.p. model for the problem is:

Minimize

$$Z = 5x + 3y$$

subjects to $2x + 4y \geq 40$

$3x + 2y \geq 50$ *and*

Both x and y are ≥ 0*.*

Problem 1.4:

A company manufactures two products, X and Y by using three machines A, B, and C. Machine A has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines B and C during the coming week is 24 hours and 35 hours respectively. One unit of product X requires one hour of Machine A, 3 hours of machine B and 10 hours of machine C. Similarly, one unit of product Y requires 1 hour, 8 hour and 7 hours of machine A, B and C respectively. When one unit of X is sold in the market, it yields a profit of Rs. 5/- per product and that of Y is Rs. 7/- per unit. Formulate a mathematical model of the problem.

Solution: The details given in the problem is given in the table below:

Machines	*Products*	*(Time required in hours)*	*Available capacity in hours*
	X	*Y*	
A	*1*	*1*	*4*
B	*3*	*8*	*24*
C	*10*	*7*	*35*
Profit per unit in Rs.	*5*	*7*	

Let the company manufactures x units of X and y units of Y, and then the L.P. model is:

Maximize

$$Z = 5x + 7y$$

subjects to $1x + 1y \leq 4$

$$3x + 8y \leq 24$$

$10x + 7y \leq 35$ and

Both x and y are ≥ 0.

Problem 1.5:

A company manufactures two products X and Y. The profit contribution of X and Y are Rs.3/- and Rs. 4/- respectively. The products X and Y require the services of four facilities. The capacities of the four facilities A, B, C, and D are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly, the requirement of product Y is 4, 5, 5, and 4 hours respectively on A, B, C and D Formulate a mathematical model of the problem.

Solution: Enter the given data in the table below:

	Products		
Machines	X	Y	Availability in hours
A	5	4	200
B	3	5	150
C	5	4	100
D	8	4	80
Profit in Rs. Per unit	3	4	

The inequalities and equations for the above data will be as follows. Let the company manufactures x units of X and y units of Y

Maximize

$$Z = 3x + 4y$$

subjects to $5x + 4y \leq 200$

$$3x + 5y \leq 150$$

$$5x + 4y \leq 100$$

$$8x + 4y \leq 80$$

And both x and y are ≥ 0.

5. Methods for The Solution of a Linear Programming Problem

Linear Programming is a method of solving the type of problem in which two or more candidates or activities are competing to utilize the available limited resources, with a view to optimize the objective function of the problem. The objective may be to maximize the returns or to minimize the costs. The various methods available to solve the problem are:

1. *The Graphical Method when we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.*

2. *The Simplex method. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic program, which can be used to solve the problem. One problem with two variables is solved by using both graphical and simplex method, so as to enable the reader to understand the relationship between the two.*

5.1 The Graphical Method for solving L.P.

In graphical method, the inequalities (structural constraints) are considered to be equations. This is because; one cannot draw a graph for inequality. Only two variable problems are considered, because we can draw straight lines in two-dimensional plane (X- axis and Y-axis). Moreover, as we have nonnegativity constraint in the problem that is all the decision variables must have positive values always the solution to the problem lies in first quadrant of the graph. Some times the value of variables may fall in quadrants other than the first quadrant. In such cases, the line joining the values of the variables must be extended in to the first quadrant. The procedure of the method will be explained in detail while solving a numerical problem. The characteristics of Graphical method are:

(i) *Generally, the method is used to solve the problem, when it involves two decision variables.*

(ii) *For three or more decision variables, the graph deals with planes and requires high imagination to identify the solution area.*

(iii) *Always, the solution to the problem lies in first quadrant.*

(iv) *This method provides a basis for understanding the other methods of solution.*

Some Solved Problems:

Problem 2.1

A company manufactures two products, X and Y by using three machines A, B, and C. Machine A has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines B and C during the coming week is 24 hours and 35 hours respectively. One unit of product X requires one hour of Machine A, 3 hours of machine B and 10 hours of machine C. Similarly, one unit of product Y requires 1 hour, 8 hour and 7 hours of machine A, B and C respectively. When one unit of X is sold in the market, it yields a profit of Rs. 5/- per product and that of Y is Rs. 7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

Solution:

The details given in the problem is given in the table below:

M	*Products*	*(Time required in hours)*	*A*
	X	*Y*	
A	*1*	*1*	*4*
B	*3*	*8*	*24*
C	*10*	*7*	*35*
Profit per unit in Rs	*5*	*7*	

Let the company manufactures x units of X and y units of Y, and then the L.P. model is:

Maximize

$$Z = 5x + 7y$$

subjects to $1x + 1y \le 4$

$$3x + 8y \le 24$$

$$10x + 7y \le 35$$

and Both x and y are ≥ 0.

As we cannot draw graph for inequalities, let us consider them as equations.

Maximize

$$Z = 5x + 7y$$

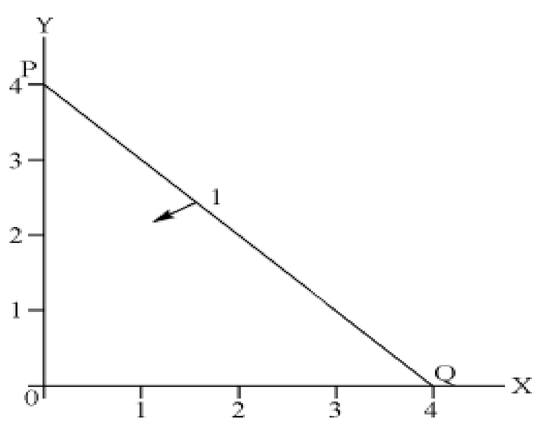
subjects to $1x + 1y = 4$

$$3x + 8y = 24$$

$$10x + 7y = 35$$

and both x and y are ≥ 0

Let us take machine A. and find the boundary conditions. If x = 0, machine A can manufacture 4/1 = 4 units of y



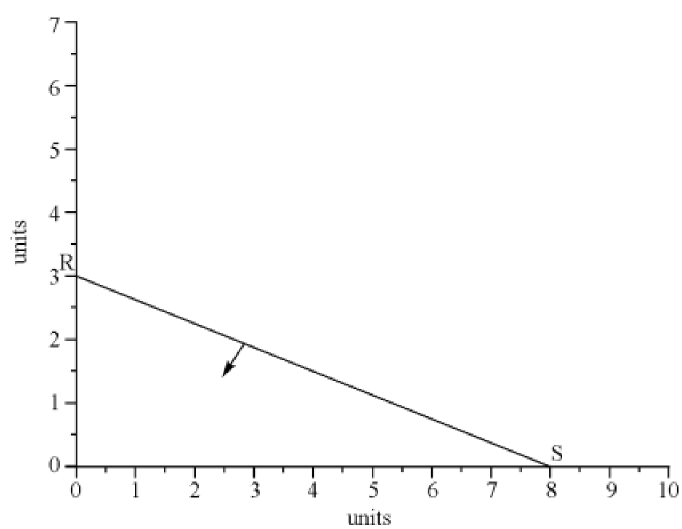
***(**Figure 1)*

Similarly, if y = 0, machine A can manufacture 4/1 = 4 units of x. For other machines:

Machine B When x = 0, y = 24/8 = 3 and when y = 0 x = 24/3 = 8
Machine C When x = 0, y = 35/10 = 3.5 and when y = 0, x = 35 / 7 = 5.

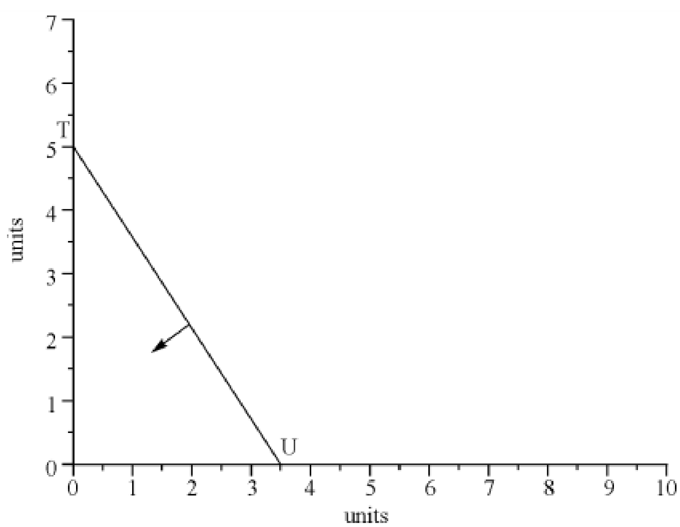
These values we can plot on a graph, taking product X on x-axis and product Y on y- axis. First let us draw the graph for machine A. In figure 2. 1 we get line 1 which represents 1x + 1y = 4. The point P on Y axis shows that the company can manufacture 4 units of Y only when does not want to manufacture X. Similarly, the point Q on X axis shows that the company can manufacture 4 units of X, when does not want to manufacture Y. In fact, triangle POQ is the capacity of machine A and the line PQ is the boundary line for capacity of machine A.

Similarly figure 2 show the Capacity line RS for machine B. and the triangle ROS shows the capacity of machine B i.e., the machine B can manufacture 3 units of product Y alone or 8 units of product X alone.

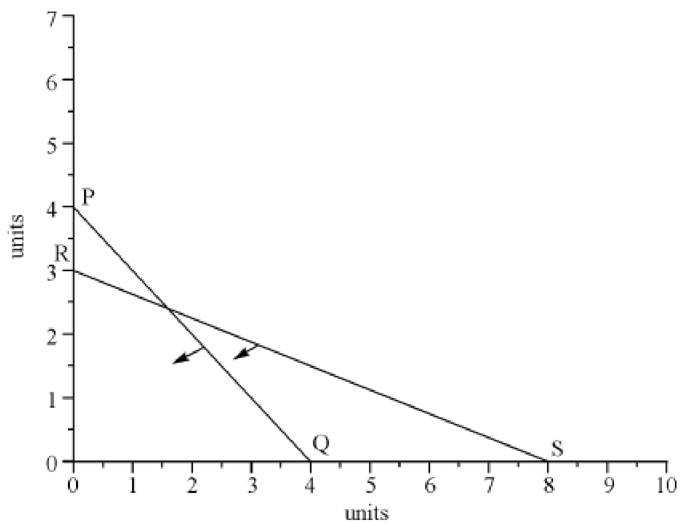


***(**Figure 2)*

The graph 3 shows that the machine C has a capacity to manufacture 5 units of Y alone or 3.5 units of X alone. Line TU is the boundary line, and the triangle TOU is the capacity of machine C. The graph is the combined graph for machine A and machine B. Lines PQ and RS intersect at M. The area covered by both the lines indicates the products (X and Y) that can be manufactured by using both machines. This area is the feasible area, which satisfies the conditions of inequalities of machine A and machine B. As X and Y are processed on A and B the number of units that can be manufactured will vary and the three will be some idle capacities on both machines. The idle capacities of machine A and machine B are shown in the figure 4.

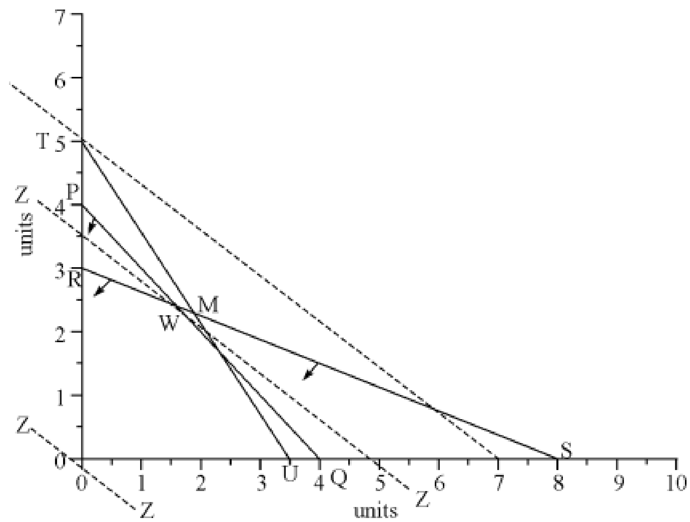


(*Figure 3)*



***(**Figure 4)*

Figure 5 shows the feasible area for all the three machines combined. This is the fact because a products X and Y are complete when they are processed on machine A, B, and C. The area covered by all the three lines PQ. RS, and TU form a closed polygon ROUVW. This polygon is the feasible area for the three machines. This means that all the points on the lines of polygon and any point within the polygon satisfies the inequality conditions of all the three machines.



(*Figure 5)*

Here we find the co-ordinates of corners of the closed polygon ROUVW and substitute the values in the objective function. In maximization problem, we select the co-ordinates giving maximum value. And in minimization problem, we select the co-ordinates, which gives minimum value. In the problem the co-ordinates of the corners are: R = (0, 3.5), O = (0,0), U = (3.5,0), V = (2.5, 1.5) and W = (1.6,2.4). Substituting these values in objective function:

Z (0,3.5) = 5 × 0 + 7 × 3.5 = Rs. 24.50, at point R

Z (0,0) = 5 × 0 + 7 × 0 = Rs. 00.00, at point O

Z (3.5,0) = 5 × 3.5 + 7 × 0 = Rs. 17.5 at point U

Z (2.5, 1.5) = 5 × 2.5 + 7 × 1.5 = Rs. 23.00 at point V

Z (1.6, 2.4) = 5 × 1.6 + 7 × 2.4 = Rs. 24.80 at point W

Hence the optimal solution for the problem is company has to manufacture 1.6 units of product X and 2.4 units of product Y, so that it can earn a maximum profit of Rs. 24.80 in the planning period.

Problem 2.2

A company manufactures two products X and Y. The profit contribution of X and Y are Rs.3/- and Rs. 4/- respectively. The products X and Y require the services of four facilities. The capacities of the four facilities A, B, C, and D are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly, the requirement of product Y is 4, 5, 4, and 4 hours respectively on A, B, C and D. Find the optimal product mix to maximize the profit.

Solution:

Enter the given data in the table below:

	Products	*(Time in hour)*	
M	*X*	*Y*	*Availability in hours*
A	*5*	*4*	*200*
B	*3*	*5*	*150*
C	*5*	*4*	*100*
D	*8*	*4*	*80*
Profit in Rs per unit:	*3*	*4*	

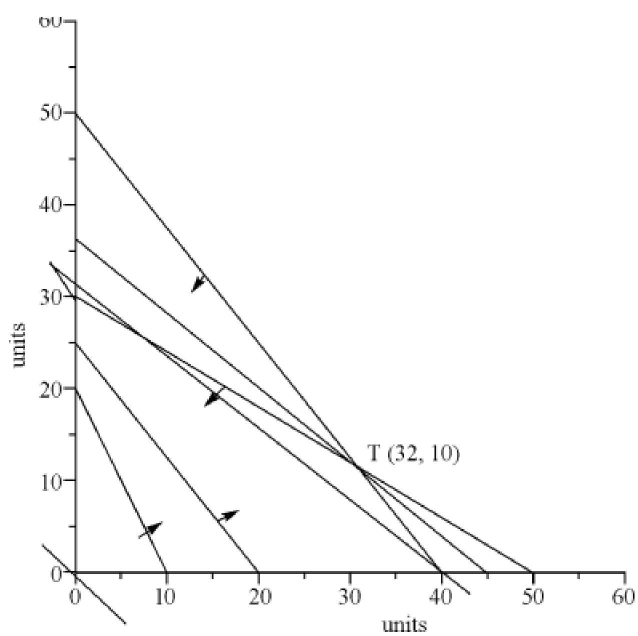
The inequalities and equations for the above data will be as follows. Let the company manufactures x units of X and y units of Y.

Maximize $Z = 3x + 4y$ S.T.	Maximize $Z = 3x + 4y$ S.T.
$5x + 4y \leq 200$	$5x + 4y = 200$
$3x + 5y \leq 150$	$3x + 5y = 150$
$5x + 4y \leq 100$	$5x + 4y = 100$

8x + 4y ≤ 80 *8x + 4y = 80*

And both x and y are ≥ 0 *And both x and y are ≥ 0*

In the graph the line representing the equation 8x + 4y is outside the feasible area and hence it is a redundant equation. It does not affect the solution. The Isoprofit line passes through corner T of the polygon and is the point of maximum profit. Therefore ZT = Z (32,10) = 3 × 32 + 4 × 10 = Rs. 136/.



SUMMARY:

1. The graphical method for solution is used when the problem deals with 2 variables.

2. The inequalities are assumed to be equations. As the problem deals with 2 variables, it is easy to write straight lines as the relationship between the variables and constraints are linear. In case the problem deals with three variables, then instead of lines we have to draw planes and it will become very difficult to visualize the feasible area.

3. If at all there is a feasible solution (feasible area of polygon) exists then, the feasible area region has an important property known as convexity Property in geometry. (Convexity means: Convex polygon consists of a set points having the property that the segment joining any two points in the set is entirely in the convex set. There is a mathematical theorem, which states, "The points which are simulations solutions of a system of inequalities of the ≤ type form a polygonal convex set". The regions will not have any holes in them, i.e., they are solids and the boundary will not have any breaks. This can be clearly stated that joining any two points in the region also lies in the region.

4. The boundaries of the regions are lines or planes.

5. There will be corners or extreme points on the boundary and there will be edges joining the various corners. The closed figure is known as polygon

6. The different situation was found when the objective function could be made arbitrarily large. Of course, no corner was optimal in that case.

5.2 The Simplex Method for solving L.P.

As discussed earlier, there are many methods to solve the Linear Programming Problem, such as Graphical Method and Simplex Method. Though we use graphical method for solution when we have two problem variables, the other method can be used when there are more than two decision variables in the problem. Among all the methods, SIMPLEX METHOD is most powerful method. It deals with iterative process, which consists of first designing a Basic Feasible Solution or a Program and proceed towards the OPTIMAL SOLUTION and testing each feasible solution for Optimality to know whether the solution on *hand is optimal or not. If not an optimal solution, redesign the program, and test for optimality until the test confirms OPTIMALITY. Hence we can say that the Simplex Method depends on two concepts known as Feasibility and optimality. The simplex method is based on the property that the optimal solution to a linear programming problem, if it exists, can always be found in one of the basic feasible solution. The simplex method is quite simple and mechanical in nature. The iterative steps of the simplex method are repeated until a finite optimal solution, if exists, is found. If no optimal solution, the method indicates that no finite solution exists.*

Problem 3.1:

A factory manufactures two products A and B on three machines X, Y, and Z. Product A requires 10 hours of machine X and 5 hours of machine Y a one our of machine Z. The requirement of product B is 6 hours, 10 hours and 2 hours of machine X, Y and Z respectively. The profit contribution of products A and B are Rs. 23/– per unit and Rs. 32 /– per unit respectively. In the coming planning period the available capacity of machines X, Y and Z are 2500 hours, 2000 hours and 500 hours respectively. Find the optimal product mix for maximizing the profit.

Solution:

The given data is:

Machines	Products		Capacity in hours
	A	B	
X	10	6	2500
Y	5	10	2000
Z	1	2	500
Profit/unit Rs	23	32	

Let the company manufactures a units of A and b units of B. Then the inequalities of the constraints (machine capacities) are:

Maximize

Z = 23 a + 32 b S.T. *(OBJECTIVE FUNCTION)*

10 a + 6 b ≤ 2500

5 a + 10 b ≤ 2000 *(STRUCTURAL CONSTRAINTS)*

1 a + 2 b ≤ 500

And both a and b are ≥ 0. (NON-NEGATIVITY CONSTRAINT)

Take machine X: One unit of product A requires 10 hours of machine X and one unit of product B require 6 units. But company is manufacturing a units of A and b units of B, hence both put together must be less than or equal to 2,500 hours. Suppose a = 10 and b = 10 then the total consumption is 10 × 10 + 6 × 10 = 160 hours. That is out of 2,500 hours, 160 hours are consumed, and 2,340 hours are still remaining idle. So, if we want to convert it into an equation then 100 + 60 + 2,340 = 2,500. As we do not know the exact values of decision variables a and b how much to add to convert the inequality into an equation. For this we represent the idle capacity by means of a SLACK VARIABLE represented by S. Slack variable for first inequality is s_1, that of second one is s_2 and that of 'n'th inequality is s_n.

Regarding the objective function, if we sell one unit of A it will fetch the company Rs. 23/– per unit and that of B is Rs. 32/– per unit. If company does not manufacture A or B, all resources remain idle. Hence the profit will be Zero rupees. This clearly shows that the profit contribution of each hour of idle resource is zero. In Linear Programming language, we can say that the company has capacity of manufacturing 2,500 units of s_1, i.e., s_1 is an imaginary product, which require one hour of machine X alone. Similarly, s_2 is an imaginary product requires one hour of machine Y alone and s_3 is an imaginary product, which requires one hour of machine Z alone. In simplex language s_1, s_2 and s_3 are idle resources. The profit earned by keeping all the machines idle is Rs.0/–. Hence the profit contributions of s_1, s_2 and s_3 are Rs.0/– per unit. By using this concept, the inequalities are converted into equations as shown below:

Maximize

$$Z = 23\,a + 32\,b + 0s_1 + 0s_2 + 0s_3$$

subjects to

$$10\,a + 6\,b + 1s_1 = 2500$$

$$5\,a + 10\,b + 1s_2 = 2000$$

$$1\,a + 2\,b + 1s_3 = 500$$

and a, b, s_1, s_2 and s_3 all ≥ 0.

In Graphical method, while finding the profit by Isoprofit line, we use to draw Isoprofit line at origin and use to move that line to reach the far-off point from the origin. This is because starting from zero rupees profit; we want to move towards the maximum profit. Here also, first we start with zero rupees profit, i.e., considering the slack variables as the basis variables (problem variables) in the initial program and then improve the program step by step until we get the optimal profit.

Let us start the first program or initial program by rewriting the entries as shown in the above simplex table

This column shows Basic or Problem variables.

This column shows objective co-efficients corresponding to basic variables in the programme

This column shows the values of the basic variables, the value of such non basic variable is = 0

This row shows C_j above each variable, the respective objective coefficient.

Variable row lists all the variable in the problem.

The numbers under non-basic variables represent substitution Ratios.

Every simplex tableau contains an identity Matrix under the basic variables.

Programme Variable or Basic variable	Profit per unit in Rs.	Quantity or Capacity	23 a	32 b	0 S_1	0 S_2	0 S_3
S_1	0	2500	10	6	1	0	0
S_2	0	2000	5	10	0	1	1
S_3	0	500	1	2	0	0	1
Z_j			0	0	0	0	0
Net Evaluation $C_j - Z_j$			23	32	0	0	0

The number in Z_j row under each column variable gives the total gross amount of **outgoing profit** when we consider the exchange between one unit of the column variable and the basic Variable.

The numbers in the net-evaluation row, under each column represent the **opportunity cost** of not having one unit of the respective column variables in the solution. In other words, the number represent the potential improvement in the objective function that will result by introducing into the programme one unit of the respective column variable.

Table: 1. Initial Programme

Solution: $a = 0$, $b = 0$, $S_1 = 2500$, $S_2 = 2000$ and $S_3 = 500$ and Z = Rs. 0.00.

Programme (Basic variables)	Profit per unit In Rs. C_b	Quantity in Units.	C_j 23 a	32 b	0 S_1	0 S_2	S_3	Replacement Ratio.
S_1	0	2500	10	6	1	0	0	2500/6 = 416.7
S_2	0	2000	5	**10**	0	1	0	2000/10 = 200
S_3	0	500	1	2	0	0	1	500/2 = 250
Z_j			0	0	0	0	0	
$C_j - Z_j$ = Opportunity cost in Rs. Net evaluation row.			23	32	0	0	0	

we have to transfer the rows of table 1 to table 2. To do this the following procedure is used.

Step 1*: To Write the incoming variable 'b' in place of outgoing variable s_2. Enter the profit of 'b' in profit column. Do not alter S_1 and S_3. While doing so DO NOT ALTER THE POSITION OF THE ROWS.*

Step 2*: DIVIDING THE ELEMENTS OF OLD COLUMN BY KEY COLUMN ELEMENTS obtains capacity column elements.*

Step 3*: Transfer of key row: DIVIDE ALL ELEMENTS OF KEY ROW BY RESPECTIVE KEY COLUMN NUMBER.*

Step 4*: Transfer of Non-Key rows: NEW ROW NUMBER = (old row number – corresponding key row number) × fixed ratio. Fixed ratio = Key column number of the row/key number.28*

Step 5*: Elements of Net evaluation row are obtained by: Objective row element at the top of the row – Σ key column element × profit column element.*

Step 6*: Select the highest positive element in net evaluation row or highest opportunity cost and mark the column by an arrow to indicate key column (incoming variable).*

Step 7*: Find the replacement ratios by dividing the capacity column element in the row by key column element of the same row and write the ratios in replacement ratio column. Select the* *limiting (lowest) ratio and mark with a tick mark to indicate key row (out going variable). The element at the intersection of key column and key row is known as key number. Continue these steps until we get:*

(i) For maximization problem all elements of net evaluation row must be either zeros or negative elements.

(ii) For Minimization problem, the elements of net evaluation row must be either zeros or positive elements.

Solution: $S_1 = 1{,}300$, $S_2 = 0$, $S_3 = 100$, $a = 0$, $b = 200$, $Z = 32 \times 200 =$ Rs. 6400.

Problem variable.	*Profit in Rs.*	*Capacity*	*C_j 23 a*	*32 b*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement Ratio (R.R)*
S_1	0	1,300	**7**	0	1	-0.6	0	1300/7 = 185.7
b	32	200	0.5	1	0	0.10	0	400
S_3	0	100	0	0	0	-0.5	1	--
Z_j			16	32	0	3.2	0	
$C_j - Z_j$	= net evaluation		7	0	0	- 3.2	0	

1. Transfer of Key row: 2000/10 , 5 /10, 10 /10, 0 /10, 1 / 10, 0 / 10
2. Transfer of Non key rows:

Rule: (Old row Number – corresponding key row number) – key column number / key number = new row no.

1st row.
$2500 - 2000 \times 6/10 = 1300$
$10 - 10 \times 6/10 = 0$
$6 - 10 \times 6/10 = 0$
$1 - 0 \times 6/10 = 1$
$0 - 1 \times 6/10 = -0.6$
$0 - 0 \times 6/10 = 0$

2nd row:
$500 - 2000 \times 2/10 = 100$
$1 - 5 \times 2/10 = 0$
$2 - 10 \times 2/10 = 0$
$0 - 0 \times 2/10 = 0$
$0 - 1 \times 2/10 = -0.2$
$1 - 0 \times 2 / 10 = 1$

Replacement ratios: 1300/7 = 185.7, 200/0.5 = 400, 100/0 = Infinity.

(table2)

Net evaluation row elements =

Column under:

$'a' = 23 - (7 \times 0 + 0.5 \times 32 + 0 \times 0) = 23 - 16 = 7$

$'b' = 32 - (0 \times 0 + 1 \times 32 + 0 \times 0) = 32 - 32 = 0$

$S_1 = 0 - (1 \times 0 + 0 \times 32 + 0 \times 0) = 0$

$S_2 = 0 - (-0.6 \times 0 + 0.1 \times 32 + -0.2 \times 0) = -3.2$

$S_3 = 0 - (0 \times 0 + 0 \times 32 + 1 \times 0) = 0$

In the above table, the net evaluation under S_2 is – 3.2. This resource is completely utilized to manufacture product B. The profit earned by manufacturing B is Rs. 6400/–. As per the law of economics, the worth of resources used must be equal to the profit earned. Hence the element 3.2 (ignore negative sign) is known as economic worth or artificial accounting price (technically it can be taken as MACHINE HOUR RATE) of the resources or shadow price of the resource. (In fact all the elements of reevaluation row under slack variables are shadow prices of respective resources). This concept is used to check whether the problem is done correctly or not. To do this MULTIPLY THE ELEMENTS IN NET EVALUATION ROW UNDER SLACK VARIABLES WITH THE ORIGINAL CAPACITY CONSTRAINTS GIVEN IN THE PROBLEM AND FIND THE SUM OF THE SAME. THIS SUM MUST BE EQUAL TO THE PROFIT EARNED BY MANUFACTRUING THE PRODUCT

Problem variable	Profit in Rs.	Capacity	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3	Replacement ratio
a	23	185.7	1	0	0.143	−0.086	0	
b	32	107.14	0	1	−0.07	0.143	0	
S_3	0	100	0	0	0	−0.02	1	
Z_j			23	32	1	2.6	0	
$C_j - Z_j$	Net evaluation.		0	0	−1.0	−2.6	0	

Transfer of key row: 1300/7 = 185.7, 7/7 = 1, 0/7 = 0, 1/7 = 0.143, –3 /5 = – 0.086 0/7 = 0

Row No. 2

200 – 1300 × 1/14 = 107.14

0.5 – 7 × 1/14 = 0

1 – 0 × 1/14 = 1

0 – 1 × 1/14 = – 0.07

0.1 – (– 0.6) × 1/14 = 0.143

0 – 0 × 1/14 = 0

Row No.3

As the fixed ratio will be zero for this row the row elements will not change.

(table3)

Net evaluation row elements:

For (a) = 23 – 1 × 23 + 0 × 32 + 0 × 0 = 0

For (b) = 32 – 0 × 23 + 1 × 32 + 0 × 0 = 0

For S_1 = 0 – 0.143 × 23 + (-0.07 × 32) + 0 × 0 = -1

For S_2 = 0 – (-0.086 × 23) + 0.143 × 32 + (-0.02 × 0) = -2.6

For S_3 = 0 – 0 × 23 + 0 × 32 + 1 × 0 = 0

Profit Z = 185.7 × 23 + 107.14 × 32 = Rs. 7,700

Shadow price = 1 × 2500 + 2.6 × 2000 = Rs. 2500 + 5200 = Rs. 7700/–

As all the elements of net evaluation row are either negative elements or zeros, the solution is optimal. Also the profit earned is equal to the shadow price.

The answer is the company has to manufacture:

185.7 units of A and 107.14 units of B and the optimal return is Z = Rs. 7,700/–

Problem. 3.2:

A company manufactures three products namely X, Y and Z. Each of the product require processing on three machines, Turning, Milling and Grinding. Product X requires 10 hours of turning, 5 hours of milling and 1 hour of grinding. Product Y requires 5 hours of turning, 10 hours of milling and 1 hour of grinding, and Product Z requires 2 hours of turning, 4 hours of milling and 2 hours of grinding. In the coming planning period, 2700 hours of turning, 2200 hours of milling and 500 hours of grinding are available. The profit contribution of X, Y and Z are Rs. 10, Rs.15 and Rs. 20 per unit respectively. Find the optimal product mix to maximize the profit.

Solution: The given data can be written in a table.

Machine	*Product* *Time required in hours per unit*			*Available hours*
	X	*Y*	*X*	
Turing	*10*	*5*	*2*	*2700*
Milling	*5*	*10*	*4*	*2200*
Grinding	*1*	*1*	*2*	*500*
Profit contribution in Rs per unit	*10*	*15*	*20*	

Let the company manufacture x units of X, y units of Y and z units of Z

Inequalities:

Maximize

$$Z = 10x + 15y + 20z$$

subjects to $10x + 5y + 2z \leq 2{,}700$

$5x + 10y + 4z \leq 2{,}200$

$1x + 1y + 2z \leq 500$

and All x, y and z are ≥ 0

Simplex format:

Maximize

$$Z = 10x + 15y + 20z + 0s_1 + 0s_2 + 0s_3$$

subjects to $10x + 5y + 2z + 1s_1 + 0s_2 + 0s_3 = 2700$

$5x + 10y + 4z + 0s_1 + 1s_2 + 0s_3 = 2200$

$1x + 1y + 2z + 0s_1 + 0s_2 + 1s_3 = 500$

And all x, y, z, S_1, S_2, S_3 *are* ≥ 0

Table I. $x = 0$, $y = 0$, $z = 0$, $S_1 = 2700$, $S_2 = 2200$, $S_3 = 500$. Profit Z = Rs. 0

Programme	*Profit*	*Capacity*	$C_j = 10$ x	15 y	20 z	0 S_1	0 S_2	0 S_3	*Replacement ratio*	*Check column.*
S_1	0	2700	10	5	2	1	0	0	2700/2 = 13500	2718
S_2	0	2200	5	10	4	0	1	0	2200/4 = 550	2220
S_3	0	500	1	1	**2**	0	0	1	500/2 = 250	505
Net evaluation			10	15	20	0	0	0		

Table: II. $x = 0$, $y = 0$, $z = 250$ units, $S_1 = 2200$, $S_2 = 1200$, $S_3 = 0$ and Z = Rs. 20 × 250 = Rs. 5,000.

Programme	*Profit*	*Capacity*	*C_j=10* *x*	*15* *y*	*20* *z*	*0* *S_1*	*0* *S_2*	*0* *S_3*	*Check column.*	*Replacement ratio*
S_1	0	2210	9	4	0	1	0	– 1	2213	552.5
S_2	0	1200	3	**8**	0	0	1	– 2	1210	150
Z	20	250	0.5	0.5	1	0	0	0.5	500	500
Net Evaluation.			0	5	0	0	0	– 10		

Profit at this stage = Rs. 20 × 250 = Rs. 5,000 and Shadow price = 10 × 500 = Rs. 5000.

Table: III. $x = 0$, $y = 150$, $z = 174.4$, $S_1 = 1600$, $S_2 = 0$, $S_3 = 0$ and Z = Rs. 5738/–

Programme	*Profit*	*Capacity*	*C_j=10* *x*	*15* *y*	*20* *z*	*0* *S_1*	*0* *S_2*	*0* *S_3*	*Check column.*	*Replacement Ratio*
S_1	0	1600	7.5	0	0	1	–0.5	0	1608	
Y	15	150	0.375	1	0	0	0.125	–0.25	151.25	
Z	20	174.4	0.311	0	1	0	–0.063	0.626	423.7	
Net Evn.			–1.85	0	0	0	–0.615	– 8.77		

As all the elements of Net evaluation row are either zeros or negative elements, the solution is optimal. The firm must produce 150 units of Y and 174.4 units of Z. The optimal profit = 15 × 150 + 20 × 174.4 = Rs.5738 /– To check the shadow price = 0.615 × 2200 + – 8.77 × 500 = 1353 + 4385 = Rs. 5738 /–.

Problem 3.3:

Minimize

$$x_1 + x_2 + x_3 = Z$$

subject to

$-x_1 + 2x_2 + x_3 \leq 1$

$-x_1 \quad + 2x_3 \geq 4$

$x_1 - x_2 + 2x_3 = 4$

$x_1, x_2, x_3 \geq 0$

Adding two slack variables, the problem in standard form becomes

Minimize

$$x_1 + x_2 + x_3 = Z$$

subject to

$-x_1 + 2x_2 + x_3 + x_4 = 1$

$-x_1 \quad + 2x_3 \quad - x_5 = 4$

$x_1 - x_2 + 2x_3 = 4$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Note that the x_4 variable can serve as a basic variable. Thus it is sufficient to add only two artificial variables, say x_6 and x_7, to the problem and at the first stage minimize the function w = x_6+x_7. The problem is then

$$
\begin{array}{llllr}
-x_1 + 2x_2 + x_3 + x_4 & & & & = 1 \\
-x_1 \quad + 2x_3 & -x_5 + x_6 & & & = 4 \\
x_1 - x_2 + 2x_3 & & & +x_7 & = 4 \\
x_1 + x_2 + x_3 & & & & = Z \\
 & & & x_6 + x_7 & = W
\end{array}
$$

Subtracting the second and third equations from the w equation gives the equation $x_2 - 4x_3 + x_5 = -8 + w$. Now the expression for w does not contain the initial basic variables x_4, x_6, and x_7, and the simplex method can be initiated. The resulting tableaux are given in Table 3.3.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	-1	2	1	1	0	0	0	1
x_6	-1	0	2	0	-1	1	0	4
x_7	1	-1	2	0	0	0	1	4
	1	1	1	0	0			0
	0	1	-4	0	1			-8
x_3	-1	2	1	1	0			1
x_6	1	-4	0	-2	-1			2
x_7	3	-5	0	-2	0			2
	2	-1	0	-1	0			-1
	-4	9	0	4	1			-4
x_3	0	$\frac{1}{3}$	1	$\frac{1}{3}$	0			$\frac{5}{3}$
x_6	0	$-\frac{7}{3}$	0	$-\frac{4}{3}$	-1			$\frac{4}{3}$
x_1	1	$-\frac{5}{3}$	0	$-\frac{2}{3}$	0			$\frac{2}{3}$
	0	$\frac{7}{3}$	0	$\frac{1}{3}$	0			$-\frac{7}{3}$
	0	$\frac{7}{3}$	0	$\frac{4}{3}$	1			$-\frac{4}{3}$

(table3.3)

The minimal value for the function w = x_6 +x_7 is 4/3, and this value is attained at the point (2/3,0, 5/3,0,0, 4/3,0). Therefore, we can conclude that the original problem has no feasible solution.

Problem 3.4:

Minimize

$$z = 2x_1 - x_2 + x_3$$

subject to

$$x_1 - 2x_2 + 3x_3 + x_4 = 6$$

$$-x_1 + x_2 + 2x_3 + \frac{2}{3}x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Adding two artificial variables, x_5 and x_6, for initial basic variables and expressing w = x_5 +x_6 in terms of the original variables, we have the system

$$x_1 - 2x_2 + 3x_3 + x_4 + x_5 \qquad = 6$$

$$-x_1 + x_2 + 2x_3 + \frac{2}{3}x_4 \qquad + x_6 = 4$$

$$2x_1 - x_2 + x_3 \qquad = Z$$

$$x_2 - 5x_3 - \frac{5}{3}x_4 \qquad = -10 + W$$

From the tableaux of Table 3.4, we see that the minimal value of the objective function z is zero, and is attained at the point (0,0,0,6). Notice that the first pivot term could have been either the 3 or 2 of the x_3 column (or term in the x_4 column, for that matter). The purpose of the second pivot step is to eliminate the *artificial variable x_6 from the basis, and this pivot could have been made at either nonzero entry in the x_6 row of the second tableau. Since both artificial variables were extracted from the basis, the original system of constraints contained no redundancies.*

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	1	-2	3	1	1	0	6
x_6	-1	1	2	$\frac{2}{3}$	0	1	4
	2	-1	1	0			0
	0	1	-5	$-\frac{5}{3}$			-10
x_3	$\frac{1}{3}$	$-\frac{2}{3}$	1	$\frac{1}{3}$			2
x_6	$-\frac{5}{3}$	$\frac{7}{3}$	0	0			0
	$\frac{5}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$			-2
	$\frac{5}{3}$	$-\frac{7}{3}$	0	0			0
x_3	$-\frac{1}{7}$	0	1	$\frac{1}{3}$			2
x_2	$-\frac{5}{7}$	1	0	0			0
	$\frac{10}{7}$	0	0	$-\frac{1}{3}$			-2
	0	0	0	0			0
x_4	$-\frac{3}{7}$	0	3	1			6
x_2	$-\frac{5}{7}$	1	0	0			0
	$\frac{9}{7}$	0	1	0			0

(table3.4)

6- Comparison Between Graphical and Simplex:

1. The graphical method is used when we have two decision variables in the problem. Whereas in Simplex method, the problem may have any number of decision variables.
2. In graphical method, the inequalities are assumed to be equations, so as to enable to draw straight lines. But in Simplex method, the inequalities are converted into equations by:
(i) Adding a SLACK VARIABLE in maximization problem and subtracting a SURPLUS VARIABLE in case of minimization problem.
3. In graphical solution the Isoprofit line moves away from the origin to towards the far-off point in maximization problem and in minimization problem, the Isocost line moves from far off distance towards origin to reach the nearest point to origin.
4. In graphical method, the areas outside the feasible area (area covered by all the lines of constraints in the problem) indicates idle capacity of resource whereas in Simplex method, the presence of slack variable indicates the idle capacity of the resources.
However, the beauty of the simplex method lies in the fact that the relative exchange profit abilities of all the non -basis variables (vectors) can be determined simultaneously and easily; the replacement process is such that the new basis does not violate the feasibility of the solution.

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