State estimation, autonomy, machine learning, & energy systems
Graded problems

Problem 1

Hint: Feel free to use a computer algebra system such as `sympy` *for this problem. Be sure to explain where you used it in your answer.* After an overhaul, a water pump reliably operates for a random time t before a failure occurs. The probability density function (PDF) for the failure time t is given by

$$f(t) = \begin{cases} \frac{2t}{\lambda^2} \exp\left(-\frac{t^2}{\lambda^2}\right) & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

where λ is a parameter, with specifically $\lambda = 500$days. The PDF is plotted below:
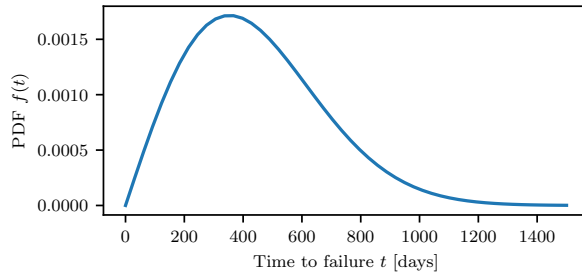
a. Confirm that this is a valid PDF (i.e. is the area under the curve 1? Is it non-negative everywhere?).

b. How long after installation should we do preventative maintenance if we wish to have the probability of unexpected failure be less than 1%? What about 10%, 50%, and 99%?

c. What is the expected lifetime (mean time between failures) for this pump? What is the probability of failure before the expected lifetime?

d. What is the variance of the pump's lifetime? What is the range of the lifetime that falls within one standard deviation of the expected value?

For the remainder of this problem, we're going to analyze the average cost associated with different preventative maintenance policies. Doing preventative maintenance costs $c_m = \$20$, while repairing a pump that has failed costs $c_r = \$100$. A pump that has been repaired after a failure has exactly the same characteristics as a pump that has just had preventative maintenance done (i.e. both "reset the lifetime clock"). We schedule preventative maintenance at time t_m. We amortize the operating cost of the pump over its lifetime and define the running cost R of the pump as the cost (repair or maintenance) divided by the running time. If the pump fails prematurely (that is, if $t \leq t_m$) we have $R = c_r/t$; otherwise we do preventative maintenance and the running cost is $R = c_m/t_m$.

e. Write a program that generates samples of t from its distribution. Evaluate the average running cost R for each of $t_m \in \{1, 10, 10^2, 10^3, 10^4\}$days. In each case, generate 10^6 samples.

Problem 2

In a substation, there are three types of transformers (from three different manufacturers: 1, 2, and 3). The probability that a transformer is operating out of specification during a day is shown in the table below.

manufacturer	1	2	3
number of transformers	8	5	20
prob. of out-of-spec	2/1000	3/1000	4/1000

An inspector chooses a machine at random, inspects it, and determines that it is outside the specifications. Compute the probability that it is from manufacturer 1.

Problem 3

A battery's state of charge at time step k is given by $q(k)$, with $q(k) = 1$ corresponding to fully charged and $q(k) = 0$ depleted. In each time step (e.g. each hour) k, the battery powers a process which consumes energy $j(k) = j_0 + v(k)$, where j_0 is the average amount of energy consumed, and $v(k)$ is a random deviation, so that

$$q(k) = q(k{-}1) - j(k{-}1)$$

We have perfect knowledge of the battery's initial charge, $q(0) = 1$, and we know that $v(k)$ is white and normally distributed as $\mathcal{N}\left(0, \sigma_v^2\right)$.

We want to know how many time steps of the process we can safely use before checking on the battery. *Note that the model allows for arbitrarily states of charge (where $q < 0$ or $q > 1$) though such states are not physically meaningful.*

a. Analytically compute the mean and variance of $q(k)$ across k.

b. After how many time steps does the mean of $q(k)$ drop below zero?

c. As a safety measure, we only want to run the process if we're confident that the battery has sufficient charge remaining to complete the process. How many time steps can I run until the battery is within 3 standard deviations of fully discharged? (Recall: the standard deviation changes over time, and at time k can be computed as $\sigma_q(k) = \sqrt{\mathrm{Var}\left[q(k)\right]}$; moreover, for a normally distributed random variable the probability of being more than three standard deviations below the mean is approximately 0.15%)

For the remainder of this problem, set $j_0 = 0.1$ and $\sigma_v = 0.05$.

d. Compare the numerical value you got for parts b and c. Comment on the difference.

e. Simulate 10^6 different evolutions, compute their mean & standard deviation across $k \in \{0, 1, \ldots, 20\}$. Make three plots against time: the mean $\mathrm{E}\left[q(k)\right]$, the standard deviation $\sqrt{\mathrm{Var}\left[q(k)\right]}$, and the fraction of sample paths where the charge $q(k)$ at time k is less than or equal to zero.

Problem 4

In this problem we estimate the behavior of a city's water network, where fresh water is supplied by a desalination plant, and consumed at various points in the network. The network contains various consumers, and various reservoirs where water is locally stored.

Our objective is to use noisy information about production and consumption levels, and noisy measurements of the amount of water in the reservoirs, to estimate the state of the network. We will consider 3 networks of increasing complexity.

(Hint – use code to compute numerical solutions, don't compute by hand.)

Let k be a time index, and let $d(k)$ be the fresh water produced by the desalination plant at time k. Each city district i has a reservoir with current level $r_i(k)$, and the district consumes a quantity of water $c_i(k)$. At each time step, we receive a noisy measurement $z_i(k) = r_i(k) + w_i(k)$ of the reservoir level $r_i(k)$, where $w_i(k)$ is the measurement error (with w zero-mean, white, and independent of all other quantities).

a. We will first investigate a very simplified system, with a single reservoir and single consumer (i.e. we only have one tank level r to keep track of). We will model our consumers as $c(k) = m + v(k)$, where m is the typical consumption, and $v(k)$ is a zero-mean uncertainty, assumed white and independent of all other quantities.
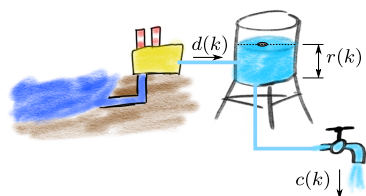
(a-i) Write down the model equations for this problem. Make explicit what is the system state, the measurement, the process noise, and the measurement noise.

(a-ii) Design a Kalman filter to estimate the level of the tank. Run the Kalman filter for ten steps, with the following problem data:

At time $k = 0$, we know $\mathrm{E}\left[r(0)\right] = 20$, with uncertainty $\mathrm{Var}\left[r(0)\right] = 25$. The desalination plant delivers a constant supply of water, so that $d(k) = 10$ for all $k \geq 0$. The consumer is predicted to use a supply $m = 7$, and our process uncertainty is $\mathrm{Var}\left[v(k)\right] = 9$. Our sensor uncertainty is $\mathrm{Var}\left[w(k)\right] = 25$.

We receive the following sequence of measurements $z(k)$:

time k	1	2	3	4	5	6	7	8	9	10
measurement $z(k)$	17.8	22.6	30.2	37.3	46.2	49.5	44.6	50.3	56.3	51.6

Using the Kalman filter, estimate the actual volume of water r for $k \in \{1, 2, \ldots, 10\}$. Also provide the associated uncertainty for your estimate of r. Provide this using graphs.

b. We now extend the previous part by also estimating the consumption, $c(k)$. We model the consumption as evolving as $c(k) = c(k-1) + n(k-1)$, where $n(k-1)$ is a zero-mean random variable, which is white and independent of all other quantities.

(b-i) Write down the model equations for this, noting that now your state has dimension 2. Make explicit what is the system state, the measurement, the process noise, and the measurement noise.

(b-ii) Again, design a Kalman filter to estimate the level of the tank. Use the same problem data as before (including as given in subproblem aa-ii), except that now $\mathrm{Var}\,[n(k)] = 0.1$. Also use $\mathrm{E}\,[c(0)] = 7$ with $\mathrm{Var}\,[c(0)] = 1$.

Using the Kalman filter, estimate the actual volume of water r for $k \in \{1, 2, \ldots, 10\}$, and the consumption rate $c(k)$, and also provide the associated uncertainty for both.

Explicitly provide the filter initialization, and then present the estimate and uncertainty using graphs. Also, provide a comment: How do your answers for r compare to the previous case?

c. Now, we extend the system to a more complex network with four reservoirs, and four consumers, as shown below.
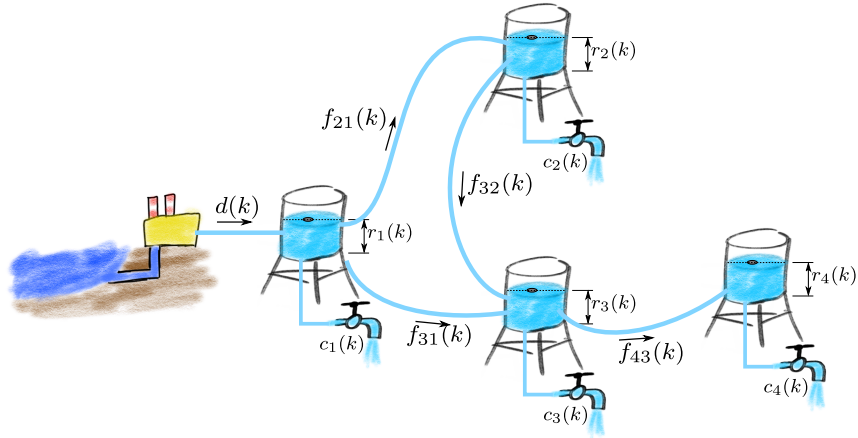
The reservoirs are connected in a network, and an automatic balancing system is in place that pumps water between the reservoirs. If two reservoirs i and j are connected, there is a balancing flow between them that is proportional to the difference in volume between the two, so that $f_{ij}(k) = \alpha \left(r_j(k) - r_i(k) \right)$, where α is a constant. For example, for reservoirs #1 and #2 in our network, we have the following dynamics:

$$r_1(k) = r_1(k\text{-}1) + d(k\text{-}1) - c_1(k\text{-}1) + \alpha \left(r_2(k\text{-}1) - r_1(k\text{-}1) \right) + \alpha \left(r_3(k\text{-}1) - r_1(k\text{-}1) \right)$$
$$r_2(k) = r_2(k\text{-}1) - c_2(k\text{-}1) + \alpha \left(r_1(k\text{-}1) - r_2(k\text{-}1) \right) + \alpha \left(r_3(k\text{-}1) - r_2(k\text{-}1) \right)$$

where, as before, $c_i(k) = c_i(k\text{-}1) + n_i(k\text{-}1)$. Note that the balancing does not change the *total* amount of water in the system, it just moves it around. Our sensor uncertainty is $\mathrm{Var}\,[w_i(k)] = 25$ for all tanks, and the consumption uncertainty is $\mathrm{Var}\,[n_i(k)] = 0.1$ for all consumers. Model the consumers as independent but identically distributed; also model the sensors as independent but identically distributed.

(c-i) Write down the model equations for this problem. Make explicit what is the system state, the measurement, the process noise, and the measurement noise. Write the solution as a linear problem, and clearly give terms of the matrices A, H, etc.

(c-ii) Design a Kalman filter to estimate the level of all the tanks, and the consumption levels. Run the Kalman filter for ten steps, with the following problem data:

At time $k = 0$, we know the following about the tanks and their consumers:

tank number i	1	2	3	4
$\mathrm{E}\left[r_i(0)\right]$	20	40	60	20
$\mathrm{Var}\left[r_i(0)\right]$	20	20	20	20
$\mathrm{E}\left[c_i(0)\right]$	7	7	7	7
$\mathrm{Var}\left[c_i(0)\right]$	1	1	1	1

The desalination plant delivers a constant supply of water, so that $d(k) = 30$ for all $k \geq 0$. The balancing is done with $\alpha = 0.3$.

We receive the following sequence of measurements $z_i(k)$:

time k	1	2	3	4	5	6	7	8	9	10
meas. $z_1(k)$	59.3	72.	64.4	83.6	84.9	94.3	84.	86.6	89.	89.1
meas. $z_2(k)$	39.1	38.4	36.2	43.4	50.5	56.3	40.3	58.5	55.4	59.6
meas. $z_3(k)$	31.1	31.2	41.6	44.4	41.	41.9	39.2	46.3	43.3	45.3
meas. $z_4(k)$	38.6	38.	32.6	18.	29.4	23.3	11.	14.6	18.4	20.5

Using the Kalman filter, estimate the actual volume of water r_i for $k \in \{1, 2, \ldots, 10\}$ for all tanks. Also provide the associated uncertainty. Provide this using graphs. Notice that the estimate uncertainty for tank 1 is *lower* than the solution you had for the simpler network – why is this?

(c-iii) We will now repeat the previous problem, except that now the sensor of tank 3 has failed, and thus no longer provides any measurements. Modify your Kalman filter from before (reduce your sensor model to remove this), and run this using the same data as before (except that you remove the measurements $z_3(k)$). Estimate the actual volume of water r_i for $k \in \{1, 2, \ldots, 10\}$ for all tanks, and also provide the associated uncertainty. How does this compare to before? That is, how much has losing the sensor affected your ability to estimate the tank levels?

Problem 5

Optional, extra credit. For the battery system of Problem 3, we now add a voltage sensor which gives us a noisy reading of the battery voltage after each time cycle. The measurement is $z(k) = h(q(k)) + w(k)$, with $w(k) \sim \mathcal{N}\left(0, \sigma_w^2\right)$, and where $h(q)$ is a nonlinear function mapping from current state of charge to voltage, plotted below, and with

$$h(q) = 4 + (q - 1)^3.$$

The battery is subject to the same discharging process as before, with

$$q(k) = q(k{-}1) - j(k{-}1)$$

where $q(0) = 1$ with no uncertainty; and $j(k) = j_0 + v(k)$ with $v(k)$ zero mean with variance σ_v^2
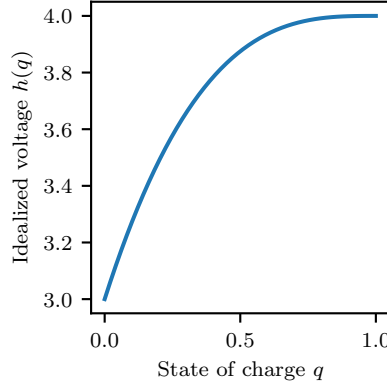
For our system, we set $j_0 = 0.1$, $\sigma_v = 0.05$, and $\sigma_w = 0.1$.

    a. Design an extended Kalman filter (EKF) to estimate the state of charge.

We note that the amount of information that the EKF gets from the measurement is determined by the function h, and specifically its slope with respect to the state, $H = \frac{\partial h}{\partial q}$. Because our system and measurements are scalar, all quantities are also scalar. We can thus reason about how useful a measurement is, by comparing the state variance after getting the measurement P_m to the variance before the measurement, P_p. Note that we're neglecting the time index (k), as all quantities are at the same time step.

    b. Show that the reduction in variance due to a measurement (i.e. a measure of its usefulness) can be described as below:

$$\frac{P_p - P_m}{P_p} = \frac{H^2 P_p}{\sigma_w^2 + H^2 P_p}$$

    where a value of 1 means that the measurement removed all uncertainty, and a value of 0 means that the measurement made no difference to the uncertainty.

    c. Using this metric, make a plot of the usefulness of the voltage measurement as a function of the estimated state of charge, for $\hat{q} \in [0, 1]$ (with usefulness as defined in the subproblem b). Set $P_p = 0.1$, and $\sigma_w^2 = 0.1$. Where is the measurement most informative? Where is it least informative?

For the remainder of the problem, given is the following sequence of measurements:

time k	1	2	3	4	5	6	7	8	9
measurement $z(k)$	4.21	3.83	3.92	3.89	3.88	3.89	3.91	3.57	3.21

    d. Run your extended Kalman filter with this data, and generate two plots: the estimated state of charge, and the variance of this estimate, across k.

    e. After 9 steps, what would the mean and variance be if you did not have the voltage measurements (use your results from Problem 3). How does this compare to your EKF output?