Exercise 4.1.
We define on $\mathbb{Z}^2$ the composition laws $\star, T, \Delta$ by:

1) $\forall x, y \in \mathbb{R}, x \star y = x^2 + y^2$.
2) $\forall x, y \in \mathbb{R}, xTy = xy + (x^2 - 1) + (y^2 - 1)$.
3) $\forall x, y \in \mathbb{R},\ x\Delta y = \log(e^x + e^y)$.

Study the algebraic properties of these laws.

Exercise 4.2.
We define on $G =]-1, 1[$ the following following composition law $\star$ defined by:

$$\forall x, y \in G : x \star y = \frac{x+y}{1+xy}$$

1) Show that the composition law $\star$ is internal in G.
2) Show that $(G, \star)$ is an abelin group.
3) Show that the function $f : G \longrightarrow \mathbb{R}^{+*}$ defined by:

$$\forall x \in G, f(x) = \frac{1+x}{1-x}$$

is a group homomorphism from $(G, \star)$ to $(\mathbb{R}^{+*}, \cdot)$. ($\mathbb{R}^{+*}$ is the set of positive real numbers and "." is the usual multiplication.)

Exercise 4.3*.
Let $(G, .)$ be a group noted multiplicatively. We notice by $Z(G)$ the subset of G defined by:

$$Z(G) = \{a \in G\ ,\ ab = ba,\ \forall b \in G\},$$

and for every element $a \in G$, the set $C(a)$ defined by:

$$C(a) = \{b \in G\ ,\ ab = ba\},$$

-Show that $Z(G)$ and $C(a)$ are subgroups of G.

Exercise 4.4.
Let $\mathbb{R}^2$ equipped with the binary operation $\circledast$ defined by

$$\forall (x, y) \in \mathbb{R}^2, \forall (x', y') \in \mathbb{R}^2, (x, y) \circledast (x', y') = (x + x', y + y' + 2xx')$$

1. Show that $(\mathbb{R}^2, \circledast)$ is a commutative group.
2. Show that $H = \{(x, x^2), x \in \mathbb{R}\}$ is a subgroup of $(\mathbb{R}^2, \circledast)$.
3. Show that the function $\phi : (\mathbb{R}, +) \to (H, \circledast)$, defined by $\phi(x) = (x, x^2)$, is a group isomorphism.

Exercise 4.5*.
We define on $G = \mathbb{Z} \times \mathbb{Z}$ an internal composition law $\star$ by:

$$\forall (m, n), (s, t) \in G, (m, n) \star (s, t) = (m + (-1)^n s, n + t).$$

1) Show that $(G, \star)$ is a non-commutative group.

2) Let $H \subset G$ be defined by:
$$H = \{(m, n) \in G, n = 0\}.$$
Show that H is a subgroup of G.

Exercise 4.6.
Let $H = \{z \in \mathbb{C} \mid z^8 = 1\}$ be a subset of $\mathbb{C}$.

1) Show that $(H, .)$ is a group (8th roots of unit).

2) Prove that the map $f : H \to H$ defined by $f(z) = z^3$ is an automorphism.

Exercise 4.7*.
We define on $\mathbb{Z}^2$ the internal composition laws denoted by $+$ and $\star$ by:
$$\forall (a, b), (c, d) \in \mathbb{Z}^2 : \quad \begin{array}{l} (a, b) \star (c, d) = (ac, ad + bc) \\ (a, b) + (c, d) = (a + c, b + d) \end{array}$$

1) Show that $(\mathbb{Z}^2, +, \star)$ is a commutative ring.

2) Show that $A = \{(a, 0), a \in \mathbb{Z}\}$ is a subring of $(\mathbb{Z}^2, +, \star)$.

Exercise 4.8*.

Let $(A, +, .)$ be a commutative ring. We put:
$$B = \left\{a \in A \ / \ a^2 = a\right\} \text{ where } a^2 = a.a).$$

1) Show that: $a \in B \Rightarrow (1 - a) \in B$.

2) We define on B the internal composition law $\star$ by:
$$\forall a, b \in B, a \star b = a + b - 2ab.$$
Show that $(B, \star, .)$ and a commutative ring.

Exercise 4.9.
Let $\mathbb{Q}\left[\sqrt{2}\right] = \left\{u + v\sqrt{2};\ u, v \in \mathbb{Q}\right\}$ and $\mathbb{Z}\left[\sqrt{2}\right] = \left\{u + v\sqrt{2};\ u, v \in \mathbb{Z}\right\}$.

1) Show that $\left(\mathbb{Q}\left[\sqrt{2}\right], +, .\right)$ is a subfield of $(\mathbb{R}, +, .)$ containing $\mathbb{Q}$.

2) Show that $\left(\mathbb{Z}\left[\sqrt{2}\right], +, .\right)$ is a subring of $\left(\mathbb{Q}\left[\sqrt{2}\right], +, .\right)$.

Exercise 4.10.
Let $\mathbb{Q}[j] = \{a + bj, a, b \in \mathbb{Q}\}$, $j \in \mathbb{C}$, $|j| = 1$ and $j^3 = 1$.

1) Show that $(\mathbb{Q}[j], \star, .)$ is a commutative field (where $+, .$ are the usual addition and multiplication).