Exercise 1.
Answer true or false.

(a) $1 \in \{\{1\},\{2\},\{1,2\}\}$, $\{1\} \in \{\{1\},\{2\},\{1,2\}\}$, $\{1\} \subseteq \{\{1\},\{2\},\{1,2\}\}$
(b) $\{1,2\} \in \{\{1,2\},\{3,4\}\},\{1,2\}\}$, $\{1,2\} \subseteq \{\{\{1,2\},\{3,4\}\},\{1,2\}\}$
(c) $\{3,4\} \in \{\{1,2\},\{3,4\}\},\{1,2\}\}$, $\{3,4\} \subseteq \{\{1,2\},\{3,4\}\},\{1,2\}\}$
(d) $\varnothing \in \{\{11,2\},\{3,4\}\},\{1,2\}\}$, $\varnothing \subseteq \{\{1,2\},\{3,4\}\},\{1,2\}\}$
(e) $\{\varnothing\} \in \{\varnothing,\{\varnothing,\{\varnothing\}\}\}$, $\{\varnothing\} \subseteq \{\varnothing,\{\varnothing,\{\varnothing\}\}\}$, $\varnothing \in \{\varnothing,\{\varnothing,\{\varnothing\}\}$
(f) $\varnothing \subseteq \{\varnothing,\{\varnothing,\{\varnothing\}\}\}$, $0 \subseteq \varnothing$, $\{\varnothing\} \subseteq \{\varnothing,\{\varnothing\}\}$, $\{\varnothing\} \subseteq \{\varnothing,\{\varnothing\}\}$, $\{\varnothing\} \in \mathbf{P}(\varnothing)$.

Exercise 2.
Let $A = [-1,3]$, $B = [2,e[$, $C = \mathbb{Z}^*$, $D = \{a,b,c\}$ and $F = \{0,4\}$ be sets. What are the following sets worth?

1- $A \cap B$, $B \cap C$, $A \cap B \cap C$, $A \cap \overline{C}$, $\overline{A} \cap \overline{B}$, $\overline{A} \cup B$, $A \triangle B$, $\overline{A} \triangle \overline{B}$, $(A-B)-C$.

2- Determine $D \times F$, $D^3 \times F$, $D \times (D \times F)$, $A \times \varnothing$ and $\mathbf{P}(\mathbf{P}(F))$.

Exercise 3.
Let E be a non-empty set and let A, B and C be non-empty subsets of E. Prove that:

1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap B)$.

2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

3) $A \cap \overline{A \cap B} = A \cap \overline{B}$.

4) $A \triangle B = \left(\overline{A} \cap B\right) \cup \left(A \cap \overline{B}\right)$.

5) $\overline{A \triangle B} = \overline{A} \triangle B = A \triangle \overline{B}$.

Exercise 4.
Let $\{A_i : i \in I\}$ be a family of sets. Prove that:
- $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$
- $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$.

Exercise 5.

Let $I_n = [n, n+1]$ and $J_n = [n, n+1], n \in \mathbf{Z}$. Define

$$\mathbb{F}_{n,m} = \{I_n \times J_m : n, m \in \mathbf{Z}\}.$$

Show $\bigcup_{n,m\in\mathbf{Z}} \mathbb{F}_{n,m} = \mathbf{R}^2$ and $\bigcap_{n,m\in\mathbf{Z}} \mathbb{F}_{n,m} = \varnothing$. Is $\mathbb{F}_{n,m}$ pairwise disjoint?

Exercise 6.
Let A, B and C be three parts of a non-empty set E. Prove that:

1) $\left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right) = A \cup B \Leftrightarrow (A \cap B = \varnothing)$.

2) $(A \cup B = B \cap C) \Leftrightarrow (A \subset B \subset C)$.

3) $(A \cap B = A \cup B) \Rightarrow A = B$.

4) $(A \cap B = A \cap C$ and $A \cup B = A \cup C) \Rightarrow B = C$.

5) $(A \triangle B = A \triangle C) \Leftrightarrow C = B$

Exercise 7.
Let $f : E \to F$ be a map and $A_1, A_2 \in P(E)$; $B_1, B_2 \in P(F)$. Prove that:

1) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$.

2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

3) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

4) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

5) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

6) $f(f^{-1}(B_1)) \subseteq B_1$,and $A_1 \subseteq f^{-1}(f(A_1))$.

Exercise 8.
Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by $f(x) = \frac{1}{1+x^2}$, $\forall x \in \mathbb{R}$.

1) Show that f is neither injective nor surjective.

2) Give a source set so that f is injective and a target set so that f is surjective.

3) In which case f is bijective.

4) Let $A = \{-1, 2, 3\}$, $B = [0, 2[$, $C = [-1.0]$.
Determine $f(A)$, $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(C)$.

Exercise 9.
Let $f : E \to F$ and $g : F \to G$ be two maps. Prove that:

1) $g \circ f$ injective $\Rightarrow f$ injective.

2) $g \circ f$ injective and f surjective $\Rightarrow g$ injective.

3) $g \circ f$ surjective $\Rightarrow g$ surjective.

Exercise 10.
Say if the map defined by: $f : \mathbb{R} \to]-1, 1[$, $f(x) = \frac{x}{1+|x|}$, is bijective or not. If yes, find the inverse map of f.

Exercise 11.
We consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by: $f(x) = \frac{3}{e^{2x}-2}$.

1) Say why f is not a map then change the source set in E so that it is.

2) Prove that $f : E \to \mathbb{R}$ is injective.

3) Say why f is not surjective, then change the target set in F so that f is surjective.

4) Say why $f : E \to F$ is bijective then give the inverse map.

Exercise 12.
We consider the two maps:

$$f: \begin{array}{l} \mathbb{R} \to [0, +\infty[\\ x \mapsto f(x) = x^2 \end{array} \qquad g: \begin{array}{l} [0, +\infty[\to \mathbb{R} \\ x \mapsto g(x) = \sqrt{4 + x^2}. \end{array}$$

1) Determine $h = g \circ f$.

2) Say why h is not injective then give a set E such $h : E \to \mathbb{R}$ is injective.

3) Give a set F such that $h : E \to F$ is bijective then determine the inverse map h^{-1}.