NATIONAL HIGHER SCHOOL OF AUTONOMOUS SYSTEMS TECHNOLOGY

First Year

Module: Analysis 1

Chapter 2: Sequences

Mathematical Sequences and their Properties

Topics Covered:

- Convergence and Divergence of Sequences
- Limit Theorems and Properties
- Monotonic and Bounded Sequences
- Recursive Sequences

Contents

1 Introduction to Sequences

1.1 Basic Definitions

Definition 1.1. A sequence is a function whose domain is the set of natural numbers $\mathbb{N}$ and range a sub-set of real numbers $\mathbb{R}$, which is defined as follow

$$\mathbb{N} \to \mathbb{R}$$

$$n \to u_n$$

We denote such a function as $(u_n)_{n\geq 0}$ or simply (u_n).
u_n is called the general term of the sequence (u_n).

Example 1. • $u_n = 1 + \frac{2}{n+1}$, $u_n = (-1)^n \cos(\frac{n\pi}{2})$.
• **Classic Sequences:**

♠ **Arithmetic Sequence with common difference** r Its general term is given by

$$u_n = a + nr \text{ and we have } \sum_{k=1}^{n} u_k = \frac{n}{2}(u_1 + u_2).$$

♠ **Geometric sequence with a common ratio** q. Its general term is given by

$$u_n = aq^n \text{ and we have } \sum_{k=1}^{n} u_k = u_1 \frac{1-q^n}{1-q} \quad (q \neq 1).$$

♠ **First-order recursive sequence** given by

$$\begin{cases} u_0 = a \in \mathbb{R} \\ u_{n+1} = f(u_n) \end{cases}$$

♠ **Second-order recursive sequence** given by

$$\begin{cases} u_0 = \alpha, \quad u_1 = \beta \\ u_{n+1} = au_n + bu_{n+1} \end{cases}$$

1.2 Increasing, Decreasing, Monotonic Sequences

Definition 1.2. We say that the sequence (u_n) is:
• increasing if and only if

$$\forall n \in \mathbb{N}, \quad u_{n+1} \geq u_n$$

• decreasing if and only if

$$\forall n \; in \mathbb{N}, \quad u_{n+1} \leq u_n$$

• strictly increasing if and only if

$$\forall n \in \mathbb{N}, \quad u_{n+1} > u_n$$

• strictly decreasing if and only if

$$\forall n \in \mathbb{N}, \quad u_{n+1} < u_n$$

• monotonic if and only if it is either increasing or decreasing.
• strictly monotonic if and only if it is either strictly increasing or strictly decreasing.

Example 2. Consider the sequence $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$, let's study its monotonicity. We have

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{(n+1)^2} > 0 \quad \forall n \geq 1.$$

The sequence (u_n) is strictly increasing.

1.3 Bounded Sequences

Definition 1.3. Let (u_n) be a sequence of real numbers. We say that the sequence (u_n) is:

• bounded above if the subset of $\mathbb{R}$, $A = \{u_n,\ n \in \mathbb{N}\}$ is bounded above , i.e.,

$$\exists M \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad u_n \leq M$$

• bounded below if the subset of $\mathbb{R}$, $A = \{u_n,\ n \in \mathbb{N}\}$ is bounded below, i.e.,

$$\exists m \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad u_n \geq m$$

• upper bounded if the subset of $\mathbb{R}$, $A = \{u_n,\ n \in \mathbb{N}\}$ is upper bounded, i.e.,

$$\exists M \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad u_n \leq M$$

• bounded if and only if it is both lower and upper bounded.

Example 3. Show that the sequence $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is upper bounded.

We have for $2 \leq k \leq n$

$$\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

Thus,

$$\begin{aligned} u_n \leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} \leq 2 \end{aligned}$$

Thus, for all $n \in \mathbb{N}^\star$, the sequence $u_n \leq 2$.

Proposition 1.1. *The sequence (u_n) is bounded if and only if the sequence $(|u_n|)$ is bounded below, i.e., there exists $M \in \mathbb{R}$ such that $|u_n| \leq M$ for all $n \in \mathbb{N}$.*

Proof. $\Longrightarrow$ The sequence (u_n) is bounded, meaning there exist M and m in $\mathbb{R}$ such that $m \leq u_n \leq M$ for all $n \in \mathbb{N}$. We take $k = \max(|m|, |M|)$, which means that $|u_n| \leq k$.

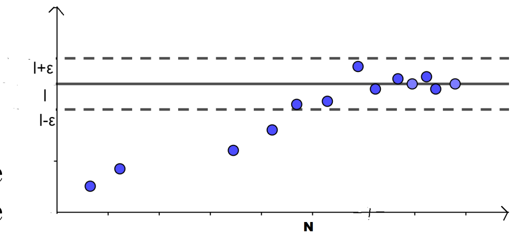
$\Longleftarrow$ The sequence $|u_n|$ is upper bounded, which means there exists $M \in \mathbb{R}$ such that $|u_n| \leq M \Leftrightarrow -M \leq u_n \leq M$, meaning that (u_n) is bounded. □

Note. We say that the sequence (u_n) is bounded (upper bounded, lower bounded) from some term onwards if there exist $N \in \mathbb{N}$ and $M \in \mathbb{R}$ (or $m \in \mathbb{R}$) such that for all $n \geq N$, we have $u_n \leq M$ (or $u_n \geq m$).

2 Convergent Sequence

Definition 2.1. We say that the sequence $(u_n)_{n\geq 0}$ has the limit $l \in \mathbb{R}$ if and only if for every

$$\boxed{\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ such that}}$$
$$\boxed{\forall n \in \mathbb{N}, \ \ n \geq N \text{ implies } |u_n - l| \leq \epsilon}$$



In other words, from a certain rank N, the terms of the sequence (u_n) get closer to the limit l (the terms of the sequence are in the interval $[l-\epsilon, l+\epsilon]$.
If such an l exists, we say that the sequence (u_n) is convergent (or the sequence (u_n) converges to l), and we write

$$\lim_{n\to\infty} u_n = l \text{ or } \lim u_n = l \text{ or } u_n \to l$$

Otherwise, we say that the sequence (u_n) is divergent.

Remark 1. • If the sequence (u_n) has infinity as its limit, we say it diverges, and the divergence is of the first kind.
• If the sequence (u_n) has no limit, we say it diverges, and the divergence is of the second kind.

Example 4. Show that the sequence $u_n = \frac{1}{n}$ $(n \in \mathbb{N}^\star)$ converges to 0.
Let $\epsilon > 0$, we want to find $N = N(\epsilon) \in \mathbb{N}$ such that $|u_n - l| \leq \epsilon$. We often reduce this to solving inequalities.
We have $|u_n - l| = \frac{1}{n} \leq \epsilon \Leftrightarrow n \geq \frac{1}{\epsilon}$ We take $N = E(\epsilon) + 1$. Thus, for $n \geq N$ we have $|u_n - 0| = \frac{1}{n} \leq \frac{1}{N} \leq \epsilon$.

Remark 2. Instead of using a non-strict inequality in the definition of a convergent sequence, we can use a strict inequality, meaning:

$$u_n \to l \Leftrightarrow \forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n \geq N \text{ we have } |u_n - l| < \varepsilon.$$

Indeed, the implication in the reverse direction $\Longleftarrow$ is obvious. For the other direction of the implication ($\Longrightarrow$), let $\varepsilon > 0$, we set $\varepsilon' = \frac{\varepsilon}{2} > 0$ (which is arbitrary), by the definition, we have

$$u_n \to l \Longrightarrow \exists N \in \mathbb{N}, \ \forall n \geq N, \ |u_n - l| \leq \varepsilon' < \varepsilon$$

Theorem 2.1. *The limit $l \in \mathbb{R}$ of a real sequence, if it exists, is unique.*

Proof. We assume by contradiction that the sequence (u_n) has two different limits l_1 and l_2 $(l_1 \neq l_2)$.
Let $\epsilon = \frac{|l_1 - l_2|}{2} > 0$, since $u_n \to l_1$ then there exists $N_1 \in \mathbb{N}$ from which we have $|u_n - l_1| < \epsilon$, and since $u_n \to l_2$ then there exists $N_2 \in \mathbb{N}$ from which we have $|u_n - l_2| < \epsilon$.
Now we consider the integer $N = \max(N_1, N_2)$, from this rank, both of the last two inequalities $|u_n - l_1| < \epsilon$, $|u_n - l_2| < \epsilon$ are satisfied. Using the triangle inequality, we obtain

$$|l_1 - l_2| = |l_1 - u_n + u_n - l_2| \leq |u_n - l_1| + |u_n - l_2| < 2\epsilon = |l_1 - l_2|$$

which is absurd. □

Theorem 2.2. *Every convergent sequence is bounded.*

Proof. Let (u_n) be a real sequence that converges to l. By definition, we have

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N}, \ \forall n \geq N, \quad |u_n - l| \leq \epsilon \Leftrightarrow -\epsilon \leq u_n - l \leq \epsilon \Leftrightarrow l - \epsilon \leq u_n \leq l + \epsilon,$$

which means that $|u_n| \leq |l + \epsilon|$ or $|u_n| \leq |l - \epsilon|$. Therefore, let $M = \max(|l + \epsilon|, |l - \epsilon|)$, we have for every $n \geq N$, $|u_n| \leq M$. The first terms of the sequence remain, so we set $M' = \max(|u_0|, |u_1|, ..., |u_{N-1}|, M)$, we have $|u_n| \leq M'$ for every $n \in \mathbb{N}$. □

3 Operations on Limits

Properties. Let (u_n) and (v_n) be two real sequences with respective limits l_1, l_2. The table below summarizes the properties of the limits of the sum, product, and quotient of two sequences

$\lim u_n$	$\lim v_n$	$\lim(u_n + v_n)$	$\lim(u_n \times v_n)$	$\lim \frac{u_n}{v_n}$
l_1	l_2	$l_1 + l_2$	$l_1 \times l_2$	$\frac{l_1}{l_2}$ $(l_2 \neq 0)$
0	l_2	l_2	0	0
l_1	0	l_1	0	∞
0	0	0	0	F.I
∞	0	∞	F.I	∞
0	∞	∞	F.I	0
l_1	∞	∞	∞	0
∞	l_2	∞	∞	∞
$+\infty$	$+\infty$	$+\infty$	$+\infty$	F.I
$-\infty$	$-\infty$	$-\infty$	$+\infty$	F.I
$+\infty$	$-\infty$	F.I	$-\infty$	F.I

4 Results on Convergent Sequences

4.1 Positive Limit

Proposition 4.1. *Let (u_n) be a convergent real sequence, such that:*

i) There exists $N \in \mathbb{N}$ from which $u_n > 0$

ii) $\lim_{n\to\infty} u_n = l$

Then $l \geq 0$.

Proof Let's assume, by contradiction, that $l < 0$. Set $\varepsilon = -l > 0$. Since $u_n \to l$, there exists $N_1 \in \mathbb{N}$ such that $|u_n - l| \leq \varepsilon$, and there exists N_2 such that for all $n \geq N_2$ we have $u_n > 0$. Let's take $N = \max(N_1, N_2)$, so for all $n \geq N$, we have:

$$l \leq u_n - l \leq -l \text{ and } u_n > 0 \Leftrightarrow 2l \leq u_n \leq 0 \text{ and } u_n > 0$$

which is absurd.

4.2 Passing to the Limit in Inequalities

Proposition 4.2. *Let (u_n) and (v_n) be two real sequences, such that:*

i) $u_n \to l$

ii) $v_n \to l'$

iii) There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $u_n \leq v_n$.

Then $l \leq l'$.

Proof. We can proceed in a similar way to the previous proof. We just need to consider the sequence $w_n = v_n - u_n$.

□

4.3 Squeeze theorem

Theorem 4.3. *Let (u_n), (v_n), and (w_n) be three real sequences, such that:*

i) $\lim v_n = \lim w_n = l$

ii) There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $v_n \leq u_n \leq w_n$.

Then $\lim u_n = l$.

Proof. We have:
- $v_n \to l$, so there exists $N_1 \in \mathbb{N}$ from which we have $|v_n - l| \leq \varepsilon \Leftrightarrow l - \varepsilon \leq v_n \leq l + \varepsilon$
- $w_n \to l$, so there exists $N_1 \in \mathbb{N}$ from which we have $|w_n - l| \leq \varepsilon \Leftrightarrow l - \varepsilon \leq w_n \leq l + \varepsilon$
- There exists $N_3 \in \mathbb{N}$ such that $n \geq N$ implies $v_n \leq u_n \leq w_n$.

Let's take $N = \max(N_1, N_2, N_3)$. Then, for all $n \geq N$, we have:

$$l - \varepsilon \leq v_n \leq u_n \leq w_n \leq l + \varepsilon$$

so

$$-\varepsilon \leq u_n - l \leq \varepsilon$$

Consequently, for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$.

□

Corollary 4.4. *Let (u_n) and (v_n) be two numerical sequences, such that:*

i) The sequence (u_n) converges to 0.

ii) The sequence (v_n) is bounded.

Then, the sequence $w_n = u_n v_n$ converges to 0.

Proof. It is enough to apply Gendarme's Theorem to the sequence (w_n). □

5 Monotone Convergence Criterion

Theorem 5.1. *Let (u_n) be a monotonic sequence, then:*

- *If it is increasing, it is convergent if and only if it is bounded. Moreover,*

$$\lim u_n = \sup\{u_n,\ n \in \mathbb{N}\}.$$

- *If it is decreasing, it is convergent if and only if it is bounded. Moreover,*

$$\lim u_n = \inf\{u_n,\ n \in \mathbb{N}\}.$$

Remark 3. We have
- If the sequence (u_n) is increasing and unbounded, then $\lim u_n = +\infty$.
- If the sequence (u_n) is decreasing and unbounded, then $\lim u_n = -\infty$.
- If the increasing sequence (u_n) converges to l, then for all $n \in \mathbb{N}$, $u_n \leq l$.
- If the decreasing sequence (u_n) converges to l, then for all $n \in \mathbb{N}$, $u_n \geq l$.

Proof. We define A as a subset of $\mathbb{R}$ by $A = \{u_n,\ n \in \mathbb{N}\}$. A is bounded, so according to the supremum property, it has a supremum l. We will show that $l = \lim u_n$.

Using the supremum property, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $l - \varepsilon < u_N \leq l$. As the sequence (u_n) is increasing, for all $n \geq N$ we have $u_n \geq u_N$. Since l is an upper bound for A, we have:

$$l - \varepsilon \leq u_N \leq u_n \leq l \Leftrightarrow -\varepsilon \leq u_n - l \leq 0 \Leftrightarrow |u_n - l| \leq \varepsilon.$$

□

Example 5. 1- We have seen in the previous examples that the sequence $u_n = \sum_{k=0}^{n} \frac{1}{n^2}$ is bounded by 2 and increasing. Therefore, according to the monotone sequence convergence criterion, the sequence (u_n) is convergent.

2- Study the nature of the numerical sequence (u_n) defined as:

$$u_n = \sum_{k=1}^{n} \frac{1}{n+k}$$

We have:

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{n+1+k} - \sum_{k=1}^{n} \frac{1}{n+k} = -\frac{1}{n+1} + \frac{1}{2n+2} + \frac{1}{2n+1} = \frac{1}{2(n+1)(2n+1)} > 0 \text{ for all } n \in \mathbb{N}$$

So, the sequence (u_n) is strictly increasing. Furthermore, we have, for all $1 \leq k \leq n$:

$$u_n \leq \sum_{k=1}^{n} \frac{1}{n+1} = \frac{n}{n+1} < 1$$

Conclusion: The sequence (u_n) is increasing and bounded, so it converges.
Using the following inequalities:

$$\ln\left(\frac{x+1}{x}\right) \leq \frac{1}{x} \leq \ln\left(\frac{x}{x-1}\right) \text{ for all } x > 1$$

Calculate the limit of (u_n).
For $1 \leq k \leq n$, we have:

$$\ln\left(\frac{n+k+1}{n+k}\right) \leq \frac{1}{n+k} \leq \ln\left(\frac{n+k}{n+k-1}\right)$$

Summing for k from 1 to n, we get:

$$\sum_{k=1}^{n} \ln\left(\frac{n+k+1}{n+k}\right) \leq u_n \leq \sum_{k=1}^{n} \ln\left(\frac{n+k}{n+k-1}\right)$$

$$\Leftrightarrow \sum_{k=1}^{n} \ln(n+k+1) - \ln(n+k) \leq u_n \leq \sum_{k=1}^{n} \ln(n+k) - \ln(n+k-1)$$

The sequences $v_n = \sum_{k=1}^{n} \ln(n+k+1) - \ln(n+k)$ and $w_n = \sum_{k=1}^{n} \ln(n+k) - \ln(n+k-1)$ are called telescopic series, and we can easily calculate their sums. In fact, we have:

$$v_n = \sum_{k=1}^{n} \ln(n+k+1) - \ln(n+k) = (\ln(n+2) - \ln(n+1)) + (\ln(n+3) - \ln(n+2)) + ...$$

$$+(\ln(2n) - \ln(2n-1)) + (\ln(2n+1) - \ln(2n)) = \ln(2n+1) - \ln(n+1) = \ln(\frac{2n+1}{n+1}) \to \ln(2)$$

Doing the same for the sequence (w_n), we find that $w_n \to \ln(2)$. According to the squeeze theorem, we deduce that the sequence (u_n) converges to $\ln(2)$.

6 Adjacent Sequences

Definition 6.1. Let (u_n) and (v_n) be two real sequences. We say that they are adjacent if and only if: 1. (u_n) is increasing.
2. (v_n) is decreasing.
3. $\lim(u_n - v_n) = 0$.

Theorem 6.1. *If (u_n) and (v_n) are two adjacent sequences, then:*
- *The sequences (u_n) and (v_n) converge to the same limit.*
- *For all $n \in \mathbb{N}$, we have $u_n \leq l \leq v_n$, where l is the common limit.*

Proof. Let (u_n) be an increasing sequence, and (v_n) be a decreasing sequence. We have:

$$u_n \leq u_{n+1} \Leftrightarrow -u_n \geq -u_{n+1} \text{ and } v_n \geq v_{n+1}$$

Thus, $v_n - u_n \geq v_{n+1} - u_{n+1}$, which means that the sequence $v_n - u_n$ is decreasing. Since $\lim(u_n - v_n) = 0$, we have $v_n - u_n \geq 0 \Leftrightarrow v_n \geq u_n$.

We now have:

$$u_n \leq v_n \leq v_0 \longrightarrow \text{The sequence } (u_n) \text{ is increasing and bounded, so it converges to } l$$

and

$$v_n \geq u_n \geq u_0 \longrightarrow \text{The sequence } (v_n) \text{ is decreasing and bounded, so it converges to } l'.$$

Since $\lim(v_n - u_n) = 0$, we have $l - l' = 0 \Leftrightarrow l = l'$.

□

Example 6. Consider the sequence $w_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Show that the sequences $u_n = w_{2n}$ and $v_n = w_{2n+1}$ are adjacent.

Monotonicity of (u_n) and (v_n): We have:

$$u_{n+1} - u_n = w_{2n+2} - w_{2n} = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \frac{1}{2n+1} - \frac{1}{2n+2} \geq 0, \quad \forall n \in \mathbb{N}^*$$

and

$$v_{n+1} - v_n = w_{2n+3} - w_{2n+1} = \sum_{k=1}^{2n+3} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k} = -\frac{1}{2n+2} + \frac{1}{2n+3} \leq 0, \quad \forall n \in \mathbb{N}^*$$

Thus, the sequence (u_n) is increasing, and the sequence (v_n) is decreasing.

Furthermore, we have:

$$u_n - v_n = \frac{1}{2n+1} \to 0$$

Therefore, the sequences (u_n) and (v_n) are adjacent, and they converge to the same limit.

7 Subsequence of a Sequence

Definition 7.1. Let (u_n) be a numerical sequence. We call a subsequence of (u_n) a sequence $(u_{\phi(n)})$ where $\phi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

Example 7. • For $u_n = \frac{1}{n}$, we have $u_{n^2} = \frac{1}{n^2}$.
• For $u_n = (-1)^n$, we have $u_{2n} = 1$ and $u_{2n+1} = -1$.
• For $u_n = \sin(\frac{2n\pi}{17})$, we have $u_{17n} = \sin(2n\pi) = 0$ and $u_{17n+1} = \sin(\frac{2\pi}{17}) \neq 0$.

Proposition 7.1. *Let (u_n) be a numerical sequence. Then, (u_n) converges to l if and only if every subsequence of (u_n) also converges to l.*

Proof. $\Longleftarrow$ This direction is obvious. We just need to take the subsequence $(u_{\phi(n)})$ with $\phi(n) = n$.

$\Longrightarrow$ To prove this direction, we need to show that for every $n \in \mathbb{N}$, we have $\phi(n) \geq n$. We will proceed by induction. For $n = 0$, we have $\phi(0) \geq 0$ because $\phi(0) \in \mathbb{N}$.

Suppose that $\phi(n) \geq n$ and let's show that $\phi(n+1) \geq n+1$. Since ϕ is strictly increasing, for $n+1 > n$, we have $\phi(n+1) > \phi(n) \geq n$. Thus, $\phi(n+1) \geq n+1$ $(\phi(n+1), n+1 \in \mathbb{N})$.

The sequence (u_n) converges to l, which means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$. But we also have $\phi(n) \geq n \geq N$, so there exists $N \in \mathbb{N}$ such that $\phi(n) \geq N$, and in this case, we have $|u_{\phi(n)} - l| \leq \varepsilon$.

This proposition is often used to show that a sequence (u_n) is divergent.

□

Corollary 7.2. *Let (u_n) be a numerical sequence. If subsequence $(u_{\psi(n)})$ diverges, then the sequence (u_n) also diverges.*

Example 8. Consider the sequence $u_n = \cos(\frac{n\pi}{3})$. If we take the subsequence (u_{3n}), we have $u_{3n} = \cos(n\pi) = (-1)^n$, which diverges. Thus, the sequence (u_n) also diverges.

Proposition 7.3. *Let (u_n) be a numerical sequence, and let $l \in \mathbb{R}$. Then:*

$$\lim u_n = l \Leftrightarrow \lim u_{2n} = \lim u_{2n+1} = l$$

Proof. $\Longrightarrow$ If (u_n) converges to l, i.e., for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$. As $2n+1 > 2n > n \geq N$, we have $|u_{2n} - l| \leq \varepsilon$ and $|u_{2n+1} - l| \leq \varepsilon$, which means that (u_{2n}) and (u_{2n+1}) converge to l.

$\Longleftarrow$ We have:

$$u_{2n} \to l \Leftrightarrow \forall \varepsilon > 0, \ \exists N_1 \in \mathbb{N}, \ \forall n \geq N_1, \ |u_{2n} - l| \leq \varepsilon$$

and

$$u_{2n+1} \to l \Leftrightarrow \forall \varepsilon > 0, \ \exists N_2 \in \mathbb{N}, \ \forall n \geq N_2, \ |u_{2n+1} - l| \leq \varepsilon$$

Let $N = \max(N_1, N_2)$. For any $n > 2N$, we can distinguish two cases:

• If n is even, i.e., $n = 2p$, for $n > 2N \Leftrightarrow p > N$, we have $|u_n - l| = |u_{2p} - l| \leq \varepsilon$.

• If n is odd, i.e., $n = 2p+1$, for $n > 2N \Leftrightarrow 2p+1 > 2N \Leftrightarrow p \geq N$, we have $|u_n - l| = |u_{2p+1} - l| \leq \varepsilon$.

□

Theorem 7.4. ***(Bolzano-Weierstrass)*** *From any bounded numerical sequence, a convergent subsequence can be extracted.*

8 Recursive Sequences

In this section, we study recurrent numerical sequences of the form:

$$\begin{cases} u_0 \text{ given} \\ u_{n+1} = f(u_n) \end{cases}$$

Theorem 8.1. ***(Fixed Point)*** *Let I be a non-empty closed interval in $\mathbb{R}$ and $f : I \to I$ be a continuous function. If the sequence (u_n) defined by the recurrence relation $u_{n+1} = f(u_n)$ converges to l, then its limit l is a fixed point of f, meaning $l = f(l)$.*

To study the convergence of the sequence (u_n), we will use the theorem of monotone sequences. First, we examine the sign of $u_{n+1} - u_n$ in several cases.

First Case: Direct Analysis

Example 9. Consider the sequence (u_n):

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \frac{u_n}{u_n^2+1}, \ \forall n \in \mathbb{N} \end{cases}$$

• Possible limits: If (u_n) converges to l, then $l = \frac{l}{l^2+1}$, which leads to $l = 0$.

• It can be easily verified that $u_n \geq 0$ for all $n \in \mathbb{N}$.

• We have $u_{n+1} - u_n = -\frac{u_n^3}{u_n^2+1} \leq 0$.

Conclusion: The sequence (u_n) is decreasing and bounded below. Therefore, by the theorem of monotone sequences, (u_n) converges, and it converges to 0.

Second Case: f is Increasing

Proposition 8.2. *Let I be a closed interval in $\mathbb{R}$. If $f : I \to I$ is continuous and increasing, then for any $u_0 \in I$, the recurrent sequence (u_n) is monotone and converges to $l = f(l)$.*

rem.
• If $u_0 \geq u_1$, then $f(u_0) \geq f(u_1) \Leftrightarrow u_1 \geq u_2 ... \Rightarrow u_n \geq u_{n+1}$, so (u_n) is decreasing.
• If $u_0 \leq u_1$, then $f(u_0) \leq f(u_1) \Leftrightarrow u_1 \leq u_2 ... \Rightarrow u_n \leq u_{n+1}$, so (u_n) is increasing.
• The assumption $f : I \to I$, i.e., that the interval I is stable under the function f, is essential for the study of the recurrent sequence (u_n). It gives meaning to the terms of the sequence (u_n).

Example 12. Consider the recurrent sequence:

$$\begin{cases} u_0 \in \mathbb{R}_+ \\ u_{n+1} = \frac{1}{6}(u_n^2 + 8), \ \forall n \in \mathbb{N} \end{cases}$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as $f(x) = \frac{1}{6}(x^2 + 8)$.
• **Possible limits:** The function f has two fixed points, 2 and 4. If (u_n) converges, it will automatically converge to one of these fixed points.
• **Stability of the interval:** The function f is increasing on $\mathbb{R}_+$.

x	0		$+\infty$
$f(x)$	$\frac{4}{3}$	$\nearrow$	$+\infty$

[Table showing that $f([0, +\infty[) = [\frac{4}{3}, +\infty[\subset [0, +\infty[$] This means that the interval $[0, +\infty[$ is stable under the function f. Furthermore, f is strictly increasing on $[0, +\infty[$, so the direction of change of the sequence (u_n) depends on the sign of $u_1 - u_0$.
• **Sign of $f(x) - x$:** We have $f(x) - x = 0$ implies $x = 2$ or $x = 4$.

x	0		2		4		$+\infty$
$f(x) - x$		$+$	0	$-$	0	$+$	

[Table showing that $f(x) - x$ is negative between 0 and 2 and positive between 2 and 4] The sign of $u_1 - u_0$ depends on the position of u_0 with respect to 2 and 4. There are three cases to consider.
i)- If $u_0 \in [0, 2]$, then $u_1 - u_0 = f(u_0) - u_0 \geq 0$, so $u_1 \geq u_0$, which makes the sequence (u_n) increasing. For all $n \in \mathbb{N}$, we have $u_n \in [0, 2]$ (it comes from stability), meaning that (u_n) is bounded below by 2. Therefore, (u_n) converges to $l \in [2, u_0[$. Since $l = 2$ or 4, then $l = 2$."
ii)- If $u_0 \in [2, 4[$, then $u_1 - u_0 = f(u_0) - u_0 \leq 0$, so $u_1 \leq u_0$, which makes the sequence (u_n) decreasing. For every $n \in \mathbb{N}$, we have $u_n \in [2, 4[$ (which comes from stability), meaning that (u_n) is bounded below by 2. Therefore, (u_n) converges to $l \in [2, u_0[$. Since $l = 2$ or 4, then $l = 2$.
iii)- If $u_0 = 4$, then $u_1 = u_0 = 4$, and the sequence (u_n) is constant and converges to 4.
iv)- If $u_0 \in]4, +\infty[$, then $u_1 - u_0 = f(u_0) - u_0 \geq 0$, so $u_1 \leq u_0$, which makes the sequence

(u_n) increasing. Since we know that if the sequence (u_n) converges, it will converge to 2 or 4, and $u_n > 4$, this is contradictory. Therefore, the sequence (u_n) diverges, and $\lim u_n = +\infty$.

Case 3: f **is decreasing**

Proposition 8.3. *Let I be a closed interval $\mathbb{R}$. Let $f : I \to I$ be a continuous function and decreasing, then*
• *$f \circ f$ is increasing, hence the subsequences $u_{2n} = f \circ f(u_{2n-2})$ et $u_{2n+1} = f \circ f(u_{2n-1})$ are monotones.*
• *(u_{2n}) converges to $l = f \circ f(l)$ and (u_{2n+1}) converges to $l' = f \circ f(l')$.*
Si $l = l'$ then the sequence (u_n) converges.

Example 13 Let the sequence

$$\begin{cases} u_0 \in \mathbb{R}_+ \\ u_{n+1} = \frac{2}{1+u_n^2}, \ \forall n \in \mathbb{N} \end{cases}$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = \frac{2}{1+x^2}$.

- **Possible limits:** The function f has a unique fixed point $x_0 = 1$. If (u_n) converges, it can only be towards 1.

- **Stability of the interval:** We have

$$f'(x) = -\frac{4x}{(1+x^2)^2} \leq 0, \quad \forall x \geq 0.$$

  Thus f is decreasing on $\mathbb{R}_+$.

x	0	$+\infty$
$f(x)$	2 ↘	0

  We have $f([0, +\infty[) = [2, 0[\subset [0, +\infty[$, so for all $u_0 \geq 0$ we have $u_n \geq 0$ for all $n \in \mathbb{N}$.

  We conclude that the subsequences (u_{2n}) and (u_{2n+1}) are monotonic. The direction of monotonicity of (u_{2n}) depends on the sign of $u_2 - u_0$ and the direction of monotonicity of (u_{2n+2}) depends on the sign of $u_3 - u_1$. For this, we will study the sign of $f(f(x)) - x$.

- **Sign of** $f(f(x)) - x$**:** We have

$$g(x) = f(f(x)) = \frac{2}{1 + \left(\frac{2}{1+x^2}\right)^2} = \frac{2(1+x^2)^2}{(1+x^2)^2 + 4} > 0.$$

  Now we compute

$$g(x) - x = -\frac{(x-1)^3(x^2+x+2)}{(1+x^2)^2+4}.$$

x	0		1		$+\infty$
$g(x)-x$		$+$	0	$-$	

i) Let $u_0 \in [0,1]$, we have $u_2 - u_0 = g(u_0) - u_0 \geq 0$ so the sequence (u_{2n}) is increasing and for $u_0 \leq 1 \Longrightarrow f(u_0) \geq f(1) = 1$, that is $u_1 \geq 1$, we therefore have $u_3 - u_1 = g(u_1) - u_1 \leq 0 \Rightarrow u_3 \leq u_1$, thus the sequence (u_{2n+1}) is decreasing.

Moreover, since $u_0 \leq u_1 \Rightarrow f(u_0) \geq f(u_1) \Rightarrow f(f(u_0)) \leq f(f(u_1))$, applying f $2n$ times we obtain $u_{2n} \leq u_{2n+1}$.

Consequently $u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq u_{2n+1} \leq \cdots \leq u_1$, we have the sequence (u_{2n}) is increasing and bounded above so it converges to $l = g(l)$ and the sequence (u_{2n+1}) is decreasing and bounded below so it converges to $l' = g(l')$. Since the function g has only one fixed point then $l = l' = 1$.

Conclusion: The subsequences (u_{2n}) and (u_{2n+1}) converge to the same limit $l = 1$, so (u_n) converges to 1.

ii) $u_0 \in [1, +\infty[$ then $u_1 = f(u_0) \in [0,1]$, same as the previous case, it suffices to swap u_1 with u_0.

Example 10. Study the nature of the sequence

$$\begin{cases} u_0 \in \mathbb{R}_+ \\ u_{n+1} = 1 + \dfrac{1}{u_n}, \forall n \in \mathbb{N} \end{cases}$$

Correction of some exercises Exercise 2.

$$a_n = \frac{\sin\Big(\exp(\sum_{k=1}^{n}\frac{k^2+2^k}{k!})\Big)}{n^2};$$

• The sequence $u_n = \sin\Big(\exp(\sum_{k=1}^{n}\frac{k^2+2^k}{k!})\Big)$ is a bounded sequence ($-1 \leq u_n \leq 1$ $\forall n \in \mathbb{N}$)

• The sequence $v_n = \frac{1}{n^2}$ converges to 0.

Then the sequence $a_n = u_n v_n$ converges to 0.

$$b_n = \frac{\sum_{k=1}^{n}\frac{1}{2^k}}{\sum_{k=1}^{n}\frac{1}{3^k}}$$

The sequences $\sum_{k=1}^{n}\frac{1}{2^k}$; $\sum_{k=1}^{n}\frac{1}{3^k}$ are geometric sums of a common ration $\frac{1}{2}$ and $\frac{1}{3}$ respectively. So we have

$$\sum_{k=1}^{n}\frac{1}{2^k} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2(1-\left(\frac{1}{2}\right)^n) \text{ and } \sum_{k=1}^{n}\frac{1}{3^k} = \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} = \frac{3}{2}(1-\left(\frac{1}{3}\right)^n\Big)$$

As we have $\lim\left(\frac{1}{2}\right)^n = \lim\left(\frac{1}{3}\right)^n = 0$. Then

$$\lim b_n = \lim \frac{4}{3}\frac{1-\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{3}\right)^n} = \frac{4}{3}.$$

$$c_n = \left(1+\frac{2}{n^2}\right)^{n^2} = \exp\left(n^2\ln\left(1+\frac{2}{n^2}\right)\right)$$

Here we use the usual limit

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

by taking $x = \frac{1}{n^2} \to 0$, so we have

$$c_n = \exp\left(n^2\Big(\frac{2}{n^2}\frac{\ln\left(1+\frac{2}{n^2}\right)}{\frac{2}{n^2}}\Big)\right) = \exp\left(2\frac{\ln\left(1+\frac{2}{n^2}\right)}{\frac{2}{n^2}}\right) \to e^2$$

The limit of c_n is finite so (c_n) is convergent.

$$d_n = \frac{1}{n^7}\sum_{k=1}^{n}k^5$$

We have

$$1 \le k \le n \Leftrightarrow 1 \le k^5 \le n^5 \Leftrightarrow \sum_1^n 1 \le \sum_1^n k^5 \le \sum_1^n n^5 \Leftrightarrow n \le \sum_1^n k^5 \le n^6 \Leftrightarrow \frac{1}{n^6} \le d_n \le \frac{1}{n}.$$

The sequences $\frac{1}{n}$ and $\frac{1}{n^6}$ converges to the same limit 0. Then by the squeeze theorem we deduce $\lim d_n = 0$ (converges).

$$e_n = \frac{\left[(5n-\frac{1}{2})^2\right]}{\left[(4n+\frac{1}{2})^2\right]} = \frac{\left[25n^2-5n+\frac{1}{4}\right]}{\left[16n^2+4n+\frac{1}{4}\right]} = \frac{25n^2-5n+\left[\frac{1}{4}\right]}{16n^2+4n+\left[\frac{1}{4}\right]} = \frac{25n^2-5n}{16n^2+4n} \to \frac{25}{16}.$$

$$f_n = \frac{1}{n^2}\sum_{k=1}^n \left[kx\right] \quad (x \in \mathbb{R}).$$

By definition and for $k = 1, ..., n$ we have

$$\begin{aligned} kx - 1 < \left[kx\right] \le kx \quad &\Leftrightarrow \quad \sum_1^n kx - \sum_1^n 1 < \sum_1^n \left[kx\right] \le \sum_1^n (kx) \\ &\Leftrightarrow \quad x\sum_1^n k - \sum_1^n 1 < \sum_1^n \left[kx\right] \le x\sum_1^n k \\ &\Leftrightarrow \quad x(\frac{n(n+1)}{2}) - n \le \sum_1^n \left[kx\right] \le x(\frac{n(n+1)}{2}) - n \\ &\Leftrightarrow \quad \frac{x}{n^2}(\frac{n(n+1)}{2}) - \frac{1}{n} \le f_n \le \frac{x}{n^2}(\frac{n(n+1)}{2}) \end{aligned}$$

As we have

$$\lim\left(\frac{x}{n^2}(\frac{n(n+1)}{2}) - \frac{1}{n}\right) = \lim \frac{x}{n^2}(\frac{n(n+1)}{2}) = \frac{x}{2}$$

and by the squeeze theorem, we get $\lim f_n = \frac{x}{2}$.

II-II- Determine $a, b \in \mathbb{R}$ such that

$$\frac{1}{k^2+3k+2} = \frac{a}{k+1} + \frac{b}{k+2}$$

By identification we get $a = 1$ and $b = -1$. We write

$$u_n = \sum_{k=0}^n \frac{1}{k^2+3k+2} = \sum_{k=0}^n \left(\frac{1}{k+1} - \frac{1}{k+2}\right).$$

The sequence (u_n) is a telescopic sum.

Telescopic sum A telescopic sum is a serie in the form:

$$\sum_{k=1}^n (a_k - a_{k+1})$$

When expanded, it looks like this:

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_n - a_{n+1})$$

As you can see, almost all intermediate terms cancel out, leaving only the first and last terms:

$$= a_1 - a_{n+1}$$

Example:
Consider the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

which can be rewritten as:

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

This expansion will lead to a telescoping effect where most terms cancel, leaving only:

$$= 1 - \frac{1}{n+1}$$

So we have

$$u_n = 1 - \frac{1}{n+2} \to 1.$$