National Higher School of Autonomous Systems Technology

Exercise Set 2 : Sequences

Exercise 1 – TD

Let (u_n), (v_n), and (w_n) be three numerical sequences. Determine whether the following statements are true or false. Justify the true statements and provide counterexamples for the false statements.

- (u_n) converges if and only if (u_n) is bounded.
- If (u_n) is monotonic, then it converges.
- If $|u_n|$ converges to 0, then (u_n) converges to 0 as well.
- If $|u_n + v_n|$ converges, then $|u_n|$ and $|v_n|$ converge.
- If $|u_n|$ and $|v_n|$ converge, then $|u_n + v_n|$ converges.
- If $u_n \leq v_n \leq w_n$ and both (u_n) and (w_n) converge, then (v_n) converges as well.
- If (u_n) has a convergent subsequence, then (u_n) converges.
- If (u_n) converges to l, then (u_{n^2}) converges to l^2.
- If (u_n) converges to l and (v_n) converges to l', then $\max(u_n, v_n)$ converges to $\max(l, l')$.
- If (u_n) is increasing and (u_{2n}) converges, then (u_n) converges.

Exercise 2

Calculate, if it exists, the limit of the following sequences :

- **Course**

$$a_n = \frac{\sin\left(\exp\left(\sum_{k=1}^{n} \frac{k^2+2^k}{k!}\right)\right)}{n^2}; \quad b_n = \frac{\sum_{k=1}^{n} \frac{1}{2^k}}{\sum_{k=1}^{n} \frac{1}{3^k}}; \quad c_n = \left(1+\frac{2}{n^2}\right)^{n^2};$$

$$d_n = \frac{1}{n^7}\sum_{k=1}^{n} k^5; \quad e_n = \frac{E\left((5n-\frac{1}{2})^2\right)}{E\left((4n+\frac{1}{2})^2\right)}; \quad f_n = \frac{1}{n^2}\sum_{k=1}^{n} E(kx) \quad (x \in \mathbb{R});$$

- **TD**

$$a_n = \frac{n^2+(-1)^n\sqrt{n}}{n^2+n+1}; \quad b_n = \frac{(n+1)!+(n-1)!}{(n+2)!}; \quad c_n = \sum_{k=2}^{n} \frac{1}{k^2-1}; d_n = \sum_{k=0}^{n} \frac{1}{C_n^k}$$

$$e_n = \frac{a^n+b^n}{a^n-b^n} \quad (a, b > 0); \quad f_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}; \quad j_n = \sum_{k=n}^{2n} \frac{k}{\sqrt{n^2+k^2}}; \quad h_n = \sqrt{n}\ln\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right)$$

•**Course**

II- Determine $a, b \in \mathbb{R}$ such that

$$\frac{1}{k^2+3k+2} = \frac{a}{k+1} + \frac{b}{k+2}$$

Consequently, deduce the limit of the sequence

$$u_n = \sum_{k=0}^{n} \frac{1}{k^2+3k+2}.$$

Exercise 3 – **TD**

Let (u_n) be a sequence such that $\lim \dfrac{u_n}{1+u_n} = 0$.
Show that $\lim u_n = 0$.

Exercise 4 – **Course**

Let $a \in \mathbb{R}$ and for $n \in \mathbb{N}$ we define

$$P_n = \prod_{k=1}^{n} \cos(\frac{a}{2^k})$$

Show that $\sin(\dfrac{a}{2^n})P_n = \dfrac{1}{2^n}\sin a$, and determine $\lim P_n$.

Exercise 5 – **TD**

Determine the limit of the numerical sequence (u_n) defined by

$$u_n = \prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)$$

Hint : For all $x \geq 0$, we have

$$x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x.$$

Exercise 6 – **TD**

1- Study the convergence of the sequence (v_n) defined by : $u_n = q^n$ $(q \in \mathbb{R})$.
2-Show that the sequence with general term $v_n = \cos n$ for all $n \in \mathbb{N}$ diverges.

Exercise 7 –

• **TD**

Show, using the criterion of monotone sequences, that the given sequences below are convergent.

$$u_n = \sum_{k=1}^{n} \frac{n+k}{n+k+1}; \qquad v_n = \prod_{k=2}^{2n} \left(1 - \frac{1}{k+1}\right).$$

•**Course**

Show, using the criterion of monotone sequences, that the given sequences below are convergent.

$$u_n = \sum_{k=1}^{n} \frac{1}{n+k}$$

Deduce the limit of the sequence (u_n) using the upper bounds.

$$\ln\left(\frac{x+1}{x}\right) \leq \frac{1}{x} \leq \ln\left(\frac{x}{x-1}\right), \quad \forall x > 1.$$

Exercise 8 – TD

Consider the numerical sequences (u_n), (v_n), and (w_n) defined for $n \in \mathbb{N}^\star$ by

$$u_n = \frac{1! + 2! + ... + n!}{(n-1)!}; \qquad v_n = \frac{1! + 2! + ... + n!}{n!}; \qquad w_n = \frac{1! + 2! + ... + n!}{(n+1)!}$$

Show that for all $n \in \mathbb{N}^\star$,

$$1 \leq v_n \leq 1 + \frac{1}{n} + \frac{n+2}{n(n+1)}.$$

Deduce the limits of the sequences (v_n), (u_n), and (w_n).

Exercise 9 – TD

Calculate, if it exists, the limit of the following numerical sequences.

$$\frac{\sin(\frac{1}{n})}{n}; \qquad \sin(\frac{2n\pi}{3}); \qquad n\sin(\frac{1}{n}); \qquad \frac{\sin n}{n}; \qquad n\sin n; \qquad a^n - (-a)^n \ (a \in \mathbb{R}).$$

Exercise 10 – TD

1. Let $a > 0$, the numerical sequence (u_n) is defined by

$$u_n = (1+a)(1+a^2)...(1+a^n)$$

• Examine the nature of the sequence (u_n) depending on the values of a.
Hint : Use the inequality $1 + x \leq e^x$ for all $x \in \mathbb{R}$.

Exercise 11 – TD

Show in each case that the sequences (u_n) and (v_n) are adjacent.

- $u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} \qquad v_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n+1}$

- $u_n = \prod_{k=1}^{n} (1 + \frac{1}{kk!}) \qquad v_n = (1 + \frac{1}{nn!})u_n \ \forall n \geq 1$

- $u_n = \sum_{k=1}^{n-1} \frac{1}{k^2(k+1)^2} \qquad v_n = u_n + \frac{1}{3n^2}\ \forall n \geq 2$

Exercise 12 – Course

Let (u_n) be a numerical sequence defined by

$$u_n = 1 - \frac{1}{2!} + \frac{1}{4!} - ... + \frac{(-1)^n}{(2n)!}$$

1- Show that the sequences (u_{2n}) and (u_{2n+1}) are adjacent.
2- Deduce the nature of the sequence (u_n).

Exercise 13 – TD

Let (u_n) be a numerical sequence such that the sub-sequences (u_{2n}), (u_{2n+1}), and (u_{3n}) converge. Show that the sequence (u_n) converges.

Exercise 14 – TD

Let (u_n) and (v_n) be two recursively defined sequences by

$$u_n = \begin{cases} u_0 > 0 \\ u_{n+1} = \sqrt{u_n v_n} \end{cases} \text{ and } v_n = \begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{u_n + v_n}{2} \end{cases}$$

Show that these two sequences are convergent and converge to the same limit.

Exercise 15

Examine the nature of the following recursively defined sequences.

TD

$$\begin{cases} u_0 = 3 \\ \\ u_{n+1} = \frac{1}{2}(u_n^2 + 1)\ (\forall n \in \mathbb{N}) \end{cases} ; \quad \begin{cases} u_0 \in \mathbb{R} \\ \\ u_{n+1} = \dfrac{u_n^3 + 6u_n}{3u_n^2 + 2}\ (\forall n \in \mathbb{N}) \end{cases}$$

Course

$$\begin{cases} u_0 \in [0, 1] \\ \\ u_{n+1} = \dfrac{2u_n}{1 + u_n^2}\ (\forall n \in \mathbb{N}) \end{cases}$$