National Higher School of Autonomous Systems Technology

First year **2024/2025**

Module: Analysis 1
Exercise Set 1 : Real Numbers

Only a few exercises will be covered during the tutorial session.

Exercise 1.
Find $a,\ b,\ c \in \mathbb{Z}$ such that for all $x \in \mathbb{R}$ one have

$$(x-a)(x-10)+1=(x+b)(x+c)$$

Exercise 2. Solve in $\mathbb{R}$ the following equations and inequations

a) $\sqrt{x+3} > x-1$ **b)** $\left|x^2+x-1\right| = 3-x$ **c)** $e^{2x}-e^{x+2}-e^{2-x}+1<0$ **d)** $\sqrt{6-x}+\sqrt{3-x}=\sqrt{x+5}+\sqrt{4-3x}$

e) $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$ **f)** $3x^2-3x-4\sqrt{x^2-x+3}=6$ **g)** $|x-2|+|2x-5|<|x+1|$

h) $\begin{cases} x^2+xy+y=3 \\ y^2+xy+x=-1 \end{cases}$

Exercise 3. Let α, β and γ be three reals such that

$$\alpha = 20+14\sqrt{2},\quad \beta = 20-14\sqrt{2},\quad \gamma=\sqrt[3]{\alpha}+\sqrt[3]{\beta}$$

Show that γ is solution of $\gamma^3-6\gamma-40=0$ and that $\gamma \in \mathbb{Q}$.

Exercise 4.

I Let A be the real number defined as

$$A=\sqrt{a+2\sqrt{a-1}}+\sqrt{a-2\sqrt{a-1}}$$

where $a \in [1,+\infty[$ rewrite A in the simple form.

II Show that

$$\frac{\sqrt[3]{54\sqrt{3}+41\sqrt{5}}}{\sqrt{3}}+\frac{\sqrt[3]{54\sqrt{3}-41\sqrt{5}}}{\sqrt{3}} \in \mathbb{Z}$$

Hint: set $u=\dfrac{\sqrt[3]{54\sqrt{3}+41\sqrt{5}}}{\sqrt{3}}$ and $v=\dfrac{\sqrt[3]{54\sqrt{3}-41\sqrt{5}}}{\sqrt{3}}$ then compute u^3+v^3 and u^3v^3.

Exercise 5.
Prove directly the following assertions

$$a)\ \forall n \in \mathbb{N}^\star \quad \sum_{k=1}^{n}(k!\times k)=(n+1)!-1$$

$$b)\ \forall n \in \mathbb{N}^\star \quad \sum_{k=1}^{n}\frac{1}{k} \leq 2\sqrt{n}$$

Exercise 6. Prove by induction that

1. For all $n \in \mathbb{N}$, 3 divides n^3-n.
2. $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}$: $|\sin(nx)| \leq n|\sin x|$.

3. $\forall n \in \mathbb{N}^*$, $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}} = 2\cos\left(\pi/2^{n+1}\right)$ (the number 2 occurring n times under the root).

4. For all $n \in \mathbb{N},\ n \geq 3$: $\displaystyle\prod_{k=3}^{n} \frac{k^2-4}{k} = \frac{(n+2)!}{12n(n-1)}$.

Exercise 7. Show the following inequalities

1. For any $x, y, z \in \mathbb{R}_+^\star$: $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} \geq \dfrac{1}{x+y+z}$

2. $\forall\, x, y \in \mathbb{R}_+ :\ 1+\sqrt{xy} \leq \sqrt{1+x} \times \sqrt{1+y}$

3. For any $x, y, z \in \mathbb{R}_+ :\ 8xyz \leq (x+y)(x+z)(y+z)$ (firstly you can verify that $(x+y)^2 \geq 4xy$)

4. $\forall x, y \in \mathbb{R}_+ :\ \frac{x^3+y^3}{2} \geq \left(\frac{x+y}{2}\right)^3$

Exercise 8. Show that, for any x, y, u, v in $\mathbb{R}$

- $|x|+|y| \leq |x+y|+|x-y|$
- $\dfrac{|x+y|}{1+|x+y|} \leq \dfrac{|x|}{1+|x|}+\dfrac{|y|}{1+|y|}$
- $||x-u|-|y-v|| \leq |x-y|+|u-v|$
- $1+|xy-1| \leq (1+|x-1|)(1+|y-1|)$

$$\bullet \max\left(|x|,|y|\right)\left|\frac{x}{|x|}-\frac{y}{|y|}\right| \leq 2|x-y|, \text{ with } x \neq 0$$

Exercise 9. Let us consider n real numbers of the same sign: $a_i > -1$, $i = 1, 2, \ldots, n$ where $n \in \mathbb{N}$. Show that

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1+a_1+a_2+\cdots+a_n.$$

Then deduce the Bernoulli's inequality

$$\forall n \in \mathbb{N}, \forall a \in \mathbb{R} \text{ with } a > -1, (1+a)^n \geq 1+na.$$

Exercise 10. Let A and B be two nonempty, bounded subsets of $\mathbb{R}$. Show that;

1. $$A \subset B \text{ then } \inf B \leq \inf A \leq \sup A \leq \sup B,$$

2. $$\min\left(\inf A, \inf B\right) = \inf\left(A \cup B\right) \leq \sup\left(A \cup B\right) = \max\left(\sup A, \sup B\right),$$

3. If $A \cap B \neq \oslash$, one have

$$\max\left(\inf A, \inf B\right) \leq \inf\left(A \cap B\right) \leq \sup\left(A \cap B\right) \leq \min\left(\sup A, \sup B\right),$$

Exercise 11. Let A be a nonempty bounded subset of $\mathbb{R}$, we consider the set $B = \{|x| : x \in A\}$.

1. Show that, B is bounded.

2. show that, $\sup B = \max\left(\left|\sup A\right|, \left|\inf A\right|\right)$.

3. Deduce that : a non empty subset A of $\mathbb{R}$ is bounded if and only if there exists a real $m \geq 0$ such that $\forall x \in A : |x| \leq m$.

Exercise 12.

$$A_1 = \{1 - n^2, n \in \mathbb{N}\} \quad A_2 = \{\sqrt{n+1} - \sqrt{n}, n \in \mathbb{N}\} \quad A_3 = \{-2 < x + \frac{1}{2x} \leq 2,\ x \in \mathbb{R}\}$$

$$A_4 = \left\{\frac{(-1)^n n}{n+1}, n \in \mathbb{N}\right\} \quad A_5 = \left\{\frac{1}{n} + \frac{1}{m}, n \in \mathbb{N}^* \text{ and } m \in \mathbb{N}^*\right\} \quad A_6 = \left\{\frac{2^n - 1}{2^n + 1}, n \in \mathbb{N}\right\}$$

$$A_7 = \left\{\frac{x-2}{x+3}, x > 0\right\} \quad A_8 = \left\{\frac{2xy}{x^2 + y^2}, x \in \mathbb{R}^\star, y \in \mathbb{R}^\star\right\} \quad A_9 = \left\{\frac{x^2 + 2x + 2}{x^2 + 2x + 4},\ x \in \mathbb{R}\right\}$$

$$A_{10} = \{(1 - \frac{1}{n}) \sin(\frac{n\pi}{2}),\ n \in \mathbb{N}^\star\} \quad A_{11} = \left\{(-1)^n + \cos(\frac{n\pi}{3}), n \in \mathbb{N}\right\}$$

These subsets, are they bounded from above? bounded from below? Give for each, when it is possible, the maximum, the minimum, the supremum and the infimum.

Exercise 13. Let A be the set defined by

$$A = \left\{\ln\left(\frac{1}{3n+1} + 1\right),\ n \in \mathbb{N}\right\}.$$

1. Show that the set A is bounded.
2. Show by using the characterization of infimum that $\inf A = 0$.
3. Give when its possible, $\max A$ and $\min A$.

Exercise 14.

1. Show that, for all $n \in \mathbb{Z}^+$

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}.$$

2. Deduce the value of the greatest integer less or equal than

$$x = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{3}} \cdots + \frac{1}{2\sqrt{1000}}.$$

Exercise 15.
Show that :

1) For all $x \in \mathbb{R}$ $\quad 0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$.

2) $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R} : \left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor$.

3) $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R} : \sum\limits_{k=1}^{k=n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor$.

4) $\forall x \in \mathbb{R} : \lfloor x \rfloor + \lfloor x + 3/2 \rfloor = \lfloor 2x \rfloor + 1$.

Exercise 16.
Resolve in $\mathbb{R}$ the following equations :

$$\lfloor 5x \rfloor = 4\lfloor x \rfloor \quad , \quad \left\lfloor \sqrt{x^2 + 1} \right\rfloor = 2 \quad , \quad \lfloor 2x \rfloor + \lfloor x + 3/2 \rfloor = \lfloor x \rfloor + 5.$$