Introduction to Combinatorics and Probability Calculations

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Learning Objectives

- Differentiate between types of arrangements (ordered/unordered, with/without repetition).
- Calculate permutations and combinations for a variety of problems.
- Apply combinatorial formulas to solve practical problems.

What is Combinatorics?

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- **Applications**: Useful in probability, computer science, and cryptography.

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$$3! = 6$$

Key Definitions

- **Discernible Elements**: a, b, c
- **Indiscernible Elements**: a, a, a
- **Ordered Arrangements**: Sequence matters (e.g., passwords).
- **Unordered Arrangements**: Sequence does not matter (e.g., team selection).

Arrangements with Repetition

Formula:

$$n^p$$

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Example: How many 4-digit PIN codes can be formed using digits 0–9?

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$$10^4 = 10,000$$

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$$A_{10}^3 = \frac{10!}{(10-3)!} = 720$$

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Without Repetition:

$$P_n = n!$$

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$$n_1 + n_2 + \cdots + n_k = n$$

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Formula:

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$$

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- 1 M, 4 I's, 4 S's, 2 P's
- Total letters: $1 + 4 + 4 + 2 = 11$

Number of distinct arrangements:

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34,650$$

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Example: Number of ways to distribute 4 identical candies among 3 children:

Total Number of Distributions

	child1	child2	child3
1	4	0	0
2	3	1	0
3	2	2	0
4	1	3	0
5	0	4	0
6	3	0	1
7	2	1	1
8	1	2	1
9	0	3	1
10	2	0	2
11	1	1	2
12	0	2	2
13	1	0	3
14	0	1	3
15	0	0	4

Result: The total number of valid ways to distribute 4 candies among 3 children is:

$$K_3^4 = C_{3+4-1}^4 = C_6^4 = 15$$

Exercises

1. How many ways can you form a 5-letter word using the English alphabet?
2. In how many ways can a group of 6 people be divided into 2 teams of 3?
3. How many unique arrangements can be made with the letters in "STATISTICS"?

Problem 1: Forming a 5-Letter Word

Question: How many ways can you form a 5-letter word using the English alphabet?

- **Case 1: Letters can be repeated**

$$\text{Total ways} = 26^5 = 11,881,376$$

- **Case 2: Letters cannot be repeated**

$$\text{Total ways} = P(26,5) = \frac{26!}{(26-5)!} = 26\times25\times24\times23\times22 = 7,893,600$$

- **Answer:**
  - With repetition: $11,881,376$
  - Without repetition: $7,893,600$

Problem 2: Dividing a Group into 2 Teams of 3

Question: In how many ways can a group of 6 people be divided into 2 teams of 3?

- Step 1: Choose 3 people out of 6:

$$C(6,3) = \frac{6!}{3!(6-3)!} = 20$$

- Step 2: Adjust for indistinguishable teams:

$$\text{Total ways} = \frac{C(6,3)}{2} = \frac{20}{2} = 10$$

- **Answer:** There are 10 ways to divide the group.

Problem 3: Arrangements of "STATISTICS"

Question: How many unique arrangements can be made with the letters in "STATISTICS"?

- Total letters: 10
- Frequencies:
  - $S:3,\ T:3,\ A:1,\ I:2,\ C:1$
- Formula for unique arrangements:

$$\text{Total arrangements} = \frac{10!}{3! \cdot 3! \cdot 2! \cdot 1! \cdot 1!}$$

- Calculation:

$$10! = 3,628,800, \quad 3! = 6, \quad 2! = 2$$

$$\text{Total} = \frac{3,628,800}{6 \cdot 6 \cdot 2} = 50,400$$

- **Answer:** There are 50,400 unique arrangements.

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- **Certain event**: Ω itself.
  Example: "Tossing the coin results in Heads or Tails."
- **Impossible event**: $\emptyset$, the empty set.
  Example: "Getting a 7 on a standard six-sided die."

Intersection of Events

Intersection ($A \cap B$): The event that A and B both occur.

$$A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}.$$

- Example: Let A: "Roll a number greater than 3" and B: "Roll an even number" on a six-sided die.

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- Example: Let A: "Roll a number greater than 3" and B: "Roll an even number" on a six-sided die.
- Then $A = \{4, 5, 6\}$ and $B = \{2, 4, 6\}$, so:

$$A \cap B = \{4, 6\}.$$

Union of Events

Union ($A \cup B$): The event that either A, B, or both occur.

$$A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}.$$

- Example: Using the same events A and B as before:

$$A = \{4, 5, 6\}, \quad B = \{2, 4, 6\},$$

$$A \cup B = \{2, 4, 5, 6\}.$$

Complement of an Event

Complement (A^c): The event that A does not occur.

$$A^c = \{\omega : \omega \notin A\}.$$

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Remark: We note :

$$A^c = \bar{A}$$

DeMorgan's Law

An important rule for complements is DeMorgan's Law:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

DeMorgan's Law : example

- Example: Using the same events A and B as before:

$$A = \{4, 5, 6\} \Rightarrow \bar{A} = \{1, 2, 3\},$$

$$B = \{2, 4, 6\} \Rightarrow \bar{B} = \{1, 3, 5\},$$

$$A \cup B = \{2, 4, 5, 6\} \Rightarrow \overline{A \cup B} = \{1, 3\} = \{1, 2, 3\} \cap \{1, 3, 5\}$$

$$A \cap B = \{4, 6\} \Rightarrow \overline{A \cap B} = \{1, 2, 3, 5\} = \{1, 2, 3\} \cup \{1, 3, 5\}$$

Set Difference ($A \setminus B$)

- The set difference $A \setminus B$ is defined as:

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

- It represents the elements that are in A but not in B.

Remark

$$A \setminus B = A \cap \bar{B}$$

Example

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$.

$$A \setminus B = \{1, 2\}.$$

Symmetric Difference ($A \triangle B$)

- The symmetric difference $A \triangle B$ is defined as:

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

- It represents the elements that are in A or B, but not in both.

Example

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$.

$$A \triangle B = \{1, 2, 5\}.$$

Uniform Probability

Definition: If all outcomes in the sample space Ω are equally likely, the probability is called uniform.

- The probability of an event A in this case is:

$$P(A) = \frac{\text{Number of favorable outcomes for } A}{\text{Total number of outcomes}}$$

- Example: Rolling a fair six-sided die:

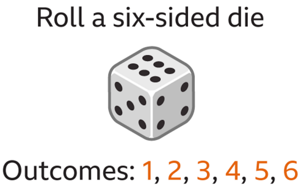

Figure: Six-sided die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $P(\text{rolling a } 4) =$

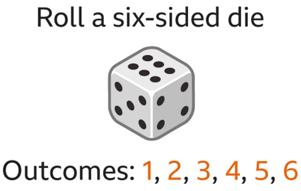

Figure: Six-sided die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $P(\text{rolling a } 4) = \frac{1}{6}$

General Definition of Probability

Kolmogorov Axioms: A probability P is a function defined on subsets of Ω that satisfies:

1. $P(A) \geq 0$, for all $A \subseteq \Omega$.
2. $P(\Omega) = 1$.
3. For disjoint events $A_1, A_2, \ldots$ ($A_i \cap A_j = \emptyset$ for $i \neq j$):

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

Example: Consider a fair die roll. Calculate the probability of rolling an even outcomes.

$$P(A) = \frac{|A|}{|\Omega|}$$

where $A = \{2, 4, 6\} = \{2\} \cup \{4\} \cup \{6\} = A_1 \cup A_2 \cup A_3$ (even outcomes), and $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Calculation:

$$
\begin{aligned}
P(A) &= \frac{|A|}{|\Omega|} = \frac{3}{6} \\
&= \frac{|A_1 \cup A_2 \cup A_3|}{|\Omega|} \\
&= \frac{|A_1|}{|\Omega|} + \frac{|A_2|}{|\Omega|} + \frac{|A_3|}{|\Omega|} \\
&= P(A_1) + P(A_2) + P(A_3) \\
&= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{2}.
\end{aligned}
$$

Some properties of probabilities

1. $P(\emptyset) = 0$,
2. $P(\bar{A}) = 1 - P(A)$,
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
4. $$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C). \end{aligned}$$
5. $P(A \cup B) \leq P(A) + P(B)$,
6. $P(A \cup B) \geq P(A) + P(B) - 1$.

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Exercise

Demonstrate the aforementioned properties.

Conditional Probabilities

Definition: The conditional probability of A given B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0.$$

- $P(A \mid B)$: Probability that A occurs, given that B has occurred.
- Example: Drawing a card(from 52-Card Deck):

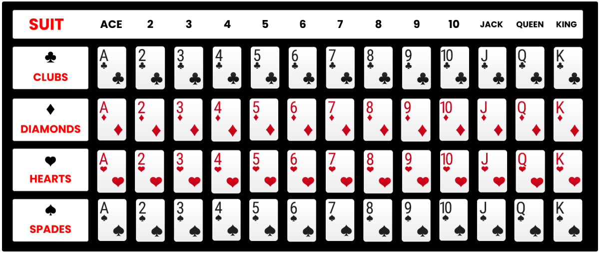

Figure: 52-Card Deck

- A: "The card is a king."
- B: "The card is black."
- $P(A \mid B) =?$

- A: "The card is a king."
- B: "The card is black."
- $P(A \mid B) = ?$
- $P(A \cap B) = \frac{2}{52}$, $P(B) = \frac{26}{52}$
- $P(A \mid B) = \frac{\frac{2}{52}}{\frac{26}{52}} = \frac{2}{26} = \frac{1}{13}$

Independent events

Definition: Two events A and B are called independent if and only if

$$P(A \cap B) = P(A)P(B)$$

When events A, B are independent, the probability of both happening can be computed by saying the event A happen first with $P(A)$ and the event B happens afterwards with $P(B)$.

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Remark: If the evensts A and B are independant, then :

$$P(A|B) = P(A) \quad \text{and } P(B|A) = P(B)$$

Independent events Lemma

Lemma

If A and B are independent then

1. A and $\bar{B}$ are independent,
2. $\bar{A}$ and B are independent,
3. $\bar{A}$ and $\bar{B}$ are independent.

Proof

1. $$\begin{aligned} P\left(A \cap \bar{B}\right) &= P(A \setminus B) \\ &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \quad \text{(since } A \text{ and } B \text{ are independent)} \\ &= P(A)(1 - P(B)) \\ &= P(A)P\left(\bar{B}\right). \end{aligned}$$

   Thus, A and $\bar{B}$ are independent.
2. Exercise,
3. Exercise.

Compound Probability, Multiplication Rule for Events

Compound Probability:
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Multiplication Rule for Events Let $E_1, E_2, \ldots, E_n$ be n events. Then:

$$P\big(E_1 \cap E_2 \cap \ldots \cap E_n\big) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2) \times \cdots \times P(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1}).$$

Urn Problem (Part a)

Problem: An urn contains 5 blue balls and 8 red balls. Each ball drawn is returned with an additional ball of the same color. If 3 balls are drawn, what is the probability that all three are blue?

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Solution: Define the events $B_1, B_2, B_3, \ldots$ where B_i is the event that the ith ball is blue. Applying the multiplication rule:

$$P(B_1 \cap B_2 \cap B_3) = P(B_1)P(B_2 \mid B_1)P(B_3 \mid B_1 \cap B_2),$$

we compute:

$$P(B_1 \cap B_2 \cap B_3) = \frac{5}{13} \cdot \frac{6}{14} \cdot \frac{7}{15}.$$

Urn Problem (Part b)

Problem: What is the probability that exactly one ball is blue?

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Solution: Denote R_i as the event that the ith ball drawn is red. Then:

$$P(\text{exactly 1 blue ball}) = P(B_1 \cap R_2 \cap R_3) + P(R_1 \cap B_2 \cap R_3) + P(R_1 \cap R_2 \cap B_3).$$

By symmetry:

$$P(\text{exactly 1 blue ball}) = 3 \cdot \frac{5 \cdot 8 \cdot 9}{13 \cdot 14 \cdot 15}.$$

Total Probability

Total Probability: If $B_1, B_2, \dots, B_n$ form a partition of Ω ($B_i \cap B_j = \emptyset$ for $i \neq j$, and $\bigcup_i B_i = \Omega$):

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i).$$

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Example: Weather Forecast

- A: "It rains tomorrow."
- B_1: "It is cloudy today."
- B_2: "It is sunny today."

Given: $P(A \mid B_1) = 0.8,\ P(A \mid B_2) = 0.2,\ P(B_1) = 0.6,\ P(B_2) = 0.4.$

Calculate $P(A)$:

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$$\begin{aligned} P(A) &= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) \\ &= 0.8 \cdot 0.6 + 0.2 \cdot 0.4 \\ &= 0.48 + 0.08 \\ &= 0.56. \end{aligned}$$

Bayes' Rule

Let $B_1, B_2, \ldots, B_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(B_i) > 0$, for all i. Then, for any event A such that $\mathbf{P}(A) > 0$, we have

$$\begin{aligned} P(B_i \mid A) &= \frac{P(B_i)\, P(A \mid B_i)}{P(A)} \\ &= \frac{P(B_i)\, P(A \mid B_i)}{P(A \mid B_1)\, P(B_1) + \cdots + P(A \mid B_n)\, P(B_n)} \\ &= \frac{P(B_i)\, P(A \mid B_i)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)} \end{aligned}$$

Example: Medical Diagnosis

Problem Statement: A medical test is designed to detect a certain disease. The test has the following properties:

- The test correctly identifies a person with the disease 98% of the time.
- The test correctly identifies a person without the disease 95% of the time.
- Approximately 1% of the population has the disease.

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- A: The person has the disease.
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Given Probabilities:

$$P(B \mid A) = 0.98 \quad \text{(Sensitivity)}$$
$$P(\bar{B} \mid \bar{A}) = 0.95 \quad \text{(Specificity)}$$
$$P(A) = 0.01 \quad \text{(Prior probability of having the disease)}$$

Example: Medical Diagnosis (Continued)

Objective: Compute the probability that a person has the disease given a positive test result, $P(A \mid B)$.

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Bayes' Theorem:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Step 1: Compute $P(B)$

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Step 1: Compute $P(B)$

$$P(B) = P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A})$$

Substituting the values:

$$P(B) = (0.98)(0.01) + (1 - 0.95)(0.99)$$

$$P(B) = 0.0098 + 0.0495 = 0.0593$$

Step 2: Compute $P(A \mid B)$

Example: Medical Diagnosis (Continued)

Objective: Compute the probability that a person has the disease given a positive test result, $P(A \mid B)$.

Bayes' Theorem:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Step 1: Compute $P(B)$

$$P(B) = P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A})$$

Substituting the values:

$$P(B) = (0.98)(0.01) + (1 - 0.95)(0.99)$$

$$P(B) = 0.0098 + 0.0495 = 0.0593$$

Step 2: Compute $P(A \mid B)$Substituting into Bayes' theorem:

$$P(A \mid B) = \frac{(0.98)(0.01)}{0.0593}$$

$$P(A \mid B) \approx 0.1652 \quad (16.52\%)$$

Some Probability Problems

Problem Statement 1

In a city:

- 60% of households subscribe to newspaper A.
- 50% subscribe to newspaper B.
- 40% subscribe to newspaper C.
- 30% subscribe to both A and B.
- 20% subscribe to both B and C.
- 10% subscribe to both A and C.
- None subscribe to all three newspapers.

Question: What percentage of households subscribe to exactly one newspaper?

Venn Diagram Representation

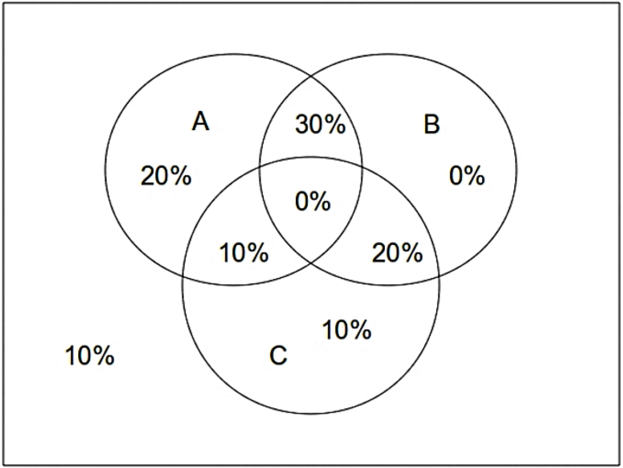

Figure: Venn Diagram Representation

Solution: Percentage of Households Subscribing to Exactly One Newspaper

From the Venn diagram:

- Households subscribing only to A: 20%.
- Households subscribing only to B: 10%.
- Households subscribing only to C: 10%.

Adding these percentages:

$$P(\text{Exactly one newspaper}) = 20\% + 10\% + 10\% = 40\%.$$

Answer: 40% of households subscribe to exactly one newspaper.

Problem Statement 2

Suppose that A and B are mutually exclusive events for which:

$$P(A) = 0.3 \quad \text{and} \quad P(B) = 0.5.$$

We address the following questions:

(a) What is the probability that A occurs but B does not?

(b) What is the probability that neither A nor B occurs?

Solution (a): Probability that A occurs but B does not

Since A and B are mutually exclusive, the only way for A to occur is when B does not occur. This means:

$$P(A \cap \bar{B}) = P(A).$$

Substituting the given value of $P(A)$:

$$P(A \cap \bar{B}) = 0.3.$$

Answer: $P(A \cap \bar{B}) = 0.3.$

Solution (b): Probability that neither A nor B occurs

Since $A \cap B = \emptyset$ (mutually exclusive events), we use Axiom 3 to calculate:

$$P(A \cup B) = P(A) + P(B) = 0.3 + 0.5 = 0.8.$$

Using DeMorgan's law, the probability that neither A nor B occurs is:

$$P(\bar{A} \cap \bar{B}) = P\left(\overline{A \cup B}\right).$$

By the complement rule:

$$P\left(\overline{A \cup B}\right) = 1 - P(A \cup B) = 1 - 0.8 = 0.2.$$

Answer: $P(\bar{A} \cap \bar{B}) = 0.2.$

Problem Statement 3

In a class:

- There are 4 male math majors.
- There are 6 female math majors.
- There are 6 male actuarial science majors.

Question: How many actuarial science girls must be present in the class so that sex and major are independent when selecting a student at random?

Solution: Setting Up the Problem

Let x represent the number of actuarial science girls. The total number of students in the class is:

$$\text{Total students} = 16 + x.$$

Probabilities are calculated as follows:

$$P(\text{Male} \cap \text{Math}) = \frac{4}{16 + x},$$

$$P(\text{Male}) = \frac{10}{16 + x} \quad (4 \text{ male math majors} + 6 \text{ male actuarial science majors})$$

$$P(\text{Math}) = \frac{10}{16 + x} \quad (4 \text{ male math majors} + 6 \text{ female math majors}).$$

Solution: Using Independence

If sex and major are independent, then:

$$P(\text{Male} \cap \text{Math}) = P(\text{Male}) \cdot P(\text{Math}).$$

Substituting the probabilities:

$$\frac{4}{16+x} = \frac{10}{16+x} \cdot \frac{10}{16+x}.$$

Simplifying:

$$\frac{4}{16+x} = \frac{100}{(16+x)^2}.$$

Cross-multiplying:

$$4 \cdot (16+x) = 100.$$

Expanding and solving for x:

$$64 + 4x = 100 \implies 4x = 36 \implies x = 9.$$

Final Answer

To ensure that sex and major are independent:

$$\boxed{x = 9}.$$

This means there must be 9 female actuarial science majors in the class.

Problem Statement 4

An urn contains:

- Urn 1: 10 balls (4 red and 6 blue).
- Urn 2: 16 red balls and an unknown number of blue balls (x).

A single ball is drawn from each urn. The probability that both balls are the same color is given as 0.44.

Question: Calculate the number of blue balls in Urn 2.

Solution: Setting Up the Problem

Define the following events:

- R_1: A red ball is drawn from Urn 1.
- B_1: A blue ball is drawn from Urn 1.
- R_2: A red ball is drawn from Urn 2.
- B_2: A blue ball is drawn from Urn 2.

Let x represent the number of blue balls in Urn 2. The total number of balls in Urn 2 is $16 + x$.

Solution: Using the Probability Formula

The probability that both balls are the same color is:

$$P\left((R_1 \cap R_2) \cup (B_1 \cap B_2)\right) = P(R_1 \cap R_2) + P(B_1 \cap B_2).$$

Since the draws are independent:

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2),$$

$$P(B_1 \cap B_2) = P(B_1) \cdot P(B_2).$$

Substituting the probabilities:

$$P(R_1) = \frac{4}{10}, \quad P(B_1) = \frac{6}{10},$$

$$P(R_2) = \frac{16}{x+16}, \quad P(B_2) = \frac{x}{x+16}.$$

Solution: Equation Setup

Combining the probabilities:

$$0.44 = P(R_1 \cap R_2) + P(B_1 \cap B_2).$$

Substituting the expressions:

$$0.44 = \frac{4}{10} \cdot \frac{16}{x+16} + \frac{6}{10} \cdot \frac{x}{x+16}.$$

Simplify:

$$0.44 = \frac{64}{10(x+16)} + \frac{6x}{10(x+16)}.$$

Combine terms:

$$0.44 = \frac{64+6x}{10(x+16)}.$$

Multiply through by $10(x+16)$ to eliminate the denominator:

$$4.4(x+16) = 64 + 6x.$$

Solution: Solving for x

Expand the equation:

$$4.4x + 70.4 = 64 + 6x.$$

Rearrange terms:

$$70.4 - 64 = 6x - 4.4x.$$

Simplify:

$$6.4 = 1.6x.$$

Solve for x:

$$x = \frac{6.4}{1.6} = 4.$$

Answer: The number of blue balls in Urn 2 is $x = 4$.

Problem Statement 5

Two dice are rolled. Define the following events:

- $A = \{\text{Sum of the two dice equals } 3\}$,
- $B = \{\text{Sum of the two dice equals } 7\}$,
- $C = \{\text{At least one of the dice shows a } 1\}$.

(a) What is $P(A \mid C)$?

(b) What is $P(B \mid C)$?

(c) Are A and C independent? What about B and C?

Solution: Part (a)

The sample space is:

$$\Omega = \{(i,j) \mid i,j = 1,2,3,4,5,6\}.$$

Each outcome is equally likely.
Event definitions:

$$\begin{aligned} A &= \{(1,2),(2,1)\}, \\ C &= \{(1,1),(1,2),(1,3),(1,4),(1,5), \\ &\quad (1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\}. \end{aligned}$$

Intersection of A and C:

$$A \cap C = \{(1,2),(2,1)\}.$$

Probability calculation:

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}.$$

Solution: Part (b)

Event B is defined as:

$$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

Intersection of B and C:

$$B \cap C = \{(1,6),(6,1)\}.$$

Probability calculation:

$$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}.$$

Solution: Part (c)

Independence analysis:

- Event A:
$$P(A) = \frac{2}{36}, \quad P(A \mid C) = \frac{2}{11}.$$
Since $P(A) \neq P(A \mid C)$, A and C are not independent.
- Event B:
$$P(B) = \frac{6}{36}, \quad P(B \mid C) = \frac{2}{11}.$$
Since $P(B) \neq P(B \mid C)$, B and C are not independent.

Conclusion: Neither A and C nor B and C are independent.

Problem Statement 6

Two factories supply light bulbs to the market:

- Factory X: Bulbs work for over 5000 hours in 99% of cases.
- Factory Y: Bulbs work for over 5000 hours in 95% of cases.
- Factory X supplies 60% of the total bulbs.

Questions:

(a) What is the chance that a purchased bulb will work for longer than 5000 hours?

(b) Given that a light bulb works for more than 5000 hours, what is the probability that it came from factory Y?

(c) Given that a light bulb does not work for more than 5000 hours, what is the probability that it came from factory X?

Solution: Part (a)

Define:

- $H = \{\text{bulb works over 5000 hours}\}$,
- $X = \{\text{bulb comes from factory } X\}$,
- $Y = \{\text{bulb comes from factory } Y\}$.

Using the Law of Total Probability:

$$P(H) = P(H \mid X)P(X) + P(H \mid Y)P(Y).$$

Substitute the values:

$$\begin{aligned} P(H) &= (0.99)(0.6) + (0.95)(0.4) \\ &= 0.594 + 0.38 \\ &= 0.974. \end{aligned}$$

Thus, the probability that a purchased bulb will work for over 5000 hours is $P(H) = 0.974$.

Solution: Part (b)

We use Bayes' Theorem to find $P(Y \mid H)$:

$$P(Y \mid H) = \frac{P(H \mid Y)P(Y)}{P(H)}.$$

Substitute the values:

$$\begin{aligned} P(Y \mid H) &= \frac{(0.95)(0.4)}{0.974} \\ &= \frac{0.38}{0.974} \\ &\approx 0.39. \end{aligned}$$

Thus, the probability that the bulb came from factory Y given it works for over 5000 hours is approximately 0.39.

Solution: Part (c)

We use the complement of H, denoted $\bar{H}$, to find $P(X \mid \bar{H})$:

$$P(X \mid \bar{H}) = \frac{P(\bar{H} \mid X)P(X)}{P(\bar{H})}.$$

Substitute the values:

$$\begin{aligned} P(X \mid \bar{H}) &= \frac{P(\bar{H} \mid X)P(X)}{1 - P(H)} \\ &= \frac{(1 - 0.99)(0.6)}{1 - 0.974} \\ &= \frac{(0.01)(0.6)}{0.026} \\ &\approx 0.23. \end{aligned}$$

Thus, the probability that the bulb came from factory X given it does not work for over 5000 hours is approximately 0.23.

Problem Statement 7

A factory production line manufactures bolts using three machines A, B, and C:

- Machine A produces 25% of the total output.
- Machine B produces 35% of the total output.
- Machine C produces the remaining 40% of the total output.

Defective rates:

- 5% of Machine A's output is defective.
- 4% of Machine B's output is defective.
- 2% of Machine C's output is defective.

If a randomly selected bolt is found to be defective, what is the probability that it came from Machine A?

Solution

Define the events:

- $D = \{\text{bolt is defective}\}$,
- $A = \{\text{bolt is from Machine A}\}$,
- $B = \{\text{bolt is from Machine B}\}$,
- $C = \{\text{bolt is from Machine C}\}$.

Using Bayes' Theorem:

$$P(A \mid D) = \frac{P(D \mid A)P(A)}{P(D)}.$$

Step 1: Compute $P(D)$

Using the Law of Total Probability:

$$P(D) = P(D \mid A)P(A) + P(D \mid B)P(B) + P(D \mid C)P(C).$$

Substitute the values:

$$\begin{aligned} P(D) &= (0.05)(0.25) + (0.04)(0.35) + (0.02)(0.40) \\ &= 0.0125 + 0.014 + 0.008 \\ &= 0.0345. \end{aligned}$$

Thus, $P(D) = 0.0345$.

Step 2: Compute $P(A \mid D)$

Substitute into Bayes' Theorem:

$$P(A \mid D) = \frac{P(D \mid A)P(A)}{P(D)}.$$

Substitute the values:

$$\begin{aligned} P(A \mid D) &= \frac{(0.05)(0.25)}{0.0345} \\ &= \frac{0.0125}{0.0345} \\ &\approx 0.362. \end{aligned}$$

Thus, the probability that the defective bolt came from Machine A is approximately 0.362.

Problem Statement 8

In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is $\frac{1}{4}$. You pick a random day.

1. What is the probability that it's not raining and there is heavy traffic and I am not late?
2. What is the probability that I am late?
3. Given that I arrived late at work, what is the probability that it rained that day?

Problem Setup

Given:

- Probability of rain: $P(R) = \frac{1}{3}$, $P(\bar{R}) = \frac{2}{3}$.
- Probability of heavy traffic given rain: $P(T \mid R) = \frac{1}{2}$.
- Probability of heavy traffic given no rain: $P(T \mid \bar{R}) = \frac{1}{4}$.
- Probability of being late:
  - $P(L \mid R \cap T) = \frac{1}{2}$.
  - $P(L \mid R \cap \bar{T}) = \frac{1}{4}$.
  - $P(L \mid \bar{R} \cap T) = \frac{1}{4}$.
  - $P(L \mid \bar{R} \cap \bar{T}) = \frac{1}{8}$.

Solution:

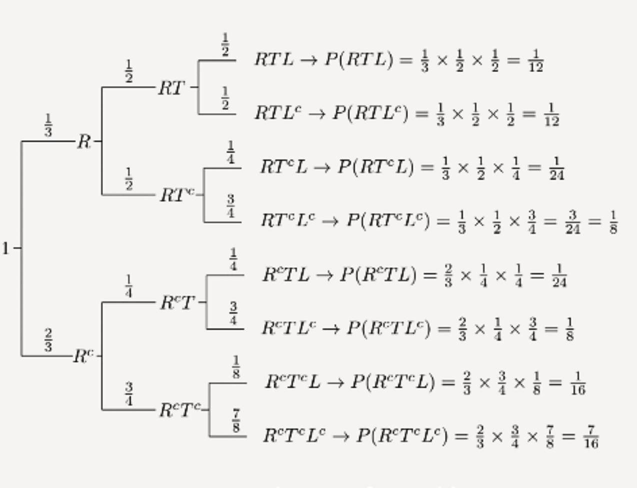

Figure: Tree Diagram of the Problem

Solution: Part 1

a. Probability that it's not raining, there is heavy traffic, and I am not late:

$$\begin{aligned} P(\bar{R} \cap T \cap \bar{L}) &= P(\bar{R})P(T \mid \bar{R})P(\bar{L} \mid \bar{R} \cap T) \\ &= \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{4} \\ &= \frac{1}{8}. \end{aligned}$$

Solution: Part 2

b. Probability that I am late:

$$\begin{aligned} P(L) &= P(R \cap T \cap L) + P(R \cap \bar{T} \cap L) + P(\bar{R} \cap T \cap L) + P(\bar{R} \cap \bar{T} \cap L) \\ &= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} \\ &= \frac{11}{48}. \end{aligned}$$

Solution: Part 3

c. Probability that it rained given that I am late:

$$
\begin{aligned}
P(R \mid L) &= \frac{P(R \cap L)}{P(L)} \\
&= \frac{P(R \cap T \cap L) + P(R \cap \bar{T} \cap L)}{P(L)} \\
&= \frac{\frac{1}{12} + \frac{1}{24}}{\frac{11}{48}} \\
&= \frac{\frac{1}{8}}{\frac{11}{48}} \\
&= \frac{1}{8} \cdot \frac{48}{11} \\
&= \frac{6}{11}.
\end{aligned}
$$