Chapter 2 : Sequences

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Definition

A sequence is a function whose domain is the set of natural numbers $\mathbb{N}$ and range a sub-set of real numbers $\mathbb{R}$, which is defined as follow

$$\mathbb{N} \to \mathbb{R}$$

$$n \to u_n$$

We denote such a function as $(u_n)_{n\geq 0}$ or simply (u_n).
u_n is called the general term of the sequence (u_n).

Examples

- $u_n = 1 + \frac{2}{n+1}$, $u_n = (-1)^n \cos(\frac{n\pi}{2})$.
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  - **Second-order recursive sequence** given by
  $$\begin{cases} u_0 = \alpha, \quad u_1 = \beta \end{cases}$$

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$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{(n+1)^2} > 0 \quad \forall n \geq 1.$$

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2. $u_n = \sum_{1}^{n} \frac{1}{n+k}$.

Definition 3

Let (u_n) be a sequence of real numbers. We say that the sequence (u_n) is :

• Upper bounded if the subset of $\mathbb{R}$, $A = \{u_n,\ n \in \mathbb{N}\}$ is bounded above , i.e.,

$$\exists M \in \mathbb{R}\,,\ \ \forall n \in \mathbb{N}\,,\ \ u_n \leq M$$

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• bounded if and only if it is both lower and upper bounded.

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Thus,

$$u_n \leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 + \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + ... +$$

$$\left(\frac{1}{n-2} - \frac{1}{n-1} \right) + \left(\frac{1}{n-1} - \frac{1}{n} \right) = 2 - \frac{1}{n} \leq 2$$

Thus, for all $n \in \mathbb{N}^\star$, the sequence $u_n \leq 2$.

Proposition.

The sequence (u_n) is bounded if and only if the sequence $(|u_n|)$ is upper bounded , i.e., there exists $M \in \mathbb{R}$ such that $|u_n| \leq M$ for all $n \in \mathbb{N}$.

Proof.

Proof. $\Longleftarrow$ The sequence $|u_n|$ is upper bounded, which means there exists $M \in \mathbb{R}$ such that $|u_n| \leq M \Leftrightarrow -M \leq u_n \leq M$, meaning that (u_n) is bounded.

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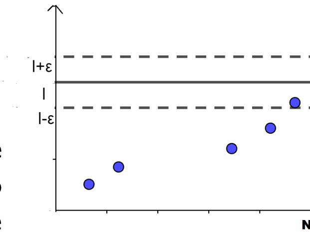
Note. We say that the sequence (u_n) is bounded (upper bounded, lower bounded) from some term onwards if there exist $N \in \mathbb{N}$ and $M \in \mathbb{R}$ (or $m \in \mathbb{R}$) such that for all $n \geq N$, we have $u_n \leq M$ (or $u_n \geq m$).

Convergent Sequence

Definition 4. We say that the sequence $(u_n)_{n\geq 0}$ has the limit $l \in \mathbb{R}$ if and only if for every

$$\boxed{\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ such that}}$$
$$\boxed{\forall n \in \mathbb{N}, \quad n \geq N \text{ implies } |u_n - l| \leq \epsilon}$$



In other words, from a certain rank N, the terms of the sequence (u_n) get closer to the limit l (the terms of the sequence are in the interval $[l-\epsilon, l+\epsilon]$.

If such an l exists, we say that the sequence (u_n) is convergent (or the sequence (u_n) converges to l), and we write

$$\lim_{n\to\infty} u_n = l \text{ or } \lim u_n = l \text{ or } u_n \to l$$

Otherwise, we say that the sequence (u_n) is divergent.

Convergent Sequence

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• If the sequence (u_n) has infinity as its limit, we say it diverges, and the divergence is of the first kind.

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Let $\epsilon > 0$, we want to find $N = N(\epsilon) \in \mathbb{N}$ such that $|u_n - l| \leq \epsilon$. We often reduce this to solving inequalities.
We have $|u_n - l| = \frac{1}{n} \leq \epsilon \Leftrightarrow n \geq \frac{1}{\epsilon}$ We take $N = E(\frac{1}{\epsilon}) + 1$.
Thus, for $n \geq N$ we have $|u_n - 0| = \frac{1}{n} \leq \frac{1}{N} \leq \epsilon$.

Convergent Sequence

Remark. Instead of using a non-strict inequality in the definition of a convergent sequence, we can use a strict inequality, meaning :

$$u_n \to l \Leftrightarrow \forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N \text{ we have } |u_n - l| < \varepsilon.$$

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Indeed, the implication in the reverse direction $\Longleftarrow$ is obvious.For the other direction of the implication ($\Longrightarrow$), let $\varepsilon > 0$, we set $\varepsilon' = \frac{\varepsilon}{2} > 0$ (which is arbitrary), by the definition, we have

$$u_n \to l \Longrightarrow \exists N \in \mathbb{N}\,,\ \forall n \geq N,\ |u_n - l| \leq \varepsilon' < \varepsilon$$

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The limit $l \in \mathbb{R}$ of a real sequence, if it exists, is unique.

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Proof. We assume by contradiction that the sequence (u_n) has two different limits l_1 and l_2 ($l_1 \neq l_2$).
Let $\epsilon = \frac{|l_1 - l_2|}{2} > 0$.
• Since $u_n \to l_1$ then there exists $N_1 \in \mathbb{N}$ from which we have $|u_n - l_1| < \epsilon$.
• Since $u_n \to l_2$ then there exists $N_2 \in \mathbb{N}$ from which we have $|u_n - l_2| < \epsilon$.
Now we consider the integer $N = \max(N_1, N_2)$, from this rank, both of the last two inequalities $|u_n - l_1| < \epsilon$, $|u_n - l_2| < \epsilon$ are satisfied. Using the triangle inequality, we obtain

$$|l_1 - l_2| = |l_1 - u_n + u_n - l_2| \leq |u_n - l_1| + |u_n - l_2| < 2\epsilon = |l_1 - l_2|$$

which is absurd.

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Proof. Let (u_n) be a real sequence that converges to l. By definition, we have

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n \geq N, \ |u_n - l| \leq \epsilon \Leftrightarrow -\epsilon \leq u_n - l \leq \epsilon \Leftrightarrow$$

$$l - \epsilon \leq u_n \leq l + \epsilon,$$

which means that $|u_n| \leq |l + \epsilon|$ or $|u_n| \leq |l - \epsilon|$. Therefore, let $M = \max(|l + \epsilon|, |l - \epsilon|)$, we have for every $n \geq N$, $|u_n| \leq M$. The first terms of the sequence remain, so we set $M' = \max(|u_0|, |u_1|, ..., |u_{N-1}|, M)$, we have $|u_n| \leq M'$ for every $n \in \mathbb{N}$.

Properties. Let (u_n) and (v_n) be two real sequences with respective limits $l_1,\ l_2$. The table below summarizes the properties of the limits of the sum, product, and quotient of two sequences

$\lim u_n$	$\lim v_n$	$\lim(u_n + v_n)$	$\lim(u_n \times v_n)$	$\lim \frac{u_n}{v_n}$
l_1	l_2	$l_1 + l_2$	$l_1 \times l_2$	$\frac{l_1}{l_2}\quad (l_2 \neq 0)$
0	l_2	l_2	0	0
l_1	0	l_1	0	∞
0	0	0	0	F.I
∞	0	∞	F.I	∞
0	∞	∞	F.I	0
l_1	∞	∞	∞	0

$\lim u_n$	$\lim v_n$	$\lim(u_n + v_n)$	$\lim(u_n \times v_n)$	$\lim \dfrac{u_n}{v_n}$
∞	l_2	∞	∞	∞
$+\infty$	$+\infty$	$+\infty$	$+\infty$	F.I
$-\infty$	$-\infty$	$-\infty$	$+\infty$	F.I
$+\infty$	$-\infty$	F.I	$-\infty$	F.I

Positive Limit

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Proposition Let (u_n) be a convergent real sequence, such that :

i) There exists $N \in \mathbb{N}$ from which $u_n > 0$
ii) $\lim_{n\to\infty} u_n = l$

Then $l \geq 0$.

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Proof Let's assume, by contradiction, that $l < 0$. Set $\varepsilon = -l > 0$. Since $u_n \to l$, there exists $N_1 \in \mathbb{N}$ such that $|u_n - l| \leq \varepsilon$, and there exists N_2 such that for all $n \geq N_2$ we have $u_n > 0$. Let's take $N = \max(N_1, N_2)$, so for all $n \geq N$, we have :

$$l \leq u_n - l \leq -l \text{ and } u_n > 0 \Leftrightarrow 2l \leq u_n \leq 0 \text{ and } u_n > 0$$

which is absurd.

Passing to the Limit in Inequalities

Proposition Let (u_n) and (v_n) be two real sequences, such that :

i) $u_n \to l$
ii) $v_n \to l'$
iii) There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $u_n \leq v_n$.
Then $l \leq l'$.

Proof We can proceed in a similar way to the previous proof. We just need to consider the sequence $w_n = v_n - u_n$.

Squeeze theorem

Theorem 3.

Let (u_n), (v_n), and (w_n) be three real sequences, such that :

i) $\lim v_n = \lim w_n = l$
ii) There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $v_n \leq u_n \leq w_n$.
Then $\lim u_n = l$.

Squeeze theorem

Proof. We have :

• $v_n \to l$, so there exists $N_1 \in \mathbb{N}$ from which we have $|v_n - l| \leq \varepsilon \Leftrightarrow l - \varepsilon \leq v_n \leq l + \varepsilon$

• $w_n \to l$, so there exists $N_1 \in \mathbb{N}$ from which we have $|w_n - l| \leq \varepsilon \Leftrightarrow l - \varepsilon \leq w_n \leq l + \varepsilon$

• There exists $N_3 \in \mathbb{N}$ such that $n \geq N$ implies $v_n \leq u_n \leq w_n$.

Let's take $N = \max(N_1, N_2, N_3)$. Then, for all $n \geq N$, we have :

$$l - \varepsilon \leq v_n \leq u_n \leq w_n \leq l + \varepsilon$$

so

$$-\varepsilon \leq u_n - l \leq \varepsilon$$

Consequently, for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$.

Squeeze theorem

Corollary Let (u_n) and (v_n) be two numerical sequences, such that :

- The sequence (u_n) converges to 0.
- The sequence (v_n) is bounded.

Then, the sequence $w_n = u_n v_n$ converges to 0.
Proof It is enough to apply Gendarme's Theorem to the sequence (w_n).

Real Sequences and Monotonicity

Theorem 4

(Convergence Criterion for Monotonic Sequences)

Let (u_n) be a monotonic sequence, then :

- If it is increasing, it is convergent if and only if it is bounded. Moreover, $\lim u_n = \sup\{u_n,\ n \in \mathbb{N}\}$.
- If it is decreasing, it is convergent if and only if it is bounded. Moreover, $\lim u_n = \inf\{u_n,\ n \in \mathbb{N}\}$.

Real Sequences and Monotonicity

Remark

- If the sequence (u_n) is increasing and unbounded, then $\lim u_n = +\infty$.

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- If the sequence (u_n) is increasing and unbounded, then $\lim u_n = +\infty$.
- If the sequence (u_n) is decreasing and unbounded, then $\lim u_n = -\infty$.
- If the increasing sequence (u_n) converges to l, then for all $n \in \mathbb{N}$, $u_n \leq l$.

Remark

- If the sequence (u_n) is increasing and unbounded, then $\lim u_n = +\infty$.
- If the sequence (u_n) is decreasing and unbounded, then $\lim u_n = -\infty$.
- If the increasing sequence (u_n) converges to l, then for all $n \in \mathbb{N}$, $u_n \leq l$.
- If the decreasing sequence (u_n) converges to l, then for all $n \in \mathbb{N}$, $u_n \geq l$.

Real Sequences and Monotonicity

Proof We define A as a subset of $\mathbb{R}$ by $A = \{u_n, \ n \in \mathbb{N}\}$. A is bounded, so according to the supremum property, it has a supremum l. We will show that $l = \lim u_n$.
Using the supremum property, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $l - \varepsilon < u_N \leq l$. As the sequence (u_n) is increasing, for all $n \geq N$ we have $u_n \geq u_N$. Since l is an upper bound for A, we have :

$$l - \varepsilon \leq u_N \leq u_n \leq l \Leftrightarrow -\varepsilon \leq u_n - l \leq 0 \Leftrightarrow |u_n - l| \leq \varepsilon.$$

Example 5

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Therefore, according to the monotone sequence convergence criterion, the sequence (u_n) is convergent.

2- Study the nature of the numerical sequence (u_n) defined as :

$$u_n = \sum_{k=1}^{n} \frac{1}{n+k}$$

We have :

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{n+1+k} - \sum_{k=1}^{n} \frac{1}{n+k} = -\frac{1}{n+1} + \frac{1}{2n+2} + \frac{1}{2n+1}$$

$$\frac{1}{2(n+1)(2n+1)} > 0 \text{ for all } n \in \mathbb{N}$$

So, the sequence (u_n) is strictly increasing.

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$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{n+1+k} - \sum_{k=1}^{n} \frac{1}{n+k} = -\frac{1}{n+1} + \frac{1}{2n+2} + \frac{1}{2n+1}$$

$$\frac{1}{2(n+1)(2n+1)} > 0 \text{ for all } n \in \mathbb{N}$$

So, the sequence (u_n) is strictly increasing. Furthermore, we have, for all $1 \leq k \leq n$:

$$u_n \leq \sum_{k=1}^{n} \frac{1}{n+1} = \frac{n}{n+1} < 1$$

Real Sequences and Monotonicity

Using the following inequalities :

$$\ln\left(\frac{x+1}{x}\right) \leq \frac{1}{x} \leq \ln\left(\frac{x}{x-1}\right) \text{ for all } x > 1$$

Calculate the limit of (u_n).

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Calculate the limit of (u_n).
For $1 \leq k \leq n$, we have :

$$\ln\left(\frac{n+k+1}{n+k}\right) \leq \frac{1}{n+k} \leq \ln\left(\frac{n+k}{n+k-1}\right)$$

Summing for k from 1 to n, we get :

$$\sum_{k=1}^{n} \ln\left(\frac{n+k+1}{n+k}\right) \leq u_n \leq \sum_{k=1}^{n} \ln\left(\frac{n+k}{n+k-1}\right)$$

$$\Leftrightarrow \sum_{k=1}^{n} \ln(n+k+1)-\ln(n+k) \leq u_n \leq \sum_{k=1}^{n} \ln(n+k)-\ln(n+k-1)$$

The sequences $v_n = \sum_{k=1}^{n} \ln(n+k+1) - \ln(n+k)$ and $w_n = \sum_{k=1}^{n} \ln(n+k) - \ln(n+k-1)$ are called telescopic series, and we can easily calculate their sums. In fact, we have :

$$v_n = \sum_{k=1}^{n} \ln(n+k+1) - \ln(n+k) = (\ln(n+2) - \ln(n+1)) +$$

$$(\ln(n+3) - \ln(n+2)) + \ldots + (\ln(2n) - \ln(2n-1)) + (\ln(2n+1) - \ln(2n))$$

$$= \ln(2n+1) - \ln(n+1) = \ln(\frac{2n+1}{n+1}) \to \ln(2)$$

Doing the same for the sequence (w_n), we find that $w_n \to \ln(2)$. According to the squeeze theorem, we deduce that the sequence (u_n) converges to $\ln(2)$.

Adjacent Sequences

Definition 4. Let (u_n) and (v_n) be two real sequences. We say that they are adjacent if and only if :

1. (u_n) is increasing.
2. (v_n) is decreasing.
3. $\lim(u_n - v_n) = 0$.

Adjacent Sequences

Theorem 5.

If (u_n) and (v_n) are two adjacent sequences, then :

- The sequences (u_n) and (v_n) converge to the same limit.
- For all $n \in \mathbb{N}$, we have $u_n \leq l \leq v_n$, where l is the common limit.

Adjacent Sequences

Proof. Let (u_n) be an increasing sequence, and (v_n) be a decreasing sequence. We have :

$$u_n \leq u_{n+1} \Leftrightarrow -u_n \geq -u_{n+1} \text{ and } v_n \geq v_{n+1}$$

Thus, $v_n - u_n \geq v_{n+1} - u_{n+1}$, which means that the sequence $v_n - u_n$ is decreasing.
Since $\lim(u_n - v_n) = 0$, we have $v_n - u_n \geq 0 \Leftrightarrow v_n \geq u_n$.
We now have :

$$u_n \leq v_n \leq v_0 \longrightarrow$$

The sequence (u_n) is increasing and bounded, so it converges to l

and

$$v_n \geq u_n \geq u_0 \longrightarrow \text{The sequence } (v_n)$$

is decreasing and bounded, so it converges to l'.

Since $\lim(v_n - u_n) = 0$, we have $l - l' = 0 \Leftrightarrow l = l'$.

Adjacent Sequences

Example 6.

Consider the sequence $w_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Show that the sequences $u_n = w_{2n}$ and $v_n = w_{2n+1}$ are adjacent.

Example 6.

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Monotonicity of (u_n) and (v_n) : We have :

$$u_{n+1} - u_n = w_{2n+2} - w_{2n} = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2} \geq 0, \quad \forall n \in \mathbb{N}^*$$

and

$$v_{n+1} - v_n = w_{2n+3} - w_{2n+1} = \sum_{k=1}^{2n+3} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k}$$

$$= -\frac{1}{2n+2} + \frac{1}{2n+3} \leq 0, \quad \forall n \in \mathbb{N}^*$$

Thus, the sequence (u_n) is increasing, and the sequence (v_n) is

Furthermore, we have :

$$u_n - v_n = \frac{1}{2n+1} \to 0$$

Therefore, the sequences (u_n) and (v_n) are adjacent, and they converge to the same limit.

Subsequence of a Sequence

Definition 6. Let (u_n) be a numerical sequence. We call a subsequence of (u_n) a sequence $(u_{\phi(n)})$ where $\phi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

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- For $u_n = (-1)^n$, we have $u_{2n} = 1$ and $u_{2n+1} = -1$.
- For $u_n = \sin(\frac{2n\pi}{17})$, we have $u_{17n} = \sin(2n\pi) = 0$ and $u_{17n+1} = \sin(\frac{2\pi}{17}) \neq 0$.

Proposition.

Let (u_n) be a numerical sequence. Then, (u_n) converges to l if and only if every subsequence of (u_n) also converges to l.

Proof $\Longleftarrow$ This direction is obvious. We just need to take the subsequence $(u_{\phi(n)})$ with $\phi(n) = n$.
$\Longrightarrow$ To prove this direction, we need to show that for every $n \in \mathbb{N}$, we have $\phi(n) \geq n$. We will proceed by induction. For $n = 0$, we have $\phi(0) \geq 0$ because $\phi(0) \in \mathbb{N}$.
Suppose that $\phi(n) \ \geq n$ and let's show that $\phi(n+1) \geq n+1$.
Since ϕ is strictly increasing, for $n+1 > n$, we have $\phi(n+1) > \phi(n) \geq n$. Thus, $\phi(n+1) \geq n+1$ $(\phi(n+1), n+1 \in \mathbb{N})$.
The sequence (u_n) converges to l, which means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$. But we also have $\phi(n) \geq n \geq N$, so there exists $N \in \mathbb{N}$ such that $\phi(n) \geq N$, and in this case, we have $|u_{\phi(n)} - l| \leq \varepsilon$.
This proposition is often used to show that a sequence (u_n) is divergent.

Corollary

Let (u_n) be a numerical sequence. If subsequence $(u_{\psi(n)})$ diverges, then the sequence (u_n) also diverges.

Example 8. Consider the sequence $u_n = \cos(\frac{n\pi}{3})$. If we take the subsequence (u_{3n}), we have $u_{3n} = \cos(n\pi) = (-1)^n$, which diverges. Thus, the sequence (u_n) also diverges.