Year 2023/2024
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National Polytechnic School
Preparatory cycle

Series of exercises 2 (Analyse Maths I)

1 Numericals seauences

Exercice 1
Let (u_n) be a numerical sequence defined defined by $u_n = f(u_{n-1},\ n \in \mathbb{N}$ such that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an increqsing map. Study the monotony of the two subsequences (u_{2n}) et (u_{2n+1}).

Exercice 2
Let (u_n) and (v_n) two numerical subsequences converging to l, l' with $l < l'$. Show that from a certain rank $u_n < v_n$.

Exercice 3
Let $a, b \in \mathbb{R}$, (u_n) and (v_n) two numerical subsequences such that

1. *$n \in \mathbb{N}$, $u_n \leq a$ and $v_n \leq b$ and*
2. *$u_n + v_n \longrightarrow a + b$.*

Show that (u_n) converges to a and (v_n) converges to b.

Exercice 4
Show that if $(u_n) \subset \mathbb{Z}$, then (u_n) converges, if and only if , (u_n) is stationary.

Exercice 5
Let (u_n) and (v_n) two numerical subsequences such that $(u_n + v_n)_n$ and $(u_n - v_n)_n$ converge. Show that (u_n) and (v_n) converge.

Exercice 6

Let $(u_n)_n$ a numerical sequence. What do you think of the following proposals? :

1. *If $(u_n)_n$ converges to a real Number l then $(u_{2n})_n$ and $(u_{2n+1})_n$ converge to l.*
2. *If $(u_{2n})_n$ and $(u_{2n+1})_n$ converge, then $(u_n)_n$ converges.*
3. *If $(u_{2n})_n$ and $(u_{2n+1})_n$ converge to the same limit l, it is the same with $(u_n)_n$.*

Exercice 7
Let $(u_n)_n$ two numerical sequences defined by : $u_0 \in]0, 1]$ and $u_{n+1} = \frac{u_n}{2} + \frac{u_n^2}{4}$.

1. *Show that : for all $n \in \mathbb{N}$, $0 < u_n \leq 1$.*
2. *Show that the sequence is monotone. Deduce the convergence of the sequence.*
3. *Compute the limit of $(u_n)_n$.*

Exercice 8
Determine the limit of the sequence $(u_n)_n$ whose general term is defined by :

$$u_n = \frac{2n + \sqrt{4n^2 + 1}}{n + \sqrt{n^2 + 1}}, \quad u_n = \frac{2n - \sqrt{4n^2 + 1}}{n - \sqrt{n^2 + 1}}, \quad u_n = (2n+1)\Big(\frac{1}{3n^2 + 1} + \frac{1}{3n^2 + 2} + \cdots + \frac{1}{3n^2 + n}\Big)?$$

$$u_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}, \quad u_n = \frac{1 \times 2 \times \cdots \times (2n+1)}{3 \times 6 \times \cdots \times (3n+3)}, \quad u_n = \sqrt{n^2 + n + 1} - \sqrt{n^2 - n + 1}.$$

Exercice 9
Prove the following formulas:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Deduce

$$\lim_{n \longrightarrow +\infty} \frac{1^2 + 2^2 + \cdots + n^2}{1^3 + 2^3 + \cdots + n^3}$$

Exercice 10
Let $u_n = 1 + \dfrac{1}{2!} + ... + \dfrac{1}{n!}$ *and* $v_n = u_n + \dfrac{1}{n!}$.

Show that (u_n) *and* (v_n) *converge to the same limit. Show that this limit is an element of* $\mathbb{R} \setminus \mathbb{Q}$.

Exercice 11
Let (u_n) *be a numerical sequence defined by* u_0 *and* $u_{n+1} = u_n^2 - 2$ *for all* n *in* $\mathbb{N}$.

1. *If* $u_0 = \sqrt{2}$ *calculate* $u_1,\ u_2,\ u_3$.
2. *Determine* u_4. *Deduce the nature of the sequence* (u_n).
3. *Examine the nature of the sequence* (u_n) *if* $u_0 = 2$
4. *Find another value of* u_0 *for which the sequence* (u_n) *is constant.*

   Study the nature of the sequence (u_n) *if* $u_0 = 1$ *or* $u_0 = 3$ *and in the case of* $u_0 = \frac{\sqrt{5}-1}{2}$.

Exercice 12
Let the numerical sequence (u_n) *defined by* u_0 *and* $u_{n+1} = u_n^2 + \frac{3}{16}$ *for all* n *in* $\mathbb{N}$.

1. *Show that* $1/4 \leq u_n \leq 3/4$ *for all* n *in* $\mathbb{N}$.
2. *Is the sequence monotone? Deduce its nature.*
3. *Determine* $\inf\{u_n,\ n \in \mathbb{N}\}$ *and* $\sup\{u_n,\ n \in \mathbb{N}\}$.

Exercice 13

1. *Let* $(u_n),\ (v_n),\ (w_n)$ *three numerical sequences, show that:*
   if $u_n \leq v_n \leq w_n$ *for all* $n \in \mathbb{N}$ *and if* $\lim\limits_{n \longrightarrow +\infty} u_n = \lim\limits_{n \longrightarrow +\infty} w_n = l$ *then* $\lim\limits_{n \longrightarrow +\infty} v_n = l$.
2. *Show that :*
   $$\int_p^{p+1} \frac{dx}{x} \leq \frac{1}{p} \leq \int_{p-1}^{p} \frac{dx}{x},\ p \in \mathbb{N}^*,\ p > 1.$$
3. *Deduce* $S_n = \sum_{k=1}^{n} \dfrac{1}{k+n}$ *for all* $n \in \mathbb{N}^*$ *then calculate* $\lim\limits_{n \longrightarrow +\infty} S_n$.
4. *Put* $S'_n = \sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k}$. *Show that* $S'_{2n} = S_n$.
5. *Show that* (S'_{2n}) *and* (S'_{2n+1}) *converge.*
6. *Deduce the convergence of* (S'_n).

Exercice 14
Consider the numerical sequence (u_n) *which converges to* 0. *Define the sequence* (v_n) *by its general term :*

$$v_n = \frac{1}{n^2} \sum_{k=1}^{n} k u_k.$$

Show that (v_n) *converges to* 0.

Exercice 15
Study the convergence of the sequence whose general term is :

$$u_n = \frac{1 + 2! + 3! + ... + n!}{n!}.$$

Exercice 16
Let (u_n) *be a sequence whose general term is :* $u_n = (1+a)(1+a^2) + ... + (1+a^n), \ \ 0 < a < 1.$

1. *Study variations of this sequence.*
2. *Prove the following inequality : for all* $x \in \mathbb{R}: \ 1 + x \leq e^x$.
3. *Deduce that* (u_n) *converges.*

Exercice 17
Consider a sequence of integers q_n *strictly increasing with* $q_0 \geq 1$. *Define* (u_n) *the sequence whose general term is*

$$u_n = \sum_{k=0}^{n} \prod_{j=0}^{k} \frac{1}{q_j}.$$

Show that (u_n) *converges.*