N. H. S. A. S. T
Preparatory Classe Department
Academic Year: 2025/2026

NHSAST_text.png

1st year Preparatory Classe
Exam of Algebra 1
Timing: 1H30

Exercise 1 (4.5 marks)

Let E be a set and $S \subseteq E$. Define the relation $\mathcal{R}$ on $\mathcal{P}(E)$ by:

$$\mathcal{R} = \{(A, B) \in \mathcal{P}(E) \times \mathcal{P}(E) \mid A \cap B = \emptyset\}.$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(E)$.
2. Determine the equivalence classes of the sets: $\{a\}$, E, S, and C_E (complement of E in E).
3. Determine the quotient set $\mathcal{P}(E)/\mathcal{R}$.

Exercise 2 (5.5 marks)

Let the map $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_n(x, y) = (2x, a^2x + y)$, where $a \in \mathbb{R}$.

1. Show f_n is bijective for all n and find its inverse.
2. Show by induction on $n \in \mathbb{N}$:

$$f_n^{(n)}(x, y) = \big(2^n x,\ (2^n - 1)a^2 x + y\big).$$

3. Show that for all $a, b \in \mathbb{R}$:

$$(f_n \circ f_n)(a, b) = (a, b).$$

Exercise 3 (6 marks)

Define on $G = \mathbb{R}^* \times \mathbb{R}$ the operation:

$$(a, b) * (c, d) = (ac,\ bc + d).$$

1. Is $*$ commutative?
2. Show $(G, *)$ is a group.
3. For $(a, b) \in G$ and $n \in \mathbb{N}^*$, compute $(a, b)^{(n)} = \underbrace{(a, b) * \cdots * (a, b)}_{n \text{ times}}$.
4. Let $H = \mathbb{R}^* \times \{0\}$. Show H is a commutative subgroup of G.
5. Let $f : (G, *) \to (H, *)$ defined by $f(a, b) = (a, 0)$.
   (a) Show f is a group homomorphism.
   (b) Is f injective? Is f an isomorphism?

Exercise 4 (4 marks)

Consider

$$P(X) = X^8 + 2X^7 + 2X^6 + 4X^5 + X^4 + 2X^3.$$

1. State the product of the roots of $P(X)$.
2. Verify that i is a root of $P(X)$ and determine its multiplicity.
3. Factor $P(X)$ over $\mathbb{C}$, then over $\mathbb{R}$.