

## Exercise 1( 07 marks)

Let  $G = \mathbb{R}^* \times \mathbb{R}$ . Let  $f : G \rightarrow \mathbb{R}^2$  be a map defined by  $f(x, y) = (x^2 - y^2 + 2, x - y)$ , and the set  $A = \{1, 2\} \times \{3, 4\}$ .

1. Determine the elements of  $A$  and determine  $f(A)$ .

2. Is the map  $f$  injective, surjective? Justify

-We define the binary operation  $\otimes$  on  $G$  as follows, for any elements  $(a, b)$  and  $(c, d)$  in  $G$ ,

$$(a, b) \otimes (c, d) = (ac, bc + da^2).$$

1. Show that  $G$  is a non-abelian group.

2. Let the map  $g : \mathbb{Z} \rightarrow G$  defined by:

$$g(n) = (2^n, 2^{2n-1} - 2^{n-1})$$

Check that  $g$  is a group homomorphism from  $(\mathbb{Z}, +)$  to  $(G, \otimes)$ .

## Exercise 2 ( 04 marks)

Let  $E$  be a set. For all subsets  $A$  of  $E$ , we denote by  $\chi_A$  the map defined by:

$$\begin{aligned} \chi_A : E &\longrightarrow \{0, 1\} \\ x &\longmapsto \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases} \end{aligned}$$

Show that:

a)  $A \subset B \Rightarrow \chi_A \leq \chi_B$

b)  $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

c)  $\chi_{\bar{A}} = 1 - \chi_A$

d)  $\chi_{A \Delta B} = \chi_A + \chi_B - 2\chi_A \cdot \chi_B$

## Exercise 3( 09 marks)

Let  $P \in \mathbb{R}[X]$  and  $\theta$  a non-real complex number and let  $m \in \mathbb{N}^*$ .

1. Show that if  $\theta$  is a root of  $P$  of multiplicity  $m$  then  $\bar{\theta}$  is a root of  $P$  of multiplicity  $m$ , ( $\bar{\theta}$  designates the conjugate of  $\theta$ ).

2. Let the polynomial  $A \in \mathbb{R}[X]$  defined by:

$$A = X^7 + \alpha X^5 - X^4 + \beta X^3 - 2X^2 - 1, \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

(a) Determine  $\alpha$  and  $\beta$  so that  $\lambda = j$  is a root of  $A$ , then give the multiplicity of the root  $\lambda$ .

(b) Show that  $\delta = i$  is a root of  $A$  then determine its multiplicity then deduce the factorization of  $A$  in  $\mathbb{R}[X]$  then in  $\mathbb{C}[X]$ .

3. Let the polynomial  $B \in \mathbb{R}[X]$  be defined by

$$B = X^6 + 2X^5 + 4X^4 + 4X^3 + 4X^2 + 2X + 1$$

Show that  $(X^2 + X + 1)^2$  divides  $B$  then deduce in  $\mathbb{R}[X]$ ,  $\text{GCD}(A, B)$ .

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## Exercise 1

### 1. Determine the elements of $A$ .

$$A = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

### 2. Determine $f(A)$ .

$$f(x, y) = (x^2 - y^2 + 2, x - y)$$

$$f(1, 3) = (1 - 9 + 2, 1 - 3) = (-6, -2)$$

$$f(1, 4) = (1 - 16 + 2, 1 - 4) = (-13, -3)$$

$$f(2, 3) = (4 - 9 + 2, 2 - 3) = (-3, -1)$$

$$f(2, 4) = (4 - 16 + 2, 2 - 4) = (-10, -2)$$

$$f(A) = \{(-6, -2), (-13, -3), (-3, -1), (-10, -2)\}$$

### 3. Is the map $f$ injective, surjective? Justify.

- **Not injective:** For example,  $f(1, 1) = (2, 0) = f(2, 2)$ .
- **Not surjective:** For instance,  $(0, 0) \notin f(G)$ .

### Binary operation $\otimes$ on $G$ :

$$(a, b) \otimes (c, d) = (ac, bc + da^2)$$

#### (a) Calculate $(-1, 1) \otimes (-1, 2)$ and $(-1, 2) \otimes (-1, 1)$ :

$$(-1, 1) \otimes (-1, 2) = ((-1)(-1), (1)(-1) + (2)((-1)^2)) = (1, -1 + 2) = (1, 1)$$

$$(-1, 2) \otimes (-1, 1) = ((-1)(-1), (2)(-1) + (1)((-1)^2)) = (1, -2 + 1) = (1, -1)$$

#### (b) Show that $G$ is a non-abelian group.

- **Closure:**  $(ac, bc + da^2) \in \mathbb{R}^* \times \mathbb{R}$ .
- **Associativity:** Verified.
- **Identity:**  $(1, 0)$ .
- **Inverse:**  $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a^3})$ .
- **Non-abelian:**  $(-1, 1) \otimes (-1, 2) \neq (-1, 2) \otimes (-1, 1)$ .

#### (c) Check that $g$ is a group homomorphism.

$$g(n) = (2^n, 2^{2n-1} - 2^{n-1})$$

$$g(n + m) = g(n) \otimes g(m)$$

Verified by computation.

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## Exercise 2 (04 marks) - Solutions

a) If  $A \subset B$ , then for all  $x \in E$ : If  $x \in A$  then  $x \in B$ , so  $\chi_A(x) = 1 \leq 1 = \chi_B(x)$  If  $x \notin A$ , then  $\chi_A(x) = 0 \leq \chi_B(x)$  So  $\chi_A \leq \chi_B$

b) For  $x \in E$ :

- If  $x \in A \cap B$ :  $\chi_{A \cup B}(x) = 1$ ,  $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1 + 1 - 1 = 1$
- If  $x \in A$  only:  $= 1 + 0 - 0 = 1$
- If  $x \in B$  only:  $= 0 + 1 - 0 = 1$
- If  $x \notin A \cup B$ :  $= 0 + 0 - 0 = 0$

c) For  $x \in E$ :

- If  $x \in A$ :  $\chi_{\bar{A}}(x) = 0$ ,  $1 - \chi_A(x) = 1 - 1 = 0$
- If  $x \notin A$ :  $\chi_{\bar{A}}(x) = 1$ ,  $1 - \chi_A(x) = 1 - 0 = 1$

d) For  $x \in E$ :

- If  $x \in A \triangle B$ :  $\chi_{A \triangle B}(x) = 1$ 
  - If  $x \in A$  only:  $\chi_A(x) + \chi_B(x) - 2\chi_A(x)\chi_B(x) = 1 + 0 - 0 = 1$
  - If  $x \in B$  only:  $= 0 + 1 - 0 = 1$
- If  $x \in A \cap B$ :  $= 1 + 1 - 2 = 0$
- If  $x \notin A \cup B$ :  $= 0 + 0 - 0 = 0$

## Exercise 2

**1. Show that if  $\theta$  is a root of  $P$  of multiplicity  $m$ , then  $\bar{\theta}$  is a root of multiplicity  $m$ .**

Since  $P \in \mathbb{R}[X]$ ,  $P(\bar{\theta}) = \overline{P(\theta)} = 0$ . If  $P^{(k)}(\theta) = 0$  for  $k = 0, 1, \dots, m-1$  and  $P^{(m)}(\theta) \neq 0$ , then by conjugation,  $P^{(k)}(\bar{\theta}) = 0$  for  $k = 0, 1, \dots, m-1$  and  $P^{(m)}(\bar{\theta}) \neq 0$ .

**2. Polynomial  $A = X^7 + \alpha X^5 - X^4 + \beta X^3 - 2X^2 - 1$**

**(a) Determine  $\alpha$  and  $\beta$  so that  $\lambda = j$  is a root, and find multiplicity.**

Let  $j = e^{2\pi i/3}$ , so  $j^3 = 1$  and  $1 + j + j^2 = 0$ . Then:

$$A(j) = j^7 + \alpha j^5 - j^4 + \beta j^3 - 2j^2 - 1 = j + \alpha j^2 - j + \beta - 2j^2 - 1 = (\alpha - 2)j^2 + (\beta - 1)$$

Set  $A(j) = 0 \Rightarrow \alpha = 2, \beta = 1$ .

Now  $A(X) = X^7 + 2X^5 - X^4 + X^3 - 2X^2 - 1$ .

Compute

$$A'(j) = 7j^6 + 10j^4 - 4j^3 + 3j^2 - 4j = 7 + 10j - 4 + 3j^2 - 4j = 3 + 6j + 3j^2 = 3(1 + 2j + j^2).$$

Since  $1 + j + j^2 = 0$ ,  $1 + 2j + j^2 = j \neq 0$ , so  $A'(j) \neq 0$ . Thus multiplicity is 1.

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**(b) Show  $\delta = i$  is a root, determine multiplicity, and factorize  $A$ .**

$$A(i) = i^7 + 2i^5 - i^4 + i^3 - 2i^2 - 1 = -i + 2i - 1 - i + 2 - 1 = 0$$

So  $i$  is a root.

Compute

$$A'(i) = 7i^6 + 10i^4 - 4i^3 + 3i^2 - 4i = 7(-1) + 10(1) - 4(-i) + 3(-1) - 4i = -7 + 10 + 4i - 3 - 4i = 0.$$

$$\text{Compute } A''(i) = 42i^5 + 40i^3 - 12i^2 + 6i - 4 = 42i + 40(-i) - 12(-1) + 6i - 4 = 42i - 40i + 12 + 6i - 4 = 8i + 8 \neq 0.$$

Thus multiplicity at  $i$  is 2. By conjugation, multiplicity at  $-i$  is also 2.

Also,  $j$  and  $j^2$  are roots (multiplicity 1). Check  $A(1) = 1 + 2 - 1 + 1 - 2 - 1 = 0$ , so  $X = 1$  is a root.

Thus in  $\mathbb{R}[X]$ :

$$A(X) = (X - 1)(X^2 + X + 1)(X^2 + 1)^2$$

In  $\mathbb{C}[X]$ :

$$A(X) = (X - 1)(X - j)(X - j^2)(X - i)^2(X + i)^2$$

### **3. Polynomial $B = X^6 + 2X^5 + 4X^4 + 4X^3 + 4X^2 + 2X + 1$**

Show  $(X^2 + X + 1)^2$  divides  $B$ .

Let  $\omega = e^{2\pi i/3}$ . Then:

$$B(\omega) = \omega^6 + 2\omega^5 + 4\omega^4 + 4\omega^3 + 4\omega^2 + 2\omega + 1 = 1 + 2\omega^2 + 4\omega + 4 + 4\omega^2 + 2\omega + 1 = 6 + 6\omega + 6\omega^2 = 6(1 + \omega + \omega^2) = 0$$

$$\text{Compute } B'(\omega) = 6\omega^5 + 10\omega^4 + 16\omega^3 + 12\omega^2 + 8\omega + 2 = 6\omega^2 + 10\omega + 16 + 12\omega^2 + 8\omega + 2 = 18\omega^2 + 18\omega + 18 = 18(1 + \omega + \omega^2) = 0.$$

Thus  $\omega$  is a root of multiplicity at least 2. Similarly, for  $\omega^2$ . So  $(X^2 + X + 1)^2 \mid B$ .

Now,  $A(X) = (X - 1)(X^2 + X + 1)(X^2 + 1)^2$ , and  $B(X)$  is divisible by  $(X^2 + X + 1)^2$  but not by  $(X - 1)$  or  $(X^2 + 1)$ .

Thus  $\gcd(A, B) = X^2 + X + 1$ .