Ecole Nationale Polytechnique
Preparatory cycle

Exercise series n ° 1 - Semester 1
2023-2024
Mathematical logic - Theory of sets

Exercise 1 : The rain and the umbrella
The following two statements are given and assumed to be true :
If it rains, Ali takes an umbrella.
"Houda never takes an umbrella if it's not raining and always takes one when it is".

The following three logical propositions are introduced : P , Q and R :
P : It's raining.
Q : Ali has an umbrella.
R : Houda has an umbrella.

1. Write the statement using these propositions and the logical connectors : $\Rightarrow$ and $\Leftrightarrow$.
2. What can we deduce from these statements in the various situations below :
   a. Ali is walking with an umbrella.
   b. Ali is walking without an umbrella.
   c. Houda is walking with an umbrella.
   d. Houda walks without an umbrella.
   e. It's not raining.
   f. It's raining.
   Justify your answers carefully.

Exercise 2 :
a.Let x, y, a, b be real numbers. Say whether the following propositions are true or false. Justify.
1. $(x \leq 2) \Rightarrow (x^2 \leq 4)$
2. $0 \leq x \leq y$ et $a \leq b \Rightarrow (xa \leq yb)$
3. $(xy \neq 0) \wedge (x \leq y) \Rightarrow \left(\frac{1}{x} \geq \frac{1}{y}\right)$

b. Complete the dotted lines with the appropriate logical connector : $\Rightarrow, \Leftarrow, \Leftrightarrow$.
1. $x \in \mathbb{R}, \quad x^2 = 4 \ldots x = 2$;
2. $z \in \mathbb{C}, \quad z = \bar{z} \ldots z \in \mathbb{R}$;
3. $x \in \mathbb{R}, \quad x = \pi \ldots e^{2ix} = 1$.

Exercise 3 :
Deny the following assertions :
1. Every right-angled triangle has a right angle ;
2. In all stables, all horses are black ;
3. For any integer x, there exists an integer y such that, for any integer z, the relation $z < x$ implies the relation $z < x+1$;
4. $\forall \varepsilon > 0, \exists \alpha > 0 \;\; s.t. \;\; (|x - 7/5| < \alpha) \Rightarrow |5x - 7| < \varepsilon)$.

Exercise 4 :

1. Show by two types of reasoning that if $a^2 + 9 = 2^n$ then a is odd.
2. Show by recurrence the property : $\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}, (1+x)^n \geq 1 + nx$.
3. Show using the contrapositive that if the integer $(n^2 - 1)$ is not divisible by 8 then the integer n is even.
4. Show that there are two infinite subsets E_1 and E_2 of $\mathbb{N}$ that are disjoint such that :
   $\forall n \in E_1 \cup E_2, (n^2 - 1)$ is a natural number multiple of 8.
5. Show that if m and n are odd integers then $m^2 + n^2$ is even but not divisible by 4.

Exercise 5 :

1. Show by two reasoning that : $x \notin \mathbb{Q} \Rightarrow 1 + x \notin \mathbb{Q}$.
2. Let $a_1, a_2, \ldots, a_9$ be real numbers arranged in ascending order such that : $a_1 + a_2 + \ldots + a_9 = 90$. Show that there are three of these numbers whose sum is greater than or equal to 30.
3. Show that the real number $\dfrac{ln2}{ln3}$ is irrational.

Exercise 6 :

Let E be a non-empty set. $P(E)$ denotes the set of parts of E.

1. Show that the following equivalence is true : $\forall A, B \in P(E) : (A \cap B = A \cup B) \Rightarrow A = B$.
2. Show that the following implications are true :
   a.$\forall A, B \in P(E)$, $(A \cap B = A \cap C) \wedge (A \cup B = A \cup C) \Rightarrow B = C$.
   b.$(A \cap B = A \cap C) \Rightarrow (A \cap \bar{B} = A \cap \bar{C})$.
   We will use direct reasoning, as well as contrapositive reasoning.

Exercise 7 : Justify the following equalities :

1. $\forall A, B \subset E;\ \ (A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$: distributivity of the $\cap$ law with respect to the $\triangle$ law
2. $\forall A, B \subset E; A \triangle B = \bar{A} \triangle \bar{B}$:Here $\bar{A}$ means $A - B$ and $\bar{B}$ means $B - A$.