Test 1 Correction: Polar Coordinates and Kinematics

National School of Autonomous Systems

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Problem Statement

A point-like object **P** moves in the xy plane. Its velocity in the polar coordinate system is given by a velocity-time graph showing the radial velocity v_r and transverse (circumferential) velocity v_θ components.

Initial Conditions:

- At $t = 0$: $r(0) = 1$ m, $\theta(0) = 0$
- Time range: $t = 0$ to $t = 7$ seconds
- Units: velocity in m/s, time in seconds

Exercise 1: Parametric Equations in Polar Coordinates

Part a) Radial Position $r(t)$

The radial position is found by integrating the radial velocity:

$$r(t) = r_0 + \int_0^t v_r(\tau)\, d\tau$$

where $r_0 = 1$ m is the initial radius.

For discrete time steps with $\Delta t = 1$ s:

$$r_{n+1} = r_n + v_{r,n} \cdot \Delta t$$

Part b) Angular Position $\theta(t)$

The angular velocity is related to transverse velocity by:

$$\dot{\theta} = \frac{v_\theta(t)}{r(t)}$$

Integrating:

$$\theta(t) = \theta_0 + \int_0^t \frac{v_\theta(\tau)}{r(\tau)}\, d\tau$$

where $\theta_0 = 0$.

For discrete time steps:

$$\theta_{n+1} = \theta_n + \frac{v_{\theta,n}}{r_n} \cdot \Delta t$$

Exercise 2: Completing the Table

Procedure:

1. **Read velocity values** from Figure 1 (velocity-time graph) at each time $t = 0, 1, 2, \ldots, 7$ s
2. **Calculate** $r(t)$ using the recursive formula above
3. **Calculate** $\theta(t)$ using the recursive formula above

Template Table for Results:

t (s)	v_r (m/s)	v_θ (m/s)	r (m)	θ (rad)
0	[from graph]	[from graph]	1.000	0.000
1	[from graph]	[from graph]	—	—
2	[from graph]	[from graph]	—	—
3	[from graph]	[from graph]	—	—
4	[from graph]	[from graph]	—	—
5	[from graph]	[from graph]	—	—
6	[from graph]	[from graph]	—	—
7	[from graph]	[from graph]	—	—

Note: The dashes (—) represent values to be calculated using the formulas above.

Exercise 3: Acceleration Components in Polar Coordinates

Formulas

The acceleration in polar coordinates has two components:

Radial Acceleration:

$$a_r = \dot{v}_r - r\dot{\theta}^2 = \frac{dv_r}{dt} - r\left(\frac{v_\theta}{r}\right)^2$$

$$a_r = \frac{dv_r}{dt} - \frac{v_\theta^2}{r}$$

Transverse Acceleration:

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = r\frac{d}{dt}\left(\frac{v_\theta}{r}\right) + 2v_r\frac{v_\theta}{r}$$

$$a_\theta = \frac{dv_\theta}{dt} + 2\frac{v_r \cdot v_\theta}{r}$$

Calculating Derivatives using Finite Differences:

$$\left.\frac{dv_r}{dt}\right|_{t_n} \approx \frac{v_{r,n+1} - v_{r,n}}{\Delta t}$$

$$\left.\frac{dv_\theta}{dt}\right|_{t_n} \approx \frac{v_{\theta,n+1} - v_{\theta,n}}{\Delta t}$$

Total Acceleration Magnitude:

$$a = \sqrt{a_r^2 + a_\theta^2}$$

Exercise 4: Vectors at $t = 2$ s

At this time point, you should:

1. **Locate position** using $r(2)$ and $\theta(2)$
2. **Draw velocity vector** with components:
   - Radial: $v_r(2)$
   - Transverse: $v_\theta(2)$
3. **Draw acceleration vector** with components:
   - Radial: $a_r(2)$
   - Transverse: $a_\theta(2)$

Use an appropriate scale (e.g., 1 cm = 0.5 m/s) and place vectors at the computed position.

Exercise 5: Tangential and Normal Accelerations

Speed:

$$v = \sqrt{v_r^2 + v_\theta^2}$$

Tangential Acceleration (along velocity):

$$a_T = \frac{dv}{dt} = \frac{v_r a_r + v_\theta a_\theta}{v}$$

Normal Acceleration (perpendicular to velocity):

$$a_N = \frac{v^2}{\rho}$$

where ρ is the radius of curvature. Alternatively:

$$a_N = \sqrt{a^2 - a_T^2}$$

Exercise 6: Trajectory Plot

To draw the trajectory:

1. Convert each (r, θ) pair to Cartesian coordinates:
   - $x = r\cos\theta$
   - $y = r\sin\theta$
2. Plot all points on an xy-plane
3. Connect the points in time order to show the path of motion
4. Mark the starting point (x_0, y_0) at $t = 0$
5. Mark intermediate times (especially $t = 2$ s) and final position

Summary of Key Relationships

Polar Velocity:

$$\vec{v} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta = v_r\,\mathbf{u}_r + v_\theta\,\mathbf{u}_\theta$$

Polar Acceleration:

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{u}_\theta$$

Relations:

- $v_r = \dot{r}$
- $v_\theta = r\dot{\theta}$
- $a_r = \ddot{r} - r\dot{\theta}^2 = \dot{v}_r - \frac{v_\theta^2}{r}$
- $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \dot{v}_\theta + 2\,\frac{v_r v_\theta}{r}$

General Approach to Solution

Step 1: Extract Data from Graph

Read the velocity values for v_r and v_θ at each second from the provided velocity-time graph.

Step 2: Calculate Positions

Use numerical integration to compute $r(t)$ and $\theta(t)$ for each time step.

Step 3: Calculate Accelerations

Use the finite difference formulas to compute a_r and a_θ.

Step 4: Draw and Plot

Create diagrams showing vectors at $t = 2$ s and the full trajectory.

Step 5: Verify

Check that units are consistent and magnitudes are reasonable.

Notes for Students

- **Significant Figures:** Maintain appropriate precision based on your measured graph data
- **Units:** Always include units in your calculations (m, s, m/s, m/s^2, rad, rad/s)
- **Graphs:** Use proper scales, label axes, and include a legend
- **Explanations:** Document your reasoning for each step
- **Communication devices:** Ensure all phones are turned off during the exam as instructed

Appendix: Conversion Formulas

If you need to convert between coordinate systems:

Polar to Cartesian:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan 2(y, x)$$

End of Correction Document