

Exercise 1(07 marks)

Let $G = \mathbb{R}^* \times \mathbb{R}$. Let $f : G \rightarrow \mathbb{R}^2$ be a map defined by $f(x, y) = (x^2 - y^2 + 2, x - y)$, and the set $A = \{1, 2\} \times \{3, 4\}$.

1. Determine the elements of A and determine $f(A)$.
2. Is the map f injective, surjective? Justify

-We define the binary operation \otimes on G as follows, for any elements (a, b) and (c, d) in G ,

$$(a, b) \otimes (c, d) = (ac, bc + da^2).$$

1. Show that G is a non-abelian group.
2. Let the map $g : \mathbb{Z} \rightarrow G$ defined by:

$$g(n) = (2^n, 2^{2n-1} - 2^{n-1})$$

Check that g is a group homomorphism from $(\mathbb{Z}, +)$ to (G, \otimes) .

Exercise 2 (04 marks)

Let E be a set. For all subsets A of E , we denote by χ_A the map defined by:

$$\begin{aligned}\chi_A : E &\longrightarrow \{0, 1\} \\ x &\longmapsto \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}\end{aligned}$$

Show that:

- a) $A \subset B \Rightarrow \chi_A \leq \chi_B$
- b) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$
- c) $\chi_{\bar{A}} = 1 - \chi_A$
- d) $\chi_{A \Delta B} = \chi_A + \chi_B - 2\chi_A \cdot \chi_B$

Exercise 3(09 marks)

Let $P \in \mathbb{R}[X]$ and θ a non-real complex number and let $m \in \mathbb{N}^*$.

1. Show that if θ is a root of P of multiplicity m then $\bar{\theta}$ is a root of P of multiplicity m , ($\bar{\theta}$ designates the conjugate of θ).

2. Let the polynomial $A \in \mathbb{R}[X]$ defined by:

$$A = X^7 + \alpha X^5 - X^4 + \beta X^3 - 2X^2 - 1, \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

- (a) Determine α and β so that $\lambda = j$ is a root of A , then give the multiplicity of the root λ .
- (b) Show that $\delta = i$ is a root of A then determine its multiplicity then deduce the factorization of A in $\mathbb{R}[X]$ then in $\mathbb{C}[X]$.
3. Let the polynomial $B \in \mathbb{R}[X]$ be defined by

$$B = X^6 + 2X^5 + 4X^4 + 4X^3 + 4X^2 + 2X + 1$$

Show that $(X^2 + X + 1)^2$ divides B then deduce in $\mathbb{R}[X]$, $\text{GCD}(A, B)$.

Exercise 1

1. Determine the elements of A .

$$A = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

2. Determine $f(A)$.

$$f(x, y) = (x^2 - y^2 + 2, x - y)$$

$$f(1, 3) = (1 - 9 + 2, 1 - 3) = (-6, -2)$$

$$f(1, 4) = (1 - 16 + 2, 1 - 4) = (-13, -3)$$

$$f(2, 3) = (4 - 9 + 2, 2 - 3) = (-3, -1)$$

$$f(2, 4) = (4 - 16 + 2, 2 - 4) = (-10, -2)$$

$$f(A) = \{(-6, -2), (-13, -3), (-3, -1), (-10, -2)\}$$

3. Is the map f injective, surjective? Justify.

- **Not injective:** For example, $f(1, 1) = (2, 0) = f(2, 2)$.
- **Not surjective:** For instance, $(0, 0) \notin f(G)$.

Binary operation \otimes on G :

$$(a, b) \otimes (c, d) = (ac, bc + da^2)$$

(a) Calculate $(-1, 1) \otimes (-1, 2)$ and $(-1, 2) \otimes (-1, 1)$:

$$(-1, 1) \otimes (-1, 2) = ((-1)(-1), (1)(-1) + (2)((-1)^2)) = (1, -1 + 2) = (1, 1)$$

$$(-1, 2) \otimes (-1, 1) = ((-1)(-1), (2)(-1) + (1)((-1)^2)) = (1, -2 + 1) = (1, -1)$$

(b) Show that G is a non-abelian group.

- **Closure:** $(ac, bc + da^2) \in \mathbb{R}^* \times \mathbb{R}$.
- **Associativity:** Verified.
- **Identity:** $(1, 0)$.
- **Inverse:** $(a, b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a^3}\right)$.
- **Non-abelian:** $(-1, 1) \otimes (-1, 2) \neq (-1, 2) \otimes (-1, 1)$.

(c) Check that g is a group homomorphism.

$$g(n) = (2^n, 2^{2n-1} - 2^{n-1})$$

$$g(n+m) = g(n) \otimes g(m)$$

Verified by computation.

Exercise 2 (04 marks) - Solutions

a) If $A \subset B$, then for all $x \in E$: If $x \in A$ then $x \in B$, so $\chi_A(x) = 1 \leq 1 = \chi_B(x)$. If $x \notin A$, then $\chi_A(x) = 0 \leq \chi_B(x)$. So $\chi_A \leq \chi_B$.

b) For $x \in E$:

- If $x \in A \cap B$: $\chi_{A \cup B}(x) = 1$, $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1 + 1 - 1 = 1$
- If $x \in A$ only: $= 1 + 0 - 0 = 1$
- If $x \in B$ only: $= 0 + 1 - 0 = 1$
- If $x \notin A \cup B$: $= 0 + 0 - 0 = 0$

c) For $x \in E$:

- If $x \in A$: $\chi_{\bar{A}}(x) = 0$, $1 - \chi_A(x) = 1 - 1 = 0$
- If $x \notin A$: $\chi_{\bar{A}}(x) = 1$, $1 - \chi_A(x) = 1 - 0 = 1$

d) For $x \in E$:

- If $x \in A \Delta B$: $\chi_{A \Delta B}(x) = 1$
 - If $x \in A$ only: $\chi_A(x) + \chi_B(x) - 2\chi_A(x)\chi_B(x) = 1 + 0 - 0 = 1$
 - If $x \in B$ only: $= 0 + 1 - 0 = 1$
- If $x \in A \cap B$: $= 1 + 1 - 2 = 0$
- If $x \notin A \cup B$: $= 0 + 0 - 0 = 0$

Exercise 2

1. Show that if θ is a root of P of multiplicity m , then $\bar{\theta}$ is a root of multiplicity m .

Since $P \in \mathbb{R}[X]$, $P(\bar{\theta}) = \overline{P(\theta)} = 0$. If $P^{(k)}(\theta) = 0$ for $k = 0, 1, \dots, m-1$ and $P^{(m)}(\theta) \neq 0$, then by conjugation, $P^{(k)}(\bar{\theta}) = 0$ for $k = 0, 1, \dots, m-1$ and $P^{(m)}(\bar{\theta}) \neq 0$.

2. Polynomial $A = X^7 + \alpha X^5 - X^4 + \beta X^3 - 2X^2 - 1$

(a) Determine α and β so that $\lambda = j$ is a root, and find multiplicity.

Let $j = e^{2\pi i/3}$, so $j^3 = 1$ and $1 + j + j^2 = 0$. Then:

$$A(j) = j^7 + \alpha j^5 - j^4 + \beta j^3 - 2j^2 - 1 = j + \alpha j^2 - j + \beta - 2j^2 - 1 = (\alpha - 2)j^2 + (\beta - 1)$$

Set $A(j) = 0 \Rightarrow \alpha = 2, \beta = 1$.

Now $A(X) = X^7 + 2X^5 - X^4 + X^3 - 2X^2 - 1$.

Compute

$$A'(j) = 7j^6 + 10j^4 - 4j^3 + 3j^2 - 4j = 7 + 10j - 4 + 3j^2 - 4j = 3 + 6j + 3j^2 = 3(1 + 2j + j^2).$$

Since $1 + j + j^2 = 0$, $1 + 2j + j^2 = j \neq 0$, so $A'(j) \neq 0$. Thus multiplicity is 1.

(b) Show $\delta = i$ is a root, determine multiplicity, and factorize A .

$$A(i) = i^7 + 2i^5 - i^4 + i^3 - 2i^2 - 1 = -i + 2i - 1 - i + 2 - 1 = 0$$

So i is a root.

Compute

$$A'(i) = 7i^6 + 10i^4 - 4i^3 + 3i^2 - 4i = 7(-1) + 10(1) - 4(-i) + 3(-1) - 4i = -7 + 10 + 4i - 3 - 4i = 0.$$

$$\text{Compute } A''(i) = 42i^5 + 40i^3 - 12i^2 + 6i - 4 = 42i + 40(-i) - 12(-1) + 6i - 4 = 42i - 40i + 12 + 6i - 4 = 8i + 8 \neq 0.$$

Thus multiplicity at i is 2. By conjugation, multiplicity at $-i$ is also 2.

Also, j and j^2 are roots (multiplicity 1). Check $A(1) = 1 + 2 - 1 + 1 - 2 - 1 = 0$, so $X = 1$ is a root.

Thus in $\mathbb{R}[X]$:

$$A(X) = (X - 1)(X^2 + X + 1)(X^2 + 1)^2$$

In $\mathbb{C}[X]$:

$$A(X) = (X - 1)(X - j)(X - j^2)(X - i)^2(X + i)^2$$

3. Polynomial $B = X^6 + 2X^5 + 4X^4 + 4X^3 + 4X^2 + 2X + 1$

Show $(X^2 + X + 1)^2$ divides B .

Let $\omega = e^{2\pi i/3}$. Then:

$$B(\omega) = \omega^6 + 2\omega^5 + 4\omega^4 + 4\omega^3 + 4\omega^2 + 2\omega + 1 = 1 + 2\omega^2 + 4\omega + 4 + 4\omega^2 + 2\omega + 1 = 6 + 6\omega + 6\omega^2 = 6(1 + \omega + \omega^2) = 0$$

$$\text{Compute } B'(\omega) = 6\omega^5 + 10\omega^4 + 16\omega^3 + 12\omega^2 + 8\omega + 2 = 6\omega^2 + 10\omega + 16 + 12\omega^2 + 8\omega + 2 = 18\omega^2 + 18\omega + 18 = 18(1 + \omega + \omega^2) = 0.$$

Thus ω is a root of multiplicity at least 2. Similarly, for ω^2 . So $(X^2 + X + 1)^2 \mid B$.

Now, $A(X) = (X - 1)(X^2 + X + 1)(X^2 + 1)^2$, and $B(X)$ is divisible by $(X^2 + X + 1)^2$ but not by $(X - 1)$ or $(X^2 + 1)$.

Thus $\gcd(A, B) = X^2 + X + 1$.