# Exam Analysis 1

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**Exercise 01 : 4 pts** Consider the set

$$A = \left\{ 2(-1)^n - (-1)^{\frac{n(n+1)}{2}} \left(2 + \frac{3}{n}\right) \ ; \ n \in \mathbb{N}^* \right\}.$$

1. Show that the set $A$ can be written as the union of four subsets.
2. Determine $\inf A$, $\sup A$, $\min A$, and $\max A$, if they exist.

**Exercise 02 :6 pts** Let $(U_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$U_0 > 0, \qquad U_{n+1} = \frac{1}{2}\left(U_n + \frac{4}{U_n}\right), \quad \forall n \in \mathbb{N}.$$

1. Show that the sequence $(U_n)$ is well defined and positive.
2. Study the monotonicity of $(U_n)$ according to the value of $U_0$.
3. Show that $(U_n)$ is convergent.
4. Determine $\lim\limits_{n\to\infty} U_n$.

**Exercise 03: 10 pts**

**Part I**

Let $k > 0$ and define the function $f_k$ by

$$f_k(x) = \begin{cases} \left(2 - \frac{x}{k}\right)^{\tan\left(\frac{x\pi}{2k}\right)}, & \text{if } k \le x < 2k \\ a, & \text{if } x < k, \end{cases}$$

where $a \in \mathbb{R}$.

1. Find the value of $a$ such that $f_k$ is continuous on $D_f$.
2. Compute the derivative $f_k'(x)$ for all $x \neq k$.
3. Study the continuity of $f_k'$ at $x = k$.
4. What can you conclude about the differentiability of $f_k$ at $x = k$?

**Part II**

Let $g$ be the function defined by

$$g(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

1. Determine the domain $D_g$ of $g$.

2. Show that

$$g'(x) = \frac{1}{\sqrt{x^2+1}}, \qquad \forall x \in D_g.$$

3. Study the bijectivity of $g$ on $D_g$ and deduce the expression of its inverse function.

4. Show that

$$\frac{x-\sqrt{x^2-1}}{\sqrt{x^2+1}} < g(x) - g\left(\sqrt{x^2-1}\right) < \frac{x-\sqrt{x^2-1}}{x},$$

then deduce

$$\lim_{x\to+\infty} x^2\left(g(x) - g\left(\sqrt{x^2-1}\right)\right).$$

**Part III**

Show that

$$\arctan\left(\sqrt{e^{2x}-1}\right) = \arccos\left(e^{-x}\right).$$

**Good luck**