National School of Autonomous Systems

1^{st} year
2022/2023

Module: Analysis 1

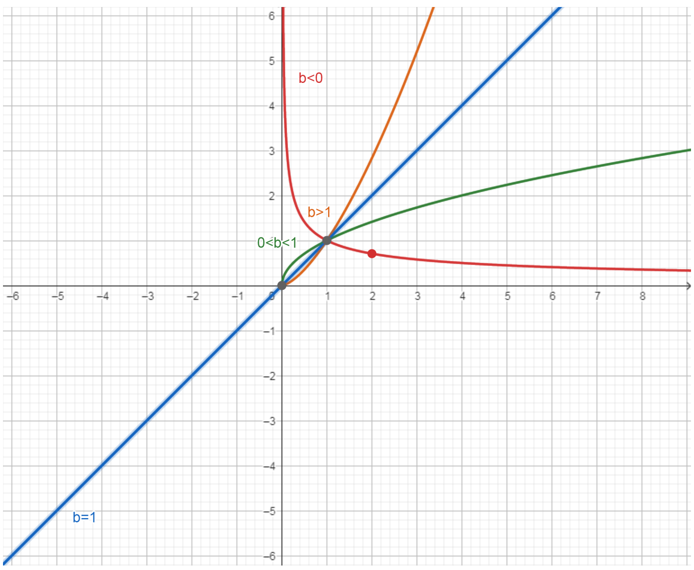
Chapter 4: Elementary Functions

1 Power Functions

Definition Let $b \in \mathbb{R}$, the function $f :]0, +\infty[\to \mathbb{R}$ defined by $f(x) = x^b$ is called a power function.

Proposition.

- We can write $f(x) = e^{b \ln x}$, so
$$f'(x) = \frac{b}{x} e^{b \ln x} = b e^{-\ln x} e^{b \ln x} = b e^{(b-1) \ln x} = b x^{b-1}.$$
- For all $x > 0$, $x^{b-1} > 0$, so $f'(x)$ has the same sign as b. Thus,
  - ⋆ f is strictly increasing if $b > 0$.
  - ⋆ f is strictly decreasing if $b < 0$.
- If $b > 0$, we observe $\lim_{x \to 0} f(x) = 0$, so we can extend f at 0 by 0.
- If $0 < b < 1$, we have $\lim_{x \to 0} f'(x) = +\infty$, the graph of f has a vertical tangent line.



Definition Let $a > 0$, the function $g_a : \mathbb{R} \to \mathbb{R}$, $x \mapsto a^x$ is called the exponential function with base a.

Proposition.

- The function g_a is differentiable on $\mathbb{R}$ and $g_a'(x) = \ln(a) a^x$.
- g_a is increasing if $a > 1$ and decreasing if $a < 1$.
- $a^{x+y} = a^x a^y$, $a^{x-y} = \dfrac{a^x}{a^y}$, $a^{nx} = (a^x)^n$, ... $\forall a, b \in \mathbb{R}_+^\star$, $x, y \in \mathbb{R}$ $n \in \mathbb{Z}$
- In particular, for $a = e$, we have $g_a = \exp$.

2 Inverse Trigonometric Functions

Recall of the Bisection Theorem
Theorem Let $f : I \to \mathbb{R}$ (I an interval) be a continuous and strictly monotonic function. Let $J = f(I)$. Then,

- f establishes a bijection from the interval I to the interval J.
- The inverse function $f^{-1} : J \to I$ is a continuous function on J, strictly monotonic, and has the same direction as f.

Proposition. Let $f : I \to J$ be a continuous, strictly monotonic, and differentiable function at the point $x_0 \in I$ such that $f(I) = J$.

The function f^{-1} is differentiable at the point $y_0 = f(x_0)$ if and only if $f'(x_0) \neq 0$, and then we have

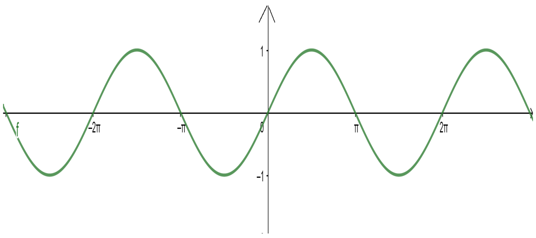
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

2.1 Arcsine Function

Consider the function

$$\begin{array}{rcl} \sin : \mathbb{R} & \to & [-1, 1] \\ x & \mapsto & \sin x \end{array}$$

- f is continuous and differentiable on $\mathbb{R}$.
- f is odd and $2\pi-$ periodic.
- f is of class $\mathcal{C}^\infty(\mathbb{R})$.
- The restriction of sin on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is strictly increasing.



Definition The function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is a bijection, and it's inverse function is called **arcsine**, denoted by arcsin

$$\begin{array}{rcl} \arcsin : [-1, 1] & \to & [-\frac{\pi}{2}, \frac{\pi}{2}] \\ y & \mapsto & \arcsin y \end{array}$$

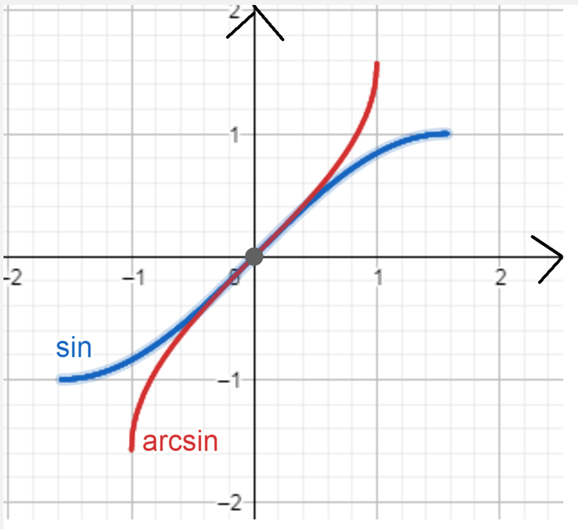
Proposition.

- $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arcsin(\sin x) = x$.
- $\forall y \in [-1, 1]$, $\sin(\arcsin y) = y$.
- arcsin is odd.

Additionally, the function arcsin satisfies the following properties:

- continuous and strictly increasing on $[-1, 1]$.
- differentiable on $]-1, 1[$ and $\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$.
- of class $\mathcal{C}^\infty(]-1, 1[)$.

y	-1		1
$\arcsin y$	$-\frac{\pi}{2}$	$\nearrow$	$\frac{\pi}{2}$



Proof
• By the bijection theorem, we conclude that the function arcsin is continuous and strictly monotonic, with the same monotonicity as sin (increasing).
• Moreover, as $\sin' x = \cos x \neq 0$ for all $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, the function arcsin is differentiable on the interval $J = \sin(]-\frac{\pi}{2}, \frac{\pi}{2}[) =]-1, 1[$. Using the theorem on the derivative of the reciprocal bijection, we have

$$\arcsin'(y) = \frac{1}{\sin'(x)} = \frac{1}{\sin'(\arcsin(y))} = \frac{1}{\cos(\arcsin y)}$$

As

$$\cos(\arcsin(y)) = \sqrt{1 - \sin^2(\arcsin(y))} = \sqrt{1 - y^2}$$

Thus

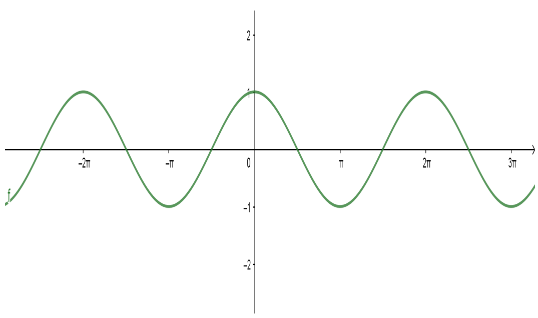
$$\arcsin'(y) = \frac{1}{\sqrt{1 - y^2}} > 0$$

2.2 Arccosine Function

Consider the function

$$\begin{aligned} \cos : \mathbb{R} &\to [-1, 1] \\ x &\mapsto \cos x \end{aligned}$$

• f is continuous and differentiable on $\mathbb{R}$.
• f is even and $2\pi-$ periodic.
• f is of class $\mathcal{C}^\infty(\mathbb{R})$.
• The restriction of cos on $[0, \pi]$ is strictly decreasing.



Definition The function $\cos : [0, \pi] \to [-1, 1]$ is a bijection. The inverse bijection is called **arccosine**, denoted by arccos

$$\begin{aligned} \arccos : [-1, 1] &\to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \\ y &\mapsto \arccos y \end{aligned}$$

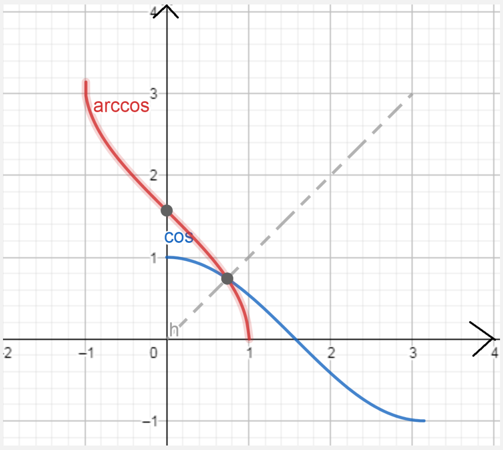
Proposition.

- $\forall x \in [0, \pi]$, $\arccos(\cos x) = x$.
- $\forall y \in [-1, 1]$, $\cos(\arccos y) = y$.

Additionally, the function arccos satisfies the following properties:

- continuous and strictly decreasing on $[-1, 1]$.
- differentiable on $]-1, 1[$ and $\arccos'(y) = \frac{-1}{\sqrt{1-y^2}}$.
- of class $\mathcal{C}^\infty(]-1, 1[)$.

y	-1	1
$\arccos y$	π $\searrow$	0



Attention

- If $x \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arcsin(\sin x) \neq x$. For example, $\arcsin(\sin \pi) = 0$.
- If $x \notin [0, \pi]$, $\arccos(\cos x) \neq x$. For example, $\arccos(\cos(-\frac{\pi}{2})) = \frac{\pi}{2}$.

Example: Calculate $\arcsin(\sin(\frac{18\pi}{5}))$ and $\arccos(\cos(\frac{7\pi}{3}))$.

- Since $\frac{18\pi}{5} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, we must find the angle $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ that corresponds to it. We have

$$\sin(\frac{18\pi}{5}) = \sin(4\pi - \frac{2\pi}{5}) = \sin(-\frac{2\pi}{5})$$

Thus

$$\arcsin(\sin(\frac{18\pi}{5})) = \arcsin(\sin(-\frac{2\pi}{5})) = -\frac{2\pi}{5}$$

- $$\arccos(\cos(\frac{7\pi}{3})) = \arccos(\cos(2\pi + \frac{\pi}{3})) = \arccos(\cos(\frac{\pi}{3})) = \frac{\pi}{3}.$$

Example: Solve the equation $\arcsin(2x) - \arcsin(\sqrt{3}x) = \arcsin(x)$.
We look for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and use the trigonometric formula $\sin(a+b) = \sin a \cos b + \sin b \cos a$.
We have

$$\sin(\arcsin(2x) - \arcsin(\sqrt{3}x)) = \sin(\arcsin(x)) \Leftrightarrow 2x\cos(\arcsin(\sqrt{3}x)) - \sqrt{3}x\cos(\arcsin(2x)) = x \qquad \textbf{(E)}$$

Since $\cos(a) = \sqrt{1 - \sin^2 a}$ $(a \in [-\frac{\pi}{2}, \frac{\pi}{2}])$, we have

$$\textbf{(E)} \Leftrightarrow 2x\sqrt{1-3x^2} - \sqrt{3}x\sqrt{1-4x^2} = x \Leftrightarrow x^2\sqrt{1-4x^2} = 0$$

Hence $x = 0$ or $x = \pm\frac{1}{2}$.

2.3 Arctangent Function

Let

$$\begin{array}{rcl} \tan : \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$$

- f is continuous and differentiable on $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\, k \in \mathbb{Z}\right\}$.
- f is odd and $\pi-$periodic.
- f is of class $\mathcal{C}^\infty \left(\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\, k \in \mathbb{Z}\right\}\right)$.
- The restriction of tan to $]-\frac{\pi}{2}, \frac{\pi}{2}[$ is strictly increasing.

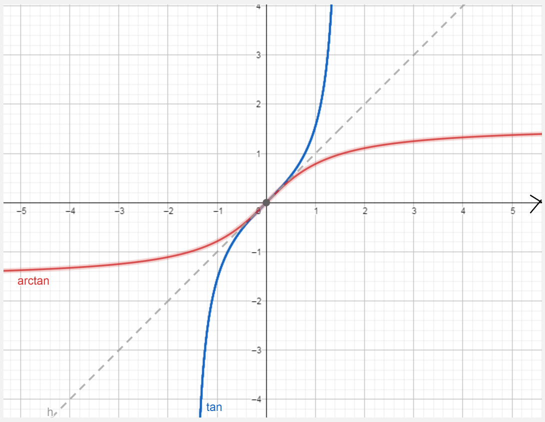
Definition: The function $\tan :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$ is a bijection. The inverse bijection is called **arctangent** denoted by arctan

$$\begin{aligned} \arctan : \mathbb{R} &\to \;]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\\ y &\mapsto \arctan y \end{aligned}$$

Proposition.

- $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\arctan(\tan x) = x$.
- $\forall y \in \mathbb{R}$, $\tan(\arctan y) = y$.
  Moreover, the function arctan satisfies the following properties:
- continuous and strictly increasing on $\mathbb{R}$.
- odd.
- differentiable on $\mathbb{R}$ with
$$\arctan'(y) = \frac{1}{1+y^2}$$
- of class $\mathcal{C}^\infty(\mathbb{R})$.

y	$-\infty$ $\qquad$ $+\infty$
$\arctan y$	$-\frac{\pi}{2} \nearrow \frac{\pi}{2}$



Proof: Since $\tan' x = \frac{1}{\cos^2 x} \neq 0$ for all $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, by the inverse function differentiation theorem,
$$\arctan'(y) = \frac{1}{\tan'(\arctan(y))} = \frac{1}{1+\tan^2(\arctan(y))} = \frac{1}{1+y^2}.$$

Proposition.

- For all $x \in [-1, 1]$, $\arcsin x + \arccos x = \frac{\pi}{2}$.
- For all $x \in \mathbb{R}^\star$, $\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}.$

Proof: It suffices to consider the function $f :]-1, 1[\to \mathbb{R}$ such that $f(x) = \arccos x + \arcsin x$. f is differentiable on $]-1, 1[$ and $f'(x) = 0$, which means $f(x) = k$ ($k \in \mathbb{R}$). To determine k, set $x = 0$, then $f(0) = \arccos(0) + \arcsin(0) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$. Similar for the second part.
Example Simplify and determine the domain of definition of the function

$$f(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$$

- f is defined if $1 - x^2 > 0$, hence $D_f =]-1, 1[$.
- Calculate f':

$$f'(x) = \left(\frac{x}{\sqrt{1-x^2}}\right)' \frac{1}{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2} = \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

Notice $f'(x) = \frac{1}{\sqrt{1-x^2}} = \arcsin'(x)$, thus $f(x) = \arcsin(x)$.

3 Hyperbolic Functions

Definition

- **Hyperbolic sine:** Denoted by sh, defined from $\mathbb{R}$ to $\mathbb{R}$ by

$$\text{sh}x = \frac{e^x - e^{-x}}{2}$$

- **Hyperbolic cosine:** Denoted by ch, defined from $\mathbb{R}$ to $\mathbb{R}$ by

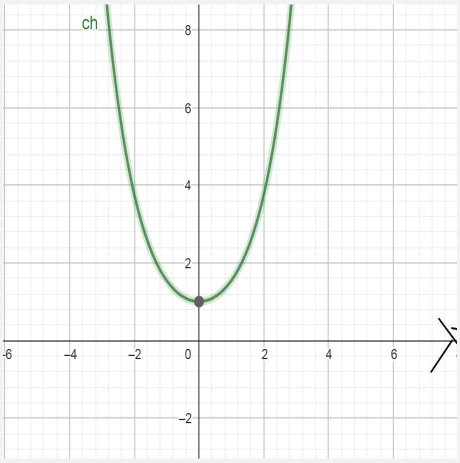
$$\text{ch}x = \frac{e^x + e^{-x}}{2}$$

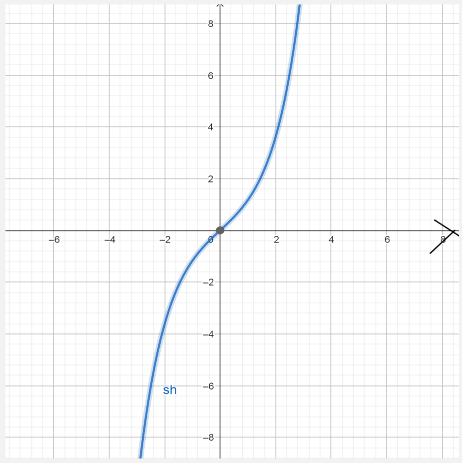
Proposition.

- ch and sh are of class $\mathcal{C}^\infty(\mathbb{R})$.
- ch is even, and sh is odd.
- $\text{ch}x + \text{sh}x = e^x$ and $\text{ch}x - \text{sh}x = e^{-x}$.
- $\text{ch}^2 x - \text{sh}^2 x = 1$.
- $\text{ch}x \geq 1$ for all $x \in \mathbb{R}$, $\text{sh}0 = 0$ and $\text{ch}0 = 1$
- $\text{sh}'x = \text{ch}x > 0$, thus sh is strictly increasing on $\mathbb{R}$, negative on $\mathbb{R}_-^\star$, and positive on $\mathbb{R}_+^\star$.
- $\text{ch}'x = \text{sh}x$, thus ch is strictly increasing on $\mathbb{R}_+^\star$, and strictly decreasing on $\mathbb{R}_-^\star$.

x	$-\infty$		0		$+\infty$
ch$x$	$+\infty$	$\searrow$	1	$\nearrow$	∞

x	$-\infty$		$+\infty$
sh$x$	$-\infty$	$\nearrow$	$+\infty$





Definition

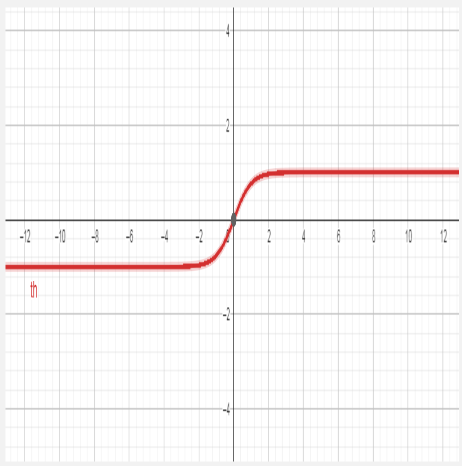
- **Hyperbolic tangent:** Denoted by th, defined from $\mathbb{R}$ to $\mathbb{R}$ by

$$\mathrm{th}x = \frac{\mathrm{sh}x}{\mathrm{ch}x} = \frac{e^{2x}-1}{e^{2x}+1}.$$

Proposition.

- th is of class $\mathcal{C}^\infty(\mathbb{R})$.
- th is odd.
- $\mathrm{th}0 = 0$, $\lim_{x\to+\infty}\mathrm{th} = 1$ and $\lim_{x\to-\infty}\mathrm{th} = -1$.
- $\mathrm{th}'x = \dfrac{1}{\mathrm{ch}^2 x} = 1 - \mathrm{th}^2 x$.

x	$-\infty$	$+\infty$
$\mathrm{th}x$	-1 $\nearrow$	$+1$



4 Inverse Hyperbolic Functions

1- Argument Hyperbolic Sine: The function sh is continuous and strictly increasing from $\mathbb{R}$ to $\mathbb{R}$, so it has an inverse denoted by argsh. We have

$$\mathrm{argsh} : \mathbb{R} \to \mathbb{R}$$

such that

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R},\ y = \mathrm{argsh}(x) \Leftrightarrow x = \mathrm{sh}(y)$$

2- Argument Hyperbolic Cosine: The function ch is continuous and strictly increasing from $[0, +\infty[$ to $[1, +\infty[$, so it has an inverse denoted by argch. We have

$$\mathrm{argch} : [1, +\infty[\to [0, +\infty[$$

such that

$$(y = \operatorname{argch}(x), \quad x \geq 1) \Leftrightarrow (x = \operatorname{ch}(y), \quad y \geq 0).$$

3- Argument Hyperbolic Tangent: The function th is continuous and strictly increasing from $\mathbb{R}$ to $]-1, 1[$, so it has an inverse denoted by argth . We have

$$\operatorname{argth} :]-1, 1[\to \mathbb{R}$$

such that

$$(y = \operatorname{argth}(x), \quad |x| < 1) \Leftrightarrow (x = \operatorname{th}(y), \quad y \in \mathbb{R}).$$

Proposition.

- The function argsh is odd, continuous on $\mathbb{R}$, and differentiable on $\mathbb{R}$, and we have

$$\operatorname{argsh}'(y) = \frac{1}{\sqrt{1+y^2}}, \quad \forall y \in \mathbb{R}.$$

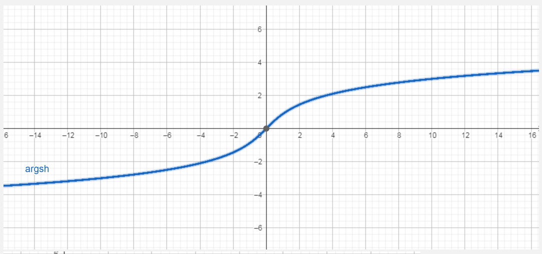
- The function argch is continuous and strictly increasing on $[1, +\infty[$, and differentiable on $]1, +\infty[$, and we have

$$\operatorname{argch}'(y) = \frac{1}{\sqrt{y^2 - 1}}, \quad \forall y \in]1, +\infty[.$$

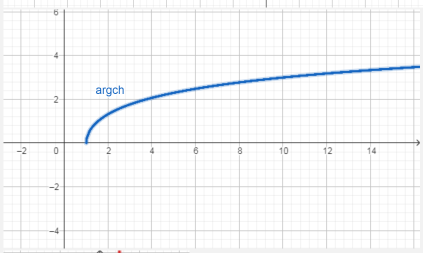
- The function argth is odd, continuous and strictly increasing on $]-1, 1[$, and differentiable on $]-1, 1[$, and we have

$$\operatorname{argth}'(y) = \frac{1}{1 - y^2}, \quad \forall y \in]-1, 1[.$$

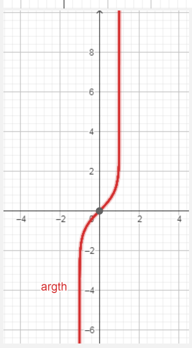
y	$-\infty$	$+\infty$
argsh$y$	$-\infty$ $\nearrow$	$+\infty$



y	$-\infty$	$+\infty$
argch$y$	0 $\nearrow$	$+\infty$



y	-1	1
argth$y$	$-\infty$ $\nearrow$	$+\infty$



Note: One can express the inverse hyperbolic functions (argsh, argch, and argth) in terms of the logarithm. The following relations hold:

$$\begin{aligned} \text{argsh}(x) &= \ln(x + \sqrt{1+x^2}), & \forall x \in \mathbb{R} \\ \text{argch}(x) &= \ln(x + \sqrt{x^2-1}), & \forall x \in [1, +\infty[\\ \text{argth}(x) &= \tfrac{1}{2}\ln\left(\tfrac{1+x}{1-x}\right), & \forall x \in]-1, 1[. \end{aligned}$$

Indeed, since

$$\text{ch}^2 y - \text{sh}^2 y = 1 \Leftrightarrow \text{ch}(y) = \sqrt{1 + \text{sh}^2 y},$$

by taking $y = \text{argsh}(x)$, we obtain $\text{ch}(\text{argsh}(x)) = \sqrt{1+x^2}$. Knowing that $e^y = \text{ch}y + \text{sh}y$, which implies $e^y = x + \sqrt{1+x^2}$, we have $\text{argsh}(x) = \ln(x + \sqrt{1+x^2})$.

Circular Trigonometry Formulas:
For all a, b, p and q in $\mathbb{R}$ we have the followigs formulas

- $\cos(a+b) = \cos a \cos b - \sin a \sin b$
- $\cos(a-b) = \cos a \cos b + \sin a \sin b$
- $\sin(a+b) = \sin a \cos b + \sin b \cos a$
- $\sin(a-b) = \sin a \cos b - \sin b \cos a$
- $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$
- $\tan(a-b) = \dfrac{\tan a + \tan b}{1 + \tan a \tan b}$
- $\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$
- $\sin(2a) = 2 \sin a \cos a$
- $\cos^2(a) = \dfrac{1 + \cos(2a)}{2}$
- $\sin^2(a) = \dfrac{1 - \cos(2a)}{2}$

Hyperbolic Trigonometry Formulas:

- $\mathrm{ch}(a+b) = \mathrm{ch}a\mathrm{ch}b + \mathrm{sh}a\mathrm{sh}b$
- $\mathrm{ch}(a-b) = \mathrm{ch}a\mathrm{ch}b - \mathrm{sh}a\mathrm{sh}b$
- $\sin(a+b) = \mathrm{sh}a\mathrm{ch}b + \mathrm{sh}b\mathrm{ch}a$
- $\sin(a-b) = \mathrm{sh}a\mathrm{ch}b - \mathrm{sh}b\mathrm{ch}a$
- $\mathrm{th}(a+b) = \dfrac{\mathrm{th}a + \mathrm{th}b}{1 + \mathrm{th}a\mathrm{th}b}$
- $\mathrm{th}(a-b) = \dfrac{\mathrm{th}a - \mathrm{th}b}{1 + -\mathrm{th}a\mathrm{th}b}$
- $\mathrm{ch}^2 a = \frac{\mathrm{ch}(2a)+1}{2}$
- $\mathrm{sh}^2 a = \frac{\mathrm{ch}(2a)-1}{2}$