2. Sets

Definition 14. *A set is any collection of objects, these objects are called elements or members of the set.*

Examples

$\mathbb{N} = \{0, 1, 2, \ldots\}$, is the set of natural numbers.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, is the set of relative integers.

$\mathbb{Q} = \left\{\frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}^*\right\}$, is the set of rational numbers.

$\mathbb{R}$, is the set of real numbers.

$\mathbb{C} = \{a + ib, a, b \in \mathbb{R}\}$, is the set of complex numbers.

$\mathbb{D} = \{a, b, ..., z\}$, is the set of alphabets.

Definition 15. *Let A be a set and a and b be objects. If a is in A, we write $a \in A$. If both a and b are in A, we write $a, b \in A$. If b is not in A, we write $b \notin A$.*

Remarks

1- The sets $\{a, b, 1\}$ and $\{1, b, a\}$are the same because the ordering does not matter.

2- The set $\{1, 1, a, a, b\}$ is also the same as $\{a, b, 1\}$, because we don't care about repetition.

Definition 16. *A set A is said to be finite (resp. infinite) if it has a finite (resp. infinite) number of elements, the number of distinct elements of A is said to be the cardinality of the set A, denoted by $|A|$.*

Examples

The set $\{0, 1\}$ is finite and the set $[0, 1]$ is infinite, $|\{0, 1\}| = 2$, $|[0, 1]| = +\infty$.

Remarks

1 - A set that contains no element is called an empty set, denoted $\emptyset$ or $\{\}$, example

$$\{x \in \mathbb{R} : x^2 + 1 = 0\} = \emptyset.$$

2 - A set which has only one element x is called singleton, denoted $\{x\}$.

2.1. **Subset, Inclusion, and Equality.**

Definition 17. *A set F is considered to be a subset (or part) of E if all elements of F belong to E, and we write $F \subseteq E$ which reads F is included in E.*

Remarks

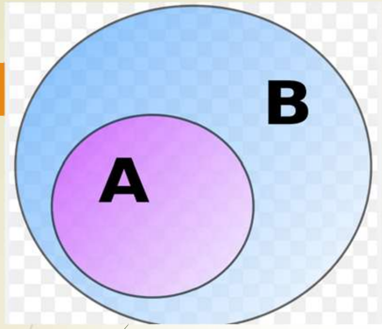
Let A and B be two subsets of a set E. Therefore:

$1 - A \subseteq B \Leftrightarrow (A \subset B \vee A = B)$.

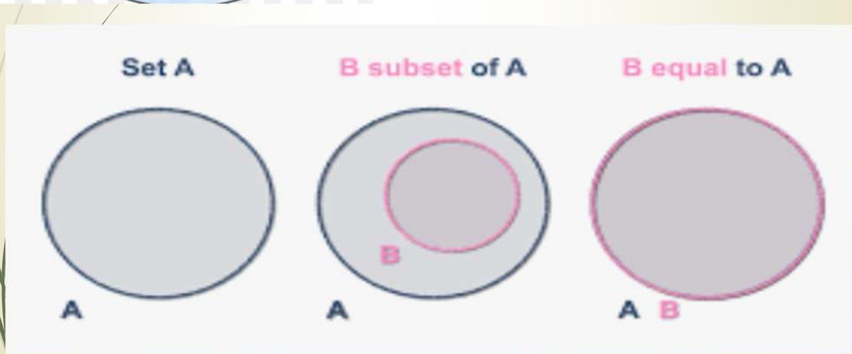
$2 - A \subseteq B \Leftrightarrow (\forall x \in E, x \in A \Rightarrow x \in B)$.

$3 - A \nsubseteq B \Leftrightarrow (\exists x \in E : x \in A \wedge x \notin B)$.

$4 - A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$.



A is a subset of **B**



Power Set

Definition 18. *The power set of a set E is the set of all its subsets, denoted by $\mathcal{P}(E)$. We clearly have:*

$$A \in \mathcal{P}(E) \Leftrightarrow A \subseteq E$$

Moreover, we always have: $\emptyset \in \mathcal{P}(E)$ and $E \in \mathcal{P}(E)$.

Example

If $E = \{a, b, c\}$, then:

$$\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, E\}$$

Note that if $|E| = n$ then $|\mathcal{P}(E)| = 2^n$.

Example of Power Set

Set A = {1,2,4,9}

P(A) = {{Ø, {1}, {2}, {4}, {9}, {1, 2}, {1, 4}, {1, 9}, {2, 4}, {2, 9}, {4, 9}, {1, 2, 4}, {1, 2, 9}, {1, 4, 9}, {2, 4, 9}, {1, 2, 4, 9}}

The power of the set **A**

A = { ☆ , ◇ }

P(A) = { Ø, { ☆ }, { ◇ }, { ☆ , ◇ } }

OR

P(A) = { {}, { ☆ }, { ◇ }, { ☆ , ◇ } }

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2.2. Union and Intersection.

Definition 19. *Let A and B be two subsets of a set E.*

- *The union of A and B, denoted $A \cup B$ is defined by:*

$$A \cup B = \{x \in E : (x \in A) \vee (x \in B)\}$$

- *The intersection of A and B, denoted $A \cap B$ is defined by:*

$$A \cap B = \{x \in E : (x \in A) \wedge (x \in B)\}$$

Remarks

1- If $A \cup B = \emptyset$ then $A = \emptyset$ and $B = \emptyset$.

2- If $A \cap B = \emptyset$ then the sets A and B are said to be disjoint.

3- If $(E_i)_{i \in I}$ is a family of subsets of a set E then:

$$\bigcup_{i \in I} E_i = \{x \in E : \exists i_0 \in I, x \in E_{i_0}\}$$

$$\bigcap_{i \in I} E_i = \{x \in E : \forall i \in I, x \in E_i\}$$

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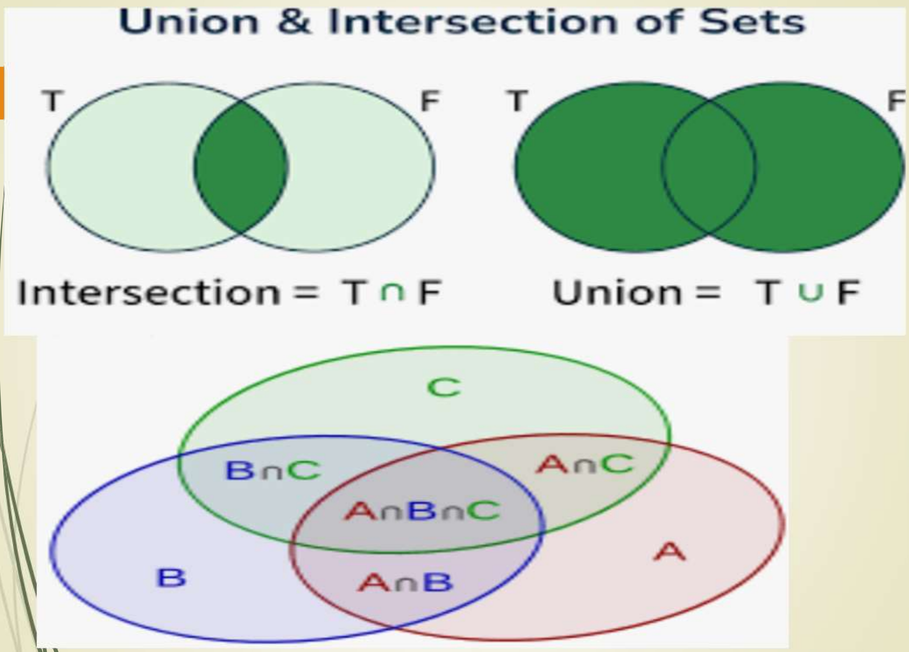


Example

If $A = \{n \in \mathbb{N} : n \text{ is even}\}$ and $B = \{n \in \mathbb{N} : n \text{ is odd}\}$ then $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$.

2.3. **Difference and Complement.**

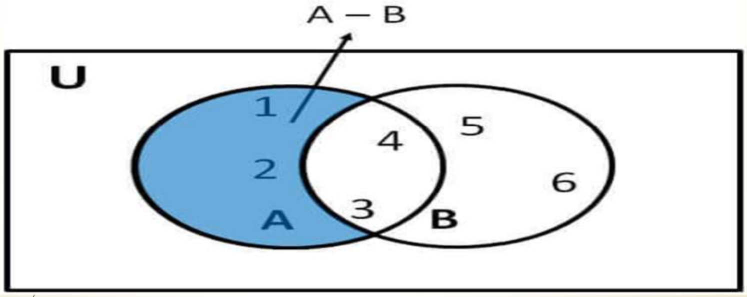
Definition 20. *Let A and B be two subsets of a set E. The set difference of B from A is the set of elements of A that are not in B, denoted A - B read A minus B.*

$$A \setminus B = A - B = \{x \in E : (x \in A) \wedge (x \notin B)\}$$

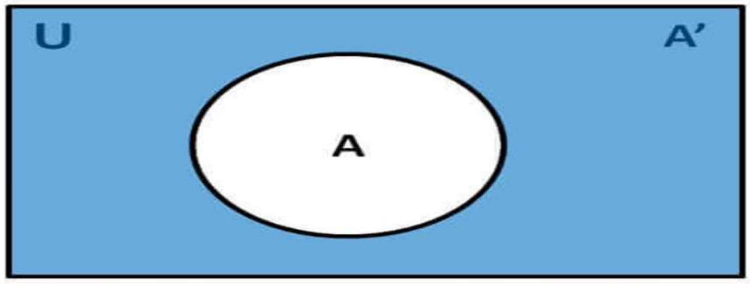
Definition 21. *Let E be a set and A a part of E. The complement of set A in set E, denoted $\complement_E A$ (or $E - A$), is the set:*

$$\complement_E A = \{x \in E : (x \in E) \wedge (x \notin A)\}$$

Difference of Sets



Complement of Set



Properties

Let E be a set and A, B and C three parts of E. Therefore:

1- $A \cup B = B \cup A, A \cap B = B \cap A.$

2- $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$

3- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

4- $A \cup A = A, \quad A \cap A = A, \quad A \cup \emptyset = A \quad A \cap \emptyset = \emptyset.$

5- $A \subseteq B \Rightarrow \complement_E B \subseteq \complement_E A.$

6 - $\complement_E (A \cup B) = \complement_E A \cap \complement_E B, \quad \complement_E (A \cap B) = \complement_E A \cup \complement_E B.$

7- $A \cup \complement_E A = E, \quad A \cap \complement_E A = 0, \quad \complement_E (\complement_E A) = A.$

Proof. Let us show the property: $\complement_E(A \cup B) = \complement_E A \cap \complement_E B$. For this we show the double inclusion. Let us show that: $\complement_E(A \cup B) \subseteq \complement_E A \cap \complement_E B$. Let $x \in \complement_E(A \cup B)$. We then have:

$$\begin{aligned} x \in \complement_E(A \cup B) &\Rightarrow x \in E \land x \notin A \cup B \\ &\Rightarrow x \in E \land (x \notin A \land x \notin B) \\ &\Rightarrow (x \in E \land x \notin A) \land (x \in E \land x \notin B) \\ &\Rightarrow x \in \complement_E A \land x \in \complement_E B \\ &\Rightarrow x \in \complement_E A \cap \complement_E B \end{aligned}$$

So we have shown the first inclusion. In the same way, we show the second inclusion, or else, by noticing that all the preceding implications are equivalences, then we directly conclude the equality. ▮

2.4. **Symmetrical difference.**

Definition 22. *Let A and B be two subsets of a set E. The symmetrical difference of A and B, denoted $A\Delta B$ is the set defined by: $A\Delta B = (A - B) \cup (B - A)$*

Example

1-If $E = \{a, b, c\}$ and if $B = \{1, b, c, 3\}$ then $E\Delta B = \{a, 1, 3\}$

Prove that: $A\Delta B = (A \cup B) - (A \cap B)$

Definition 23. *A family $(E_i)_{i \in I}$ of subsets of a set E is a partition of E Iff:*

$-E_i \neq \emptyset, \forall i \in I.$

$-E_i \cap Ej = \emptyset, \forall i, j \in I, i \neq j.$

$-E = \bigcup_{i \in I} E_i$

Examples

1 - If $A \subseteq E$ then A and $\complement_E A$ form a partition of E.

2 - In the previous example, it is clear that A the set of even integers and B the set of odd integers form a partition of $\mathbb{N}$.

Cartesian product

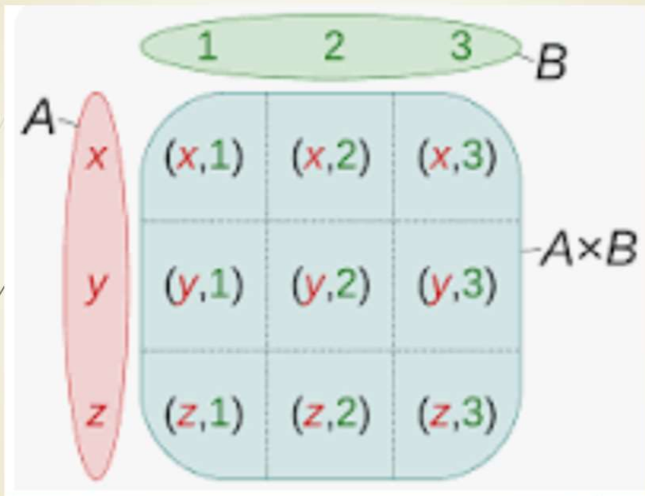
Definition 24. *Let E and F be two sets. The Cartesian product of E and F is the set, denoted $E \times F$, defined by:*

$$E \times F = \{(x, y) : x \in E, y \in F\}$$

Example

For $A = \{1, 2\}$ and $B = \{0, 3, 5\}$, we have:

$$A \times B = \{(1,0), (1,3), (1,5), (2,0), (2,3), (2,5)\}.$$



Remarks

Let $n \in \mathbb{N}^*$ and $E, E_1, \ldots, E_n, A, B, C$ be sets and $x, y \in E$. So we have:

$1-(x_1,y_1)=(x_2,y_2) \Leftrightarrow x_1=x_2 \wedge y_1=y_2$.

$2-$ If $x \neq y$ then $(x,y) \neq (y,x)$ but $\{x,y\}=\{y,x\}$.

$3-E_1 \times E_2 \times \ldots \times E_n = \{(x_1,x_2,\ldots,x_n) : x_i \in E_i, 0 \leq i \leq n\}$. In this case the element $(x_1,x_2,\ldots,x_n)$ is called an n-tuple or (an *ordered n-tuple* or just a *list*)

$4-$ The sets $E \times E, E \times E \times E, \ldots$ are denoted $E^2, E^3, \ldots$

$5-(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$6-(A \cap B) \times C = (A \times C) \cap (B \times C)$.

3. Maps

Definition 25. *Let E and F be two sets. A map f from E to F is any subset $G \subseteq E \times F$, such that for all $x \in E$ there exists a unique $y \in F$ such that $(x, y) \in G$. In most cases, $y = f(x)$ represents y. We write $f\colon E \to F$ to say that f is a map from E to F. In this case, we call E the* **source** *set, and we call F the* **target** *set. We also say that y is the image of x by the map f, and x is the antecedent of y.*

Remarks

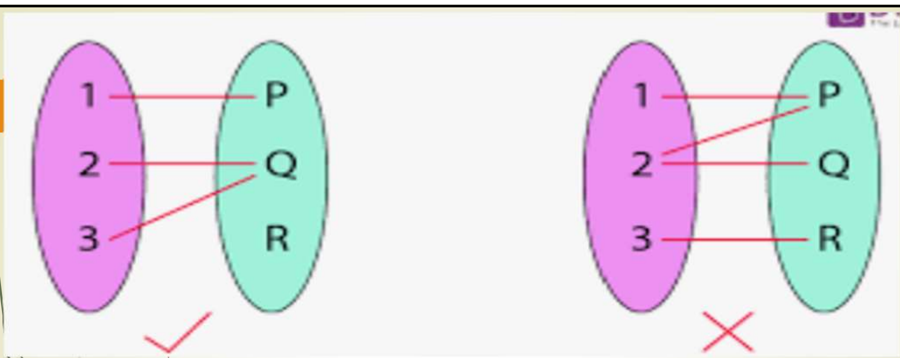
1-f is a map if and only if:

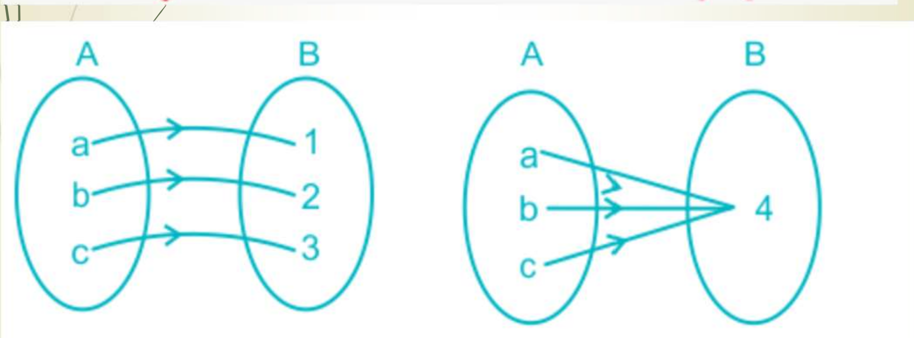
$$\forall x_1, x_2 \in E, x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$$

2 - The map

$$\begin{aligned} f : E &\to E \\ x &\longmapsto f(x) = x \end{aligned}$$

is called identity map, denoted $f = id_E$ or I_E.





3-Let A subsets of E, the characteristic map of A $\chi_A : E \to \{0,1\}$ is defined by

$$\chi_A = \begin{cases} 0 \text{ if } x \notin A \\ 1 \text{if } x \in A \end{cases}$$

4 - Two maps f and g are equal iff they have the same source set E and the same target set F and moreover $f(x) = g(x), \forall x \in E$.

5 - We denote by E^F the set of all maps from E into F.

3.1. Restriction and Extension of a map.

Definition 26. *Let E, F be two sets, $A \subseteq E$ and $f : E \longrightarrow F$ be a map. the restriction of a map f is a new map, denoted by $f_{/A}$, and we have:*

$$f(x) = f_{/A}(x), \forall x \in A$$

Definition 27. *Let E, F be two sets, $A \subseteq E$ and $f : A \longrightarrow F$ a map. The mapping $g : E \longrightarrow F$, where $f(x) = g(x), \forall x \in A$ is called an extension of f.*

Definition 28. *Let E and F be two sets and $f : E \longrightarrow F$ be a map. We call graph of f the subset of $E \times F$, denoted $G(f)$ and defined by:*

$$G(f) = \{(x, f(x)), x \in E\} \subset E \times F$$

Examples

1-Let the two maps, f and h, defined as follows:

$$f : \mathbb{R} \to \mathbb{R},$$
$$x \longmapsto |x| + 1$$
$$h : (0,1[\to \mathbb{R}$$
$$x \longmapsto x + 1$$

We can see that h is the restriction of f to the interval $(0, 1[$.

2-We have two maps, f and g, defined as follows:

$$f : [0,+\infty) \to \mathbb{R},$$
$$x \longmapsto x+1$$
$$g : \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto |x|+1$$

The map g is an extension of f to $\mathbb{R}$.

3.2. Injection- Surjection- Bijection.

Definition 29. *Let E, F be two sets and $f : E \longrightarrow F$ be a map.*
- We say that f is injective (one to one) if and only if:

$$\forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- We say that f is surjective (onto) if and only if:

$$\forall y \in F, \exists x \in E, y = f(x)$$

- We say that f is bijective if and only if f is injective and surjective.

Remarks

1-f is injective if and only if:

$$\forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

2-f is bijective if and only if:

$$\forall y \in F, \exists! x \in E, y = f(x)$$

3-If f is injective then $|E| \leq |F|$,if the sets E, F are finite.

4-If f is surjective then $|E| \geq |F|$,if the sets E, F are finite .

3.3. Composition and inverse of maps.

Definition 30. *Let E, F and G be three sets and $f : E \longrightarrow F, g : F \longrightarrow G$ be two maps. We call composition of f and g the map, denoted $g \circ f$, defined by:*

$$g \circ f : E \longrightarrow G$$
$$x \longmapsto (g \circ f)(x) = g(f(x))$$

Remarks

1- $(h \circ g) \circ f = h \circ (g \circ f)$.

2- $g \circ f \neq f \circ g$, in general.

Example

Let

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \qquad g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$
$$x \longmapsto f(x) = x^2 + 1 \qquad x \longmapsto g(x) = \sqrt{x+1}$$

So

$$(g \circ f)(x) = g(f(x)) = g\left(x^2+1\right) = \sqrt{x^2+1+1} = \sqrt{x^2+2}, \quad x \in \mathbb{R}_+$$
$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x+1}) = (\sqrt{x+1})^2 + 1 = x + 1 + 1 = x + 2, \quad x \in \mathbb{R}_+$$

We can clearly see that $g \circ f \neq f \circ g$.

Definition 31. *Let E, F be two sets and $f : E \longrightarrow F$ a bijective map. The inverse map of f, denoted $f^{-1} : F \longrightarrow E$, is defined by:*

$$y = f(x) \Leftrightarrow x = f^{-1}(y), \quad \forall x \in E$$

Remarks

1- If $f : E \longrightarrow F$ is a bijective mapping then $f^{-1} : F \longrightarrow E$ is too and we have:

$$f \circ f^{-1} = id_F \wedge f^{-1} \circ f = id_E$$

2- If $f : E \longrightarrow F$ and $g : F \longrightarrow E$ are two bijective mappings then:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Example

Consider the following map:

$$f : \mathbb{R}^* \longrightarrow \mathbb{R} - \{1\}$$
$$x \longmapsto f(x) = \frac{x+2}{x}$$

- Let us show that f is injective. Let $x_1, x_2 \in \mathbb{R}^*$, such that $f(x_1) = f(x_2)$. We have:

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow \frac{x_1+2}{x_1} = \frac{x_2+2}{x_2} \\ &\Rightarrow (x_1+2)\,x_2 = (x_2+2)\,x_1 \\ &\Rightarrow x_1x_2 + 2x_2 = x_2x_1 + 2x_1 \\ &\Rightarrow 2x_2 = 2x_1 \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

So $\forall x_1, x_2 \in \mathbb{R}^*, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Consequently, f is injective. - Let us show that f is surjective. Let $y \in \mathbb{R} - \{1\}$ and find $x \in \mathbb{R}^*$ such that $y = f(x)$. We have:

$$\begin{aligned} y = f(x) &\Rightarrow y = \frac{x+2}{x} \\ &\Rightarrow xy = x + 2 \\ &\Rightarrow x(y-1) = 2 \\ &\Rightarrow x = \frac{2}{y-1} \end{aligned}$$

As $y \neq 1$ then x exists and $x \neq 0$. So $\forall y \in \mathbb{R} - \{1\}, \exists x \in \mathbb{R}^* : y = f(x)$. Consequently, f is surjective. The map f is therefore bijective and consequently f^{-1} exists and it is given by:

$$f^{-1} : \mathbb{R} - \{1\} \longrightarrow \mathbb{R}^*$$
$$x \longrightarrow f^{-1}(x) = \frac{2}{x-1}$$

If the source and target sets of f are $\mathbb{R}$, i.e. $f:\mathbb{R}\longrightarrow\mathbb{R}$, then:

- f is not a map because the element $x=0$ does not have an image. So we replace the source set with $\mathbb{R}^*$ so that f becomes a map (we take the domain of f).

- f is not surjective because the element $y=1$ does not have an antecedent. Therefore, we replace the target set by $\mathbb{R}-\{1\}$ so that f becomes surjective.

- If f is not injective, the source set will be replaced by the set E, which contains all elements, to verify the implication of injection.

3.4. Direct and Inverse image by a map.

Definition 32. *Let E,F be two sets and $f:E\longrightarrow F$ be a map and $A\subseteq E, B\subseteq F$, then:*

- *The set:* $f(A)=\{y\in F : \exists x\in A, y=f(x)\}$

$$=\{f(x)\in F, x\in A\}$$

is called the direct image of A by f.

- *The set:* $f^{-1}(B)=\{x\in E : \exists y\in B, y=f(x)\}$

$$=\{x\in E : f(x)\in B\}$$

is called the inverse image of B.

Remarks

1- We can calculate the inverse image of a set without requiring f to be bijective.

For example : If $y\in F$ then $f^{-1}(\{y\})\neq f^{-1}(y)$. Indeed: if f is bijective, then $f^{-1}(\{y\})$ is a subset of E, which is the inverse image of $\{y\}$ and $f^{-1}(y)$ is an element of E which is the image of y by f^{-1}. And if f is not bijective, then $f^{-1}(\{y\})$ is the inverse image of $\{y\}$ and $f^{-1}(y)$ does not make sense.

2- Let $x \in E$ and $y \in F$. So:

$$y \in f(A) \Leftrightarrow \exists x \in A : y = f(x),$$
$$x \in A \Leftrightarrow f(x) \in f(A),$$
$$x \in f^{-1}(B) \Leftrightarrow f(x) \in B.$$

Properties

Let E, F, G be three sets and $f : E \longrightarrow F, g : F \longrightarrow G$ be two maps. We have:

1. If $g \circ f$ is injective then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.
3. If $g \circ f$ is injective and f is surjective then g is injective.
4. If $g \circ f$ is surjective and g is injective then f is surjective.

Proof. We will only show properties 2 and 3. It is easy to show properties 1 and 4.

a - Let us show property 2. Suppose $g \circ f$ is surjective. Let $z \in G$ and search for $y \in F$ such that $z = g(y)$. Since $g \circ f : E \longrightarrow G$ is surjective and $z \in G$ then:

$$\exists x \in E, z = (g \circ f)(x)$$

that is to say :

$$\exists x \in E, z = g(f(x))$$

Proof. We will only show properties 2 and 3. It is easy to show properties 1 and 4.

a - Let us show property 2. Suppose $g \circ f$ is surjective. Let $z \in G$ and search for $y \in F$ such that $z = g(y)$. Since $g \circ f : E \longrightarrow G$ is surjective and $z \in G$ then:

$$\exists x \in E, z = (g \circ f)(x)$$

that is to say :

$$\exists x \in E, z = g(f(x))$$

By setting $y = f(x) \in F$, we then have:

$$\exists y \in F, z = g(y)$$

So: $\forall z \in G, \exists y \in F, z = g(y)$. Consequently, g is surjective.

b - Let us show property 3. Suppose that $g \circ f$ is injective and f is surjective. Let $y_1, y_2 \in F$ be such that $g(y_1) = g(y_2)$. We then have, on the one hand: $y_1, y_2 \in F \Rightarrow \exists x_1, x_2 \in E / y_1 = f(x_1) \wedge y_2 = f(x_2)$ (because f is surjective). And on the other hand, we have:

$$\begin{aligned} y_1, y_2 \in F &\Rightarrow \exists x_1, x_2 \in E / y_1 = f(x_1) \wedge y_2 = f(x_2) \\ &\Rightarrow \exists x_1, x_2 \in E / g(y_1) = g(f(x_1)) \wedge g(y_2) = g(f(x_2)) \quad \text{(because } g \text{ is a map)} \\ &\Rightarrow \exists x_1, x_2 \in E / g(y_1) = (g \circ f)(x_1) \wedge g(y_2) = (g \circ f)(x_2) \end{aligned}$$

Since $g(y_1) = g(y_2)$ we deduce that $(g \circ f)(x_1) = (g \circ f)(x_2)$ Hence $x_1 = x_2$, because $g \circ f$ is injective. Hence, $f(x_1) = f(x_2)$ because f is a map and therefore $y_1 = y_2$. So: $\forall y_1, y_2 \in F, g(y_1) = g(y_2) \Rightarrow y_1 = y_2$. Consequently, g is injective. ∎

Properties

Let E, F be two sets and $f : E \longrightarrow F$ be a map. Let A_1, A_2 be two parts of E and B_1, B_2 be two parts of F. We then have:

1. $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, $\quad B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$, $\quad f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
3. $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$, $\quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$