

**Exercise 1.**

1) Say whether the following statements are true or false.

a)  $(\sqrt{4} = -2) \vee (\sqrt{3} \in \mathbb{Q})$  ( $\mathbb{Q}$  the set of rational numbers.),

b)  $(x \in \emptyset) \Rightarrow (x \in \mathbb{Q})$ .

2) Write a) as an implication.

3) Write the negation of the preceding propositions and give their contrapositives.

**Exercise 2.** Let  $P, Q, R$  be logical propositions.

1) Prove that the following propositions are tautologies:

1.  $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$

2.  $\neg(P \Leftrightarrow Q) \Leftrightarrow (P \Leftrightarrow \neg Q)$

3.  $[P \Rightarrow (Q \vee R)] \Leftrightarrow [(P \wedge \neg Q) \Rightarrow R]$

2) Provide complete truth tables for:

1.  $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

2.  $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

3) Write the contrapositive and converse of:

$$(P \wedge Q) \Rightarrow (R \vee S)$$

**Exercise 3.**

A) Let  $R, S$  and  $T$  be logical propositions. Are the following logical equivalences true?

1)  $(R \Rightarrow (\bar{S} \wedge T)) \Leftrightarrow (\overline{(T \Rightarrow S) \wedge R})$ .

2)  $(\bar{S} \vee (T \Rightarrow R)) \Leftrightarrow ((S \Rightarrow \bar{T}) \wedge R)$ .

B) Give the truth table of the following compound propositions.

1)  $(R \Rightarrow (S \wedge T)) \Rightarrow (R \Rightarrow S) \wedge (R \Rightarrow T)$ .

2)  $(\neg R \vee S) \Rightarrow (S \wedge (R \Leftrightarrow S))$ .

3)  $R \wedge (R \Rightarrow \neg S) \wedge S$ .

**Exercise 4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Define the predicates:

$$A : \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \mathbb{R}, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

$$B : \exists M > 0, \forall x \in \mathbb{R}, |f(x)| \leq M$$

$$C : \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, f(y) > f(x)$$

1) Express in natural language the meaning of the predicates B and C.

2) Provide the negation of each predicate.

3) Among the following implications, which are always true?

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- 1)  $A \Rightarrow B$ , 2)  $B \Rightarrow A$ , 3)  $C \Rightarrow \neg B$ , 4)  $\neg C \Rightarrow B$

**Exercise 5.** Let us consider a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We have the following predicates:

$$P : (\forall x \in \mathbb{R}, f(x) = 0); Q : (\exists x \in \mathbb{R}, f(x) = 0);$$

$$R : (\forall x \in \mathbb{R}, f(x) < 0) \vee (\forall x \in \mathbb{R}, f(x) > 0).$$

Which of the following implications are true.

- 1)  $(P \Rightarrow Q)$ ; 2)  $(Q \Rightarrow P)$ ; 3)  $(Q \Rightarrow R)$ ; 4)  $(\neg R \Rightarrow Q)$ ; 5)  $(\neg P \Rightarrow \neg R)$ .

**Exercise 6.** Which of the following relationships represent logical propositions and those which are not.

1)  $\forall x \in [1, 2], x^2 - 3x + 2 \geq 0$ .

2)  $x \in \mathbb{R}, -x^2 + 2x + 2 > 0$ .

3)  $x$  is a rational number.

4)  $\exists x \in \mathbb{N}^* : \forall m \in \mathbb{N}, x < m$ .

5)  $\forall x, y \in \mathbb{R}, (xy \neq 0 \wedge x \leq y) \Rightarrow \left(\frac{1}{x} \geq \frac{1}{y}\right)$ .

6)  $x + y \geq 0$ , when  $x \leq 0$ .

**Exercise 7.** Give the negation of the following propositions and determine which one is true.

1)  $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, (x \geq 2) \wedge (x + y > 10)$ .

2)  $\forall x, y \in \mathbb{R}, (x \geq 0) \Rightarrow (x + y < 9)$ .

3)  $\forall n \in \mathbb{N}, (n < 2) \Rightarrow (n^2 = n)$ .

4)  $\exists M \in \mathbb{R}_+ : \forall n \in \mathbb{N}, |e^{\sin(n)}| \leq M$ .

5)  $\exists x \in \mathbb{R}^*, \forall y \in \mathbb{R}^*, \forall z \in \mathbb{R}^*, z - 2xe^y = 0$ .

**Exercise 8.** Let  $f$  and  $g$  be two real functions defined on  $\mathbb{R}$ . Translate the following expressions in terms of quantifiers and then give their negations.

- $f$  is an increasing function and greater than  $g$ .
- $f + g$  is an odd function.
- There exists a real number  $M$  such that  $fg$  is less than  $M$ .
- $f$  is bounded.

**Exercise 9.**

A) Show by induction that

1)  $\forall n \in \mathbb{N}^*, 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ .

2)  $\forall n \geq 2, n! \leq \left(\frac{n+2}{2}\right)^n$ .

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3)  $\forall n \in \mathbb{N}, 10^n - 1$  is a multiple of 9.

4)  $\forall n \in \mathbb{N}, 2 - 2.7 + 2.7^2 + \cdots + 2.(-7)^n = \frac{1 - (-7)^{n+1}}{4}$ .

**B)** Show that  $\sqrt{5}$  and  $\frac{\ln 2}{\ln 3}$  are not rational numbers.

**C)** Show that  $\forall n \in \mathbb{N}, n(n+1)(n+2)$  is a multiple of 3.

**D)** Prove that every integer  $n \geq 2$  can be written as a product of prime numbers.

**E)** Let the sequence be defined by:

$$u_0 = 1, u_1 = 1, u_{n+1} = u_n + u_{n-1} \quad \text{for } n \geq 1$$

Prove that for all  $n \in \mathbb{N}, u_n \leq 2^n$ .

**F)** Prove that every integer  $n \geq 8$  can be written in the form  $n = 3a + 5b$  with  $a, b \in \mathbb{N}$ .

**G)** Prove that:

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$