NATIONAL HIGHER SCHOOL OF AUTONOMOUS SYSTEMS TECHNOLOGY

المدرسة الوطنية العليا لتكنولوجيا الأنظمة المستقلة

First Year

Module: Analysis 1

Chapter 3: Real Functions with a real variable

Limits-Continuity-Derivability

Topics Covered:

- Functions and their properties
- Limits at a point
- Continuity of a function
- Some fundamental theorems of continuous functions
- Derivability of a function
- Rolle and Mean Value theorems

Contents

1 Generalities

Definition 1. Let I and J be two sets in $\mathbb{R}$, generally intervals or a union of intervals. We have

$$f : I \to J$$

f defines a relation from J (called the domain) to I (called the codomain), where each element x in I (called the pre-image) has at most one element $y = f(x)$ (called the image) in J. In other words, elements of I cannot have more than one image.

Definition 2. (**Domain of Definition**) Let $f : I \to J$, the set of definition of f, denoted $\mathcal{D}_f$, is the set of elements in I that have exactly one image in J under the function f. It is written as

$$\begin{array}{rccc} f: & \mathcal{D}_f & \to & J \\ & x & \mapsto & f(x) \end{array}$$

The notation $f : \mathcal{D}_f \to J$ is read as "f goes from $\mathcal{D}_f$ to J" and $x \mapsto f(x)$ is read as "x maps to $f(x)$."

Definition 3. (**Graph of a function**) Let $f : \mathcal{D}_f \to J$. The graph of f or the representative curve of f, denoted $\mathcal{G}_f$, is a subset of the set $\mathcal{D}_f \times J$ given by

$$\mathcal{G}_f = \Big\{(x, f(x)),\ \ x \in \mathcal{D}_f\Big\}.$$

2 Some Properties of Real Functions

2.1 Even and Odd Functions

Definition 4. Let $f : I \to J$ be a real function such that the domain I is symmetric with respect to the origin, i.e., for all $x \in I$, $-x \in I$. We say that:
- f is even if for all $x \in I$, $f(-x) = f(x)$.
- f is odd if for all $x \in I$, $f(-x) = -f(x)$.

Remark:
- Any constant function on I is even.
- If f is even, its graph is symmetric with respect to the y-axis.
- If f is odd, its graph is symmetric with respect to the origin.

Example 1.
The functions $\cos x : \mathbb{R} \to \mathbb{R}$ and $x^{2x} : \mathbb{R} \to \mathbb{R}$ are even.
The functions $\sin x : \mathbb{R} \to \mathbb{R}$, $x^{2x+1} : \mathbb{R} \to \mathbb{R}$, and $\tan x : \mathbb{R} - \{\frac{\pi}{2} + k\pi\} \to \mathbb{R}$ are odd ($k \in \mathbb{Z}$).
If $f : I \to \mathbb{R}$ is even and $g : I \to \mathbb{R}$ is odd, then

$$(fg)(-x) = f(-x)g(-x) = -fg(x).$$

2.2 Periodic Functions

Definition 5. Let $f : I \to \mathbb{R}$ be a real function. Let $T \in \mathbb{R}_+^*$ such that for all $x \in I$, $x + T \in I$. f is called T-periodic if for all $x \in I$, $f(x + T) = f(x)$.
Remark If f has periods T and T', then f has period nT for all $n \in \mathbb{N}^*$ and period $T + T'$.

Example 2. Study the periodicity of the functions $f(x) = \sin^2 x$, $f(x) = \sin(\frac{2}{3}x)$, and $f(x) = x - \lfloor x \rfloor$. We have:

$$\bullet\ \sin^2(x) = \frac{1-\cos(2x)}{2}$$

since cos is 2π-periodic, then $\sin^2 x$ is π-periodic.

$$\bullet\ \sin(\frac{2}{3}x) = \sin(\frac{2}{3}x + 2\pi)) = \sin(\frac{2}{3}(x + 3\pi)) = f(x + 3\pi).$$

Thus, f is 3π-periodic.

$$\bullet\, f(x+1) = x + 1 - \lfloor x+1 \rfloor = x + 1 - \lfloor x \rfloor - 1 = f(x).$$

So, f is 1-periodic.

2.3 Composition of Functions

Definition 6. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that $f(I) \subset J$. The composite function of f and g, denoted $g \circ f$ (read as g circle f), is defined for all $x \in I$ by

$$(g \circ f)(x) = g(f(x)).$$

It has the form

$$g \circ f : I \to J \to \mathbb{R} \quad : x \mapsto f(x) \mapsto g(f(x)).$$

2.4 Order Relation

Definition 7. Let $f, g, h : I \to \mathbb{R}$ be three real functions. We have

$$\bullet\ f \leq g \Leftrightarrow \forall x \in I\ \ f(x) \leq g(x)$$

$$\bullet f \leq g \text{ and} g \leq h \Longrightarrow f \leq h \ \ \text{(transitivity)}$$

Definition 8. (absolute value of a real function) Let $f : I \to \mathbb{R}$. We define $|f| : I \to \mathbb{R}$ as

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) \leq 0 \end{cases}$$

It can also be defined as

$$|f| = \sup(f, -f).$$

2.5 Bounded Function, Upper Bounded, Lower Bounded, Bounded

Definition 9. Let $f : I \to \mathbb{R}$ be a real function. f is called:
- upper-bounded if and only if there exists $M \in \mathbb{R}$ such that for every $x \in I$, $f(x) \leq M$.
- lower-bounded if and only if there exists $m \in \mathbb{R}$ such that for every $x \in I$, $f(x) \geq m$.
- bounded if and only if there exist $M, m \in \mathbb{R}$ such that for every $x \in I$, $m \leq f(x) \leq M$.

Remark f is called bounded if and only if $|f|$ is upper-bounded.

2.6 Monotony of Functions

Definition 10. Let $f : I \to \mathbb{R}$ be a function. We say that:
1. f is increasing if and only if $\forall (x_1, x_2) \in I^2$, $x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2)$.
2. f is decreasing if and only if $\forall (x_1, x_2) \in I^2$, $x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2)$.
3. f is strictly increasing if and only if $\forall (x_1, x_2) \in I^2$, $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$.
4. f is strictly decreasing if and only if $\forall (x_1, x_2) \in I^2$, $x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$.
5. f is monotonic if and only if f is increasing or decreasing.
6. f is strictly monotonic if and only if f is strictly increasing or strictly decreasing.

Remark: Every strictly monotonic function is injective. Be cautious! The converse is not true.

Example 3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } (x \neq -1 \text{ and } x \neq 1) \\ 1 & \text{if } x = -1 \\ -1 & \text{if } x = 1 \end{cases}$$

f is injective but not monotonic.

Proposition 1. Let $f, g : I \to \mathbb{R}$ be two real functions. We have:
- If f is increasing, then $-f$ is decreasing.
- If f is increasing and $\lambda \in \mathbb{R}_+$, then λf is increasing.
- If f is increasing and $\lambda \in \mathbb{R}_-$, then λf is decreasing.
- If f and g are increasing, then $f + g$ is also increasing.
- If f and g are increasing and $fg \geq 0$, then fg is also increasing.
- If f and g are increasing or decreasing, then $f \circ g$ is increasing.
- If f is increasing and g is decreasing, then $f \circ g$ is decreasing.

Remark. If f is increasing and g is decreasing, we cannot conclude anything about $f+g$ because if we take $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^3$, which is increasing, and $g : \mathbb{R} \to \mathbb{R}$ $g(x) = -x$, which is decreasing, then $(f + g)(x) = x^3 - x$ is neither increasing nor decreasing.

3 Limits

3.1 Function Defined in the Neighborhood of a Point

Definition 11(Neighborhood of a Point) Let V be a subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. We say that:
- V is a neighborhood of x_0 if and only if there exists $\varepsilon > 0$ such that $]x_0-\varepsilon, x_0+\varepsilon[\subset V$.
- V is a neighborhood of $+\infty$ if and only if there exists $A \in \mathbb{R}$ such that $]A, +\infty[\subset V$.
- V is a neighborhood of $-\infty$ if and only if there exists $A \in \mathbb{R}$ such that $]-\infty, A[\subset V$.

We denote the set of neighborhoods of x_0 by $\mathcal{V}(x_0)$.

Definition 12. We say that the function $f : D_f \to \mathbb{R}$ is defined in the neighborhood of a point $x_0 \in \mathbb{R}$ if and only if there exists a neighborhood V of x_0 such that $V \subset D_f$. We write $f \in \mathcal{V}(x_0)$.

3.2 Finite Limit of f

Definition 13. Let $x_0 \in \overline{\mathbb{R}}$ (i.e., $x \in \mathbb{R} \cup \{-\infty, +\infty\}$). Let $f : D_f \to \mathbb{R}$ be a function defined in the neighborhood of x_0 except possibly at x_0.

We say that f has a finite limit $l \in \mathbb{R}$ at x_0 if and only if:

- $x_0 \in \mathbb{R}$: We say that f has a limit $l \in \mathbb{R}$ at x_0 if and only if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D_f, |x - x_0| \leq \delta \Longrightarrow |f(x) - l| \leq \varepsilon.$$

- $x_0 = +\infty$: We say that f has a limit $l \in \mathbb{R}$ at $+\infty$ if and only if

$$\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in D_f, x \geq A \Longrightarrow |f(x) - l| \leq \varepsilon.$$

- $x_0 = -\infty$: We say that f has a limit $l \in \mathbb{R}$ at $-\infty$ if and only if

$$\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in D_f, x \leq A \Longrightarrow |f(x) - l| \leq \varepsilon.$$

We write $\lim\limits_{x \to x_0} f(x) = l$ or $f(x) \underset{x \to x_0}{\longrightarrow} l$.

Remark. We can also use strict inequalities in the above definition.

$\lim\limits_{x \to x_0} f(x) = l$ means that $f(x)$ can be made arbitrarily close to l (in a neighborhood of l as small as we want) provided x is close enough to x_0 (in a neighborhood of x_0 as small as we want).

Example 4. 1. Show, using the definition of limit, that $\lim\limits_{x \to 0} x^2 = 0$.

We want to show that the proposition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in D_f$, $|x| \leq \delta \Longrightarrow |x^2| \leq \varepsilon$, is true. That is, find a $\delta > 0$ such that $|x| \leq \delta \Longrightarrow |x^2| \leq \varepsilon$.

We have

$$|x^2| \leq \varepsilon \Leftrightarrow |x|^2 \leq \varepsilon \Leftrightarrow |x| \leq \sqrt{\varepsilon}.$$

This means that the inequality $|x^2| \leq \varepsilon$ is satisfied if $|x| \leq \sqrt{\varepsilon}$, so we can take $\delta = \sqrt{\varepsilon} > 0$.

2. Let $x_0 > 0$, show, using the definition of limit, that $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$.

We have

$$\sqrt{x} - \sqrt{x_0} = \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}.$$

Since $\sqrt{x} + \sqrt{x_0} > \sqrt{x_0}$, then

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{|x - x_0|}{\sqrt{x_0}}.$$

The inequality $|\sqrt{x} - \sqrt{x_0}| \leq \varepsilon$ is satisfied if $\dfrac{|x - x_0|}{\sqrt{x_0}} \leq \varepsilon \Leftrightarrow |x - x_0| \leq \sqrt{x_0}\varepsilon$. To show $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$, it suffices to take $\delta = \sqrt{x_0}\varepsilon > 0$.

3. Show that $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$.

We want to show that: For every $\varepsilon > 0$, there exists $A \in \mathbb{R}$ such that for all $x \in \mathbb{R}^*$, $x \leq A \Longrightarrow |\frac{1}{x}| \leq \varepsilon$. We have

$$|\frac{1}{x}| = \frac{1}{-x} \leq \varepsilon \Leftrightarrow -x \geq \frac{1}{\varepsilon} \Leftrightarrow x \leq -\frac{1}{\varepsilon}.$$

Thus, we can take $A = -\frac{1}{\varepsilon}$.

3.3 Infinite Limit of f

Definition 14. Let $x_0 \in \overline{\mathbb{R}}$. Let $f : D_f \to \mathbb{R}$ be a function defined in the neighborhood of x_0 except possibly at x_0.

We say that f tends to $+\infty$ as x approaches x_0 if and only if:

- $x_0 \in \mathbb{R}$: We say that f tends to $+\infty$ as a limit at x_0 if and only if

$$\forall B \in \mathbb{R},\ \exists \delta > 0,\ \forall x \in D_f, |x - x_0| \leq \delta \Longrightarrow f(x) \geq B.$$

- $x_0 = +\infty$: We say that f tends to $+\infty$ as a limit at $+\infty$ if and only if

$$\forall B \in \mathbb{R},\ \exists A \in \mathbb{R},\ \forall x \in D_f, x \geq A \Longrightarrow f(x) \geq B.$$

- $x_0 = -\infty$: We say that f tends to $+\infty$ as a limit at $-\infty$ if and only if

$$\forall B \in \mathbb{R},\ \exists A \in \mathbb{R},\ \forall x \in D_f, x \leq A \Longrightarrow f(x) \geq B.$$

We write $\lim\limits_{x \to x_0} f(x) = +\infty$ or $f(x) \underset{x \to x_0}{\longrightarrow} +\infty$.

Similarly, we can define the limit $-\infty$ as follows: For every $B \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in D_f$, $|x - x_0| \leq \delta \Longrightarrow f(x) \leq B$.

We write $\lim\limits_{x \to x_0} f(x) = -\infty$ or $f(x) \underset{x \to x_0}{\longrightarrow} -\infty$.

Remark: We also define the limit $-\infty$ as follows:

- $x_0 \in \mathbb{R}$: We say that f tends to $-\infty$ as a limit at x_0 if and only if

$$\forall B \in \mathbb{R},\ \exists \delta > 0,\ \forall x \in D_f, |x - x_0| \leq \delta \Longrightarrow f(x) \leq B.$$

- $x_0 = +\infty$: We say that f tends to $-\infty$ as a limit at $+\infty$ if and only if

$$\forall B \in \mathbb{R},\ \exists A \in \mathbb{R},\ \forall x \in D_f, x \geq A \Longrightarrow f(x) \leq B.$$

- $x_0 = -\infty$: We say that f tends to $-\infty$ as a limit at $-\infty$ if and only if

$$\forall B \in \mathbb{R},\ \exists A \in \mathbb{R},\ \forall x \in D_f, x \leq A \Longrightarrow f(x) \leq B.$$

We write $\lim\limits_{x \to x_0} f(x) = -\infty$ or $f(x) \underset{x \to x_0}{\longrightarrow} -\infty$

Example 5.

1-Show that $\lim\limits_{x \to +\infty} \sqrt{x} = +\infty$.

We want to show that

$$\forall B > 0,\ \exists A \in \mathbb{R},\ \forall x \in [0, +\infty[, x \geq A \Longrightarrow \sqrt{x} \geq B.$$

We have $\sqrt{x} \geq B$ is satisfied if $x \geq B^2$, so it suffices to take $A = B^2$.

3.4 Left and Right Limits

Definition 15. Let V be a subset of $\mathbb{R}$, and $x_0 \in \mathbb{R}$. We say that

- V is a right neighborhood of x_0 if there exists $\varepsilon > 0$ such that $[x_0, x_0 + \varepsilon] \subset V$.
- V is a left neighborhood of x_0 if there exists $\varepsilon > 0$ such that $[x_0 - \varepsilon, x_0] \subset V$.

Definition 16.
• Let $f : D_f \to \mathbb{R}$ be a function defined in a right neighborhood of $x_0 \in \mathbb{R}$ except possibly at x_0. We say that f has a right-hand limit of l at x_0 if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f, |x - x_0| \leq \delta \text{ and } x > x_0 \Longrightarrow |f(x) - l| \leq \varepsilon.$$

We write $\lim\limits_{x \to x_0^+} f(x)$ or $\lim\limits_{x \overset{>}{\to} x_0} f(x)$
• Let $f : D_f \to \mathbb{R}$ be a function defined in a left neighborhood of $x_0 \in \mathbb{R}$ except possibly at x_0. We say that f has a left-hand limit of l at x_0 if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f, |x - x_0| \leq \delta \text{ and } x < x_0 \Longrightarrow |f(x) - l| \leq \varepsilon.$$

We write $\lim\limits_{x \to x_0^-} f(x)$ or $\lim\limits_{x \overset{<}{\to} x_0} f(x)$

Proposition 2. Let f be a function defined on $\mathcal{V}(x_0)\backslash\{x_0\}$. f has a limit $l \in \overline{\mathbb{R}}$ at x_0 if and only if it has a right-hand limit and a left-hand limit at x_0, i.e.,

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = l.$$

Example 6. Compute, if it exists, $\lim\limits_{x \to 0} x\sqrt{1 + \frac{1}{x^2}}$. We have

$$x\sqrt{1 + \frac{1}{x^2}} = \frac{x}{|x|}\sqrt[2]{x^2 + 1} = \begin{cases} \sqrt[2]{x^2 + 1} & \text{if } x > 0 \\ -\sqrt[2]{x^2 + 1} & \text{if } x < 0 \end{cases}$$

Since $\lim\limits_{x \to 0^+} x\sqrt{1 + \frac{1}{x^2}} = \lim\limits_{x \to 0^+} \sqrt[2]{x^2 + 1} = 1$ and $\lim\limits_{x \to 0^-} x\sqrt{1 + \frac{1}{x^2}} = \lim\limits_{x \to 0^+} -\sqrt[2]{x^2 + 1} = -1$, then $\lim\limits_{x \to 0} x\sqrt{1 + \frac{1}{x^2}}$ does not exist.

Proposition 3(Gendarme's Theorem for Functions): Let f, g, and h be three functions defined in the neighborhood of x_0 except possibly at x_0 such that

$$f(x) \leq g(x) \leq h(x).$$

Assume that $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} h(x)$ exist and are equal. Then $\lim\limits_{x \to x_0} g(x)$ exists and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x).$$

Example 7. Compute the limit $\lim\limits_{x \to 1} (x - 1) \sin\left(\frac{1}{x^2 - 1}\right)$. We have

$$-1 \leq \sin\left(\frac{1}{x^2 - 1}\right) \leq 1 \Leftrightarrow -(x - 1) \leq \sin\left(\frac{1}{x^2 - 1}\right) \leq (x + 1) \ \ (x > 0)$$

Since

$$\lim_{x\to 1^+}(-(x-1)) = \lim_{x\to 1^+}(x-1) = 0,$$

then by the sandwich theorem, we deduce that

$$\lim_{x\to 1^+}(x-1)\sin\left(\frac{1}{x^2-1}\right) = 0$$

We do the same for the left limit of 1; we find $\lim_{x\to 1^-}(x-1)\sin\left(\frac{1}{x^2-1}\right) = 0$. Therefore,

$$\lim_{x\to 1}(x-1)\sin\left(\frac{1}{x^2-1}\right) = 0.$$

Remark:

- If $\lim_{x\to x_0} h(x) = +\infty$ then $\lim_{x\to x_0} g(x) = +\infty$.
- If $\lim_{x\to x_0} f(x) = -\infty$ then $\lim_{x\to x_0} g(x) = -\infty$.

3.5 Using Sequences to Express a Function Limit

Theorem 1. Let f be a function defined on $\mathcal{V}(x_0)\backslash\{x_0\}$. We say that $\lim_{x\to x_0} f(x)$ exists and is equal to $l \in \mathbb{R}$ if and only if for any sequence (x_n) $(x_n \neq 0)$ converging to x_0, the sequence $f(x_n)$ converges to l.
We can derive two corollaries from this theorem that are used to show that a function limit does not exist.
Corollary 1. Let f be a function defined on $\mathcal{V}(x_0)\backslash\{x_0\}$. If there exists a sequence (x_n) converging to x_0 $(x_n \neq 0)$, and $f(x_n)$ diverges, then $\lim_{x\to x_0} f(x)$ does not exist.
Corollary 2. Let f be a function defined on $\mathcal{V}(x_0)\backslash\{x_0\}$. If there exist two sequences (x_n) and (y_n) such that $x_n \neq x_0$ and $y_n \neq y_0$ converging to x_0, and $f(x_n)$ and $f(y_n)$ converge but to two different limits, then $\lim_{x\to x_0} f(x)$ does not exist.
Example 8. Show that $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.
Consider the sequence $x_n = \frac{1}{2\pi n}$; we have $\forall n \geq 1$, $x_0 \neq 0$ and $\lim_{n\to\infty} x_n = 0$ such that

$$f(x_n) = \cos\left(\frac{1}{x_n}\right) = \cos(2\pi n) = 1 \underset{n\to\infty}{\longrightarrow} 1$$

Now consider $y_n = \frac{1}{(2n+1)\pi}$; we have $\forall n \geq 1$, $y_0 \neq 0$ and $\lim_{n\to\infty} y_n = 0$ such that

$$f(y_n) = \cos\left(\frac{1}{y_n}\right) = \cos((2n+1)\pi) = -1 \underset{n\to\infty}{\longrightarrow} -1$$

According to the second corollary, $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.
Example 9. Show that $\lim_{x\to +\infty}\sin(\cos x)$ does not exist.
Take the sequences $x_n = 2n\pi \to +\infty$ and $y_n = (2n+1)\pi \to +\infty$ such that

$$f(x_n) = \sin(\cos x_n) = \sin(\cos(2n\pi)) \to \sin 1$$

and

$$f(y_n) = \sin(\cos y_n) = \sin(\cos((2n+1)\pi)) \to 0$$

As $\sin 1 \neq 0$, $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.

3.6 Algebraic Operations on Limits

1. Case of finite limits

Proposition 4. Let $f, g : I \to \mathbb{R}$ be two functions defined in a neighborhood of x_0 except possibly at x_0 and $l, l', \lambda \in \mathbb{R}$. We have

- $\lim\limits_{x \to x_0} f(x) = l \Longrightarrow \lim\limits_{x \to x_0} |f(x)| = |l|$.
- $\lim\limits_{x \to x_0} f(x) = 0 \longmapsto \lim\limits_{x \to x_0} |f(x)| = 0$.
- $\lim\limits_{x \to x_0} f(x) = l \Longrightarrow \lim\limits_{x \to x_0} \lambda f(x) = \lambda l$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = l \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = l' \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} (f+g)(x) = l + l'$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = l \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = l' \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} (f+g)(x) = l + l'$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = l \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = l' \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} (fg)(x) = ll'$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} g(x) = l \\ \text{et} \\ l' \neq 0 \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} \frac{1}{g(x)} = \frac{1}{l'}$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = l \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = l' \; (l' \neq 0) \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{l}{l'}$

2. Case of infinite limits

Proposition 5. Let $f, g : I \to \mathbb{R}$ be two functions defined in in a neighborhood of x_0 except possibly at x_0 and $l, l' \in \mathbb{R}$. We have

- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = +\infty \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = +\infty \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} (f+g)(x) = +\infty$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = +\infty \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = l \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} (f+g)(x) = +\infty$
- $\left.\begin{array}{c} \lim\limits_{x \to x_0} f(x) = +\infty \\ \text{et} \\ \lim\limits_{x \to x_0} g(x) = +\infty \end{array}\right\} \Longrightarrow \lim\limits_{x \to x_0} (fg)(x) = +\infty$

- $$\left.\begin{array}{c}\lim\limits_{x\to x_0} f(x)=l\\ \text{et}\\ \lim\limits_{x\to x_0} g(x)=+\infty\end{array}\right\} \Longrightarrow \lim_{x\to x_0}(fg)(x)=\begin{cases}+\infty & \text{si } l>0\\ -\infty & \text{si } l<0\end{cases}$$

Indeterminate Forms

1. If $\lim\limits_{x\to x_0} f(x)=+\infty$ and $\lim\limits_{x\to x_0} g(x)=-\infty$ then $\lim\limits_{x\to x_0}(f+g)(x)=?$
2. If $\lim\limits_{x\to x_0} f(x)=\infty$ and $\lim\limits_{x\to x_0} g(x)=0$ then $\lim\limits_{x\to x_0}(fg)(x)=?$
3. $\lim\limits_{x\to x_0} f(x)=0$ and $\lim\limits_{x\to x_0} f(x)=0$ then $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=?$

Comparative growth

1. $\lim\limits_{x\to+\infty}\dfrac{\ln^\beta x}{x^\alpha}=0,\ \forall\alpha>0,\ \forall\beta\in\mathbb{R}.$
2. $\lim\limits_{x\to 0^+} x^\alpha|\ln x|^\beta=0,\ \forall\alpha>0,\ \forall\beta\in\mathbb{R}.$
3. $\lim\limits_{x\to+\infty}\dfrac{e^{\alpha x}}{x^\beta}=+\infty,\ \forall\alpha>0,\ \forall\beta\in\mathbb{R}.$
4. $\lim\limits_{x\to+\infty} x^\beta e^{-\alpha x}=0,\ \forall\alpha>0,\ \forall\beta\in\mathbb{R}.$

Example 10. Calculate $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[4]{x^3}-\sqrt[3]{x^2}}=\frac{0}{0}$ (**F.I**).
We perform the change of variable $x=y^{12}$ (when x goes 1, y goes to 1). We have $\sqrt{x}=x^{\frac{1}{2}}=y^6$, $\sqrt[3]{x}=x^{\frac{1}{3}}=y^4$, $\sqrt[4]{x^3}=x^{\frac{3}{4}}=y^9$and $\sqrt[3]{x^2}=x^{\frac{2}{3}}=y^8$. We get the limit

$$\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[4]{x^3}-\sqrt[3]{x^2}}=\lim_{y\to 1}\frac{y^6-y^4}{y^9-y^8}=\lim_{y\to 1}\frac{y^2-1}{y^5-y^4}=\lim_{y\to 1}\frac{(y-1)(y+1)}{y^4(y-1)}=\lim_{y\to 1}\frac{y+1}{y^4}=2.$$

4 Continuity

Definition 17. Let $f: I\to\mathbb{R}$ be a function defined in the neighborhood of x_0. f is said to be continuous at the point x_0 if

$$\lim_{x\to x_0} f(x)=f(x_0)$$

which means

$$\forall\varepsilon>0,\ \exists\delta>0,\forall x\in I;\ |x-x_0|\le\delta\Longrightarrow|f(x)-f(x_0)|\le\varepsilon$$

If f is not continuous at x_0, we say it is discontinuous at x_0.
Proposition 6. Let $f: I\to\mathbb{R}$ be a function defined in the neighborhood of x_0. f is continuous at x_0 if and only if for every sequence (x_n) such that $x_n\neq 0$ and $\lim x_n=x_0$, the sequence of images $f(x_n)$ converges to $f(x_0)$.

4.1 Continuity from the Left and Right

Definition 18. Let f be a function defined in the neighborhood of x_0. f is said to be

• continuous from the left of x_0 if and only if $\lim\limits_{x\to x_0^-} f(x) = f(x_0)$.

• continuous from the right of x_0 if and only if $\lim\limits_{x\to x_0^+} f(x) = f(x_0)$.

• continuous at x_0 if and only if f is continuous from the left and right of x_0, i.e., $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = f(x_0)$.

Example 11.

• The function $E(x)$ is continuous from the right at every point in $\mathbb{R}$ but is discontinuous at points $k \in \mathbb{Z}$.

• Let $g(x) = e^{\frac{1}{x}}$. We have $\lim\limits_{x\to x_0^-} g(x) = 0$ and $\lim\limits_{x\to x_0^+} g(x) = +\infty$, so g is discontinuous at 0.

4.2 Continuity on an Interval

Definition 19. Let $f : I \to \mathbb{R}$ be a function. f is said to be continuous on the interval I if and only if f is continuous at every point in I.

Intuitively, "A function is continuous on an interval if and only if its graph can be drawn without lifting the hand".

We express this definition with quantifiers:

$$\boxed{\forall x_0 \in I,\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in I,\ |x - x_0| \leq \delta \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.}$$

We denote $\mathcal{C}^0(I)$ or $\mathcal{C}(I)$ as the set of functions continuous on I.

Example 12.

- The constant function is continuous on $\mathbb{R}$.
- Functions $x \mapsto x^n$ $(n \in \mathbb{N})$, $x \mapsto \sin x$, $x \mapsto \cos x$, $x \mapsto e^x$ are continuous on $\mathbb{R}$.

4.3 Operations on Continuous Functions

Propositions 7. Let $f, g : I \to \mathbb{R}$ be two functions continuous at $x_0 \in I$. Then,

• **Linearity:** For any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is continuous at x_0.

• **Product:** The function fg is continuous on I.

• **Inverse:** If g does not vanish on I, then the function $\frac{1}{g}$ is continuous at x_0.

• **Quotient:** If g does not vanish on I, then the function $\frac{f}{g}$ is continuous at x_0.

Example 13.

• A polynomial function is continuous on $\mathbb{R}$.

• Any rational function of the form $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials is continuous on the interval where Q does not vanish.

• Let $f, g : I \to \mathbb{R}$ be two continuous functions. We define $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$, both of which are also continuous on I.

Proposition 8. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions with $f(I) \subset J$. If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0)$, then the function $g(f(x))$ is continuous at x_0.

4.4 Continuity extension

Definition 20. Let I be an interval in $\mathbb{R}$ and $x_0 \in I$ $(\mathcal{V}(x_0) \subset I)$. Let $f : I \setminus \{x_0\} \to \mathbb{R}$. If f has a finite limit $l \in \mathbb{R}$ at x_0 we say then that extendable by continuity on x_0. We then define the function $\tilde{f} : I \to \mathbb{R}$ as follows:

$$\tilde{f}(x) \begin{cases} f(x) & \text{if } x \neq x_0 \\ l & \text{if } x = x_0. \end{cases}$$

which is the extension of f in I. The function $\tilde{f}$ is continuous on I.

Example 14. • As we have $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ (exists and finite) then the function $f(x) = \dfrac{\sin x}{x}$ is extendable by continuity at 0, and we have

$$\tilde{f}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if} \quad x \in \mathbb{R}^\star \\ 1 & \text{if} \quad x = 0 \end{cases}$$

• The function $g(x) = e^{-\frac{1}{x^2}}$ is extendable by continuity at 0, and we have

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if} \quad x \in \mathbb{R}^\star \\ 0 & \text{if} \quad x = 0 \end{cases}$$

The function $\tilde{g}$ is defined and continuous on $\mathbb{R}$.

4.5 Fundamental Theorems on Continuous Functions

4.6 Intermediate Value Theorems

Theorem 2. Let $f : [a, b] \to \mathbb{R}$ be a function. If
• f is continuous on $[a, b]$.
• $f(a)f(b) \leq 0$
Then there exists a real number $c \in [a, b]$ such that $f(c) = 0$.
Proof:
• If $f(a)f(b) = 0$, then $c = a$ or $c = b$.
• If $f(a)f(b) < 0$, i.e., $f(a)$ and $f(b)$ have opposite signs.

- Let $c = \frac{a+b}{2}$ (center of the interval $[a, b]$), if $f(c) \neq 0$, then it has the same sign as $f(a)$ or $f(b)$.
  In the first case, we consider the interval

  $$[a_1, b_1] = [\frac{a+b}{2}, b]$$

  In the second case, we consider the interval

  $$[a_1, b_1] = [a, \frac{a+b}{2}]$$

  In both cases, the size of the interval $[a_1, b_1]$ is $\frac{b-a}{2}$.

- We repeat this process, obtaining the interval $[a_2, b_2]$ of size $\frac{a+b}{2^2}$.
- Repeating this process n times, we get the interval $[a_n, b_n]$ of size $\frac{b-a}{2^n}$.
- The sequences (a_n) and (b_n) satisfy the bounds

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

such that $f(a_n)f(b_n) \leq 0$. We have (a_n) is an increasing sequence and b_n is a decreasing sequence.

- Moreover, we have

$$\lim b_n - a_n = \lim(\frac{b-a}{2^n}) = 0$$

Thus, the sequences (a_n) and (b_n) are convergent to the same limit c.

- Since f is a continuous function on $[a, b]$ and as $a_n, b_n \in [a, b]$, by taking the limit in the inequality $f(a_n)f(b_n) \leq 0$, we obtain $f^2(c) \leq 0$; hence $f(c) = 0$.

Remark:
• The point c in the intermediate value theorem is not necessarily unique. For example, consider the function $f(x) = x^3 - x$ on the interval $[-2, 2]$, which satisfies the conditions of the intermediate value theorem. We can factorize it as $f(x) = x(x-1)(x+1)$, and it has three different roots.
• One may even encounter cases where the equation $f(x) = 0$ has an infinite number of roots. Consider the function $f : [-1, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We have $f(x) = 0 \Leftrightarrow x^2 \sin(\frac{1}{x}) = 0$, which has $x_0 = 0$ and $x_k = \frac{1}{k\pi}$ $(k \in \mathbb{Z})$ as solutions.
Corollary 3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If y_0 is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y_0$.
Proof: Assume that $f(a) \leq f(b)$. Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - y_0$, as f is continuous, then g is also continuous.
Moreover, we have $g(a) = f(a) - y_0 \leq 0$ and $g(b) = f(b) - y_0 \geq 0$. By the IVT, there exists $c \in [a, b]$ such that $g(c) = 0 \Leftrightarrow f(c) = y_0$.

4.7 Continuous Function on an Interval

Theorem 3. The image of an interval I under a continuous function is an interval.
Proof: Let f be a continuous function on the interval I. We want to show that $f(I)$ is an interval, i.e., for any $(y_1, y_2) \in (f(I))^2$, we have $[y_1, y_2] \subset f(I)$.
Let $(y_1, y_2) \in (f(I))^2$, then there exist two elements x_1, $x_2 \in I$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Let $y \in [y_1, y_2]$ (arbitrary), according to the previous corollary, there exists $c \in [a, b]$ such that $y = f(c)$, and thus $y \in f(I)$.
Proposition 9. If f is a strictly monotonic continuous function on an interval I, the following table gives $f(I)$

	[a,b]	[a,b[	]a,b]	]a,b[
$f \nearrow$	$[f(a), f(b)]$	$[f(a), \lim\limits_{x\to b} f(x)[$	$]\lim\limits_{x\to a} f(x), f(b)]$	$]\lim\limits_{x\to a} f(x), \lim\limits_{x\to b} f(x)[$
$f \searrow$	$[f(b), f(a)]$	$[f(b), \lim\limits_{x\to a} f(x)[$	$]\lim\limits_{x\to b} f(x), f(a)]$	$]\lim\limits_{x\to b} f(x), \lim\limits_{x\to a} f(x)[$

Example 15. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2 + x + 1$.
$f(x) : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \dfrac{1}{x^2+1}$

Proposition 10. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded and attains its bounds.
Which means that f is bounded ($m \leq f(x) \leq m$) and there exist $c, d \in [a, b]$ such that $f(c) = m$ and $f(d) = M$.
Proof: Let's show that f has an upper bound M on $[a, b]$, and M is a maximum, i.e.,

$$\exists c \in [a, b] \;\; f(c) = M$$

• Suppose, by contradiction, that f is not bounded above, i.e.,

$$\forall M \in \mathbb{R}, \, \exists x \in [a, b] \;\; f(x) > M$$

Set $M = n$, and construct a sequence (x_n) taking values in the interval $[a, b]$ such that

$$f(x_n) > n$$

The sequence (x_n) is bounded ($x_n \in [a, b]$), so it has a convergent subsequence $(x_{\phi(n)})$. Let $c \in \mathbb{R}$ be its limit. As $(x_{\phi(n)})$ is a subsequence of (x_n), we have

$$a \leq x_{\phi(n)} \leq b$$

by taking the limit, we get $a \leq c \leq b$.
Moreover, since f is a continuous function,

$$\lim f(x_{\phi(n)}) = f(c)$$

But as $f(x_n) > n \to +\infty$, we have $f(x_{\phi(n)}) \to +\infty$., which is a contradiction.
• Now, let's show that f attains its upper bound. Set $M = \sup\limits_{[a,b]} f(x)$, according to the supremum property, we have

$$\forall \varepsilon > 0, \; \exists x \in [a, b] \; M - \varepsilon \leq f(x) \leq M.$$

For $n \in \mathbb{N}^*$, set $\varepsilon = \frac{1}{n}$, and construct a sequence (x_n) taking values in $[a, b]$ such that

$$M - \frac{1}{n} \leq f(x_n) \leq M$$

The sequence (x_n) being bounded, it has a convergent subsequence $(x_{\phi(n)})$ converging to a limit $c_1 \in [a, b]$. Since f is continuous, we have $f(x_{\phi(n)}) \to f(c_1)$.
Moreover, the subsequence $(x_{\phi(n)})$ satisfies the bounds

$$M - \frac{1}{n} \leq M - \frac{1}{\phi(n)} \leq f((x_{\phi(n)}) \leq M$$

Taking the limit, we obtain

$$M \leq f(c_1) \leq M$$

Hence, $M = f(c_1)$.

4.8 Monotonic Bijection Theorem

Lemma 1. Let $f : I \to \mathbb{R}$ (I an interval) be a strictly monotonic function. Then the function f is injective.

Proof: Without loss of generality, assume f is strictly increasing. Let $x, y \in I$ such that $x \neq y$. If $x < y$, then $f(x) < f(y)$, and if $x > y$, then $f(x) > f(y)$; in both cases, $f(x) \neq f(y)$. Thus, f is injective.

Theorem 4. Let $f : I \to \mathbb{R}$ (I an interval) be a continuous and strictly monotonic function. Denote $J = f(I)$. Then

- f establishes a bijection from the interval I to the interval J.
- The inverse bijection $f^{-1} : J \to I$ is a continuous function on J, strictly monotonic, and has the same direction as f.

Proof:

- According to the previous lemma, f is injective.
- Since $J = f(I)$, f is surjective. Thus, f is bijective.
- Suppose, for example, that f is strictly increasing. Let $f^{-1} : J \to I$ be its inverse function. We will show that f^{-1} is also strictly increasing. Take $(X, Y) \in J^2$ such that $X < Y$. Assuming $f^{-1}(Y) < f^{-1}(X)$ leads to a contradiction. Since f is increasing, $f(f^{-1}(Y)) < f(f^{-1}(X))$, implying $Y < X$, which is false.
- It remains to show that the function f^{-1} is continuous. Let $y \in J$.

Consider the function $f^{-1} : J \to I$. Let $X_0 \in J$, and $x_0 \in I$, we want to show that f^{-1} is continuous at X_0. Without loss of generality, suppose f^{-1} is strictly increasing, and x_0 is in the interior of I.

For any $\varepsilon \geq \varepsilon' > 0$, since $x_0 \in I$ and I is an interval, there exists ε' such that

$$[x_0 - \varepsilon', x_0 + \varepsilon'] \subset I$$

Set $Y_1 = x_0 - \varepsilon'$, and $Y_2 = x_0 + \varepsilon'$, since f is strictly increasing, we have

$$Y_1 < X_0 < Y_2.$$

We need to show that the following proposition is true

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \ \forall X \in J,\ |X - X_0| \leq \delta \Longrightarrow |f^{-1}(X) - f^{-1}(X_0)| \leq \varepsilon.$$

We have

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \ \forall X \in J,\ |X - X_0| \leq \delta \Longrightarrow |f^{-1}(X) - f^{-1}(X_0)| \leq \varepsilon.$$

As f is strictly increasing, then

$$f(x_0 - \varepsilon') < X < f(x_0 + \varepsilon') \Leftrightarrow Y_1 < X < Y_2 \Leftrightarrow Y_1 - X_0 < X - X_0 < Y_2 - X_0$$

It suffices to take $\delta = \min(Y_2 - X_0, X_0 - Y_1) > 0$.

Example 16. Consider the function $f : [2, +\infty[\to \mathbb{R}$ defined by $f(x) = \sqrt{x^2 - 4x + 8}$.

- Show that the inverse function $f^{-1} : J \to I$ of f exists, specifying the intervals I and J.
- Demonstrate that f^{-1} is continuous and provide its expression.

Answer

- The function f is defined as the composition of two functions (a polynomial and a square root) that are continuous on $[2,+\infty[$, so f is continuous on this interval.
- The functions $g(x) = x^2 - 4x + 8$ and $\sqrt{x}$ are strictly increasing on $[2,+\infty[$, so$f$ is strictly increasing (a composition of two increasing functions).
- We have $f([2,+\infty[) = [2,+\infty[$. Thus, the function $f : [2,+\infty[\to [2,+\infty[$ establishes a bijection, and therefore, f^{-1} exists.
- According to the monotone bijection theorem, the function $f^{-1} : [2,+\infty[\to [2,+\infty[$ is continuous and strictly increasing. For $y \in [2,+\infty[$, we have

  $$y = \sqrt{x^2 - 4x + 8} \implies y^2 = x^2 - 4x + 8 = (x-2)^2 + 4 \implies x = \pm\sqrt{y^2 - 4} + 2.$$

  Since x belongs to $[2,+\infty[$, we get

  $$\begin{aligned} f^{-1} : [2,+\infty[&\to [2,+\infty[\\ y &\mapsto \sqrt{y^2-4} + 2. \end{aligned}$$

5 Differentiability

Throughout this section, we consider f to be a function defined on I with $x_0 \in I$.
Definition
• We say that f is differentiable at x_0 if the function τ_{x_0}, called the incremental ratio of f at x_0 defined on $I \setminus \{x_0\}$ by

$$\tau_{x_0}(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

has a finite limit at x_0.
This limit is called the derivative of f at x_0 and is denoted by $f'(x_0)$ or $\dfrac{df}{dx}(x_0)$.
• We say that f is differentiable on I if it is differentiable at every point x_0 in I. In this case, the function $f' : I \to \mathbb{R}$ is called the derivative of f.
Example 17.
• Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We have

$$\lim_{x\to 0} \tau_{x_0}(x) = \lim_{x\to 0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to 0} \sin(\frac{1}{x})$$

The function $\sin(\frac{1}{x})$ does not have a limit at 0, so f is not differentiable at 0.
• Consider the function $f(x) = \sqrt{x}$ and study the differentiability of f in its domain. Let $x_0 \in [0,+\infty[$.

- If $x_0 > 0$, we have

  $$\lim_{x\to x_0} \tau_{x_0}(x) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x\to x_0} \frac{1}{\sqrt{x + x_0}} = \frac{1}{2\sqrt{x_0}}$$

- If $x_0 = 0$, we have

$$\lim_{x\to 0} \tau_0(x) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x}} = +\infty$$

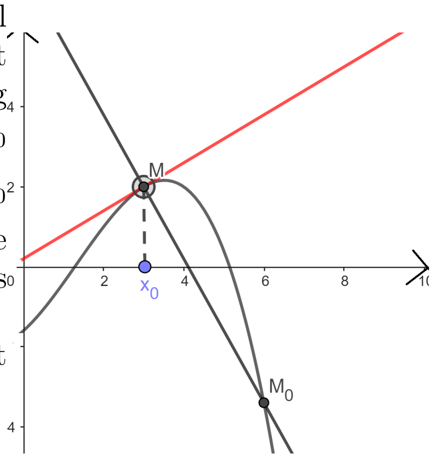
Conclusion: The function $\sqrt{x}$ is differentiable on $]0, +\infty[$.
Definition (Right and Left Derivative) Let $f : I \in \mathbb{R}$ be a function and $x_0 \in I$, we say that

- f is differentiable to the right of x_0 if and only if the rate τ_{x_0} has a finite limit as x approaches x_0 from the right, and we denote $f'_d(x_0)$ as the right limit of τ_{x_0}.
- f is differentiable to the left of x_0 if and only if the rate τ_{x_0} has a finite limit as x approaches x_0 from the left, and we denote $f'_g(x_0)$ as the left limit of τ_{x_0}.
- f is differentiable at the point x_0 if and only if it is differentiable to the right of x_0 and to the left of x_0, and we have $f'_d(x_0) = f'_g(x_0)$.

Example 18. Study the differentiability of $f(x) = |x|$ at 0. We have
f is differentiable to the right of 0 with $f'_d(0) = 1$, and f is differentiable to the left of 0 with $f'_g(0) = -1$. Thus, f is not differentiable at 0.

5.1 Geometric Interpretation of the Derivative

Let $\mathcal{C}$ be the curve representing the function f in an orthonormal coordinate system (Ox, Oy) in the plane. Let M_0 be the point $(x_0, f(x_0))$, and M be the point $(x, f(x))$ (both M_0 and M belong to $\mathcal{C}$). Let D_x be the line connecting the two points M and M_0 with a slope of $\dfrac{f(x) - f(x_0)}{x - x_0}$. Stating that f is differentiable at x_0 means that as x tends towards x_0, the line D_x tends towards a line passing through the point $(x_0, f(x_0))$ with a slope of $f'(x_0)$. This line is called the tangent to $\mathcal{C}$ at the point M_0. Thus,
If f is differentiable at $x_0 \in I$, then the curve $\mathcal{C}$ has, at the point $M_0(x_0, f(x_0))$, a tangent line with the equation



$$y = f'(x_0)(x - x_0) + f(x_0).$$

Remark

- The tangent at $M_0(x_0, f(x_0))$ is horizontal if and only if $f'(x_0) = 0$.
- If f is continuous at x_0 but not differentiable at this point, and if the rate of change τ_{x_0} tends to $+\infty$ or $-\infty$, then the graph of f admits a vertical tangent.
- If f is left-differentiable at x_0 (respectively right-differentiable at x_0), then the graph of f admits a left half-tangent (respectively a right half-tangent).

5.2 Properties of the Derivative

5.2.1 Differentiability and Continuity

Proposition 11. If f is differentiable in x_0, then it is continuous in x_0.
Proof f is differentiable in x_0 then the function τ_{x_0} is extendable by continuity in x_0 by $f'(x_0)$. We have

$$f(x) - f(x_0) = \tau_{x_0}(x)(x - x_0)$$

By a limit transition, we find $\lim_{x \to x_0} f(x) = f(x_0)$, thus f is continuous in x_0.

Remark

• **Attention!** The converse is false. A function can be continuous at x_0 without being differentiable at this point. If we take, for example, the function $x \mapsto |x|$ which is continuous at 0 and not differentiable at this point.

• If f is not differentiable at x_0 then it is discontinuous at this point.

Example We consider the function $x \mapsto E(x)$

- Isn't continuous at the points $k \in \mathbb{Z}$, so it is not differentiable at these points.
- Has a right derivative null at the points $k \in \mathbb{Z}$ (in the neighborhood of k it coincides with a constant).
- Does not have a left derivative at $k \in \mathbb{Z}$ because it is not continuous to the left of this point.

5.3 Algebraic Operations on Derivable Functions

Proposition 12. Let f and g be two functions defined on I and differentiable at a point $x_0 \in I$. Then

- For any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is differentiable at x_0, and
$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0)$$
- The function fg is differentiable at x_0, and
$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$
- If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is differentiable at x_0, and
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$
In particular, $\frac{1}{g}$ is differentiable at x_0, and
$$\left(\frac{1}{g}\right)'(x_0) = \frac{-1}{g^2(x_0)}$$

Proof. We will demonstrate only the second property. We calculate

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) + \frac{g(x) - g(x_0)}{x - x_0} f(x_0) \right) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

5.4 Derivative of the Composition of Two Functions

Proposition 13. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that $f(x_0) \in J$. Suppose that

- f is differentiable at the point $x_0 \in I$.
- g is differentiable at the point $f(x_0) \in J$.

Then the function $g \circ f$ is differentiable at the point x_0, and

$$(g \circ f)'(x_0) = f'(x_0) \times g'(f(x_0)).$$

Proof. Let ϕ be the function defined for $y \in f(I)$ by

$$\phi(y) = \begin{cases} \dfrac{g(y) - g(f(x_0))}{y - f(x_0)} & \text{if } x \neq f(x_0) \\ g'(f(x_0)) & \text{if } x = f(x_0) \end{cases}$$

As

$$\lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} = g'(f(x_0)) = \phi(f(x_0)) \text{ (since } g \text{ is differentiable at } f(x_0))$$

then ϕ is continuous at $f(x_0)$. Now, we calculate the limit of the incremental ratio

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \times \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \phi(f(x)) \times \frac{f(x) - f(x_0)}{x - x_0} \\ &= g'(f(x_0)) \times f'(x_0). \end{aligned}$$

Example. Determine the domain of definition of the function $f(x) = (1+x)^x$ and give its derivative.
We write f in the form $f(x) = e^{x \ln(1+x)}$, which is defined on $]-1, +\infty[$. We have

$$f'(x) = (x \ln(1+x))' e^{x \ln(1+x)} = (\ln(1+x) + \frac{x}{1+x})(1+x)^x.$$

5.5 Derivation of the Inverse Bijection

Proposition 14. Let $f : I \to J$ be a function that is continuous, strictly monotonic, and differentiable at the point $x_0 \in I$ such that $f(I) = J$. The function f^{-1} is differentiable at the point $y_0 = f(x_0)$ if and only if $f'(x_0) \neq 0$, and in that case, we have

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proof By the monotone bijection theorem, f^{-1} exists and is continuous on J. Now, let's show that it is differentiable. For $y_0 \in J$, we have

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{\frac{f(f^{-1}(y)) - f(f^{-1}(y_0))}{f^{-1}(y) - f^{-1}(y_0)}}$$

Since f is differentiable at $x_0 = f^{-1}(y_0)$ and $f'(x_0) \neq 0$, we obtain

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(f^{-1}(y_0))}$$

Example
I-Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + x - 2$.
Show that f^{-1} exists and then calculate $(f^{-1})'(0)$.
The function f is the sum of two elementary functions that are continuous and strictly increasing on $\mathbb{R}$ (x^3 and $x-2$), so f is continuous and strictly increasing on $\mathbb{R}$. Moreover, $f(\mathbb{R}) = \mathbb{R}$, thus f is a bijection.
Since $f(1) = 0$, we calculate

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{4}$$

II-Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4x + \sin^4 x$.
• Show that f^{-1} exists and then determine the equation of the tangent to the curve f^{-1} at 0.

1 f is defined by the sum and product of two elementary functions (x and $\sin x$) that are continuous on $\mathbb{R}$, so f is continuous on $\mathbb{R}$.

2 We have $f'(x) = 4 + 4\cos x \sin x^3$ using the trigonometric identity $\sin(2x) = 2\cos x \sin x$, we get $f'(x) = 4 + 2\sin(2x)\sin^2(x) > 0$ for all $x \in \mathbb{R}$. Thus, f is strictly increasing on $\mathbb{R}$.

3 As $\lim\limits_{x \to +\infty} f(x) = +\infty$ and $\lim\limits_{x \to -\infty} f(x) = -\infty$, then $f(\mathbb{R}) = \mathbb{R}$. Therefore, and according to the monotone bijection theorem, f^{-1} exists.

Since $f(0) = 0$, we have

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{4}$$

Thus, the equation of the tangent line to f^{-1} at 0 is

$$y_T = (f^{-1})'(0)(x - 0) + (f^{-1})'(0) = \frac{x}{4}$$

III-Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = e^x$, calculate $f^{-1}(x)$ for all $x \in \mathbb{R}$.
Let $g = f^{-1}$, then

$$f(g(x)) = x \Longrightarrow g'(x) f'(g(x)) = 1 \Longrightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))} = \frac{1}{x}.$$

Proposition 15. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $c \in [a, b]$ where f reaches its maximum or minimum. Then c must be one of the following points:
• a or b
• Stationary points of f ($f'(x) = 0$).
• Points in the interval $]a, b[$ where $f'(x)$ does not exist.

6 Rolle's Theorem and the Mean Value Theorem

6.1 Extremum of a Differentiable Function

Proposition 16. Let f be a differentiable function on I, and let $x_0 \in I$ such that • x_0 is in the interior of I.
• The point x_0 is a local extremum (maximum or minimum) of the function f on I.
• The function f is differentiable at x_0.
Then $f'(x_0) = 0$.
Proof.

- x_0 is in the interior of I, meaning there exists $\delta' > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset I$.
- f has a local extremum at the point x_0, assumed to be a local maximum, i.e.,

$$\exists \delta'' > 0, \quad \forall x \in I, \quad |x - x_0| \leq \delta'' \Longrightarrow f(x) \leq f(x_0).$$

Let $\delta = \min(\delta', \delta'') > 0$. Then,

$$0 < x - x_0 \leq \delta \Longrightarrow \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

and therefore

$$f'_d(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

Similarly,

$$-\delta \leq x - x_0 < 0 \Longrightarrow \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

and thus

$$f'_g(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

- Since f is differentiable at x_0, we have

$$f'(x_0) = f'_d(x_0) = f'_g(x_0) = 0.$$

Remark
1-**Attention!!** The condition $f'(x_0) = 0$ is only a necessary condition (it is not sufficient). For example, consider the function $x \mapsto x^3$ at $x_0 = 0$; the derivative is zero at this point, but x_0 is not an extremum.
2-A function may have a local extremum at x_0 without being differentiable at that point, such as the function $x \mapsto |x|$ at 0.
3-If x_0 is an endpoint of the interval, a function may have an extremum at x_0 without its derivative being zero. For example, take the function $f(x) = x$ on the interval $[0, 1]$, where 0 is a minimum and 1 is a maximum.
Example
1-Determine the extrema of the polynomial $P(x) = x^5 - 5x^4 + 2$ on $[-4, 6]$.
First, find the stationary points, i.e., the roots of P'. We have $P'(x) = 0 \Longrightarrow 5x^4 - 20x^3 = 5x^3(x-4) = 0$. P has two stationary points $x_0 = 0$ and $x_1 = 4$. Check if these points are extrema:

$$P(-4) = -2302, \quad P(0) = 2, \quad P(4) = -254, \quad P(6) = 1298$$

Therefore, the extrema of f are at -4 (minimum) and 6 (maximum).
2-Same question for the function $f(x) = \frac{1}{x}$ on $[1, 2]$.
We have $f'(x) = -\frac{1}{x^2} \neq 0$. f has no stationary points. Since f is decreasing on $[1, 2]$, we have

$$\max_{x\in[0,1]} f(x) = f(1) = 1 \text{ and } \min_{x\in[0,1]} f(x) = f(2) = \frac{1}{2}.$$

6.2 Rolle's Theorem

Theorem 5. Let $f : [a; b] \to \mathbb{R}$ be a function such that

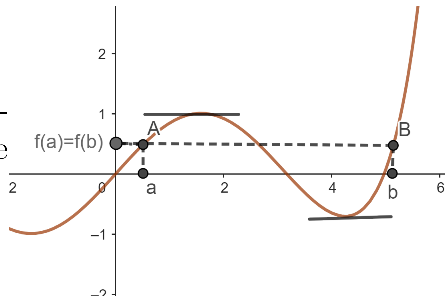
- f is continuous on the interval $[a, b]$.
- f is differentiable on the open interval (a, b).
- $f(a) = f(b)$.

Then, $\exists c \in (a, b)$ such that $f'(c) = 0$.
Proof • Since f is a continuous function on the interval $[a, b]$, then $f([a, b])$ is an interval $[m, M]$ with $m \leq M$.

1 If $m = M$, f is constant on $[a, b]$, so its derivative is zero on (a, b).

2 If $M < m$, and since $f(a) = f(b)$, one of the two values must be different from m or M. Suppose $f(a) \neq m$. The minimum of f on $[a, b]$ is attained at a point $c \in [a, b]$ such that $c \neq a$ and $c \neq b$, meaning it is in the interior of $[a, b]$. Then, according to the previous proposition, we have $f'(c) = 0$.

Remark The Rolle's Theorem states that the graph of the function C_f satisfying the conditions of this theorem has at least one horizontal tangent.



6.3 Mean Value Theorem (M.V.T)

Theorem Let $f : [a; b] \to \mathbb{R}$ be a function such that

- f is continuous on the interval $[a, b]$.
- f is differentiable on the open interval $]a, b[$.

Then, $\exists c \in]a, b[$ such that $f(b) - f(a) = f'(c)(b - a)$.
Proof We will construct a function g using the function f that satisfies the conditions of Rolle's theorem.

Let's consider the function $g : [a; b] \to \mathbb{R}$ defined by $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$.
The function g is continuous on $[a, b]$ and differentiable on $]a, b[$ such that $g(a) = g(b)$.

Then, according to Rolle's theorem, $\exists c \in]a, b[$ such that $g'(c) = 0$, i.e., $g'(c) = f'(c) - k = 0$. Thus

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remark Under the conditions of the M.V.T, let (C_f) be the graph of f, if A and B are two points on (C_f) with abscissas a and b, then there exists a point on (C_f) where the tangent is parallel to the line segment (AB).

Example

1 Show that for all $x > 0$

$$\frac{1}{x+1} \leq \ln(x+1) - \ln x \leq \frac{1}{x}$$

2 Deduce $\lim_{n\to\infty} \frac{1}{\ln n} \sum_{p=1}^{n} \frac{1}{p}$

- For $x > 0$, consider the function

$$\begin{aligned} f : [x, x+1] &\rightarrow \mathbb{R} \\ x &\mapsto \ln x \end{aligned}$$

The function f satisfies the conditions of the Mean Value Theorem, so there exists $c \in]x, x+1[$ such that

$$f'(c) = \frac{f(x+1) - f(x)}{x+1-x} = \ln(x+1) - \ln x$$

On the other hand, we have $f'(c) = \frac{1}{c}$ and since $x < c < x+1$, then $\frac{1}{x+1} \leq f'(c) \leq \frac{1}{x}$. Consequently, we obtain

$$\frac{1}{x+1} \leq \ln(x+1) - \ln x \leq \frac{1}{x} \tag{1}$$

From (1), for $p \in \mathbb{N} \setminus \{0, 1\}$

$$\frac{1}{p} \leq \ln(p) - \ln(p-1) \text{ and } \frac{1}{p} \geq \ln(p+1) - \ln p$$

Thus, we get the enclosure

$$\ln(p+1) - \ln p \leq \frac{1}{p} \leq \ln(p) - \ln(p-1)$$

By summing for p from 2 to n

$$\sum_{p=2}^{n} (\ln(p+1) - \ln p) \leq \sum_{p=2}^{n} \frac{1}{p} \leq \sum_{p=2}^{n} (\ln(p) - \ln(p-1))$$

The sequences $\sum_{p=2}^{n} (\ln(p+1) - \ln p)$ and $\sum_{p=2}^{n} (\ln(p) - \ln(p-1))$ are telescopic sums, then

$$\sum_{p=2}^{n} (\ln(p+1) - \ln p) = \ln(n+1) - \ln 2 \text{ and } \sum_{p=2}^{n} (\ln(p) - \ln(p-1)) = \ln n - \ln 1$$

So

$$\frac{1}{\ln n}(\ln(n+1) - \ln 2 + 1) \leq u_n \leq \frac{1}{\ln n}(\ln n + 1)$$

The sequences $v_n = \frac{1}{\ln n}(\ln(n+1) - \ln 2 + 1)$ and $w_n = \frac{1}{\ln n}(\ln n + 1)$ tend to the same limit $l = 1$, according to the sandwich theorem, we deduce that $\lim u_n = 1$.

6.4 Applications

6.4.1 Function Variation

Proposition 17. Let $f : I \to \mathbb{R}$ be a differentiable function on I. We have

- $\forall x \in I,\ \ f'(x) \geq 0 \iff f$ is increasing
- $\forall x \in I,\ f'(x) \leq 0 \iff f$ is decreasing
- $\forall x \in I,\ f'(x) > 0 \implies f$ is strictly increasing
- $\forall x \in I,\ f'(x) < 0 \implies f$ is strictly decreasing
- $\forall x \in I,\ f'(x) = 0 \iff f$ is constant.

Proof
$\Longleftarrow$ If f is increasing, then the rate of change of f is positive, as well as its limit $f'(x_0)$.
$\Longrightarrow$ Let $x \in I$, we have $f'(x) \geq 0$. Take $x_1,\ x_2 \in I$ such that $x_1 < x_2$. f is continuous on the interval $[x_{1,2}]$ and differentiable on $]x_1, x_2[$. Then, according to the Mean Value Theorem (MVT), there exists $c \in]x_1, x_2[$ such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \Longrightarrow f(x_1) - f(x_2) = (x_1 - x_2)f'(c) \geq 0$$

hence $f(x_1) \geq f(x_2)$, and f is increasing on I.
Note If f is strictly increasing, it does not necessarily imply $\forall x \in I\ f'(x) > 0$. Take, for example, the function $f(x) = x^3$; f is strictly increasing, but the derivative vanishes.

6.5 Higher Derivatives

Definition (Second Derivative) The function f is twice differentiable on I if f' is differentiable at every point in I. Its derivative is called the second derivative of f and is denoted by f'' or $f^{(2)}$.
Definition (Derivative of Order n) Let f be a function defined on I, we set $f^{(0)} = f$. We define by recurrence the n^{th} derivative of f, denoted $f^{(n)}$, as the derivative of $f^{(n-1)}$, if it exists.
Note
• If $f^{(n)}$ exists, then the derivatives of order less than n are continuous on I.
• If f and g are n times differentiable on I, then the functions $f + b$, fg, $\frac{f}{g}$ ($g \neq 0$ on I), and $g \circ f$ are also n times differentiable.
Proposition 18 (Leibniz's Formula) If f and g are n times differentiable, Leibniz's formula allows us to calculate the n^{th} derivative of the product. We have

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Definition (C^n Functions) We say that a function $f : I \to \mathbb{R}$ is of class C^n if and only if
• f is n times differentiable on I.
• The function $f^{(n)}$ is continuous on I.
* We denote $C^0(I)$ as the set of continuous functions on I.

$*$ We denote $C^n(I)$ $(n \geq 1)$ as the set of n times differentiable functions on I.
$*$ We denote $C^\infty(I)$ as the set of indefinitely differentiable functions on I.

Example

- Calculate the n^{th} derivative of the functions $\sin x$, $\ln x$, and $x^3 \ln x$.
- Let $f(x) = \sin x$, $f \in C^\infty(\mathbb{R})$. We have

$$\begin{aligned} f(x) = \sin x \quad &= \quad \sin(x + 0\frac{\pi}{2}) \longrightarrow f'(x) = \cos x = \sin(x + 1\frac{\pi}{2}) \\ &\longrightarrow \quad f''(x) = -\sin x = \sin(x + 2\frac{\pi}{2}) \\ &\longrightarrow \quad f^{(3)}(x) = -\cos x = \sin(x + 3\frac{\pi}{2}) \end{aligned}$$

By recursion, we obtain the n^{th} derivative of sin given by

$$\sin^{(n)}(x) = \sin(x + \tfrac{n\pi}{2}), \quad \forall n \in \mathbb{N}$$

The function $\ln x$ is of class $C^\infty(]0, +\infty[)$. We have

$$f(x) = \ln x \longrightarrow f'(x) = \frac{1}{x} \longrightarrow f''(x) = -\frac{1}{x^2} \longrightarrow f^{(3)}(x) = \frac{1 \cdot 2}{x^3} \longrightarrow f^{(4)} = \frac{-1 \cdot 2 \cdot 3}{x^4}$$

By recursion, we obtain the n^{th} derivative of ln given by

$$(\ln(x))^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n} \quad \forall n \in \mathbb{N}^\star$$

- For the function $h(x) = x^3 \ln x$, we will use Leibniz's formula to determine its n^{th} derivative. Letting $f(x) = x^3$ and $g(x) = \ln x$, we have

$$\begin{aligned} (x^3 \ln x)^{(n)} &= \binom{n}{0} x^3 \frac{(-1)^{n-1}(n-1)!}{x^n} + \binom{n}{1} 3x^2 \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \\ &+ \binom{n}{2} 6x \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} + 6\binom{n}{3} \frac{(-1)^{n-4}(n-4)!}{x^{n-3}} \\ &= \frac{(-1)^{n-1}(n-1)!}{x^{n-3}} + 3n \frac{(-1)^{n-2}(n-2)!}{x^{n-3}} \\ &+ 3n(n-1) \frac{(-1)^{n-3}(n-3)!}{x^{n-3}} + n(n-1)(n-2) \frac{(-1)^{n-4}(n-4)!}{x^{n-3}} \\ &= \frac{(-1)^n(n-4)!}{x^{n-3}} \Big(-(n-1)(n-2)(n-3) + 3n(n-2)(n-3) \\ &- 3n(n-1)(n-3) + n(n-1)(n-2) \Big) \\ &= \frac{(-1)^n(n-4)!}{x^{n-3}} \Big(-(n-1)(n-2)(n-3) + n(n-1)(n-2) \\ &+ 3n(n-2)(n-3) - 3n(n-1)(n-3) \Big) \\ &= \frac{(-1)^n(n-4)!}{x^{n-3}} \big(3(n-1)(n-2) - 3n(n-3) \big) \\ &= \frac{6(-1)^n(n-4)!}{x^{n-3}}. \end{aligned}$$

6.6 L'Hôpital's Rule

Proposition 19. Let $f, g :]a, b[\to \mathbb{R}$ be two differentiable functions such that g and g' do not vanish on $]a, b[$. Additionally, suppose

$$1)\ \lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \alpha \quad \text{where } \alpha = 0,\ -\infty \text{ or } +\infty$$

$$2)\ \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \beta \text{ where } \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$$

Note This rule remains valid if x tends toward b^-, $-\infty$, or $+\infty$.