National School of Autonomous Systems

1^{st} year
2022/2023

Module: Analysis 1

Chapter 1: Real Numbers

Some Mathematical Notations

$x \in A$	Reads as "x belongs to A" or "x is an element of set A." $\in$ is the membership symbol.
$A \subset B$	Reads as "A is included in B" or "A is a subset of B." $\subset$ is the inclusion symbol.
$A \cup B$	Reads as "A union B," meaning the elements that are in A or in B. $\cup$ is the union symbol.
$A \cap B$	Reads as "A intersection B" or simply "A intersect B," meaning the elements that are in both A and B. $\cap$ is the intersection symbol.
$\emptyset$	Reads as "empty set."
$\forall x \in A$	This is a quantifier that means "for all elements x in A" or "for any x in A."
$\exists x \in A$	This is a quantifier that means "there exists at least one element x in A."
$P \Rightarrow Q$	Reads as "P implies Q" or "P entails Q," meaning that if P is true, then Q is also true.
$P \Leftrightarrow Q$	Reads as "P is equivalent to Q" or "P if and only if Q," indicating that P and Q are true or false together.
$A^{\star}$	This represents the set A without the element zero.

1 Set of Numbers

Definition 1. A set is a collection of objects gathered according to a common property.
Example 1.
Let $\mathbb{N}$ denote the set of natural numbers, which is defined as

$$\mathbb{N} = \{0, 1, 2, \ldots\} \text{ and } \mathbb{N}^{\star} = \mathbb{N} \setminus 0.$$

Let A be the set defined as $A = \{2n;\ n \in \mathbb{N}\} = \{0, 2, 4, \ldots\}$, which represents the set of even natural numbers.
Let B be the set defined as $B = \{2m + 1;\ m \in \mathbb{N}\} = \{1, 3, 5, \ldots\}$, representing the set of odd natural numbers.
Remark 1. Equations of the form $a + x = b$, where $a, b \in \mathbb{N}$ and $a > b$, do not have solutions in $\mathbb{N}$." For this reason, a new set denoted as $\mathbb{Z}$ has been introduced:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, \ldots\},$$

which is called the set of integers.
Remark 2. Equations of the form $ax = b$, where a and b are integers and are coprime, do not have solutions in $\mathbb{Z}$. Hence, the set $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \left\{\frac{a}{b}; a \in \mathbb{Z}, b \in \mathbb{Z}^{\star}\right\},$$

and it is known as the set of rational numbers.
Remark 3. Equations of the form $x^n = a$ ($n \in \mathbb{N}$, $a \in \mathbb{Q}$) do not always have solutions in $\mathbb{Q}$.
Proposition 1. The number $\sqrt{2}$ is not a rational number.
Proof. To prove this proposition, we will use a proof by absurd, assuming that $\sqrt{2}$ is a rational number, which leads us to a contradiction. If $\sqrt{2} \in \mathbb{Q}$, it means there exist two coprime integers a and b ($b \neq 0$) such that $\sqrt{2} = \frac{a}{b}$. Therefore, $2 = \frac{a^2}{b^2}$, which implies $a^2 = 2b^2$, and thus a^2 is divisible by 2, so a is divisible by 2. This means there exists an integer $p \in \mathbb{Z}$ such that $a = 2p$. This leads to $2 = \frac{a^2}{b^2}$, which further simplifies to $4p^2 = 2b^2$, and it follows that $b^2 = 2p^2$, which means b^2 is divisible by 2, so b is divisible by 2.
Conclusion: The greatest common divisor (GCD) of a and b is at least 2, which contradicts the fact that a and b are coprime.
Example 2. Solve the equation $x + \sqrt{2}y = 6$ in $\mathbb{Q}$.
For $y \in \mathbb{Q}^\star$, we have: $x + \sqrt{2}y = 6 \Leftrightarrow \sqrt{2} = \frac{6-x}{y}$. Since $6 - x \in \mathbb{Q}$ and $y \in \mathbb{Q}^\star$, it follows that $\sqrt{2} \in \mathbb{Q}$, which is absurd. The only remaining case is when $y = 0$, and in this case, we find $x = 6$. The equation $(*)$ has a unique solution in $\mathbb{Q}$, which is $x = 6$ and $y = 0$.
Example. Show that there exists no rational number whose square is 8.
We assume by absurd that there exists a rational number whose square is 8. Let p, q two integers such that

$$8 = (\frac{p}{q})^2 \Rightarrow 2 < \frac{p}{q} < 3 \Rightarrow 2q < p < 3q \Rightarrow 0 < p - 2q < q \Rightarrow 0 \leq \frac{p-2q}{q} < 1$$

Thus $\frac{p-2q}{q}$ is not an integer (it's a rational number).
Furthermore, we have

$$(\frac{p}{q})(p - 2q) = \frac{p^2}{q} - 2p = \frac{p^2}{q^2}q - 2p = 8q - 2p \in \mathbb{Z}\,.$$

which is a contradiction!
Remark 4. There are other well-known numbers that are not rational, called irrational numbers, such as π, e, $\sqrt{3}$, and more. Thus, we have the set that includes both rational and irrational numbers, denoted as $\mathbb{R}$, which is called the set of real numbers. We have the following inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

2 Algebraic Structure of $\mathbb{R}$

2.1 Addition in $\mathbb{R}$

The application

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} \\ (x, y) & \to & x + y \end{array}$$

satisfies the following properties:
i)-**Associativity:** $\forall (x, y, z) \in \mathbb{R}^3$, we have

$$(x + y) + z = x + (y + z)$$

ii)- **Identity Element**: For all $x \in \mathbb{R}$ there exists $e \in \mathbb{R}$ such that

$$x + e = x \Rightarrow e = 0 (\text{ 0 is the additive identity})$$

iii) Symmetric Element: For all $x \in R^\star$ there exists $x^\star$ in $\mathbb{R}$ such that:

$$x + x^\star = e \Rightarrow x^\star = -x (\text{ -x is the additive inverse})$$

iv) Commutativity: For all $x, y \in \mathbb{R}$

$$x + y = y + x$$

These properties (1), (2), (3), and (4) define what is called a commutative group structure on the set $\mathbb{R}$. We say that $(\mathbb{R}, +)$ is a commutative group.

2.2 Multiplication in $\mathbb{R}$

The application:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} \\ (x,y) & \to & x \times y \end{array}$$

satisfies the following properties
i)-**Associativity:** $\forall (x,y,z) \in \mathbb{R}^3$:

$$(x \times y) \times z = x \times (y \times z)$$

ii)- **Identity Element:** $\forall x \in \mathbb{R}, \exists e \in \mathbb{R}$ such that

$$x \times e = x \Rightarrow e = 1(\text{ l'élement neutre pour la multiplication})$$

iii)- **Symmetric Element:** $\forall x \in \mathbb{R}^\star$, $\exists x^\star \in \mathbb{R}$ such that

$$x \times x^\star = e \Rightarrow x^\star = \frac{1}{x}$$

$\frac{1}{x}$ is the multiplication inverse .
iv)- **Commutativity:** For all $x, y \in \mathbb{R}$, we have

$$x \times y = y \times x$$

v) **Distribution of Multiplication over Addition:** $\forall (x,y,z) \in \mathbb{R}^3$

$$(x+y) \times z = (x \times z) + (y \times z).$$

Conclusion: The aforementioned axioms establish that $(\mathbb{R}, +, \times)$ forms a commutative field.

3 Order Relation in $\mathbb{R}$

The set of real numbers $\mathbb{R}$ is equipped with an order relation denoted by "$\leq$," which means for any $x, y \in \mathbb{R}$, we have $x \leq y$. Whether this is true or false depends on the values of x and y. This order relation satisfies the following properties:
Reflexivity: For every $x \in \mathbb{R}$, $x \leq x$.
Antisymmetry: For all $x, ; y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.
Transitivity: For all $x, ; y, ; z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
Additionally, this order relation is total, meaning that for all $x, y \in \mathbb{R}$:

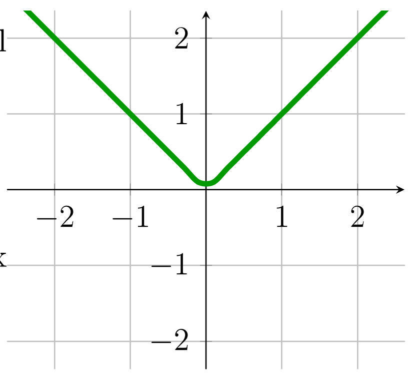
$$\forall x,\ y \in \mathbb{R}, \quad (x \leq y) \text{ or } (y \leq x)$$

4 Absolute Value

Definition 2. For all real number x we can associate a non-negative real number defined by the following:

$$|x| = \left\{ \begin{array}{ll} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0. \end{array} \right.$$

$|x|$ is called the absolute value of $x$which represents the distance between x and the origin (0) on the real number line and is always non-negative.
The graph of the absolute value function is shown in the figure below:



We can also define the absolute value as follow

$$|x| = \max(x, -x).$$

Properties
1)
$$|x| = 0 \Leftrightarrow x = 0$$
2)
$$-|x| \leq x \leq |x|$$
3)$\forall a > 0$:
$$i)\ |x| \leq a \Leftrightarrow -a \leq x \leq a$$
$$ii)\ |x| \geq a \Leftrightarrow x \leq -a \text{ or } x \geq a$$
4)
$$|xy| = |x||y|, \quad |\frac{x}{y}| = \frac{|x|}{|y|} \quad (y \neq 0)$$
5)Triangular inequality
$$|x + y| \leq |x| + |y|$$
6) Second triangular inequality
$$||x| - |y|| \leq |x - y|$$
Proof.
i) **Triangular inequality**
One have for all $x,\ y \in \mathbb{R}$
$$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy = |x|^2 + |y|^2 + 2xy$$
we know that $\forall a \in \mathbb{R}$, $a \leq |a|$ then
$$|x + y|^2 \leq |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$$
hence
$$|x + y| \leq |x| + |y|$$
ii) Second triangular inequality
we write $|x| = |(x - y) + y|$ by the triangular inequality, we get
$$|x| \leq |x - y| + |y| \Leftrightarrow |x| - |y| \leq |x - y|$$
We do the same for $|y|$, one finds
$$|y| \leq |y - x| + |x| \Leftrightarrow |y| - |x| \leq |y - x| = |x - y|$$
as $||x| - |y|| = \max(|x| - |y|, |y| - |x|)$ so $||x| - |y|| \leq |y - x| = |x - y|$
Proposition 2. $\forall \varepsilon > 0$, $|x| \leq \varepsilon \Rightarrow x = 0$.
Proof. We will prove this result by absurd, assuming that $|x| \leq \varepsilon$ et $x \neq 0$.
ε being arbitrary, we can take for example $\varepsilon = \frac{|x|}{2} > 0$, one has $|x| \leq \varepsilon \Leftrightarrow |x| \leq \frac{|x|}{2} \Leftrightarrow 1 < \frac{1}{2}$
which is absurd.
Exercise Show the following properties

- $\forall x \in \mathbb{R}, \quad |x|^n = |x^n|$
- $\forall x_1, x_2, ..., x_n \in \mathbb{R}, \quad \left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|$

5 Interval

Definition 3. A non-empty subset I of $\mathbb{R}$ is called an interval if $\forall(a,b) \in I \times I$ satisfying $a \leq b$, the relation $a \leq x \leq y$ implies $x \in I$.
Example 3: let A be a non-empty subset of $\mathbb{R}$ defined as

$$A = \left\{\frac{1}{n},\ \ n \in \mathbb{N}^\star\right\}$$

A is not an interval. for example, if we take $a = \frac{1}{2}$ et $b = \frac{1}{3}$ which are elements of A and $x = \frac{2}{5}$ which lies between a and b but it is not in A.

We distinguish several forms of intervals:
1-Bounded intervals

- **Open interval**

$$]a, b[= \{x \in \mathbb{R},\ \ a < x < b\}$$

- **Closed interval**

$$[a, b] = \{x \in \mathbb{R},\ \ a \leq x \leq b\}$$

- **Half-open interval (left-open)**

$$]a, b] = \{x \in \mathbb{R},\ \ a < x \leq b\}$$

- **Half-open interval (right-open)**

$$[a, b[= \{x \in \mathbb{R},\ \ a \leq x < b\}$$

2-Unbounded intervals

- **Open interval**

$$]a, +\infty[= \{x \in \mathbb{R},\ \ x > a\}$$
$$]-\infty, b[= \{x \in \mathbb{R},\ \ x < b\}$$

- **Closed Interval**

$$[a, +\infty[= \{x \in \mathbb{R},\ \ x \geq a\}$$
$$]-\infty, b] = \{x \in \mathbb{R},\ \ x \leq b\}$$

5.1 Neighbourhood

Definition 4. Let V be a non-empty subset $\mathbb{R}$. We say that V is a neighbourhood of $x \in \mathbb{R}$ if it contains an open interval $]a, b[$ containing x which means a $a < x < b$. In other words, there exists a positive number $r > 0$ such that

$$]x - r, x + r[\subset V.$$

The set of neighbourhood of x is denoted by $\mathcal{V}(x)$.
Example
• The sets $\mathbb{R}$, $]-1, 1[$ are neighbourhood of 0.
• the sets $\{0\}$, $]0, 1[$, $[0, 1[$, $]-1, 0[\cup]0, 1]$ are not neighbourhood of 0.

6 Upper bound, lower bound, the greatest and smallest members

Definition 8.
• A non-empty subset A of $\mathbb{R}$ is said to be bounded above if there exists an element $M \in \mathbb{R}$ such that $\forall x \in A$, $x \leq M$. M is called an upper bound of A. we write

$$\boxed{M \in \mathbb{R} \text{ is an upper bound of } A \text{ if } \forall x \in A,\ x \leq M}$$

• A non-empty subset A of $\mathbb{R}$ is said to be bounded below if there exists an element $m \in \mathbb{R}$ such that $\forall x \in A$, $x \geq m$. m is called a lower bound of$A$. We write

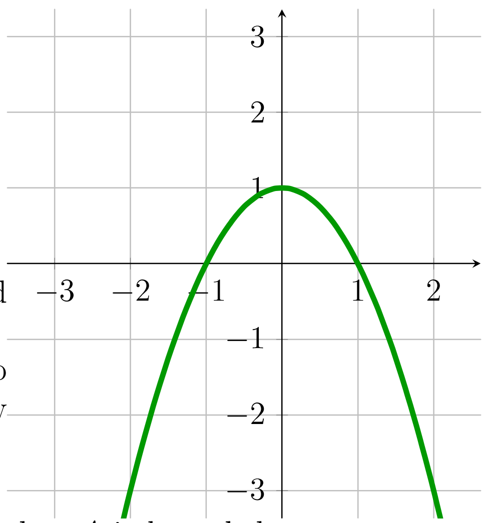
$$\boxed{m \in \mathbb{R} \text{ is a lower bound } A \text{ if } \forall x \in A,\ x \geq m}$$

• A non-empty subset$A$ of $\mathbb{R}$ is said to bo bounded if it is both bounded above and bounded below.
Remark 5. If a set is bounded above (respectively below) that it has an infinitely upper bounds (respectively lower bounds).
Attention Note that M et do not necessarily belong to the set A.
Example
• The set $A_1 = [1, 6[$ is bounded above, because $M = 7$ is an upper bound of A (6 is also an upper bound of A). The set $[6, +\infty[$ is the set of upper bounds of A.
A is bounded below, because $m = 0$ is a lower bound of A (1 is also a lower bound of A). The set $]-\infty, 1]$ is the set of lower bounds of A.
• The set $A_2 = [0, +\infty[$ is not bounded above and bounded below by 0.
• The set $\mathbb{Z}$ is neither bounded above nor bounded below.
• Let the set $A_3 = \{\frac{n}{2n+1},\ \ n \in \mathbb{N}\}$.
We observe that for all $n \in \mathbb{N}$,we have $n < 2n + 1$ which means $\dfrac{n}{2n+1} < 1$ for all $n \in \mathbb{N}$. Therefore A_3 is bounded above . Furthermore, this set is bounded below by 0. So it is a bounded set.

Example 4:
Let A be a non-empty subset of $\mathbb{R}$ defined by

$$A = \{1 - x^2,\ \ \forall x \in \mathbb{R}\}$$

Show that A is bounded above and unbounded below.
One has $\forall x \in \mathbb{R}$, $x^2 \geq 0$ so $1 - x^2 \leq 1$. it follows that the set A is bounded above by 1.
We can make a graphical representation of the function $f(x) = 1 - x^2$ to confirm that 1 is indeed an upper bound of A and A is unbounded below (see the figure below).



To show that A is unbounded below, we ill proceed by absurd. Suppose that A is bounded below, which means there exists $m \in \mathbb{R}$ such that $1 - x^2 \geq m$ for all $x \in \mathbb{R}$. This implies that $x^2 \leq 1 - m$ for all $x \in \mathbb{R}$, which is absurd.
Definition 9.
• Let A be a non-empty subset of $\mathbb{R}$. We called the largest element of A an element $M \in \mathbb{R}$ which satisfies the following two conditions

$$\begin{cases} M \in A \\ M \text{ is an upper bound of A} \end{cases}$$

• Let A be a non-empty subset of $\mathbb{R}$. We called the smallest element of A an element $M \in \mathbb{R}$ which satisfies the following two conditions

$$\begin{cases} m \in A \\ m \text{ is a lower bound of A} \end{cases}$$

The largest element of set A, when it exists, is denoted as $\max(A)$, and the smallest element, when it exists, is denoted as $\min(A)$.
Remark 6. For a non-empty subset of $\mathbb{R}$ to have a largest element (respectively, a smallest element), it is necessary for the subset to be bounded above (respectively, bounded below), but this condition is not sufficient. In other words, a bounded subset (respectively, lower-bounded subset) does not always have a largest element (respectively, a smallest element).
Proposition 3. If the largest element or the smallest element of a non-empty subset of $\mathbb{R}$ exists, then they are unique.
Proof. We assume that there exist M and M' in $\mathbb{R}$ that are both the largest elements of the non-empty set A in $\mathbb{R}$, meaning that:

$$\begin{cases} M \in A \\ \forall x \in A, \quad x \leq M \end{cases} \text{and} \begin{cases} M' \in A \\ \forall x \in A, \quad x \leq M' \end{cases}$$

this implies that $M' \leq M$ et $M \leq M'$ (M and M' are elements of A) therefore $M = M'$.
Example 5:
$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ do not have a maximum.
$\mathbb{N}$ possesses a minimum, which is 0.
$A_1 = [0, 1[$ has a minimum $\min A = 1$ but $\max A$ does not exist.
$A_2 = \{\frac{1}{n}, \quad n \in \mathbb{N}\}$ one has $\max A_2 = 1$ et le $\min A_2$ does not exist.
$A_3 = \{x^2 \leq 2, \quad x \in \mathbb{R}\}$ possesses a maximum which is $\sqrt{2}$. We can write A_3 as an interval $[-\sqrt{2}, \sqrt{2}]$.
$A_4 = \{\dfrac{2xy}{x^2+y^2}, \ x \in \mathbb{R}^\star, y \in \mathbb{R}^\star\}$
$A_5 = \{\dfrac{2^n-1}{2^n+1}, \quad n \in \mathbb{N}\}$

7 Supremum and Infimum

Definition 10. Let A be a non-empty subset of $\mathbb{R}$.
A real number M is called the supremum of A, denoted as $M = \sup(A)$, if and only if:

M is an upper bound of A, meaning that for all $x \in A$, $x \leq M$. If M' is an upper bound of A, then $M \leq M'$.

In other words, M is the smallest among all the upper bounds of A.
A real number m is called the infimum of A, denoted as $m = \inf(A)$, if and only if:

m is a lower bound of A, meaning that for all $x \in A$, $x \geq m$. If m' is a lower bound of A, then $m \geq m'$.

In other words, m is the largest among all the lower bounds of A.
Remark 7. 1. For a non-empty subset of $\mathbb{R}$ to have a supremum, it must be bounded above; similarly, for it to have an infimum, it must be bounded below.
2. If a non-empty subset of $\mathbb{R}$ has a supremum (respectively, an infimum), it is unique.
3. The supremum of a non-empty set A in $\mathbb{R}$ is not necessarily an element of A, and the same applies to the infimum.
Example 6. • Let $A = [5, 7]$. In this case, $\sup A = 7$ and $\inf A = 5$.

• Consider the set $B = \{\frac{x^2+2}{x^2+1}, x \in \mathbb{R}\}$. We can express $\frac{x^2+2}{x^2+1}$ as $1 + \frac{1}{1+x^2}$. Since $0 < \frac{1}{1+x^2} \leq 1$ for all $x \in \mathbb{R}$, we have $1 < \frac{x^2+2}{x^2+1} \leq 2$. Consequently, the set B is bounded (bounded above by 2 and bounded below by 1).
Since 2 is an element of B (for $x = 0$), we have $\sup B = \max B = 2$".
Now, you asked whether $\inf B = 1$. The answer will be provided later in the course.
Proposition 4. (Axiom of the Supremum) Every non-empty, bounded subset of $\mathbb{R}$ has a supremum.
Remark 8. This property does not hold in $\mathbb{Q}$ (the rational numbers).
Example. Consider A as a non-empty subset of $\mathbb{Q}$ given by:

$$A = \{x^2 < 2, \quad x \in \mathbb{Q}\}$$

It's clear that this set is bounded above by numbers like $\frac{7}{2}$, etc. Now, let's show that it doesn't have a supremum in $\mathbb{Q}$.
Suppose M is a rational number that's an upper bound of A. Let's define $M' = \frac{M^2+2}{2M}$. We'll demonstrate that M' is also a rational number, which is an upper bound of A, and is less than M. This means that for every upper bound M of A, there exists another upper bound M' that is smaller, indicating that A doesn't have a supremum in $\mathbb{Q}$.
M'* as an upper bound of *A : We can show that $(M')^2 > 2$ by noting that:

$$(M')^2 - 2 = \frac{(M^2+2)^2}{4M^2} = \frac{(M^2-2)^2}{4M^2}$$

Since M is a rational number, $M^2 - 2 \neq 0$ (because $M \neq \sqrt{2}$), which means that $(M')^2 - 2 > 0$.
M'* is less than *M : We can prove that $M - M' > 0$ as follows:

$$M - M' = M - \frac{M^2+2}{2M} = \frac{M^2-2}{2M} > 0$$

Here, we use the fact that M is an upper bound of A, which implies $M^2 > 2$.

7.1 Caracterization of Supremum and Infimum

Proposition 5. Let A be a non-empty subset of $\mathbb{R}$.
If A is bounded above by a real number M, then

$M = \sup(A)$ if and only if for every $\varepsilon > 0$, there exists $x \in A$ such that $x \in]M - \varepsilon, M[$.

If A is bounded below by a real number m, then

$m = \inf(A)$ if and only if for every $\varepsilon > 0$, there exists $x \in A$ such that $x \in]m, m + \varepsilon[$.

Proof. Let's prove the first part (1) as an example: If M is the supremum of A, it's the smallest upper bound. So, for any $\varepsilon > 0$, if $M - \varepsilon$ were an upper bound of A, we'd have $M \leq M - \varepsilon$, which is not true. Therefore, there must exist an $x \in A$ such that $M - \varepsilon < x < M$.
Conversely, if for every $\varepsilon > 0$, there exists $x \in A$ such that $x \in]M - \varepsilon, M[$, then M is the supremum. Suppose it's not, and there exists M' such that $M' < M$. Set $\varepsilon = M - M' > 0$. By the property, there exists $x \in A$ such that $x \in]M - \varepsilon, M[=]M', M[$. This means M' is not an upper bound, which is a contradiction. Therefore, M must be the supremum.
Example 7:
Let's consider the set A as a non-empty subset of $\mathbb{R}$, defined as:

$$A = \left\{\frac{n-1}{n}, \quad n \in \mathbb{N}^\star\right\}.$$

For any $n \in \mathbb{N}^\star$, we have $\frac{n-1}{n} = 1 - \frac{1}{n}$. Since $0 < \frac{1}{n} \leq 1$, we have $A \subset [0, 1[$ (Note: A is not an interval). The set A is bounded below by 0 (which is the smallest lower bound), so $\inf A = \min A = 0$.

A is bounded above by 1. The question is whether this upper bound is the supremum of A? Suppose, by contradiction, that 1 is not the supremum of A, meaning that there exists $M \in \mathbb{R}$ which is an upper bound of A and $M < 1$. Let $M = 1 - \varepsilon$ where $\varepsilon > 0$. Then, for all $n \in \mathbb{N}^\star$, we have $\frac{n-1}{n} \leq 1 - \varepsilon$, which implies $1 - \frac{1}{n} \leq 1 - \varepsilon$, or $n \leq \frac{1}{\varepsilon}$ for all $n \in \mathbb{N}^\star$. This proposition means that the set $\mathbb{N}^\star$ is bounded above by $\frac{1}{\varepsilon}$, which is absurd.
Now, let's show, using the characteristic of the infimum, that the infimum of the previous example is 1.
Suppose $\varepsilon > 0$, and we want to show that there exists $\alpha_x \in B$ such that $\alpha_x \in]1, 1 + \varepsilon[$. We have:

$$\alpha_x < 1 + \varepsilon \Leftrightarrow \frac{x^2 + 2}{x^2 + 1} < 1 + \varepsilon \Leftrightarrow \frac{1}{1 + x^2} < \varepsilon \Leftrightarrow x^2 > \frac{1}{\varepsilon} - 1.$$

As a result:

- If $0 < \varepsilon < 1$, we can take $x > \sqrt{\frac{1}{\varepsilon} - 1}$.
- If $\varepsilon \geq 1$, the inequality $x^2 > \frac{1}{\varepsilon} - 1$ is satisfied for all $x \in \mathbb{R}$.

8 Archimedean Property

Theorem 1 The set of real numbers $\mathbb{R}$ is Archimedean, which means that

$$\boxed{\forall x \in \mathbb{R}_+^\star,\ \forall y \in \mathbb{R}, \exists n \in \mathbb{N},\ \ nx \geq y}$$

Proof. We will assume, for the sake of contradiction, that $\mathbb{R}$ is not Archimedean, meaning

$$\exists x \in \mathbb{R}_+^\star, \exists y \in \mathbb{R},\ nx < y\ \ (\star)$$

Let A be a subset of $\mathbb{R}$ defined as

$$A = \{nx,\ \ n \in \mathbb{N}\}$$

A is non-empty and bounded by y (according to $\star$). Since A is a bounded subset of $\mathbb{R}$, it has a supremum, denoted as $M \in \mathbb{R}$. For all $n \in \mathbb{N}$, we have $nx \leq M$, and as $n + 1 \in \mathbb{N}$, then $(n+1)x \leq M \Leftrightarrow nx \leq M - x$, which means that $M - x$ is an upper bound for A. But since $x > 0$, we have $M - x < M$, meaning that M is not the smallest upper bound, which is contradictory.

8.1 Floor Function

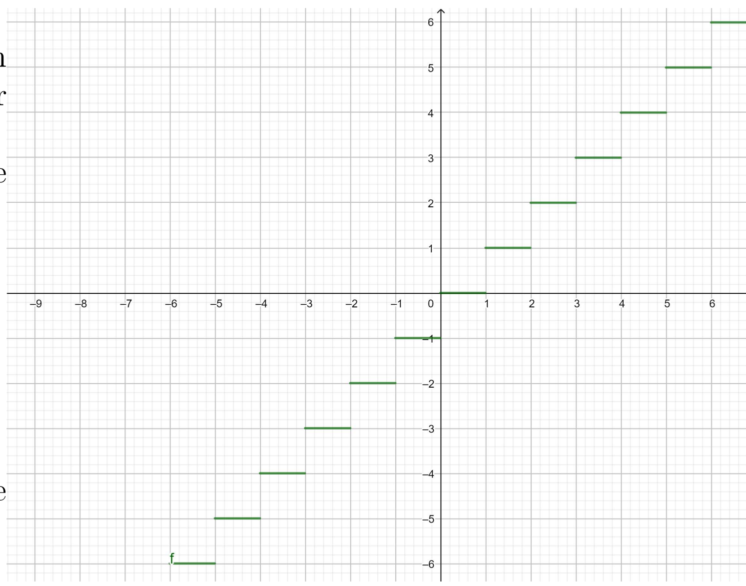
Definition 11. Let $x \in \mathbb{R}$, the greatest integer less than or equal to x is called the floor of x, denoted as $E(x)$ or $[x]$.
Proposition 6. For any $x \in \mathbb{R}$, there exists a unique integer n such that

$$n \leq x < n + 1$$

Consequently, $E(x)$ is the unique integer that satisfies

$$\boxed{E(x) \leq x < E(x) + 1}$$

The graph representing the floor function is shown in the figure on the right.



Remark 9:
• If $x \in \mathbb{Z}$, then $E(x) = x$.

• If x is a positive real number, its floor is the integer obtained by truncating its decimal part.
• If x is a negative real number, its floor is the integer less than the number obtained by removing its decimal part.

Example 8: $E(e) = 2$, $E(\sqrt{2}) = 1$, $E(-\pi) = -4$, and $E(-5.2) = -6$.

Proof. (of Proposition 6) Let A be the subset of $\mathbb{Z}$ defined as

$$A = \{n \leq x, \quad n \in \mathbb{Z}\}.$$

A is not empty:
• If $x \geq 0$, we have $0 \in \mathbb{Z}$, so $A \neq \emptyset$.
• If $x < 0$, we have $-x > 0$. According to the Archimedean property for $x = 1 \in \mathbb{R}_*^+$, $y = -x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx \geq y \Leftrightarrow n \geq -x \Leftrightarrow -n \leq x$, and as $-n \in \mathbb{Z}$, we have $A \neq \emptyset$.
A is bounded: According to the Archimedean property, for any $x \in \mathbb{R}$, $\exists M \in \mathbb{N}$, $M \geq x$ (It suffices to take $x = 1$ and $y = x$). For all $n \in A$, $n \leq x \leq M$, i.e., $n \leq M$ for all $n \in A$, which means that A is bounded. We have the following result:
Result: Every non-empty and bounded subset of $\mathbb{Z}$ has a greatest element. A is a bounded subset of $\mathbb{Z}$, so there exists $p \in \mathbb{Z}$, $p = \max(A)$.
Since $p \in A$, $p \leq x$ but $p+1$ is not in A, meaning $p + 1 > x$. Hence,

$$\forall x \in \mathbb{R}, \quad \exists p \in \mathbb{Z}, \quad p \leq x < p + 1.$$

Uniqueness of the floor function: We assume, for the sake of contradiction, that there exist p and $q \in \mathbb{Z}$ such that $p \neq q$ $(p < q)$. We have

$$\begin{cases} p \leq x < p+1 \\ q \leq x < q+1 \end{cases}$$

Result: For any two integers a and b, if $a < b$, then $a + 1 \leq b$.
This implies $x < p + 1 \leq q$, i.e., $x < q$, which is contradictory.
Properties Let $x, y \in \mathbb{R}$, we have:
i) $x - 1 < E(x) \leq x$

ii) $E(x) + E(-x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$

iii) $x \leq y \Rightarrow E(x) \leq E(y)$

iv) $\forall k \in \mathbb{Z}, \quad E(x + k) = E(x) + k$

v) $E(x) + E(y) \leq E(x + y) < E(x) + E(y) + 1$.
Proof.
ii) **First case:** If $x \in \mathbb{Z}$, we have $-x \in \mathbb{Z}$, which gives $E(x) = x$ and $E(-x) = -x$. Thus,

$$E(x) + E(-x) = 0.$$

Second case: If $x \in \mathbb{R} \setminus \mathbb{Z}$, we have, by definition,

$$E(x) < x < E(x) + 1 \Leftrightarrow -E(x) - 1 < -x < -E(x),$$

and

$$E(-x) < -x < E(-x) + 1,$$

which means that the real number x is bounded by the two integers $-E(x)-1$ and $-E(x)$, and $E(-x)$ and $E(-x) + 1$. By uniqueness of the floor function, we obtain $-E(x) - 1 = E(-x)$, which results in

$$E(x) + E(-x) = 1.$$

iii) We have:

$$E(x) \leq x < E(x) + 1$$

and

$$E(y) \leq y \leq E(y) + 1$$

and since $x \leq y$, then

$$E(x) \leq x \leq y < E(y) + 1 \Rightarrow E(x) < E(y) + 1.$$

Since both $E(x)$ and $E(y)+1$ are integers, it follows that $E(x) < E(y) + 1 \Rightarrow E(x) \leq E(y)$.
iv) By definition, we have:

$$E(x+k) \leq x + k < E(x+k) + 1$$

and

$$E(x) \leq x < E(x) + 1 \Leftrightarrow E(x) + k \leq x + k < E(x) + 1 + k.$$

Thus, $E(x) + k$ satisfies the characteristics of the floor function for $x + k$ for any $k \in \mathbb{Z}$, and therefore

$$E(x) + k = E(x+k).$$

Example 9: Show that for all $(m, n) \in \mathbb{Z}^2$, prove that

$$E\left(\frac{n+m}{2}\right) + E\left(\frac{n-m+1}{2}\right) \in \mathbb{Z}.$$

We will consider two cases:
First case: If we assume that $n + m$ is even, then $\frac{n+m}{2} \in \mathbb{Z}$, which gives $E\left(\frac{n+m}{2}\right) = \frac{n+m}{2}$. Furthermore, we have:

$$\frac{n-m+1}{2} = \frac{n+m-2m+1}{2} = \frac{n+m}{2} - m + \frac{1}{2}.$$

Since $\frac{n+m}{2} - m \in \mathbb{Z}$ and based on the previous proposition, we have:

$$E\left(\frac{n+m}{2} - m + \frac{1}{2}\right) = \frac{n+m}{2} - m.$$

Therefore,

$$E\left(\frac{n+m}{2}\right) + E\left(\frac{n-m+1}{2}\right) = \frac{n+m}{2} + \frac{n+m}{2} - m = n \in \mathbb{Z}.$$

Second case: If $n + m$ is odd, we observe that:

$$n - m + 1 = n + m - 2m + 1 = (n + m + 1) - (2m),$$

which is even since it is the sum of two even integers. Thus, $E\left(\frac{n-m+1}{2}\right) = \frac{n-m+1}{2}$. We also have:

$$E\left(\frac{n+m}{2}\right) = E\left(\frac{n-m+1+2m-1}{2}\right) = E\left(\frac{n-m+1}{2} + m - \frac{1}{2}\right) = \frac{n-m+1}{2} + m - 1.$$

Hence,

$$E\left(\frac{n+m}{2}\right) + E\left(\frac{n-m+1}{2}\right) = \frac{n-m+1}{2} + m - 1 + \frac{n-m+1}{2} = n \in \mathbb{Z}.$$