Exercise 5.1.

1) Show that i is a double root of $P(x) = x^6 + x^5 + 3x^4 + 2x^3 + 3x^2 + x + 1$.

2) Factor P in $\mathbb{C}[x]$ and in $\mathbb{R}[x]$.

Exercise 5.2.
Find a polynomial of degree 3 which admits two simple roots $x_0 = 1$ and $x_1 = 2$ and that the remainder of its Euclidean division by $x^2 + 1$ is $R(x) = 1$.

Exercise 5.3.
1. Perform the Euclidean division of P by Q
   (a) $P = 3x^4 + 3x^2 + 1, Q = x^2 + 2x + 3 - 1$
2. Perform the division in ascending powers of P by Q to the order k.
   (a) $P = 1 - 2x + x^3 + x^3, Q = 2 + 2x + x^2, k = 3$.

Exercise 5.4.

Let P and $Q \in \mathbb{R}[x]$ two polynomials such that $P(x) = x^4 - 5x^3 + 4x^2 - 3x + 3$ and $Q(x) = x^3 - x^2 - x + 1$.

1) Find D the greatest common divisor of P and Q.

2) Find U and V two polynomials of $\mathbb{R}[x]$ such that $PU + QV = D$.

Exercise 5.5.
Let P be a polynomial of $\mathbb{R}[x]$ such that $P(i) = 0$.

1) Show that there exists a polynomial Q of $\mathbb{R}[x]$ verifying $\forall x \in \mathbb{R} : P(x) = (x^2 + 1) Q(x)$.

2) Factor $P(x) = x^5 + x^3 - x^2 - 1$ in $\mathbb{C}[x]$ and in $\mathbb{R}[x]$.

Exercise 5.6.
Let $n \in \mathbb{N}$ and P, Q two polynomials in $\mathbb{R}[x]$ such that :

$$P(x) = (x-1)^{n+2} + x^{2n+1} + 1 \text{ and } Q(x) = x^2 - x + 1.$$

1) Show that $-j$ is a root of Q.

2) Calculate $P(-j)$.

3) Deduce the remainder of the division of P by Q .

Exercise 5.7.
Let $P(x) = 2x^3 + x^2 + x - 1 \in \mathbb{R}[x]$.

1) Show that, if P admits a rational root written under irreducible form $\frac{a}{b}$ then $a|1$ and $b|2$.

2) Deduce all rational roots of P.

3) Find all roots of P in $\mathbb{R}$.

Exercise 5.8.

A Let $P(x) = x^5 + 4x^3 - x^2 - 4 \in \mathbb{R}[x]$.

1) Verify that the complex numbers $2i$ and j are roots of the polynomial P.
2) Find the other two complex roots of P.
3) Factor P into irreducible factors in $\mathbb{C}[x]$ then in $\mathbb{R}[x]$.

B Let $Q(x) = x^{6n+2} + x^{3n+1} + 1 \in \mathbb{R}[x], n \in \mathbb{N}^*$.

1) Verify that j is a root of Q.
2) For $n = 1$, give two polynomials Q_1, Q_2 in $\mathbb{R}[x]$ such that $Q = Q_1 Q_2$ with $d^\circ Q_1 = 2$ and $d^\circ Q_2 = 6$.

1 Solutions

Solution to exercise 3.1.

1) We have: $P(i) = 0$, $P'(i) = 0$ and $P''(i) \neq 0$. Then i is a double root of P.

2) Since i is a double root of P and $P \in \mathbb{R}[x]$ then $-i$ is a double root of P so we can write P in the form:
$$P(x) = (x-i)^2 (x+i)^2 Q(x) \text{ where } \left(d^\circ Q = 2\right).$$
We can deduce Q using the Euclidean division of $P(x)$ by $\left(x^2+1\right)^2$. We get:
$$\begin{array}{r|l} x^6 + x^5 + 3x^4 + 2x^3 + 3x^2 + x + 1 & \\ -(x^6 + 2x^4 + x^2) & \\ \hline x^5 + x^4 + 2x^3 + 2x^2 + x + 1 & \underline{x^4 + 2x^2 + 1} \\ -(x^5 + 2x^3 + x) & x^2 + x + 1 \\ \hline x^4 + 2x^2 + 1 & \\ -(x^4 + 2x^2 + 1) & \\ \hline 0 & \end{array}$$
then $Q(x) = x^2 + x + 1$. The roots of Q are j and j^2 then factoring Q into $\mathbb{C}[x]$ is:
$$P(x) = (x-i)^2 (x+i)^2 (x-j)\left(x-j^2\right) \text{ où } \left(j = -\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$
and factoring Q into $\mathbb{R}[x]$ is:
$$P(x) = \left(x^2+1\right)^2 \left(x^2+x+1\right)$$
because the polynomials x^2+1 and x^2+x+1 are irreducible in $\mathbb{R}[x]$.

Solution de l'exercice 3.1.

1) We have: $P(i) = 0$, $P'(i) = 0$ and $P''(i) \neq 0$. Then i is a double root of P.

2) Since i is a double root of P and $P \in \mathbb{R}(x)$ then $-i$ is a double root of P so we can éwrite P in the form:
$$P(x) = (x-i)^2 (x+i)^2 Q(x) \text{ où } \left(d^\circ Q = 2\right).$$
We can deduce Q using the Euclidean division of $P(x)$ by $\left(x^2+1\right)^2$. We obtain:
$$\begin{array}{r|l} x^6 + x^5 + 3x^4 + 2x^3 + 3x^2 + x + 1 & \\ -(x^6 + 2x^4 + x^2) & \\ \hline x^5 + x^4 + 2x^3 + 2x^2 + x + 1 & \underline{x^4 + 2x^2 + 1} \\ -(x^5 + 2x^3 + x) & x^2 + x + 1 \\ \hline x^4 + 2x^2 + 1 & \\ -(x^4 + 2x^2 + 1) & \\ \hline 0 & \end{array}$$

then $Q(x) = x^2 + x + 1$. The roots of Q are j and j^2 then factoring Q into $\mathbb{C}[x]$ is:

$$P(x) = (x-i)^2 (x+i)^2 (x-j) \left(x - j^2\right) \text{ où } (\; j = -\frac{1}{2} + \frac{\sqrt{3}}{2} i)$$

and factoring Q into $\mathbb{R}[x]$ is:

$$P(x) = \left(x^2+1\right)^2 \left(x^2+x+1\right)$$

because the polynomials x^2+1 and x^2+x+1 are irreducible in $\mathbb{R}[x]$.

Solution to exercise 3.1.

1) We have: $P(i) = 0$, $P'(i) = 0$ and $P''(i) \neq 0$. Then i is a double root of P.

2) Since i is a double root of P and $P \in \mathbb{R}(x)$ then $-i$ is a double root of P so we can éwrite P in the form:

$$P(x) = (x-i)^2 (x+i)^2 Q(x) \text{ où } \left(d^{\circ} Q = 2\right).$$

We can deduce Q using the Euclidean division of $P(x)$ by $\left(x^2+1\right)^2$. We obtain:

$$
\begin{array}{c|c}
x^6+x^5+3x^4+2x^3+3x^2+x+1 & \\
-(x^6+2x^4+x^2) & x^4+2x^2+1 \\
\cline{1-1} \cline{2-2}
x^5+x^4+2x^3+2x^2+x+1 & x^2+x+1 \\
-(x^5+2x^3+x) & \\
\cline{1-1}
x^4+2x^2+1 & \\
-(x^4+2x^2+1) & \\
\cline{1-1}
0 &
\end{array}
$$

then $Q(x) = x^2 + x + 1$. The roots of Q are j and j^2 then factoring Q into $\mathbb{C}[x]$ is:

$$P(x) = (x-i)^2 (x+i)^2 (x-j) \left(x - j^2\right) \text{ where } (\; j = -\frac{1}{2} + \frac{\sqrt{3}}{2} i)$$

and factoring Q into $\mathbb{R}[x]$ is:

$$P(x) = \left(x^2+1\right)^2 \left(x^2+x+1\right)$$

because the polynomials x^2+1 and x^2+x+1 are irreducible in $\mathbb{R}[x]$.

Solution to exercise 3.2.

Since the remainder of the Euclidean division of $P(x)$ by x^2+1 is 1, then $\exists (a,b) \in \mathbb{R}^2$ such that:

$$P(x) = (ax+b)\left(x^2+1\right) + 1$$

then 1 and 2 are roots of P, therefore:

$$\begin{cases} (a+b)\left(1^2+1\right)+1 = 0 \\ (2a+b)\left(2^2+1\right)+1 = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{3}{10} \\ b = -\frac{4}{5} \end{cases}.$$

SO

$$\begin{aligned} P(x) &= \left(\frac{3}{10}x + -\frac{4}{5}\right)(x^2+1)+1 \\ &= \frac{3}{10}x^3 - \frac{4}{5}x^2 + \frac{3}{10}x + \frac{1}{5}. \end{aligned}$$

Solution to exercise 3.3.

1) We have:

$$\begin{array}{r|l} x^4-5x^3+4x^2-3x+3 & \underline{x^3-x^2-x+1} \\ -\underline{(x^4-x^3-x^2+x)} & \quad x-4 \\ -4x^3+5x^2-4x+3 & \\ -\underline{(-4x^3+4x^2+4x-4)} & \\ x^2-8x+7 & \end{array}.$$

From where‘ $P(x) = (x-4)\,Q(x) + (x^2-8x+7)$, then we have:

$$\begin{array}{r|l} x^3-x^2-x+1 & \underline{x^2-8x+7} \\ -\underline{(x^3-8x^2+7x)} & \quad x+7 \\ 7x^2-8x+1 & \\ -\underline{(7x^2-56x+49)} & \\ 48x-48 & \end{array}.$$

So $Q(x) = (x^2-8x+7)(x+7) + 48x - 48$. We have:

$$\begin{array}{r|l} x^2-8x+7 & \underline{48x-48} \\ -\underline{(x^2-x)} & \tfrac{1}{48}x - \tfrac{7}{48} \\ -7x+7 & \\ -\underline{(-7x+7)} & \\ 0 & \end{array}$$

Then $D = \gcd(P,Q) = x-1$.

2) We have:

$$x-1 = \frac{1}{48}Q(x) - \frac{1}{48}\left(x^2-8x+7\right)(x+7)$$

And

$$\left(x^2-8x+7\right) = P(x) - (x-4)\,Q(x)$$

SO

$$x-1 = \frac{1}{48}Q(x) - \frac{1}{48}\left(P(x) - (x-4)\,Q(x)\right)(x+7).$$

SO

$$x-1 = \frac{1}{48}\left(1+(x-4)(x+7)\right)Q(x) - \frac{1}{48}(x+7)\,P(x),$$

henceù $U(x) = -\frac{1}{48}(x+7)$ and $V(x) = \frac{1}{48}(x^2+3x-27)$.

Solution to exercise 3.4.

1) We have: $P(x) \in \mathbb{R}[x]$ and $P(i) = 0$ then $P(-i) = 0$. So we can factor P in $\mathbb{C}[x]$ as follows:

$$P(x) = (x+i)(x-i)Q(x) \text{ with } Q(x) \in \mathbb{C}[x].$$

Hence $P(x) = (x^2+1)Q(x)$.

* $*$ We show that $Q(x) \in \mathbb{R}[x]$. We have: $\overline{P(x)} = P(\overline{x})$, therefore

$$\overline{(x^2+1)Q(x)} = \left(\overline{x}^2+1\right)Q(\overline{x}) \Rightarrow \left(\overline{x}^2+1\right)\overline{Q(x)} = \left(\overline{x}^2+1\right)Q(\overline{x})$$

it therefore follows that $\overline{Q(x)} = Q(\overline{x})$. Then $Q(x) \in \mathbb{R}[x]$.

2) We have: $P(i) = 0$, then we can factor P as follows:

$$P(x) = \left(x^2+1\right)\left(x^3-1\right)$$

so $P(x) = (x-i)(x+i)(x-1)(x-j)(x-j^2)$ in $\mathbb{C}[x]$,
and $P(x) = (x^2+1)(x-1)(x^2+x+1)$ in $\mathbb{R}[x]$.

Solution to exercise 3.5.

1) Like $j^2 = -j-1$, therefore

$$Q(-j) = j^2 + j + 1 = 0$$

henceù $-j$ is a root of Q.

2) We have: $P(-j) = (-j-1)^{n+2} + (-j)^{2n+1} + 1$ is like $-j-1 = j^2$, so

$$\begin{aligned} P(-j) &= j^{2n+4} - j^{2n+1} + 1 \\ &= j^{2n+1} - j^{2n+1} + 1 = 0 \end{aligned}$$

hence $P(-j) = 1$.

3) We assume that:

$$P(x) = Q(x)U(x) + R(x) \text{ with } d^\circ(R) < d^\circ(Q)$$

henceù $R(x) = ax + b$ with $(a,b) \in \mathbb{R}^2$. As $P(-j) = 0$ then $R(-j) = 1$ therefore $R(x) = 1$.

Solution to exercise 3.6.

1) P admits a rational root written in form irreducible $\frac{a}{b}$ then:

$$\begin{aligned} P\left(\frac{a}{b}\right) = 0 &\Rightarrow 2\left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 1 = 0 \\ &\Rightarrow a\left(2a^2 + ba + b^2\right) = b^3 \\ &\Rightarrow a|b^3 \end{aligned}$$

like $a \wedge b = 1$ then according to the Gaussian theorem we have $a|1$. Then we have:

$$\begin{aligned} 2\left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 1 = 0 &\Rightarrow 2a^3 = b\left(b^2 - a^2 - ab\right) \\ &\Rightarrow b|2a^3 \end{aligned}$$

like $a \wedge b = 1$ then according to the Gaussian theorem we have $b|2$.

2) If $\frac{a}{b}$ is a root of P, then $a \in \{-1, 1\}$ and $b \in \{-1, -2, 1, 2\}$ from where

$$\frac{a}{b} \in \left\{-1, 1, -\frac{1}{2}, \frac{1}{2}\right\}.$$

Like $P(-1) \neq 0$, $P(1) \neq 0$, $P\left(-\frac{1}{2}\right) \neq 0$ and $P\left(\frac{1}{2}\right) = 0$, then $\frac{1}{2}$ is a rational root of P.

3) As $\frac{1}{2}$ is a root of P then we can éwrite P in the form:

$$\begin{aligned} P(x) &= \left(x - \frac{1}{2}\right)\left(ax^2 + bx + c\right) \\ &= ax^3 + \left(b - \frac{1}{2}a\right)x^2 + \left(c - \frac{1}{2}b\right)x - \frac{1}{2}c. \end{aligned}$$

By identification, we obtain:

$$a = 2, b = 2 \text{ and } c = 2.$$

Hence the other roots are the roots of polynomial $2\left(x + x^2 + 1\right)$ i.e. j and j^2.

Solution to exercise 3.7.

A/ 1) We have: $P(2i) = 0$, $P(j) = 0$.

2) Since $P \in \mathbb{R}[x]$ then the other two complex roots of P are $-2i$ and $\overline{j}$.

3) We have:

$$P(x) = (x - i)(x + i)(x - j)\left(x - \overline{j}\right)(ax + b)$$

so by identification we obtain : $a = 1$ and $b = -1$ therefore

$$P(x) = (x - 2i)(x + 2i)(x - j)\left(x - \overline{j}\right)(x - 1).$$

Then factoring P into $\mathbb{R}[x]$ is:

$$P(x) = \left(x^2 + 4\right)\left(x^2 + x + 1\right)(x - 1).$$

B/ We have: $Q \in \mathbb{R}[x], n \in \mathbb{N}^*$.

1) We have:

$$\begin{aligned} Q(j) &= j^{6n+2} + j^{3n+1} + 1 \\ &= j \times j^{6n} + j^{3n} \times j + 1 \\ &= j^2 + j + 1 = 0 \end{aligned}$$

therefore j is a root of Q.

2) For $n = 1$, $Q(x) = x^8 + x^4 + 1$ and since j is a root of Q and $Q(-x) = Q(x)$ then:

$$Q(j) = Q\left(\overline{j}\right) = 0.$$

Hence:

$$\begin{aligned} Q(x) &= (x - j)\left(x - \overline{j}\right)Q_2(x), \text{ with } d^\circ Q_2 = 6 \\ &= \left(x^2 + x + 1\right)Q_2(x). \end{aligned}$$

By Euclidean division we obtain:

$$
\begin{array}{c|c}
x^8 + x^4 + 1 & \\
-\underline{(x^8 + x^7 + x^6)} & x^2 + x + 1 \\
\cline{2-2}
-x^7 - x^6 + x^4 + 1 & x^6 - x^5 + x^3 - x + 1 \\
-\underline{(-x^7 - x^6 - x^5)} & \\
x^5 + x^4 + 1 & \\
-\underline{(x^5 + x^4 + x^3)} & \\
-x^3 + 1 & \\
\underline{-(-x^3 - x^2 - x)} & \\
x^2 + x + 1 & \\
\underline{-(x^2 + x + 1)} & \\
0 &
\end{array}
$$

$Q_2(x) = x^6 - x^5 + x^3 - x + 1.$