

Exercise 1.

- 1) Say whether the following statements are true or false.
 - a) $(\sqrt{4} = -2) \vee (\sqrt{3} \in \mathbb{Q})$ (\mathbb{Q} the set of rational numbers.) ,
 - b) $(x \in \emptyset) \Rightarrow (x \in \mathbb{Q})$.
- 2) Write a) as an implication.
- 3) Write the negation of the preceding propositions and give their contrapositives.

Exercise 2. Let P, Q, R be logical propositions.

- 1) Prove that the following propositions are tautologies:

1. $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
2. $\neg(P \Leftrightarrow Q) \Leftrightarrow (P \Leftrightarrow \neg Q)$
3. $[P \Rightarrow (Q \vee R)] \Leftrightarrow [(P \wedge \neg Q) \Rightarrow R]$

- 2) Provide complete truth tables for:

1. $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$
2. $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

- 3) Write the contrapositive and converse of:

$$(P \wedge Q) \Rightarrow (R \vee S)$$

Exercise 3.

A) Let R, S and T be logical propositions. Are the following logical equivalences true?

- 1) $(R \Rightarrow (\bar{S} \wedge T)) \Leftrightarrow ((\bar{T} \Rightarrow S) \wedge \bar{R})$.
- 2) $(\bar{S} \vee (T \Rightarrow R)) \Leftrightarrow ((S \Rightarrow \bar{T}) \wedge R)$.

B) Give the truth table of the following compound propositions.

- 1) $(R \Rightarrow (S \wedge T)) \Rightarrow (R \Rightarrow S) \wedge (R \Rightarrow T)$.
- 2) $(\neg R \vee S) \Rightarrow (S \wedge (R \Leftrightarrow S))$.
- 3) $R \wedge (R \Rightarrow \neg S) \wedge S$.

Exercise 4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Define the predicates:

$$\begin{aligned} A &: \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \mathbb{R}, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \\ B &: \exists M > 0, \forall x \in \mathbb{R}, |f(x)| \leq M \\ C &: \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, f(y) > f(x) \end{aligned}$$

- 1) Express in natural language the meaning of the predicates B and C.
- 2) Provide the negation of each predicate.
- 3) Among the following implications, which are always true?

$$1) A \Rightarrow B, 2) B \Rightarrow A, 3) C \Rightarrow \neg B, 4) \neg C \Rightarrow B$$

Exercise 5. Let us consider a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. We have the following predicates:

$$P : (\forall x \in \mathbb{R}, f(x) = 0); Q : (\exists x \in \mathbb{R}, f(x) = 0);$$

$$R : (\forall x \in \mathbb{R}, f(x) < 0) \vee (\forall x \in \mathbb{R}, f(x) > 0).$$

Which of the following implications are true.

$$1) (P \Rightarrow Q); 2) (Q \Rightarrow P); 3) (Q \Rightarrow R); 4) (\neg R \Rightarrow Q); 5) (\neg P \Rightarrow \neg R).$$

Exercise 6. Which of the following relationships represent logical propositions and those which are not.

$$1) \forall x \in [1, 2], x^2 - 3x + 2 \geq 0.$$

$$2) x \in \mathbb{R}, -x^2 + 2x + 2 > 0.$$

3) x is a rational number.

$$4) \exists x \in \mathbb{N}^* : \forall m \in \mathbb{N}, x < m.$$

$$5) \forall x, y \in \mathbb{R}, (xy \neq 0 \wedge x \leq y) \Rightarrow \left(\frac{1}{x} \geq \frac{1}{y} \right).$$

$$6) x + y \geq 0, \text{ when } x \leq 0.$$

Exercise 7. Give the negation of the following propositions and determine which one is true.

$$1) \exists x \in \mathbb{R} : \forall y \in \mathbb{R}, (x \geq 2) \wedge (x + y > 10).$$

$$2) \forall x, y \in \mathbb{R}, (x \geq 0) \Rightarrow (x + y < 9).$$

$$3) \forall n \in \mathbb{N}, (n < 2) \Rightarrow (n^2 = n).$$

$$4) \exists M \in \mathbb{R}_+ : \forall n \in \mathbb{N}, |e^{\sin(n)}| \leq M.$$

$$5) \exists x \in \mathbb{R}^*, \forall y \in \mathbb{R}^*, \forall z \in \mathbb{R}^*, z - 2xe^y = 0.$$

Exercise 8. Let f and g be two real functions defined on \mathbb{R} . Translate the following expressions in terms of quantifiers and then give their negations.

- f is an increasing function and greater than g .
- $f + g$ is an odd function.
- There exists a real number M such that fg is less than M .
- f is bounded.

Exercise 9.

A) Show by induction that

$$1) \forall n \in \mathbb{N}^*, 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

$$2) \forall n \geq 2, n! \leq \left(\frac{n+2}{2}\right)^n.$$

3) $\forall n \in \mathbb{N}, 10^n - 1$ is a multiple of 9.

4) $\forall n \in \mathbb{N}, 2 - 2 \cdot 7 + 2 \cdot 7^2 + \cdots + 2 \cdot (-7)^n = \frac{1 - (-7)^{n+1}}{4}$.

B) Show that $\sqrt{5}$ and $\frac{\ln 2}{\ln 3}$ are not rational numbers.

C) Show that $\forall n \in \mathbb{N}, n(n+1)(n+2)$ is a multiple of 3.

D) Prove that every integer $n \geq 2$ can be written as a product of prime numbers.

E) Let the sequence be defined by:

$$u_0 = 1, u_1 = 1, u_{n+1} = u_n + u_{n-1} \quad \text{for } n \geq 1$$

Prove that for all $n \in \mathbb{N}, u_n \leq 2^n$.

F) Prove that every integer $n \geq 8$ can be written in the form $n = 3a + 5b$ with $a, b \in \mathbb{N}$.

G) Prove that:

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$