Ecole Nationale Polytechnique
Preparatory cycle

Exercise series n ° 2 - Semester 1
2022-2023
Functions - applications

Exercice 1 :
Are the following applications well defined ? if yes, are they injective ? surjective ? bijective ?

1. $f : \{0, 1, 2\} \to \{1, 8, -1, 24\}$ such that $f(0) = -1$, $f(1) = 24$, $f(2) = 1$.
2. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n) = -n$.
3. $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = n + 1$.
4. $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = n - 1$.
5. $f : \mathbb{N} \to \{-1, +1\}$ which for each n of $\mathbb{N}$ associates 1 if n is even, and -1 if n is odd.

For each of apllications $1), 2), 3), 4)$ and 5) of the previous exercice, calculate :

$$f(\{2\}), f(\{0, 2\}), f^{-1}(\{1\}), f^{-1}(\{-1, 1\})$$

Exercice 2.
Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{2x}{1 + x^2}.$$

1. Is f injective ? surjective ?
2. Show that $f(\mathbb{R}) = [-1, 1]$.
3. Show that the restriction $g : [-1, 1] \to [-1, 1]$, $g(x) = f(x)$ is a bijection.
4. Find this result by studying the variations of f.

Exercice 3 :
Let h the application from $\mathbb{R}$ to $\mathbb{R}$ defined by : $h(x) = \dfrac{4x}{x^2 + 1}$.

1. Verify that for all non zero real a we have : $h(a) = h(\frac{1}{a})$. Is the application h injective ? Justify.
2. Let f the function defined on the interval $I = [1; +\infty[$ by $f(x) = h(x)$.
   a) Show that f is injective.
   b) Verify that : $\forall x \in I; f(x) \leq 2$.
   c) Show that f is a bijection of I on $]0, 2]$ and find $f^{-1}(x)$.

Exercice 4 :
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, a function defined by $f(x, y) = (2x + y, x + y)$.

1. Is f injective ? surjective ? bijective ?
2. Let $A = \{(2, 1)\}$ and $B = \{x, y \ \in \mathbb{R}^2 | y = 0\}$ two subsets of $\mathbb{R}^2$.
   Determine $f(\mathbb{R}^2)$, $f^{-1}(\mathbb{R}^2)$, $f^{-1}(A)$ and $f(B)$.

Exercice 5 :
For each $(a, b, c, d) \in \mathbb{Z}^4$ such that $ad - bc = 1$, we consider the application $f : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by

$$f(n, m) = (an + bm, cn + dm).$$

Let F the set of all these applications.

1. Justify that f is a bijection.
2. Justify that F is stable by composition of applications.

Exercice 6 :
Let $f : \mathbb{R} \to \mathbb{C}$ with $f(t) = e^{it}$. Change the domain and the codomain so that (the restriction of) f becomes bijective.

Exercice 7 :
(1) Show that $f : \mathbb{R} \to]-1; 1[$ defined by :
$f(x) = \frac{x}{1+|x|}$ is bijective and determine its inverse.
(2) Let g the application from $\mathbb{R}$ to $]-1; 1[$ defined by :

$$f(x) = sin(\pi x)$$

a) Is this application injective ? surjective ? bijective ?
b) Show that the restriction of f in $]\frac{-1}{2}; \frac{1}{2}[$ is a bijection from $]\frac{-1}{2}; \frac{1}{2}[$ to $]-1; 1[$.

Exercice 8 :
Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined $\forall (x, y) \in \mathbb{R}^2$ by

$$f(x, y) = \frac{x + y}{1 - xy}$$

1. Show that the restriction of f in $]-1, 1[\times]-1, 1[$ is an application.
2. Is f injective ? Justify.
3. Justify that $f(\tan x, \tan y) = \tan(x + y)$, for all x and y not equal to $\frac{\pi}{2} + k\pi, \forall k \in \mathbb{Z}$, such that : $\tan x, \tan y$ are in $]-1, 1[$.
4. Determine $f(\mathbb{R}^2 -]-1, 1[^2)$ as well as $f(]-1, 1[^2)$.
5. Conclude.