National Higher School of Autonomous Systems Technology

Exercise Set 3 : Real Functions with a Real Variable

Exercise 1 – Domain of Definition

Determine the domain of definition of the following functions

$$f_1(x) = \frac{x^2+1}{E(x)}; \quad f_2(x) = \sqrt{\frac{1-x}{1+x}}\ ; \ f_3(x) = \ln(\ln(x-1)); \quad f_4(x) = \frac{1}{\sin(3x)} \quad f_5(x) = \frac{\tan x}{\cos(\pi x)}.$$

Exercise 2 – Limits by Definition

Show, by using the limit definition, that

$$\lim_{x\to 2}(4x+5) = 13; \quad \lim_{x\to x_0}\sqrt{x} = \sqrt{x_0}; \quad \lim_{x\to 1^-}\frac{1}{1-x^2} = +\infty; \quad \lim_{x\to+\infty}(\sqrt{x^2+2}-\sqrt{x^2+1}) = 0.$$

Exercise 3 – Limits

Compute, if it exists, the limit of the following functions

$$\lim_{x\to-\infty}\frac{2^x+1}{2^{\frac{1}{x}}+1}; \quad \lim_{x\to 2}\frac{|x-2|}{x^2-3x+2}; \quad \lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3} \ (a\neq 0); \quad \lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right); \quad \lim_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt[3]{x+1}-1}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x^m}-\sqrt{1-x^m}}{x^n}(\text{for } m,n\in\mathbb{N}); \quad \lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}; \quad \lim_{x\to 0}\frac{1-\cos x}{x^2}; \quad \lim_{x\to 1}\frac{1+\cos(\pi x)}{\sin^2(\pi x)};$$

$$\lim_{x\to 0}\frac{\ln(\cos(3x))}{\ln(\cos(2x))}; \quad \lim_{x\to 0}\frac{\sin^2(6x)-\sin^2(4x)}{\tan^2(x)}; \quad \lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}; \quad \lim_{x\to 0}\frac{x\sin(\alpha x)}{1-\cos(\beta x)} \ (\alpha,\beta\in\mathbb{R}^\star)$$

$$\lim_{x\to+\infty}\frac{x^{\ln x}}{(\ln x)^x}; \quad \lim_{x\to+\infty}e^{x-\sin x}; \quad \lim_{x\to 0}(1-4x)^{\frac{1}{x}}; \quad \lim_{x\to+\infty}\frac{(x+3)^x}{x^{x+1}}; \quad \lim_{x\to 0}\frac{x}{a}E\left(\frac{b}{x}\right) \ (a,b>0).$$

$$\lim_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin(px)-\cos(px)}(p\in\mathbb{R}); \quad \lim_{x\to+\infty}(\sqrt{x^4+x^2}-\sqrt{x^4+1});$$

Exercise 4 – Limits

Show that the following limits do not exist

$$\lim_{x\to 0}\cos\left(\frac{1}{x}\right); \ \lim_{x\to 0}\cos(\ln x): \ \lim_{x\to+\infty}\sin\left(\frac{x^3}{x^2+1}\right); \ \lim_{x\to 0}\frac{1}{x}-E(\frac{1}{x})$$

Exercise 5 – Continuity at a point

Study the continuity of the following functions at the indicate point x_0

$$f_1(x) = \begin{cases} x+\dfrac{\sqrt{x^2}}{x} & \text{si } x\neq 0 \\ 0 & \text{si } x=0 \end{cases} \quad (x_0=0) \qquad f_2(x) = \begin{cases} \frac{1}{x}e^{-\frac{1}{x^2}} & \text{si } x<0 \\ 0 & \text{si } x=0 \\ x\sin\left(\dfrac{\cos(x)}{\sqrt{x}}\right) & \text{si } x>0 \end{cases} \quad (x_0=0)$$

$$f_3(x) = xE\left(\frac{1}{x}\right) \quad (x_0 = \frac{1}{n},\ n\in\mathbb{N}^\star)$$

Exercise 6 – Continuity on a set

On which part of $\mathbb{R}$ the following functions are continuous :

$$f_1(x) = x(x - E(x))^2, \quad f_2(x) = E(x) + \sqrt{x - E(x)}, \quad f_3(x) = E(x)\sin \pi x,$$

$$f_4(x) = \begin{cases} \sin x \sin \frac{1}{x} & \text{si} \quad x \neq 0 \\ 0 & \text{si} \quad x = 0 \end{cases}$$

Exercise 7 – **Continuity on a set**

Let f and g be two functions defined in $\mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{e^{2x} - e^2}{e^x - e} & \text{if } x < 1 \\ a & \text{if } x = 1 \\ \dfrac{b\sin(x-1)}{x^3 - 1} & \text{if } x > 1 \end{cases} \quad \text{and } g(x) = \begin{cases} \sqrt{x^4+1} - (x^2 + c) + \dfrac{1 - \cos(bx)}{x^2} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$$

find $a,\ b,\ c,\ d \in \mathbb{R}$ such that f and g are continuous in $\mathbb{R}$ and $\lim\limits_{x \to +\infty} g(x) = -3$.

Exercise 8 – **Extension by continuity**

Can the following functions be extended by continuity over the interval I ,

(i) $f(x) = \dfrac{x + 1 + |x+5|}{|3 - x| + 2x}$ where $I = \mathbb{R}$ **(ii)** $g(x) = \dfrac{x^3 - 2x - x + 2}{1 - |x|}$ where $I = \mathbb{R}$;

(iii) $k(x) = (x - 3)E(x)$ where $I = [0,\ 3]$ **(v)** $h(x) = x\sin(\frac{1}{x}) + \cos(\frac{1}{\sqrt{|x|}})$ where $I = \mathbb{R}$

(vi) $l(x) = x\left|x + \dfrac{1}{x}\right|$ where $I = \mathbb{R}$.

Exercise 9 – **IVT**

1-Show that the equation $x^3 + x - 5 = 0$ has a root in $[1, 2]$.
2-For which values of $\alpha \in \mathbb{R}$ does the equation $x^2 + \sqrt{x} - \alpha = 0$ a unique root in the interval $[0, 1]$.

Exercise 10 – **Fundamental theorems**

- $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and periodic. Show that f is bounded. Is the condition that f is continuous necessary ?
- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic function. Show that there $x_0 \in \mathbb{R}$ such that $f(x_0 + \frac{T}{2}) = f(x_0)$.
- Let $f : [a, b] \longrightarrow [a, b]$ be a continuous function. Show that there exists $c \in [a, b]$ such that $f(c) = c$.
- Let f be a continuous function in the interval $[0, 1]$ such that $f(0) = f(1)$. Show that there exists at least $c \in \left[0, \frac{1}{2}\right] : f(c) = f\left(c + \frac{1}{2}\right)$.
- Let $f : \mathbb{R} \to \mathbb{R}$ b a continuous function such that $\lim\limits_{x \to -\infty} f(x) = -1$ et $\lim\limits_{x \to +\infty} f(x) = 1$. Show that f vanish.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim\limits_{x \to -\infty} f(x) = -\infty$ et $\lim\limits_{x \to +\infty} f(x) = +\infty$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function . Show that there exists $c \in \mathbb{R}$ such that $f(c) = g(c)$.

Exercise 11 – **Theorem of bijection monotonic functions**

Study the continuity and monotonicity of the following functions, then determine their inverse functions

$a)-f(x) = \sqrt{\dfrac{1-x}{x}}$ $\quad b)-g(x) = \begin{cases} \dfrac{|x|}{x}\sqrt{|x|} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$ $\quad c) h : [0, +\infty[\to [1, +\infty[$ where $h(x) = \dfrac{2e^{2x}}{1 + e^x}$

II-Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f(x) = E(x) + (x - E(x))^2 \text{ et } g(x) = E(x) + \sqrt{x - E(x)}$$

1-Verify that $E(f(x)) = E(g(x)) = E(x)$.

2-Deduce $g = f^{-1}$.

Exercise 12 – Differentiability at a point

Let f and g be Two differentiable functions at the point a. Calculate the following limits.

$$\lim_{x\to a}\frac{af(x)-xf(a)}{x-a} \quad \text{and} \quad \lim_{x\to a}\frac{f(x)g(a)-f(a)g(x)}{x-a}$$

Exercise 13 – Differentiability at a point

Let $f:[0,1]\to\mathbb{R}$ be a differentiable function on $[0,1]$ and $g:[0,1]\to\mathbb{R}$ defined by

$$g(x)=\begin{cases} f(2x) & \text{if } x\in[0,\frac{1}{2}] \\ f(2x-1) & \text{else}\end{cases}$$

Find a necessary and sufficient condition for g to be differentiable on [0, 1].

Exercise 14 – Differentiability at a point

For which values of $k\in\mathbb{Z}$ the functions f and g differentiable at $x_0=0$

$$f(x)=\begin{cases}\sin(x^k) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases} \quad ; \quad g(x)=\begin{cases} x^k\sin(\frac{1}{x}) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

Exercise 15 – Differentiability at a point

Examine the differentiability of the following functions at the specified point x_0

(1) $f_1(x)=x\left|\sin x\right|, x_0=0$ **(2)** $f_2(x)=\sqrt{|x-1|},\ x_0=1$ **(3)** $f_3(x)=xE(x),\ x_0=0$

Exercise 16 – Derivative

Calculate the derivative of each of the following functions :

$$f_1(x)=\sin(\ln x)\ ;\ f_2(x)=\frac{1}{\sqrt{1+x^2}};\ f_3(x)=\tan\left(\frac{1-x}{1+x}\right)$$

$$f_4(x)=\ln\left(\frac{1+\sin x}{\cos x}\right)\ f_5(x)=\sqrt[3]{\cos^2 x+1};\ f_6(x)=\cos^2\left(\ln\frac{1}{x}\right);$$

Exercise 17 – Twice differentiable

Let $f:\mathbb{R}\to\mathbb{R}$ be the function

$$f(x)=\begin{cases}(x^x)^x & \text{if } x>0\\ 1 & \text{if } x\leq 0\end{cases}$$

1-Show that f is differentiable over $\mathbb{R}$ and compute it's derivative f'.
2- Is f in $\mathcal{C}^1(\mathbb{R})$.
3- Is f twice differentiable over $\mathbb{R}$.

Exercise 18 – n-th derivatives

Compute the n-th derivatives of the following functions :

$$f_1(x)=\frac{1}{x+1},\quad f_2(x)=\frac{1+x}{1-x},\quad f_3(x)=\ln(1+x),\ f_4(x)=\sin^2 x,\quad f_5(x)=\left(1+x^2\right)e^x.$$

Exercise 19 – Rolle's Theorem

Can we apply Rolle's Theorem in the following cases

$$f_1(x)=|x-1|\ \text{ sur } I=[0,2]\ ;\ f_2(x)=\sqrt[3]{(x-2)^2}\text{ on } I=[0,4]\quad f_3(x)=\begin{cases}x,\text{ si } & x\leq 1\\ 2-x,\ \text{si } x>1\end{cases}\ \text{ sur }\ I=[0,2].$$

Exercise 20 – Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and differentiable on $]a, b[$. Let h be the function

$$\begin{array}{rcl} h : [a,b[& \to & \mathbb{R} \\ x & \mapsto & (f(x) - f(a))^{\frac{1}{e^{x-b}}} \end{array}$$

1. Show that h can be extendable by continuity on $[a, b]$.

2. Show that there exists $c \in]a, b[$ such that $f'(c) = \dfrac{f(c) - f(a)}{(c - b)^2}$.

<u>Exercise 21</u> – Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be a twice derivable function such that

$$f(a) = f'(a) \text{ and } f(b) = f'(b).$$

Show that there exists $c \in]a, b[$ such that $f(c) = f''(c)$.

<u>Exercise 22</u> – MVT

Use the mean value theorem to prove the following inequalities

1. $\forall x \in]-1, +\infty[,\ \dfrac{x}{1+x} \leq \ln(1+x) \leq x$.
2. $\forall x, y \in \mathbb{R},\ |\sin x - \sin y| \leq |x - y|$
3. $\forall x \in]0, \frac{\pi}{4}[,\ |\tan x - \sin x| < 2|x|$

<u>Exercise 23</u> – MVT

We take 100 as an approximate value of $\sqrt{10001}$. Using the mean value theorem, provide an upper bound for the error committed.

<u>Exercise 24</u> – Hospital's rule

Using Hospital's rule, calculate the following limits :

$$\lim_{x\to 1} \frac{x^P - 1}{x^q - 1}\ (p, q \in \mathbb{N}^*)\ ;\ \lim_{x\to 0} \frac{\sqrt[p]{1 + x^P} - 1}{\sqrt[q]{1 + x^q} - 1}\ (p, q \in \mathbb{N}^*)\ ;\ \lim_{x\to+\infty} \frac{\sqrt[p]{1 + x} + 1}{\sqrt[q]{1 + x} - 1}\ (p, q \in \mathbb{N}^*)\ ;\ \lim_{x\to 0} \frac{\tan(2x)}{\sin x};\ .$$

<u>Exercise 25</u> – Extrema

Find the extrema of the following functions :

(1) $f(x) = x^5 - 5x + 2$ on the segment $[-4; 6]$.

(2) $f(x) = x \sin \frac{1}{x}$ on the segment $\left[\frac{\pi}{2}, 2\pi\right]$.

(3) $f(x) = x^{\ln(\frac{1}{x})}$ on the segment $[\frac{1}{2}, 4]$.