Logic, sets and maps

1. Logic

1.1. Propositions.

Definition 1. *A mathematical* **proposition** *(or statement) is a declarative sentence that is true or false, not both at the same time.*

Remark 1. *Let P be a proposition. If P is true (resp. false), we attribute to P the value 1(resp. 0) or the letter T (resp. F).*

Examples

- $1 < 2$ is a true proposition.
- $5 + 3 = 7$ is a false proposition.
- $x \leq 2$ is not a proposition.
- What is your name is not a proposition.

- Let us go to Annaba is not a proposition.

Definition 2. *An **axiom** is a proposition assumed to be true and can use as a basis for proving theorems.*

Examples

- Through two points passes one and only one straight line.
- Every element is equal to itself.

Definition 3. *A **theorem** any proposition that we prove true.*

Example

- The set of prime numbers is infinite

Definition 4. *A **corollary** is any proposition that can be proved as a consequence of a theorem that has just been proved*

Example

- The set of odd numbers is infinite

Definition 5. *A **lemma** any theorem used to prove other theorems.*

Definition 6. *A **conjecture** any mathematical assertion proposed to be true, but that has not been proved.*

1.2. Symbols and logical connectors (Operators).

Let P and Q be two propositions,

the logical connectors allow to define a new propositions using P and Q. The usual logical connectors are: not, and, or, if then, if and only if. A proposition that is constructed using other propositions is called a **compound proposition**.

Negation

Definition 7. *Negation (not or , ¬) is an operator used to modify the truth value of a proposition. The negation of the proposition P is the proposition denoted $\overline{P}$ (read not P) which is true when P is false and false when P is true.*

The truth table of $\overline{P}$ is:

P	$\overline{P}$
1	0
0	1

Conjunction and Disjunction

Definition 8. *A logical conjunction of two propositions P and Q is the proposition denoted $P \wedge Q$ (read P and Q), which is true if P and Q are both true and is false otherwise.*

Definition 9. *A logical disjunction of two propositions P and Q is the proposition denoted $P \vee Q$ (read P or Q), which is true if at least one of the two propositions P or Q is true.*

The truth tables of $P \wedge Q$ and $P \vee Q$ are:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples

- $(1 = 4) \vee (2 - 1 = 0)$ is a false proposition.
- $(1 \neq 2) \vee (1 < 6)$ is a true proposition.
- $(5^2 \geq 0) \wedge (\frac{-2}{5}$ is not an integer $)$ is a true proposition.
- $(5 < 0) \wedge (-1$ is an integer $)$ is a false proposition.

Conditional (Implication)

Definition 10. *A logical conditional of two propositions P and Q is the proposition $\bar{P} \vee Q$, denoted by $P \Rightarrow Q$, (read "P implies Q").*

The truth table of $P \Rightarrow Q$ is:

P	Q	$\bar{P}$	P $\Rightarrow$ Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Examples

Consider the following propositions

A : If $1 + 4 = 5$ then the capital of Algeria is Algiers.

S: If the sun is a star then $1 + 2 = 5$.

T : If $2 = 3 \Rightarrow 4 = 5$.

U: If earth is not a planet then $7 > 3$

The propositions A, T and U are true. The proposition S is a false .

Remark 2.

1-The conditional statement $P \Rightarrow Q$ is false only if P is true and Q is false.

2-The conditional statement $P \Rightarrow Q$ can also be read: if P then Q or it is only necessary P for than Q.

3-In the conditional statement $P \Rightarrow Q$, P called **antecedent** *and Q called* **consequent**.

4-The converse of $P \Rightarrow Q$is $Q \Rightarrow P$.

5-The contrapositive of $P \Rightarrow Q$ is $\bar{Q} \Rightarrow \bar{P}$

Biconditional

Definition 11. *A logical biconditional (equivalence) of two propositions P and Q is the proposition $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ denoted as $P \Leftrightarrow Q$, and which reads P if and only if Q.*

The truth table of $P \Leftrightarrow Q$ is:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Remark 3.

1-The biconditional $P \Leftrightarrow Q$ is true only when P and Q are both true or both false .

2-The proposition $P \Leftrightarrow Q$ also reads "P is a necessary and sufficient condition for Q" or "P is equivalent to Q"

Definition 12.

1. *A **tautology** is a statement that is always true independently of its components.*
2. *A **contradiction** is a statement that is always false independently of its components.*
3. *A **predicate** is a mathematical statement or assertion that contains variables that are not defined or specified, which leaves the truth value of the statement undefined.*

Examples

1-The proposition $(1 < 5) \Leftrightarrow (-1 < 0)$ is a true proposition.

2-The proposition $(1 + 4 = 6) \Leftrightarrow$ "Kouba is in Algiers" is a false proposition.

3-The proposition P or $\neg P$ is a tautology.

4. The proposition $P \wedge \neg P$ is a contradiction.

5- x is an integer and $x < 5$ is a predicate.

Properties of logical connectors

Let P, Q and R be three propositions. So:

(1) $(\neg(\neg P)) \Leftrightarrow P$, $(P \wedge P) \Leftrightarrow P$ and $(P \vee P) \Leftrightarrow P$

(2) $Q \vee R \Leftrightarrow R \vee Q$ and $Q \wedge R \Leftrightarrow R \wedge Q$

(3) $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$,

(4) $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ and $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$,

(5) $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$,

(6) $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ and$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$.

(7) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ and $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.

Proof. Each of these can be proved by the truth-table. ∎

1.3. Quantifiers.

Some notions of set theory

A set is any collection of objects, these objects are called elements . The elements in a set can be defined by a condition, or by a common characteristic, or by enumeration,... A set is denoted by a capital letter: A, B, C and D....etc, and the elements in a set are designated by lower-case letters: x, y and z...etc. we write $x \in S$ if x is an element of a set S and we say x belongs to the set S: We write $x \notin S$ if x is not an element of S and we say x does not belong to S.

Definition 13.
In mathematics, the expressions "for all" and "There exists or it exists " are denoted by the symbols $\forall$ and $\exists$ called respectively universal quantifier and existential quantifier.

Remarks

Let $P(x)$ be a predicate depending on x.

- $\forall x \in E, P(x)$ is true when, for all $x \in E$, the proposition $P(x)$ is true.

- $\exists x \in E, P(x)$ is true when there is at least one $x \in E$ such that the proposition $P(x)$ is true.

- The symbol $\exists$ means: it exists at least.
- The symbol $\exists!$ means: there exists a unique.
- The order of the two quantifiers $\forall$, $\exists$ are important; for example, the following two propositions are different :

$\forall a \in \mathbb{N}, \exists b \in \mathbb{N}, a < b$ (this is a true proposition).

$\exists b \in \mathbb{N}, \forall a \in \mathbb{N}, a < b$ (this is a false proposition).

Properties

Let $P(x)$, $Q(x)$ and $L(x, y)$ be predicates. We have:

(1) $\neg(\forall x \in E, P(x)) \Leftrightarrow \exists x \in E, \neg P(x)$

(2) $\neg(\exists x \in E, P(x)) \Leftrightarrow \forall x \in E, \neg P(x)$

(3) $\exists x \in E, (P(x) \wedge Q(x)) \Rightarrow (\exists x \in E, P(x)) \wedge (\exists x \in E, Q(x))$

(4) $\forall x \in E, P(x) \vee \forall x \in E, Q(x) \Rightarrow \forall x \in E, (P(x) \vee Q(x))$

(5) $\exists x \in E, (P(x) \vee Q(x)) \Leftrightarrow (\exists x \in E, P(x)) \vee (\exists x \in E, Q(x))$

(6) $\forall x \in E, (P(x) \wedge Q(x)) \Leftrightarrow (\forall x \in E, P(x)) \wedge (\forall x \in E, Q(x))$

(7) $\forall x \in E, \forall y \in E, L(x, y) \Leftrightarrow \forall y \in E, \forall x \in E, L(x, y)$

(8) $\exists x \in E, \forall y \in E, L(x,y) \Rightarrow \forall y \in E, \exists x \in E, L(x,y)$

1.4. **Methods of Proof.**

Proof by Cases

To show a proposition $(P(x), \forall x \in E = E_1 \cup \cdots \cup E_n)$, we separates the following reasoning that $x \in E_1, x \in E_2, \ldots, x \in E_n$.

Example

To prove that $n^2 + n$ is even, we can use proof by cases.

Proof. Proof by cases: **Case 1:** If n is even. We assume that n is an even integer and can be expressed as $n = 2k$ for some integer k. Let us substitute the value of n into the expression n^2+n. Therefore, $n^2+n = (2k)^2+2k = 4k^2+2k = 2(2k^2+k)$. As $2k^2+k$ is an integer, we can deduce that n^2+n is even in this instance.

Case 2: Assuming n is an odd integer, we can write $n = 2k+1$ for some integer k. Substituting this into the expression n^2+n, we get $n^2+n = (2k+1)^2+2k+1 = 4k^2+4k+1+2k+1 = 4k^2+6k+2 = 2(2k^2+3k+1)$. As $2k^2+3k+1$ is an integer, we can conclude that n^2+n is even.

Since n^2+n is even in both Case 1 (when n is even) and Case 2 (when n is odd), we have covered all possible cases for n. Therefore, we can deduce that for all integers n, n^2+n is even. ▮

Proof by Contrapositive

To show the implication $P \Rightarrow Q$, it is equivalent to show its contrapositive $\bar{Q} \Rightarrow \bar{P}$ which is sometimes easier and faster.

Example

Show that if a^2 is even then a is even.

Let $a \in \mathbb{N}, a \geq 2$. Show that if a^2 is even then a is even. Using contrapositive reasoning, we show that if a is odd then a^2 is odd. Suppose therefore that a is odd, we then have:

$$\begin{aligned} a \text{ odd } &\Rightarrow a = 2k+1, k \in \mathbb{N} \\ &\Rightarrow a^2 = 4k^2 + 4k + 1 \\ &\Rightarrow a^2 = 2\left(2k^2 + 2k\right) + 1 \\ &\Rightarrow a^2 = 2h + 1, h = 2k^2 + 2k \in \mathbb{N} \\ &\Rightarrow a^2 = 2h + 1, h \in \mathbb{N} \\ &\Rightarrow a^2 \text{ odd} \end{aligned}$$

Hence: $(a \text{ odd }) \Rightarrow (a^2 \text{ odd })$ and this is equivalent to $(a^2 \text{ even }) \Rightarrow (a \text{ even})$.

Deductive Proof **Direct Proof**

To prove that the proposition $P \Rightarrow Q$ is true, we assume that P is true, and we use various properties to establish that Q is true. So this method is based on the tautology $P \wedge (P \Rightarrow Q) \Rightarrow Q$.

There are two methods to prove that $P \Leftrightarrow Q$:

1- We prove that $P \Rightarrow Q$. Then, we show that $Q \Rightarrow P$.

2- We go from P to Q by equivalences.

Examples

- Show that a is even then a^2 is even.
- Show that a is even if and only if a^2 is even.

Proof. Let us assume that a is an even number. By definition, an even number is a number that can be expressed as $2k$, where k is an integer. So, we can write $a = 2k$, where k is an integer. Now, let's calculate a^2:

$$a^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

We can write $a^2 = 2m$, where $m = 2k^2$, which is an integer. Therefore, if a is even, then a^2 is also even. This completes the proof; if we show the second implication, we prove the equivalence.

Proof by Contradiction (Indirect Proof)

To prove the truth of the proposition P, we use reasoning by contradiction, which involves assuming that P is false and following the logical implications until we reach a contradiction. A contradiction is a proposition of the form $R \wedge \neg R$.

Example

Let us show that $\sqrt{2} \notin \mathbb{Q}$

Assume that $\sqrt{2} \in \mathbb{Q}$. We then have: $\sqrt{2} = \frac{a}{b}, a, b \in \mathbb{N}, b \neq 0$ with $\text{gad}(a, b) = 1$. Therefore:

$$
\begin{aligned}
a = b\sqrt{2} \Rightarrow a^2 &= 2b^2 \\
&\Rightarrow a^2 \text{ is even} \\
&\Rightarrow a \text{ is even} \qquad \text{(from the previous example)} \\
&\Rightarrow a = 2k, k \in \mathbb{N} \\
&\Rightarrow (2k)^2 = 2b^2 \quad \left(\text{ because a } = 2\text{k and } a^2 = 2b^2\right) \\
&\Rightarrow 4k^2 = 2b^2 \\
&\Rightarrow 2k^2 = b^2 \\
&\Rightarrow b^2 \text{ is even} \\
&\Rightarrow b \text{ is even} \\
&\Rightarrow b = 2h, h \in \mathbb{N}
\end{aligned}
$$

Hence $a = 2h$ and $b = 2k$ with $h, k \in \mathbb{N}$. So 2 divides a and b consequently $\gcd(a, b) \neq 1$ and since $\gcd(a, b) = 1$ then we have a contradiction. So, we have $\sqrt{2} \notin \mathbb{Q}$.

Proof by Induction

To prove the validity of the property P_n for all natural numbers $n \geq n_0$, the principle of induction can be applied as follows.

First, check that the property holds for $n = n_0$, which means that P_{n_0} is true, this step is known as the base case.

Second, assume that P_i is true for all i from n_0 to n. This step is called the induction hypothesis, where $P_{n_0}, P_{n_0+1}, \ldots, P_n$ are true.

Inductive step: Show that P_{n+1} is true by assuming that P_n is true. In other words, prove this implication (P_n true) $\Rightarrow$ (P_{n+1} true).

By following these steps it can be deduced that P_n is true for all $n \geq n_0$ by the principle of induction.

Example

Let us demonstrate the following property : $\forall n \geq 1, \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Base case: when $n = n_0 = 1$, we obtain:

$$1 = \frac{1(1+1)}{2} = 1$$

Hence, the property is true for $n = 1$.

Inductive hypothesis: Assume that the property holds up to order n, i.e., we have:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$