Solutions to "Solution Cours.pdf" Exercises

Source: uploaded file. :contentReference[oaicite:1]index=1

Exercise 1

Consider the vector field

$$\mathbf{F}(x,y,z) = (\,2xy + z^3\,)\,\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}.$$

(Problem statement: see source. :contentReference[oaicite:2]index=2)

(a) Check whether $\nabla \times \mathbf{F} = 0$

Compute the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}.$$

Hence the components are

$$\begin{aligned}
(\nabla \times \mathbf{F})_x &= \partial_y(3xz^2) - \partial_z(x^2) = 0 - 0 = 0, \\
(\nabla \times \mathbf{F})_y &= \partial_z(2xy + z^3) - \partial_x(3xz^2) = 3z^2 - 3z^2 = 0, \\
(\nabla \times \mathbf{F})_z &= \partial_x(x^2) - \partial_y(2xy + z^3) = 2x - 2x = 0.
\end{aligned}$$

Therefore $\nabla \times \mathbf{F} = \mathbf{0}$, so $\mathbf{F}$ is conservative (in any simply connected domain).

(b) Find the scalar potential Φ with $\mathbf{F} = \nabla\Phi$

Here the scalar potential Φ is called potential energy $\Phi = -U(x,y,z)$, so that $\mathbf{F} = -\nabla U(x,y,z)$ We seek $U(x,y,z)$ such that

$$-\frac{\partial U}{\partial x} = 2xy + z^3, \qquad -\frac{\partial U}{\partial y} = x^2, \qquad -\frac{\partial U}{\partial z} = 3xz^2.$$

Integrate $\partial U/\partial y = -x^2$ with respect to y:

$$U(x,y,z) = -x^2 y + g(x,z),$$

where g is an arbitrary function of x and z. Differentiate this expression with respect to x:

$$\frac{\partial U}{\partial x} = -2xy + \frac{\partial g}{\partial x}.$$

Comparing with the required $\partial U/\partial x = -2xy - z^3$ gives

$$\frac{\partial g}{\partial x} = -z^3 \quad \Longrightarrow \quad g(x,z) = -xz^3 + h(z),$$

with $h(z)$ arbitrary. Finally require $\partial U/\partial z = -3xz^2$:

$$\frac{\partial U}{\partial z} = -3xz^2 + h'(z) \stackrel{!}{=} -3xz^2 \Longrightarrow h'(z) = 0.$$

Hence h is constant and we may take it zero. A valid potential is

$$\boxed{U(x, y, z) = -x^2y - xz^3 + C, \qquad (C \text{ constant}).}$$

Exercise 2

Consider $\mathbf{F}(x, y) = (2xy^2)\,\mathbf{i} + (2x^2y)\,\mathbf{j}$ and the motion of a particle from $A(0, 0)$ to $B(3, 3)$ in the plane. (Problem statement: see source. :contentReference[oaicite:3]index=3)

(1) Work along various paths

First check whether $\mathbf{F}$ is conservative. Compute the curl in 2D (or the z-component of the 3D curl):

$$\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(2xy^2) = 4xy - 4xy = 0.$$

So $\mathbf{F}$ is conservative. Find a potential $\Phi(x, y)$:

$$-\frac{\partial U}{\partial x} = 2xy^2 \implies U(x, y) = -x^2y^2 + g(y).$$

Differentiate w.r.t. y:

$$-\frac{\partial U}{\partial y} = -2x^2y + g'(y) \stackrel{!}{=} -2x^2y \Longrightarrow g'(y) = 0.$$

Thus a potential is $U(x, y) = -x^2y^2 + C$.

For any path from A to B the work is path independent:

$$W_{A\to B} = -(U(B) - U(A)) = -(U(3, 3) - U(0, 0)) = -(-(3^2)(3^2) - 0) = 9 \cdot 9 = 81.$$

Now answer subparts:

(a) Along the segment $A \to B$: $W = +81$.

(b) Along the segment $B \to A$: $W = -81$.

(c) Along the closed path $A \to B \to A$: $W = 81 + (-81) = 0$ (zero for any closed path since conservative).

(d) Along $(0, 0) \to (3, 0) \to (3, 3)$: by path independence $W = +81$.

(e) Along $(3, 3) \to (0, 3) \to (0, 0)$: this goes from B to A so $W = -81$.

(2) Curl

As computed above the curl vanishes:

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0}.}$$

(3) Conclusion

Because $\nabla \times \mathbf{F} = 0$ (in the domain considered) the force is conservative and the work between two fixed endpoints is independent of the path.

Exercise 3 (roller coaster loop)

A frictionless roller coaster track contains a vertical circular loop of radius $R = 20.0$ m. The car just barely makes the loop: at the top of the loop the normal force is zero (riders feel weightless). (Problem statement: see source. :contentReference[oaicite:4]index=4)

(1) Speed at the top (point C)

At the top the required centripetal acceleration is supplied solely by gravity:

$$\frac{mv_C^2}{R} = mg \quad \Longrightarrow \quad v_C^2 = gR.$$

Hence

$$\boxed{v_C = \sqrt{gR}\,.}$$

Numerically (with $g = 9.81$ m/s^2, $R = 20.0$ m):

$$v_C = \sqrt{9.81 \times 20} = \sqrt{196.2} \approx 14.00 \text{ m/s}.$$

(2) Speed at the bottom (point A)

Use energy conservation between bottom A (height 0) and top C (height $2R$):

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_C^2 + mg(2R) \quad \Longrightarrow \quad v_A^2 = v_C^2 + 4gR.$$

With $v_C^2 = gR$ this gives $v_A^2 = 5gR$, so

$$\boxed{v_A = \sqrt{5gR}\,.}$$

Numerically:

$$v_A = \sqrt{5 \times 9.81 \times 20} = \sqrt{981} \approx 31.32 \text{ m/s}.$$

(3) Speed halfway up the loop (point B)

If B is at height R above the bottom, then

$$v_B^2 = v_A^2 - 2gR = 5gR - 2gR = 3gR,$$

so

$$\boxed{v_B = \sqrt{3gR}\,.}$$

Numerically:

$$v_B = \sqrt{3 \times 9.81 \times 20} \approx 24.27 \text{ m/s}.$$

(4) Height difference between A and D if $v_D = 10.0$ m/s

Let the vertical difference h_{AD} be the height of D above A. Using energy from A to D:

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_D^2 + mgh_{AD} \quad \Rightarrow \quad h_{AD} = \frac{v_A^2 - v_D^2}{2g}.$$

With $v_A^2 = 5gR$ (from above) and $v_D = 10.0$ m/s:

$$h_{AD} = \frac{5gR - (10.0)^2}{2g} = \frac{5gR}{2g} - \frac{100}{2g} = \frac{5R}{2} - \frac{50}{g}.$$

Numerically (with $R = 20.0$ m, $g = 9.81$ m/s^2):

$$h_{AD} = \frac{5 \times 20}{2} - \frac{50}{9.81} = 50 - 5.097 \approx 44.90 \text{ m}.$$

(Interpretation: D would be ≈ 44.9 m above A.)

Exercise 4 (pendulum)

A bob of mass $m = 20$ g $= 0.020$ kg is attached to a massless string of length $L = 0.40$ m. The bob is released from rest at angle $\theta_0 = 45°$ from the vertical; neglect air resistance. (Problem statement: see source. :contentReference[oaicite:5]index=5)

(1) Free-body diagram

At any instant the forces on the bob are:

- The weight: $\mathbf{W} = m\mathbf{g}$ (vertically downward).
- The tension: $\mathbf{T}$ along the string directed toward the pivot O.

(The tension is always radial; a sketch should show $\mathbf{T}$ along the string and $\mathbf{W}$ downward.)

(2) Work done by the weight from initial position to the vertical

The bob drops by a vertical distance

$$\Delta h = L(1 - \cos\theta_0).$$

The work done by gravity (weight) when the bob moves from initial release to the bottom (vertical) equals the loss of gravitational potential energy (positive):

$$\boxed{W_{\text{weight}} = mg\Delta h = mgL(1 - \cos\theta_0)\,.}$$

Numerically, with $m = 0.020$ kg, $g = 9.81$ m/s^2, $L = 0.40$ m, $\cos 45^\circ = \frac{\sqrt{2}}{2} \approx 0.7071067$:

$$\Delta h = 0.40(1 - 0.7071067) \approx 0.11716 \text{ m},$$

$$W_{\text{weight}} \approx 0.020 \times 9.81 \times 0.11716 \approx 0.0230 \text{ J}.$$

(3) Work done by the tension

The tension is always radial while the displacement of the bob along the arc is tangential; hence the tension does no work:

$$\boxed{W_{\mathrm{T}} = 0.}$$

(4) Angular momentum of the bob relative to O at the bottom

The speed at the bottom can be found using energy:

$$\frac{1}{2}mv^2 = mg\Delta h \quad \Rightarrow \quad v = \sqrt{2g\Delta h}.$$

Using the numbers above:

$$v \approx \sqrt{2 \times 9.81 \times 0.11716} \approx 1.516 \text{ m/s}.$$

The angular momentum magnitude about O at the bottom (radius $r = L$, momentum tangential) is

$$\boxed{L_O = mvL \approx 0.020 \times 1.516 \times 0.40 \approx 0.0121 \text{ kg}\cdot\text{m}^2/\text{s}.}$$

Exercise 5

A block of mass M is pushed along a horizontal surface by a force $\mathbf{F}$ that makes an angle θ with the *vertical* (so the vertical component is $F\cos\theta$ upward and the horizontal component is $F\sin\theta$ to the right). The coefficients of static and kinetic friction are equal, μ.

(a) Free-body and magnitudes of forces

Vertical forces (taking upward positive):

$$N + F\cos\theta - Mg = 0 \quad \Longrightarrow \quad \boxed{N = Mg - F\cos\theta\,.}$$

Horizontal forces (to the right positive): horizontal component of the push $F_x = F\sin\theta$ and friction f (opposes motion). If the block moves right the kinetic friction magnitude is

$$\boxed{f_k = \mu N = \mu\big(Mg - F\cos\theta\big)\,.}$$

(b) Work done when the block moves a distance B to the right

The displacement is purely horizontal by distance B.

- Work done by the applied force $\mathbf{F}$: only the horizontal component does work:

$$W_F = F\sin\theta \cdot B.$$

- Work done by friction:

$$W_{\text{fric}} = -\,f_k \cdot B = -\mu\big(Mg - F\cos\theta\big)B.$$

- Work done by the normal force N: $W_N = 0$ (normal is perpendicular to displacement).
- Work done by gravity: $W_g = 0$ (gravity is vertical; displacement is horizontal).

(c) Change of kinetic energy: relation between V_1 and V_2

Use the work–energy theorem for mass M:

$$\frac{1}{2}MV_2^2 - \frac{1}{2}MV_1^2 = W_{\text{net}} = W_F + W_{\text{fric}}.$$

Substitute the works:

$$\frac{1}{2}M(V_2^2 - V_1^2) = \Big[F\sin\theta - \mu\big(Mg - F\cos\theta\big)\Big]B.$$

Hence

$$\boxed{V_2^2 = V_1^2 + \frac{2B}{M}\Big[F\sin\theta - \mu\big(Mg - F\cos\theta\big)\Big].}$$

(d) Decide whether V_2 is larger, equal or smaller than V_1

Compare the sign of the net work:

$$W_{\text{net}} = \Big[F\sin\theta - \mu\big(Mg - F\cos\theta\big)\Big]B.$$

- If $F\sin\theta > \mu\big(Mg - F\cos\theta\big)$ then $W_{\text{net}} > 0 \Rightarrow V_2 > V_1$.
- If equality holds then $W_{\text{net}} = 0 \Rightarrow V_2 = V_1$.
- If $F\sin\theta < \mu\big(Mg - F\cos\theta\big)$ then $W_{\text{net}} < 0 \Rightarrow V_2 < V_1$.

End of solutions.