Exercise 1. Determine if the following statements are true or false. Justify briefly.

1. Every reflexive and transitive relation is an order relation.
2. The relation $\mathcal{R}$ on $\mathbb{Z}$ defined by $a\mathcal{R}b \iff a-b$ is odd, is an equivalence relation.
3. In a total order, every pair of elements is comparable.
4. The divisibility relation on $\mathbb{N}^*$ is a total order.

Exercise 2. For each binary relation $\mathcal{R}$ on E below, determine whether it is **reflexive**, **symmetric**, **antisymmetric**, or **transitive**. Specify if it is an order relation or an equivalence relation.

1. $E=\mathbb{R}$, $a\mathcal{R}b \iff a \leq b$
2. $E=\mathbb{Z}$, $a\mathcal{R}b \iff a+b$ is even
3. $E=\mathbb{N}^*$, $a\mathcal{R}b \iff a \mid b$
4. $E=\mathcal{P}(X)$, $A\mathcal{R}B \iff A \subset B$
5. $E=\mathbb{R}^2$, $(x,y)\mathcal{R}(z,t) \iff x=z$

Exercise 3. Let $E=\mathbb{Z}$ and $\mathcal{R}$ be defined by:

$$a\mathcal{R}b \iff a^2 \equiv b^2 \pmod 3$$

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Determine the equivalence classes of 0, 1, and 2.
3. Give the quotient set $E/\mathcal{R}$.

Exercise 4. Let $E=\{1,2,3,4,6,12\}$ with the divisibility relation:

$$a \preceq b \iff a \mid b$$

1. Show that $\preceq$ is an order relation.
2. Is this order total?

Exercise 5. Let $E=\mathbb{R}^2$ and define:

$$(x_1,y_1)\mathcal{R}(x_2,y_2) \iff x_1^2+y_1^2=x_2^2+y_2^2$$

1. Show this is an equivalence relation.
2. Describe the equivalence class of $(1,0)$.
3. Describe the equivalence class of $(0,0)$.

Exercise 7. Define the binary relation Δ on $\mathbb{R}$ as follows:

$$\forall x,y \in \mathbb{R}, \quad x\Delta y \Leftrightarrow x^3-y^3=x-y$$

1. Show that Δ is an equivalence relation.
2. Find the equivalence classes of 2, (-1), and of a where $a \in \mathbb{R}$.

Exercise 8. Define the binary relation S on $\mathbb{R}^2$ as follows:

$$\forall (x,y),(x',y') \in \mathbb{R}^2, \quad (x,y)S(x',y') \Leftrightarrow |x-x'| \leq y'-y$$

Show that S is an order relation. Is this order total?