worksheet 1 (Analyse maths I)

1 Set of Real Numbers

Exercice 1

1. *Proof that if* $r \in \mathbb{Q}$ *and* $x \notin \mathbb{Q}$ *then* $r + x \notin \mathbb{Q}$ *and if* $r \neq 0$ *then* $rx \notin \mathbb{Q}$.
2. *Proof that between two rational numbers there is always an irrational number.*

Exercice 2
Proof that $\frac{\ln 3}{\ln 2}$ *is irrational.*

Exercice 3
Soit $M = 0.2015201520152015\cdots$ *Give the rational whose decimal writing is* M.

Exercice 4
1. *Proof that* $2^{n-1} \leq n! \leq n^{n-1}$.
2. *Proof that for all natural numbers* $n \geq 1$ *and for all positive real numbers* $x_1, ..., x_n$, *we have*

$$\prod_{k=1}^{n}(1 + x_k) \geq 1 + \sum_{k=1}^{n} x_k.$$

Exercice 5
Proof the following inequalities

1. *For all* $x, y \in \mathbb{R}_+$: *we have* $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.
2. *For all* $x, y \in \mathbb{R}_+$: *we have* $\sqrt{xy} \leq \frac{x+y}{2}$.
3. *For all* $x, y \in \mathbb{R}_+$: *we have* $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$.

Exercice 6
Let E *the set of affine functions defined on* $\mathbb{R}$. *Let* $\preceq$ *the binary relation defined on* E *as following : for* f *and* g *from the set* E *such that* $f(x) = ax + b$ *and* $g(x) = a'x + b'$, *we have*

$$(f \leq g) \Longleftrightarrow [(a < a')or\ (a = a'\ et\, b \leq b')].$$

Is the binary relation $\preceq$ *is a total order relation?*

2 The upper and lower bounds, Maximum and minimum element

Exercice 7

Let $f(x) = \frac{x^2+2x+1}{x^2+2x+4}$, $x \in \mathbb{R}$. *Determine* $sup f$ *and* $inf f$ *on* $\mathbb{R}$.

Exercice 8
Determine (if it exists) : The upper and lower bounds, the supremum, the infimum, the maximal and minimal element of the following set

$$A = [2, 3[, \quad B = [0, 1] \cap \mathbb{Q}, \quad C =]0, 1[\cap \mathbb{Q}, \quad \mathbb{N}, \quad D = \{(-1)^n + \frac{1}{n^2},\ n \in \mathbb{N}\}, \qquad E = \{-\frac{1}{x}/1 < x < 2\}$$

Exercice 9
Let a, b *be any two real numbers. Show that :*

1. $\sup(a,b) = \max(a,b) = \frac{1}{2}(a+b+\mid a-b \mid)$
2. $\inf(a,b) = \min(a,b) = \frac{1}{2}(a+b-|a-b|)$

Exercice 10

1. *Let A a non-empty bounded above subset of $\mathbb{R}$. Suppose that the supremum of the set A is strictly positive. Proof that there exists a strictly positive element of A.*
2. *Let a a positive real number. Proof that for all $\varepsilon > 0 : a < \varepsilon \Longrightarrow a = 0$.*

Exercice 11
Let A and B two nonempty bounded subset of $\mathbb{R}$, true or false?

1. *If $A \subset B$ then $\sup A \leq \sup B$.*
2. *If $A \subset B$ then $\inf A \leq \inf B$.*
3. $\sup(A \cup B) = \max(\sup A, \sup B)$.
4. $\inf(A \cup B) = min(\inf A, \inf B)$.
5. *If $A \cap B \neq \emptyset$ then $\sup(\inf A, \inf B) \leq \inf(A \cap B)$.*
6. *If $A \cap B \neq \emptyset$ then $\sup(A \cap B) \leq \inf(\sup A, \sup B)$.*
7. *On désigne par $A + B = \{a + b$ tel que $a \in A, b \in B\}$ then*
$$\sup(A + B) = \sup(A) + \sup(B).$$
8. *We designate by $(-A) = \{-x/x \in A\}$, then :*
$$\sup(-A) = -\inf A \text{ et } \inf(-A) = -\sup A.$$

Exercice 12
Let A, E and F three sets such that $A \subseteq E \subseteq F$. Suppose that A have a supremum into E and into F. then $\sup_E(A) \geq \sup_F(A)$ is it true or false.

3 Greatest Integer, floor function

Exercice 13
Proof the following results

1. *For all $x \in \mathbb{R}$, we have $E(x + 1) = E(x) + 1$.*
2. *For all t $x, y \in \mathbb{R}$, we have $x \leq y \Longrightarrow E(x) \leq E(y)$.*

Exercice 14 (Sum)
Let $x \in \mathbb{R}$. Note by $E(x)$ the greatest Integer of x.

1. *Proof that $E(a) + E(b) \leq E(a + b) \leq E(a) + E(b) + 1,\ a, b \in \mathbb{R}$.*
2. *Proof that $E(a + b) + 1 \leq E(a) + E(b) + 2,\ a, b \in \mathbb{R}$.*
3. *Compute for all $(m, n) \in \mathbb{Z}^2$ the sum $E(\frac{n+m}{2}) + E(\frac{n-m+1}{2})$.*

Exercice 15 (Product and division)
Let $x \in \mathbb{R}$. Let us note by $E(x)$ the greatest Integer of x. Proof that
$$E(x) = E\Big(\frac{E(n\,x)}{n}\Big)$$

Exercice 16 *Let the function f defined by $f(x) = E(2x) - 2E(x)$.*

1. *Compute $f(x)$ for $x \in [0, \frac{1}{2}[$ and for $x \in [1/2, 1[$. Deduce that for all $x \in \mathbb{R}$*
$$0 \leq E(2x) - 2E(x) \leq 1.$$
2. *Proof that $0 \leq E(2x) - 2E(x) \leq 1$.*

Exercice 17 *Solve the equation*
$$E(2x + 3) = E(x + 2).$$
Study the function $g(x) = E(2x) + 1,\ x \in \mathbb{R}$.

4 Absolute value

Exercice 18
Proof that for all $x, y \in \mathbb{R}$, we have :

1. $\mid x \mid + \mid y \mid \leq \mid x + y \mid + \mid x - y \mid$.
2. $1+ \mid xy - 1 \mid \leq (1+ \mid x - 1 \mid)(1+ \mid y - 1 \mid)$.
3. $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$.

Exercice 19 (Cauchy Shwartz Inequality)

Let x_i and y_i, with $i = 1, 2, ..., n$ and $n \in \mathbb{N}$, $2n$ real numbers. Proof that

$$\mid \sum_{i=1}^{n} x_i y_i \mid \leq \sum_{i=1}^{n} \mid x_i \mid\mid y_i \mid \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$$

Exercice 20 (Minkowski Inequality)
Let x_i and y_i, with $i = 1, 2, ..., n$ and $n \in \mathbb{N}$, $2n$ real numbers. Proof that

$$\sqrt{\sum_{i=1}^{n} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{n} (x_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i)^2}.$$