Binary Relations on a Set

Binary relations are a fundamental concept in mathematics and computer science. They allow us to define relationships between the elements of a set.

Definition 1.1. Let E be a set. A **binary relation** on E is a subset of the Cartesian product $E \times E$. In other words, a binary relation $\mathcal{R}$ on E is a set of ordered pairs (a, b), where $a, b \in E$. We write $\mathcal{R} \subseteq E \times E$.

Remark 1.1. If an ordered pair (a, b) belongs to the relation $\mathcal{R}$, we say that *"a is related to b"* and we denote this by $a\mathcal{R}b$.

Example 1.1.

1. Let $E = \mathbb{R}$. Define $a\mathcal{R}b \iff a^2 = b^2$.
2. Let $E = \mathbb{Z}^*$ (the set of non-zero integers). Define $a\mathcal{R}b \iff \exists k \in \mathbb{Z}^* : b = ka$. (This is the divisibility relation).

Properties of Binary Relations

A binary relation can be characterized by several fundamental properties.

Property 2.1 (Reflexivity). The relation $\mathcal{R}$ is **reflexive** if every element is related to itself:

$$\forall a \in E,\ a\mathcal{R}a.$$

Property 2.2 (Symmetry). The relation $\mathcal{R}$ is **symmetric** if whenever a is related to b then b is related to a:

$$\forall a, b \in E,\ a\mathcal{R}b \implies b\mathcal{R}a.$$

Property 2.3 (Antisymmetry). The relation $\mathcal{R}$ is **antisymmetric** if whenever a is related to b and b is related to a, then a must be equal to b:

$$\forall a, b \in E,\ (a\mathcal{R}b \wedge b\mathcal{R}a) \implies a = b.$$

Property 2.4 (Transitivity). The relation $\mathcal{R}$ is **transitive** if whenever a is related to b and b is related to c, then a is related to c:

$$\forall a, b, c \in E,\ (a\mathcal{R}b \wedge b\mathcal{R}c) \implies a\mathcal{R}c.$$

3 Order Relations

Definition 3.1 (Order Relation). A binary relation $\mathcal{R}$ on E is an **order relation** if it is reflexive, antisymmetric, and transitive.

Definition 3.2 (Total Order). An order relation $\mathcal{R}$ is a **total order** if every pair of elements in E is comparable:

$$\forall a, b \in E,\ a\mathcal{R}b \quad \text{or} \quad b\mathcal{R}a.$$

If there exists at least one pair of elements that is not comparable, the relation is called a **partial order**.

Example 3.1.

1. The relation "$\leq$" on $\mathbb{R}$ is a **total order**. It is reflexive ($a \leq a$), antisymmetric (if $a \leq b$ and $b \leq a$ then $a = b$), transitive (if $a \leq b$ and $b \leq c$ then $a \leq c$), and total (for any a, b, either $a \leq b$ or $b \leq a$).

2. The relation "$\subseteq$" (subset) on the power set $\mathcal{P}(A)$ of a set A is a **partial order**. It is reflexive, antisymmetric, and transitive. However, it is not a total order if A has more than one element (e.g., for $A = \{1, 2\}$, the sets $\{1\}$ and $\{2\}$ are not comparable).

3. The divisibility relation on $\mathbb{Z}^*$ (from Example 1.1) is a partial order on $\mathbb{N}^*$ but not on $\mathbb{Z}^*$ (antisymmetry fails for negative numbers, e.g., $2 \mid -2$ and $-2 \mid 2$ but $2 \neq -2$).

4 Equivalence Relations

Definition 4.1 (Equivalence Relation)**.** A binary relation $\mathcal{R}$ on E is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 4.2 (Equivalence Class)**.** Let $\mathcal{R}$ be an equivalence relation on a set E. For an element $a \in E$, the **equivalence class** of a, denoted $\dot{a}$ or $[a]$, is the set of all elements in E that are related to a:

$$\dot{a} = [a] = \{b \in E \mid b\mathcal{R}a\}.$$

The element a is called a **representative** of the equivalence class $[a]$.

Definition 4.3 (Quotient Set)**.** Let E be a set and $\mathcal{R}$ an equivalence relation on E. The **quotient set** of E by $\mathcal{R}$, denoted $E/\mathcal{R}$, is the set of all equivalence classes of the elements of E:

$$E/\mathcal{R} = \{[a] \mid a \in E\}.$$

Example 4.1. Let $E = \mathbb{R}$ and define the relation $\mathcal{R}$ by:

$$a\mathcal{R}b \iff a^2 - b^2 = a - b.$$

1. **Show that $\mathcal{R}$ is an equivalence relation.**
2. **Find the equivalence classes of 0, 1, and 2.**
3. **Find the quotient set $E/\mathcal{R}$.**