ITGD4207 Operations Research

Chapter 5

Part 2

Linear Programming

Assignment Model

Linear Programming Assignment Model

- The Assignment Model
- The Hungarian algorithm
- Examples 1,2 and 3

The Assignment Model

" The best person for job" is a description of the assignment model.

The general assignment model with n workers and n jobs is presented below:

$$
\begin{array}{cc|cccc|}
 & & \multicolumn{4}{c}{\text{Jobs}} \\
 & & 1 & 2 & \ldots. & n \\
\hline
 & 1 & c_{11} & c_{12} & \ldots & c_{1n} \\
\text{Workers} & 2 & c_{21} & c_{22} & \ldots & c_{2n} \\
 & & & & & \\
 & n & c_{n1} & c_{n2} & \ldots & c_{nn} \\
\hline
\end{array}
$$

The element c_{ij} is the "cost" of assigning the worker i to the job j. More meaningfully, it may be thought of as the time taken by the worker i to complete the job j. There is no loss of generality in assuming that the number of workers = the number of jobs. If there are more workers, we may introduce dummy jobs and if there are less workers, we may introduce dummy workers. **One important assumption we make is that each worker is assigned to one and only one job.** And each job is done by one and only one worker.

The assignment model is actually a special case of the transportation model in which the sources are workers, the destinations are jobs, and the number of sources = the number of destinations. Also from the last assumption we get that the availabilities at each source equals 1 and the demand at each destination equals 1. x_{ij} =1 if worker i assigned to job j and = 0 otherwise. Though we can apply the transportation algorithm to solve the assignment model, a special simple algorithm, called the *Hungarian algorithm*, is used to solve such models.

Consider the assignment problem:

4	0	1	2
3	5	2	0
0	4	1	5
1	2	0	3

The solution is obvious:

$W1 \rightarrow J2$, $W2 \rightarrow J4$, $W3 \rightarrow J1$, $W4 \rightarrow J3$

We note that in any assignment problem, if a constant *c* is subtracted from the costs of any row (or column), the optimum solution does not change, but the assignment cost decreases by *c*. This is because in each row we have to assign ‘1’ to one and only one cell.

The Hungarian algorithm exploits this fact and tries to get a zero in each row (column) by subtracting the minimum value in that row (column) to get at least one zero.

The Hungarian Algorithm

- Objective of Algorithm: To subtract a sufficiently large cost from the rows and columns in such a way that an optimal (assignment) can be found by inspection.

  Steps to accomplish this:

  1. Identify the smallest element in each row (column)
  2. Subtract this quantity from each element in that row (column)
  3. From the resulting costs, try to make a feasible assignment using only the cells with zero cost (at least one will exist in each row (column)). If possible, the assignment is optimal.
  4. If not,
     a) Draw the minimum number of lines in such a way that all the zeros are covered
     b) Select the smallest element that is not covered by the lines
     c) Subtract this number from all the elements that are not covered
     d) Add this number to all covered elements that are at the intersection of two lines
     e) Return to Step 3

Example1

Consider the assignment problem:

				Row Min
8	6	5	7	$p_1 = 5$
6	5	3	4	$p_2 = 3$
7	8	4	6	$p_3 = 4$
6	7	5	6	$p_4 = 5$

Step 1: From each entry of a row, we subtract the minimum value in that row and get the following reduced cost matrix:

3	1	0	2
3	2	0	1
3	4	0	2
1	2	0	1
$q_1=1$	$q_2=1$	$q_3=0$	$q_4=1$

Column Minimum

Step 2: From each entry of a column, we subtract the minimum value in that column and get the following reduced cost matrix:

2	0	0	1
2	1	0	0
2	3	0	1
0	1	0	0

Step 3: Now we test whether an assignment can be made as follows. If such an assignment is possible, it is the optimal assignment.

(a) Examine the first row. If there is only one zero in that row, surround it by a square ($\boxed{0}$) and cross ($\times$) all the other zeros in the **column** passing through the surrounded zero.

Next examine the other rows and repeat the above procedure for each row having only one zero.

If a row has more than one zero, do nothing to that row and pass on to the next row.

Step 3(a) gives the following table.

2	0	~~0~~	1
2	1	~~0~~	0
2	3	$\boxed{0}$	1
0	1	~~0~~	0

Step 3(b): Now repeat the above procedure for columns. (Remember to interchange row and column in that step.)

Step 3(b) gives the following table.

2	$\boxed{0}$	~~0~~	1
2	1	~~0~~	$\boxed{0}$
2	3	$\boxed{0}$	1
$\boxed{0}$	1	~~0~~	~~0~~

If there is now a surrounded zero in each row and each column, the optimal assignment is obtained.

In our example, there is a surrounded zero in each row and each column and so the optimal assignment is:

Worker 1 is assigned to Job 2

Worker 2 is assigned to Job 4

Worker 3 is assigned to Job 3

Worker 4 is assigned to Job 1

And the optimal cost =
$p_1+p_2+p_3+p_4+q_1+q_2+q_3+q_4= 20$

If the final stage is reached (that is all the zeros are either surrounded or crossed) and if there is no surrounded zero in each row and column, it is not possible to get the optimal solution at this stage. We have to do some more work. Again we illustrate with a numerical example.

Solve the following unbalanced assignment problem (Only one job to one man and only one man to one job):

7	5	8	4
5	6	7	4
8	7	9	8

Since the problem is unbalanced, we add a dummy worker 4 with cost 0 and get the following starting cost matrix:

7	5	8	4	$p_1=4$
5	6	7	4	$p_2=4$
8	7	9	8	$p_3=7$
0	0	0	0	$p_4=0$

Applying Step 1, we get the reduced cost matrix

3	1	4	0
1	2	3	0
1	0	2	1
0	0	0	0

Now Step 2 is Not needed. We now apply Step 3(a) and get the following table.

3	1	4	$\boxed{0}$
1	2	3	$\cancel{0}$
1	$\boxed{0}$	2	1
$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Now all the zeros are either surrounded or crossed but there is no surrounded zero in Row 2. Hence assignment is NOT possible. We go to Step 4.

Step 4 (a) We now draw minimum number *m* of vertical and horizontal lines to cover all the zeros.

Thus we draw 3 (2 vertical and one horizontal) lines to cover all the zeros and get the following table.

3	1	4	$\boxed{0}$
1	2	3	$\cancel{0}$
1	$\boxed{0}$	2	1
$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Step 4(b) Select the smallest element, say, u, from among all elements uncovered by all the lines.

In our example, $u = 1$.

Step 4(c) Now subtract this u from all uncovered elements but **add this to all elements** that lie at the intersection of two lines. Doing this, we get the table:

2	1	3	0
0	2	2	0
0	0	1	1
0	1	0	1

Step 5: Reapply Step 3.

We thus get the table

2	1	3	$\boxed{0}$
$\boxed{0}$	2	2	$\cancel{0}$
$\cancel{0}$	$\boxed{0}$	1	1
$\cancel{0}$	1	$\boxed{0}$	1

Thus the optimum allocation is:

W1 $\rightarrow$ J4 W2 $\rightarrow$ J1 W3 $\rightarrow$ J2 W4 $\rightarrow$ J3

Hence Job 3 is not done by any (real) worker.

And the optimal cost = 16 (= $p_1+p_2+p_3+p_4+u$)

Example2

Solve the Assignment Model

3	8	2	10	3	$p_1 = 2$
8	7	2	9	7	$p_2 = 2$
6	4	2	7	5	$p_3 = 2$
8	4	2	3	5	$p_4 = 2$
9	10	6	9	10	$p_5 = 6$

Applying Step 1, we get the reduced cost matrix

1	6	0	8	1
6	5	0	7	5
4	2	0	5	3
6	2	0	1	3
3	4	0	3	4

$q_1 = 1$ $q_2 = 2$ $q_3 = 0$ $q_4 = 1$ $q_5 = 1$

Applying Step 2, we get the reduced cost matrix

0	4	0	7	0
5	3	0	6	4
3	0	0	4	2
5	0	0	0	2
2	2	0	2	3

Applying Step 3, we get the table

$\boxed{0}$	4	$\cancel{0}$	7	$\cancel{0}$
5	3	$\boxed{0}$	6	4
3	$\boxed{0}$	$\cancel{0}$	4	2
5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
2	2	$\cancel{0}$	2	3

Now all the zeros are either surrounded or crossed but there is no surrounded zero in Row 5. Hence assignment is NOT possible. We go to Step 4.

$\boxed{0}$	4	$\cancel{0}$	7	$\cancel{0}$
5	3	$\boxed{0}$	6	4
3	$\boxed{0}$	$\cancel{0}$	4	2
5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
2	2	$\cancel{0}$	2	3

Step 4(a) Thus we draw 4 (3 vertical and one horizontal) lines to cover all the zeros and get the above table.

Step 4(b) The smallest element, say, u, from among all elements uncovered by all the lines is 2

Step 4(c) Now subtract this u(2) from all uncovered elements but **add this to all elements** that lie at the intersection of two lines. Doing this, we get the table:

0	6	2	9	0
3	3	0	6	2
1	0	0	4	0
3	0	0	0	0
0	2	0	2	1

We reapply Step 3.

~~0~~	6	2	9	$\boxed{0}$
3	3	$\boxed{0}$	6	2
1	$\boxed{0}$	~~0~~	4	~~0~~
3	~~0~~	~~0~~	$\boxed{0}$	~~0~~
$\boxed{0}$	2	~~0~~	2	1

Thus the optimum allocation is:

W1 $\rightarrow$ J5, W2 $\rightarrow$ J3, W3 $\rightarrow$ J2, W4 $\rightarrow$ J4, W5 $\rightarrow$ J1

And the optimal cost = 21

Example3

Solve the Assignment Model

3	9	2	3	7	$p_1 = 2$
6	1	5	6	6	$p_2 = 1$
9	4	7	10	3	$p_3 = 3$
2	5	4	2	1	$p_4 = 1$
9	6	2	4	5	$p_5 = 2$

Applying Step 1, we get the reduced cost matrix

1	7	0	1	5
5	0	4	5	5
6	1	4	7	0
1	4	3	1	0
7	4	0	2	3

$q_1 = 1$ $q_2 = 0$ $q_3 = 0$ $q_4 = 1$ $q_5 = 0$

Applying Step 2, we get the reduced cost matrix

0	7	0	0	5
4	0	4	4	5
5	1	4	6	0
0	4	3	0	0
6	4	0	1	3

Applying Step 3, we get the table

0	7	~~0~~	0	5
4	$\boxed{0}$	4	4	5
5	1	4	6	$\boxed{0}$
0	4	3	0	~~0~~
6	4	$\boxed{0}$	1	3

Remark: Now in Rows 1 and 4 (and also in columns 1 and 4) there is no single zero. Now we modify the step 3 for this situation as follows.

In row 1 we arbitrarily surround a zero and cross all the remaining zeros in that row and in column 1. Thus we get the table:

$\boxed{0}$	7	$\cancel{0}$	$\cancel{0}$	5
4	$\boxed{0}$	4	4	5
5	1	4	6	$\boxed{0}$
$\cancel{0}$	4	3	$\boxed{0}$	$\cancel{0}$
6	4	$\boxed{0}$	1	3

W		J
1	$\rightarrow$	1
2	$\rightarrow$	3
3	$\rightarrow$	5
4	$\rightarrow$	4
5	$\rightarrow$	3

Opt. Cost = 11

Now we are left with only one zero in row 4 and surround it with a square and thus get the optimum allocation: