ITGD4207 Operations Research

Chapter 3

Part-3

Linear Programming

Graphical and Algebraic Solution

Linear Programming
Algebraic Solution

- Artificial Starting Solution
- Artificial Variables
- Big M Method
- Two-Phase Method

Artificial Starting Solution

- When the linear programming problem being considered has constraints of the type $\geq$ or equality constraints, then a starting basic feasible solution is not readily available, as was the case for the LP problem with only $\leq$ constraints.
- Example (Inspector Assignment Problem)

$$\begin{aligned}
&\text{Minimize } Z = 40X_1 + 36X_2\\
&\text{Subject to:}\\
&5X_1 + 3X_2 \geq 45\\
&X_1 \leq 8\\
&X_2 \leq 10\\
&X_1, X_2 \geq 0
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
&-Z + 40X_1 + 36X_2 = 0 \quad (0)\\
&5X_1 + 3X_2 - \boldsymbol{S_1} = 45 \quad (1)\\
&X_1 + \boldsymbol{S_2} = 8 \quad (2)\\
&X_2 + \boldsymbol{S_3} = 10 \quad (3)\\
&X_1, X_2, S_1, S_2, S_3 \geq 0
\end{aligned}$$

Notice that setting the slack and surplus variables as the initial basic variables will not provide an initial basic feasible solution (why?)

Artificial Variables

- In order to obtain an initial basic feasible solution, artificial variables are introduced. An artificial variable is introduced for each constraint of the form $\geq$ or = .

For instance, Equation (1) becomes:

$$5X_1 + 3X_2 - \mathbf{S_1} + A_{\mathbf{1}} = 45 \qquad (1)$$

- Where $A_{\mathbf{1}}$ is the artificial variable. This artificial variable plays the role of providing an initial basic variable; however, unlike the slack variables, it does not have any physical meaning and must be driven out of the basic solution. Two methods exist to accomplish this: The Big M and the Two-Phase methods.

Big M Method

- A very large negative penalty (represented by -**M**) is assessed to each artificial variable (assuming objective is maximized). This way, the Simplex Method will tend to force the artificial variable out of the basic solution.
- Once the artificial variables leave the solution, an initial basic feasible solution will be obtained. Once this is achieved, the role of the artificial variables has been fulfilled and they can be dropped from the model.
- Example (Inspection Problem):

$$-\mathbf{z} + 40X_1 + 36X_2 + MA_{\mathbf{1}} = 0 \quad (0)$$

$$5X_1 + 3X_2 - \mathbf{S_1} + A_{\mathbf{1}} = 45 \quad (1)$$

$$X_1 + \mathbf{S_2} = 8 \quad (2)$$

$$X_2 + \mathbf{S_3} = 10 \quad (3)$$

Setting up the Objective Function

- Since A_1 is a basic variable, its coefficient in the objective function must be zero. To accomplish this, we solve for A_1 in Equation (1) and substitute it in Equation (0). We can also accomplish this by row operations; i.e. we multiply Row 1 by M and subtract it from Row 0.
- From (1): $A_1 = 45-5X_1 - 3X_2 + S_1$

  Then Equation (0) (Objective function) becomes:

  $$\mathbf{-z} + 40X_1 + 36X_2 + M(45-5X_1 - 3X_2 + S_1) = 0$$

  $$\mathbf{-z} + (40 - 5M)X_1 + (36 - 3M)X_2 + MS_1 = -45M$$

  Then the modified problem becomes:

  $$\mathbf{-z} + (40 - 5M)X_1 + (36 - 3M)X_2 + MS_1 = -45M \quad \mathit{(0)}$$

  $$5X_1 + 3X_2 - \mathbf{S_1} + \mathbf{A_1} = 45 \quad \mathit{(1)}$$

  $$X_1 + \mathbf{S_2} = 8 \quad \mathit{(2)}$$

  $$X_2 + \mathbf{S_3} = 10 \quad \mathit{(3)}$$

- We can now solve this problem by using the simplex method

Inspector Allocation Problem Solution

Iter	Basic Var.	Eq.	Z	X_1	X_2	S_1	A_1	S_2	S_3	Right Side (Sol)
0	z	0	-1	$40-5M$	$36-3M$	M	0	0	0	$-45M$
	A_1	1	0	5	3	-1	1	0	0	45
	S_2	2	0	1	0	0	0	1	0	8
	S_3	3	0	0	1	0	0	0	1	10
1	z	0	-1	0	$36-3M$	M	0	$-40+5M$	0	$-5M-320$
	A_1	1	0	0	3	-1	1	-5	0	5
	X_I	2	0	1	0	0	0	1	0	8
	S_3	3	0	0	1	0	0	0	1	10
2	z	0	-1	0	0	12	$-12+M$	20	0	-380
	X_2	1	0	0	1	-1/3	1/3	-5/3	0	5/3
	X_I	2	0	1	0	0	0	1	0	8
	S_3	3	0	0	0	1/3	-1/3	5/3	1	25/3

Notice that at the end of Iteration 1, we have found a basic feasible solution (why?). Thus, at this point, the artificial variable has accomplished its objective and it can be dropped from the tableau.

Two-Phase Method

Another approach to handling the artificial variables

- Phase I

  Consists of finding an initial basic feasible solution to the original problem. For this purpose, an artificial objective function is created which is the sum of all the artificial variables. The artificial objective is then minimized using the Simplex Method. If the minimum value of the artificial problem is zero, then it follows that all of the artificial variables are zero, and an initial basic feasible solution for the original problem has been obtained. Otherwise, a basic feasible solution for the original problem does not exist (the problem has no feasible solutions- referred to as <u>Inconsistent</u>).

- Phase II

  The basic feasible solution found at the end of Phase I is optimized with respect to the original objective function, i.e., the final tableau of Phase I becomes the first tableau for Phase II (the objective row is changed).

Example

- Example:

  Minimize $z = -3x_1 + x_2 + x_3$

  Subject to:

  $x_1 - 2x_2 + x_3 \leq 11$

  $-4x_1 + x_2 + 2x_3 \geq 3$

  $2x_1 - x_3 = -1$

  $x_1, x_2, x_3 \geq 0$

- The first phase of the two-phase method is given by the problem:

  Minimize $w = A_1 + A_2$

  Subject to:

  $x_1 - 2x_2 + x_3 + S_1 = 11$ (1)

  $-4x_1 + x_2 + 2x_3 - S_2 + A_1 = 3$ (2)

  $-2x_1 + x_3 + A_2 = 1$ (3)

  $x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

(Note A_1 , A_2 are solely for the purpose of developing a basic solution)

Example (cont.)

Objective function in standard Form is

$$\text{Maximize } -w + A_1 + A_2 = 0 \qquad (0)$$

$$x_1 - 2x_2 + x_3 + S_1 = 11 \qquad (1)$$

$$-4x_1 + x_2 + 2x_3 - S_2 + A_{\mathbf{1}} = 3 \qquad (2) \longrightarrow A_{\mathbf{1}} = 3 + 4x_1 - x_2 - 2x_3 + S_2$$

$$-2x_1 + x_3 + A_{\mathbf{2}} = 1 \qquad (3) \longrightarrow A_{\mathbf{2}} = 1 + 2x_1 - x_3$$

$$x_1, x_2, x_3, S_1, S_2, A_{\mathbf{1}}, A_{\mathbf{2}} \geq 0$$

Substituting $A_{\mathbf{1}}$, $A_{\mathbf{2}}$ into (0) we get:

$$-w + (3 + 4x_1 - x_2 - 2x_3) + (S_2 + 1 + 2x_1 - x_3) = 0$$

$$-w + 6x_1 - x_2 - 3x_3 + S_2 = -4 \qquad (0)$$

$$x_1 - 2x_2 + x_3 + S_1 = 11 \qquad (1)$$

$$-4x_1 + x_2 + 2x_3 - S_2 + A_{\mathbf{1}} = 3 \qquad (2)$$

$$-2x_1 + x_3 + A_{\mathbf{2}} = 1 \qquad (3)$$

$$x_1, x_2, x_3, S_1, S_2, A_{\mathbf{1}}, A_{\mathbf{2}} \geq 0$$

First Phase Solution

Iter	Basic Var.	Eq.	w	X_1	X_2	X_3	S_1	S_2	A_1	A_2	Rt Sd (Sol)
0	w	0	-1	6	-1	-3	0	1	0	0	-4
	S_1	1	0	1	-2	1	1	0	0	0	11
	A_1	2	0	-4	1	2	0	1	1	0	3
	A_2	3	0	-2	0	1	0	0	0	1	1
1	w	0	-1	0	-1	0	0	1	0	3	-1
	S_1	1	0	3	-2	0	1	0	0	-1	10
	A_1	2	0	0	1	0	0	-1	1	-2	1
	X_3	3	0	-2	0	1	0	0	0	1	1
2	w	0	-1	0	0	0	0	0	1	1	**0**
	S_1	1	0	3	0	0	1	-2	2	-5	12
	X_2	2	0	0	1	0	0	-1	1	-2	1
	X_3	3	0	-2	0	1	0	0	0	1	1

At this point, we have found a feasible solution (w = 0): X_2=1, X_3=1. Now we need to solve the original problem (Second Phase)

Second Phase Solution

- Now we need to solve the original (real) problem; so the Phase I w row is discarded and the actual objective is restored.
  - Max $-z \;\; -3x_1 + x_2 + x_3 = 0$
- Note that x_2 , x_3 are basic variables (their coefficient in the objective function must be zero), so they must be removed ("priced out") using the other equations (solving or using row operations).
- First, we illustrate with equations (from the last tableau of the first phase)

$$3x_1 - \quad + S_1 - 2S_2 = 12 \qquad (1)$$

$$x_2 \quad - S_2 = 1 \qquad (2) \qquad x_2 = 1 + S_2$$

$$-2x_1 \quad + x_3 = 1 \qquad (3) \longrightarrow \; x_3 = 1 + 2x_1$$

Thus the objective function becomes:

$$-z \;\; -3x_1 + (1 + S_2) + (1 + 2x_1) = 0$$

$$-z - x_1 + S_2 = -2 \qquad (0)$$

- If we use row operations, we can accomplish the same result.

Second Phase Solution

It	Basic Var.	Eq.	z	X_1	X_2	X_3	S_1	S_2	Rt Sd (Sol)
0	z	0	-1	-1	0	0	0	1	-2
	S_1	1	0	3	0	0	1	-2	12
	X_2	2	0	0	1	0	0	-1	1
	X_3	3	0	-2	0	1	0	0	1
1	z	0	-1	0	0	0	1/3	1/3	2
	X_1	1	0	1	0	0	1/3	-2/3	4
	X_2	2	0	0	1	0	0	-1	1
	X_3	3	0	0	0	1	2/3	-4/3	9

This is the solution to the original problem