Operations Research

Chapter 3

Part-2

Linear Programming

Graphical and Algebraic Solution

Linear Programming Algebraic Solution

- Introduction to the Simplex Method
- Example of use of Slack and Surplus Variables
- Conceptual Outline of the Steps of the Simplex Algorithm
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- Steps of the Simplex Algorithm
- Using the Simplex procedure for Minimization problems
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Introduction to the Simplex Method

- ***Simplex Method .-*** *An algebraic, iterative method to solve linear programming problems.*
- *The simplex method identifies an initial basic solution (Corner point) and then systematically moves to an adjacent basic solution, which improves the value of the objective function. Eventually, this new basic solution will be the optimal solution.*
- *The simplex method requires that the problem is expressed as a standard LP problem (see module on Standard Form). This implies that all the constraints are expressed as equations by adding slack or surplus variables.*
- *The method uses Gaussian elimination (sweep out method) to solve the linear simultaneous equations generated in the process.*

Example of use of Slack and Surplus Variables

$$6X_1 + 3X_2 \le 10 \qquad (1)$$

$$3X_1 + X_2 = 7 \qquad (2)$$

$$7X_1 + 4X_2 + X_3 \ge 10 \qquad (3)$$

- Since (1) and (3) are inequalities we need to add a slack (1) and subtract a surplus variable (3) accordingly. Then the inequalities can be expressed as equations of the form:

$$6X_1 + 3X_2 + S_1 = 10 \qquad (1')$$

$$3X_1 + X_2 = 7 \qquad (2')$$

$$7X_1 + 4X_2 + X_3 - S_3 = 10 \qquad (3'),$$

Where S_1 is a slack variable and S_3 is a surplus variable. What is the physical meaning of these variables?

Example Reddy Mikks Problem

Original Reddy Mikks Problem

Maximize $z = 3X_E + 2X_I$

Subject to:

$X_E + 2X_I \leq 6$ (1)
$2X_E + X_I \leq 8$ (2)
$-X_E + X_I \leq 1$ (3)
$X_I \leq 2$ (4)

$X_E, X_{I,} \geq 0$

Reddy Mikks Problem with slack variables

Maximize $z = 3X_E + 2X_I + 0S_1 + 0S_2 + 0S_3 + 0S_4$

Subject to:

$X_E + 2X_I + S_1 = 6$ (1)
$2X_E + X_I + S_2 = 8$ (2)
$-X_E + X_I + S_3 = 1$ (3)
$X_I + S_4 = 2$ (4)

$X_E , X_{I,}, S_{1,}, S_{2,}, S_{3,}, S_4 \geq 0$

Example Inspection Problem

Original Inspection Problem

Minimize $Z = 40 X_1 + 36X_2$

Subject to:

$5X_1 + 3X_2 \geq 45$

$X_1 \leq 8$

$X_2 \leq 10$

$X_1, X_2 \geq 0$

Inspection Problem with slack and surplus variables

Minimize $Z = 40 X_1 + 36X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to:

$5X_1 + 3X_2 - S_1 = 45$

$X_1 + S_2 = 8$

$X_2 + S_3 = 10$

$X_1, X_2\ S_{1,}, S_{2,}, S_{3,} \geq 0$

Conceptual Outline of the Steps of the Simplex Algorithm

- **Step 0:** Using the standard form determine a starting basic feasible solution by setting n-m non-basic variables to zero.
- **Step 1**: Select an entering variable from among the current non-basic variables, which gives the largest per-unit improvement in the value of the objective function.
  If none exists stop; the current basic solution is optimal.
  Otherwise go to Step 2.
- **Step 2**: Select a leaving variable from among the current basic variables that must now be set to zero (become non-basic) when the entering variable becomes basic.
- **Step 3**: Determine the new basic solution by making the entering variable, basic; and the leaving variable, non-basic, and return to Step 1.

Simplex Method in Tableau Format

- The tableau format allows us to represent the problem compactly and more easily solve it.
- We merely record the coefficients of the problem only.
- In order to use the Tableau Method we need to do three things first:
  - Represent the LP problem in standard form
  - Represent the objective function as an equation

$$z - \sum_{j} c_j \chi_j = 0;$$

where z is the value of the objective function.

  - Determine the initial basic solution

Setting up the Reddy-Mikks Problem

- *For Instance the Reddy-Mikks Problem:*

  Maximize $z = 3X_E + 2X_I$

  Subject to:

  $X_E + 2X_I \leq 6 \quad (1)$

  $2X_E + X_I \leq 8 \quad (2)$

  $-X_E + X_I \leq 1 \quad (3)$

  $X_I \leq 2 \quad (4)$

  $X_E, X_I, \geq 0$

☞ Is expressed as:

$$z - 3X_E - 2X_I - 0S_1 - 0S_2 - 0S_3 - 0S_4$$

Subject to:

$X_E + 2X_I + S_1 = 6 \quad (1)$

$2X_E + X_I + S_2 = 8 \quad (2)$

$-X_E + X_I + S_3 = 1 \quad (3)$

$X_I + S_4 = 2 \quad (4)$

$X_E, X_I, S_1, S_2, S_3, S_4 \geq 0$

Maximization is implied by the (+1)z since it is the objective function of the standard form.

Because they provide an immediate basic feasible solution we choose the slack variables as the initial basic variables

Simplex Tableau

- The tabular form of the simplex method uses a simplex **tableau** to display the system of equations yielding the current basic feasible solution.
- There are different ways to design the tableau a common design is as follows

Initial Tableau for the R-M problem (Iteration 0)

Header

Objective Function

Constraints

Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	Rt Side (Sol)
z	0	1	-3	-2	0	0	0	0	0
S_1	1	0	1	2	1	0	0	0	6
S_2	2	0	2	1	0	1	0	0	8
S_3	3	0	-1	1	0	0	1	0	1
S_4	4	0	0	1	0	0	0	1	2

Steps of the Simplex Algorithm

- Step 0: Optimality test: the current basic feasible solution is optimal if every coefficient in row 0 of the tableau is non-negative. If it is, stop; if not perform an additional iteration.
- Step 1: Determine the entering variable by selecting the variable having the most negative coefficient in row 0. The corresponding column is called the **pivot column**.
- Step 2: Determine the leaving variable by applying the minimum ratio test:
  - Pick out each coefficient in the **pivot column** having a positive value.
  - Divide the right hand of each row by each of these positive coefficients.
  - Determine the smallest of these ratios
  - The basic variable for the row corresponding to the smallest ratio is the leaving variable. Replace this variable with the entering variable in the next tableau.
  - The row with the smallest ratio is called the **pivot row**. The number in the intersection of the pivot column and row is the **pivot element**.
- Step 3: Solve for the new BFS by using elementary row operations.

Iteration 0 (Steps 0,1,2)

Iter-ation	Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)
0	z	0	1	-3	-2	0	0	0	0	0
	S_1	1	0	1	2	1	0	0	0	6
	S_2	2	0	2	1	0	1	0	0	8
	S_3	3	0	-1	1	0	0	1	0	1
	S_4	4	0	0	1	0	0	0	1	2

6/1

8/2

Pivot Element

Entering Variable: Select the variable with the most negative coefficient in the objective function

Leaving Variable: Smallest ratio obtained by dividing the RHS of each row by the positive coefficients of the entering variable (Column)

Iteration 0 (Step 3)

- To get the new BFS we perform the following steps
  1. Divide the pivot row by the pivot element and call it the “new pivot row”. Copy the result in the tableau for the next iteration
  2. For every row of the current tableau (excluding the pivot row) subtract the product of its pivot-column coefficient times the *new pivot row*. Copy the result in the corresponding row of the next-iteration tableau. Make sure to properly identify the new basic variable.
- M-R Example: New pivot 1:

Iter-ation	Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)
1	z	0	1							
	S_1	1	0							
	X_E	2	0	1	1/2	0	1/2	0	0	8/2
	S_3	3	0							
	S_4	4	0							

M-R Example: New Row 1

	Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)	
Original Equation 1	S_1	1	0	1	2	1	0	0	0	6	
Minus (1) times new pivot row	S_2	2	0	1	1/2	0	1/2	0	0	8/2	*(-1)*
New Equation 1	S_1	1	0	0	1.5	1	-0.5	0	0	2	

Resulting Tableau (Partial)

Iter-ation	Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)
1	z	0	1							
	S_1	1	0	0	1.5	1	-0.5	0	0	2
	X_E	2	0	1	0.5	0	0.5	0	0	4
	S_3	3	0							
	S_4	4	0							

Resulting Tableau

Original Equation 0

Minus (-3) times new pivot row

New Equation 0

Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)
Z	0	1	-3	-2	0	0	0	0	0
S_2	2	0	1	1/2	0	1/2	0	0	8/2
Z	0	1	0	-0.5	0	1.5	0	0	12

Iteration 1

Iteration	Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)	
1	z	0	1	0	-0.5	0	1.5	0	0	12	
	S_1	1	0	0	1.5	1	-0.5	0	0	2	*2/1.5*
	X_E	2	0	1	0.5	0	0.5	0	0	4	*4/0.5*
	S_3	3	0	0	1.5	0	0.5	1	0	5	*5/1.5*
	S_4	4	0	0	1	0	0	0	1	2	*2/1.0*

Entering Variable: Select the variable with the most negative coefficient in the objective function

Leaving Variable: Smallest ratio obtained by dividing the RHS of each row by the positive coefficients of the entering variable (Column)

Iteration 2

Iter-ation	Basic Var.	Eq.	Z	X_E	X_I	S_1	S_2	S_3	S_4	RHS (Sol)
2	**z**	0	1	0	0	0.33	1.33	0	0	**12.66**
	X_I	1	0	0	1	0.66	-0.33	0	0	**1.33**
	X_E	2	0	1	0	-0.33	0.66	0	0	**3.33**
	S_3	3	0	0	0	-1.0	1.0	1	0	3
	S_4	4	0	0	0	-0.66	0.33	0	1	0.66

Since all the coefficients in the objective function are positive we stop: We have found the optimal Solution

Using the Simplex procedure for Minimization problems

- A minimization problem can be converted to a maximization problem just by multiplying the objective function by (-1).
- Once this is done the problem is solved exactly the same as the maximization problem
- Example:

Minimize $z = x_1 - 3x_2 - 2x_3$	Maximize $(-z) = -x_1 + 3x_2 + 2x_3$	Max. $-z + x_1 - 3x_2 - 2x_3 = 0$
Subject to:	Subject to:	Subject to:
$3x_1 - x_2 + 2x_3 \leq 7$	$3x_1 - x_2 + 2x_3 \leq 7$	$3x_1 - x_2 + 2x_3 + S_1 = 7$
$-2x_1 + 4x_2 + 2x_3 \leq 12$	$-2x_1 + 4x_2 + 2x_3 \leq 12$	$-2x_1 + 4x_2 + 2x_3 + S_2 = 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$	$-4x_1 + 3x_2 + 8x_3 \leq 10$	$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$
$x_1, x_2 \geq 0$	$x_1, x_2 \geq 0$	$x_1, x_2, S_1, S_2, S_3, \geq 0$

Simplex Solution of Minimization Problem

Iteration	Basic Var.	Eq	Z	X_1	X_2	X_3	S_1	S_2	S_3	RHS (Sol)
0	z	0	-1	1	-3	-2	0	0	0	0
	S_1	1	0	3	-1	2	1	0	0	7
	S_2	2	0	-2	4	0	0	1	0	12
	S_3	3	0	-4	3	8	0	0	1	10
1	z	0	-1	-0.5	0	-0.5	0	0.75	0	9
	S_1	1	0	2.5	0	2.5	1	0.25	0	10
	X_2	2	0	-0.5	1	0.5	0	0.25	0	3
	S_3	3	0	-2.5	0	6.5	0	-0.75	1	1
2	z	0	-1	0	0	0	0.2	0.8	0	11
	X_1	1	0	1	0	1	0.4	0.1	0	4
	X_2	2	0	0	1	1	0.2	0.3	0	5
	S_3	3	0	0	0	9	1.0	-0.5	1	11

Solution

- Efficiency of the simplex algorithm
  - One way to assess the efficiency of the simplex algorithm is to count the number of iteration needed to arrive to the optimal solution and to compare this against the total number of corner point solutions given by the formula:

$$\binom{n}{m} = \frac{n!}{(n-m)!\,m!}$$

  - *(*n*) = number of variables, (*m*) = number of equations*

Corner Point Solution

- Previously we learned that a corner point solution corresponds to a solution of the subset of equations corresponding to the constraints meeting at that corner point (vertex)
- Thus if we have m constraints and n dimensions we use only n equations to find the corner point solution.