Linear Programming: Sensitivity Analysis and Interpretation of Solution

- Introduction to Sensitivity Analysis
- Graphical Sensitivity Analysis
- Sensitivity Analysis: Computer Solution
- Limitations of Classical Sensitivity Analysis

Introduction to Sensitivity Analysis

- In the previous chapter we discussed:
  - objective function value
  - values of the decision variables
  - reduced costs
  - slack/surplus
- In this chapter we will discuss:
  - changes in the coefficients of the objective function
  - changes in the right-hand side value of a constraint

Introduction to Sensitivity Analysis

- Sensitivity analysis (or post-optimality analysis) is used to determine how the optimal solution is affected by changes, within specified ranges, in:
  - the objective function coefficients
  - the right-hand side (RHS) values
- Sensitivity analysis is important to a manager who must operate in a dynamic environment with imprecise estimates of the coefficients.
- Sensitivity analysis allows a manager to ask certain what-if questions about the problem.

Graphical Sensitivity Analysis

- For LP problems with two decision variables, graphical solution methods can be used to perform sensitivity analysis on
  - the objective function coefficients, and
  - the right-hand-side values for the constraints.

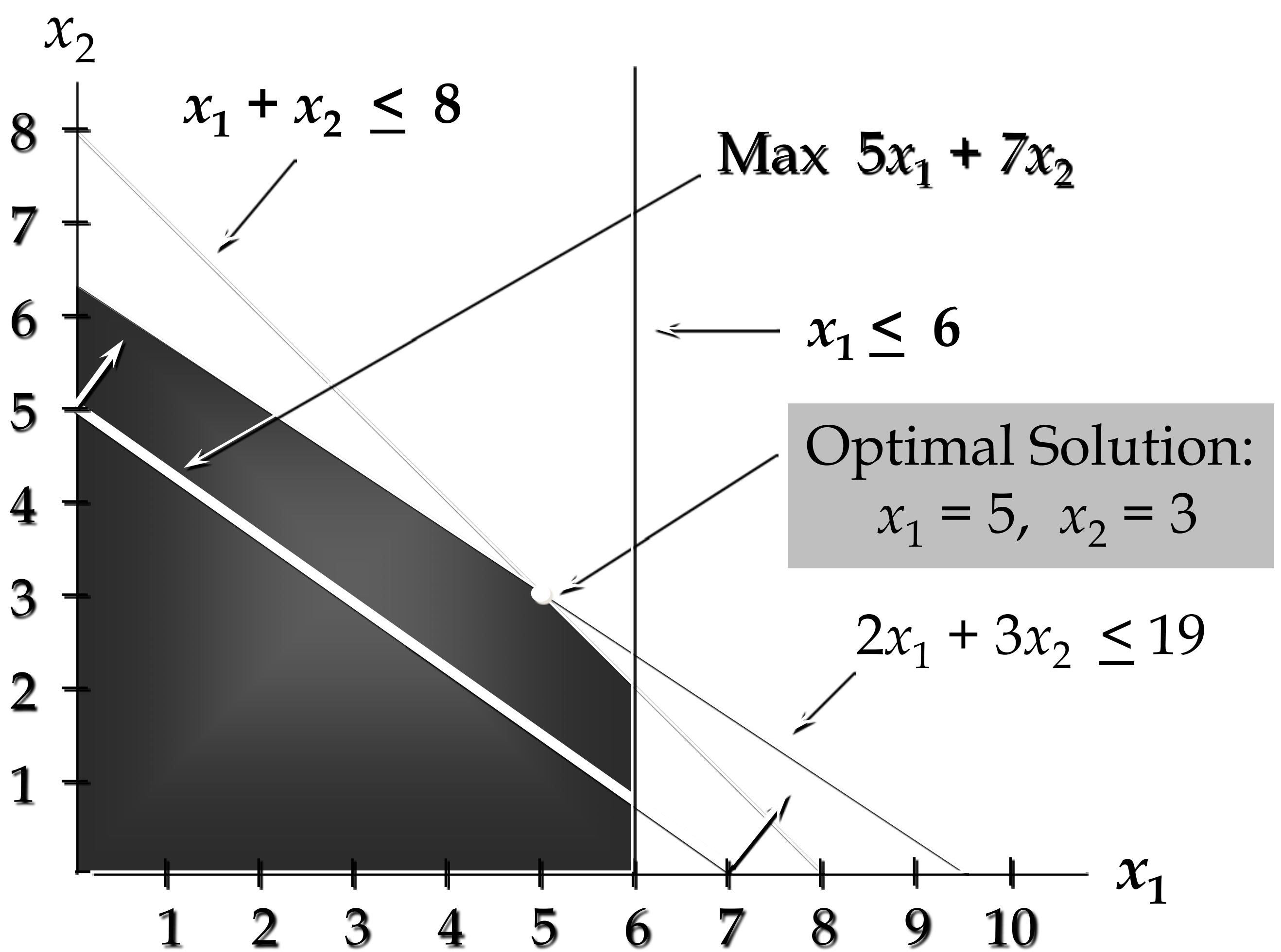
Example 1

- LP Formulation

$$
\begin{aligned}
\text{Max} \quad & 5x_1 + 7x_2 \\
\text{s.t.} \quad & x_1 \leq 6 \\
& 2x_1 + 3x_2 \leq 19 \\
& x_1 + x_2 \leq 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Example 1

- Graphical Solution

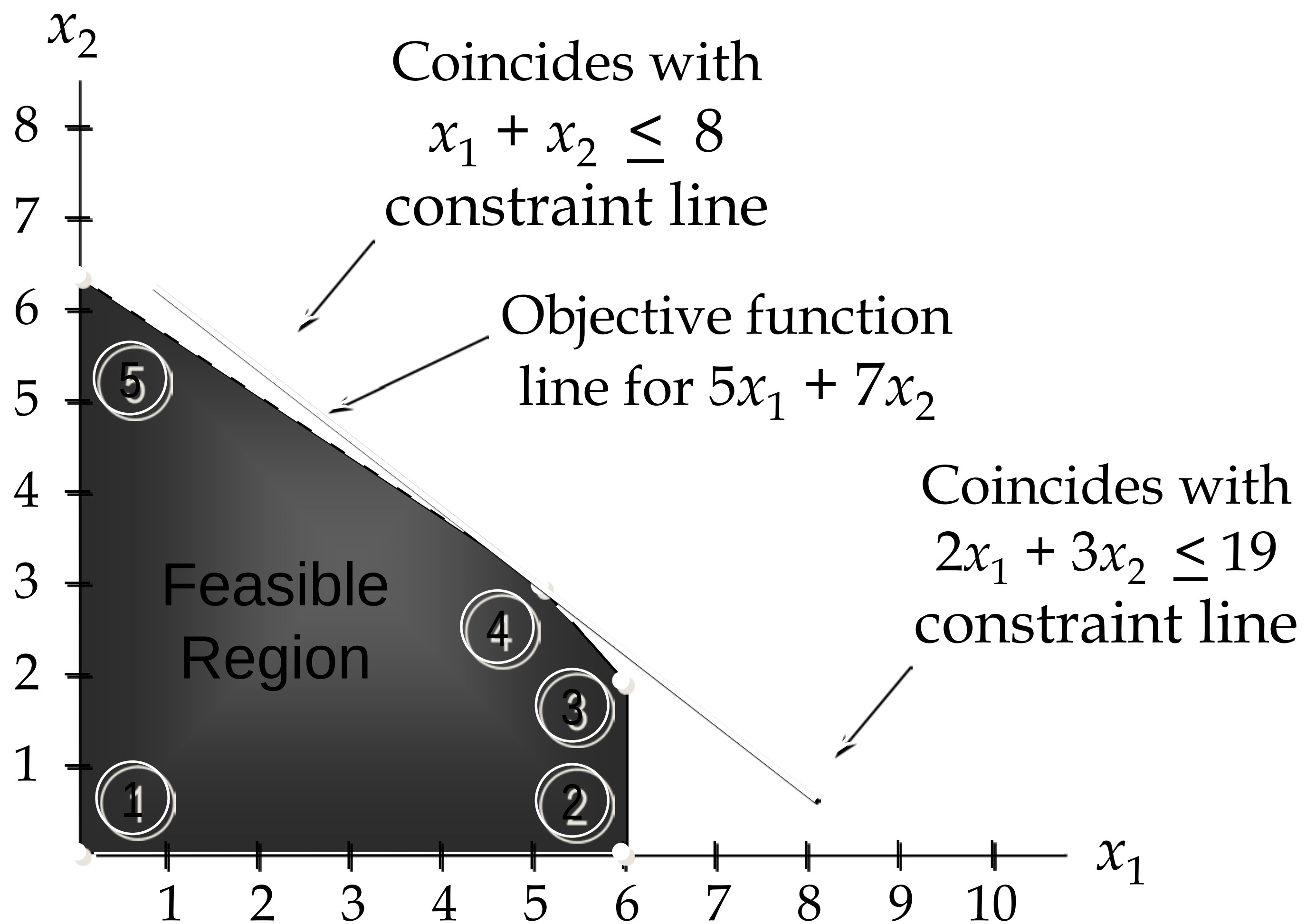


Objective Function Coefficients

- Let us consider how changes in the objective function coefficients might affect the optimal solution.
- The range of optimality for each coefficient provides the range of values over which the current solution will remain optimal.
- Managers should focus on those objective coefficients that have a narrow range of optimality and coefficients near the endpoints of the range.

Example 1

- Changing Slope of Objective Function



Range of Optimality

- Graphically, the limits of a range of optimality are found by changing the slope of the objective function line within the limits of the slopes of the binding constraint lines.
- Slope of an objective function line, Max $c_1x_1 + c_2x_2$, is $-c_1/c_2$, and the slope of a constraint, $a_1x_1 + a_2x_2 = b$, is $-a_1/a_2$.

Example 1

- Range of Optimality for c_1

The slope of the objective function line is $-c_1/c_2$. The slope of the first binding constraint, $x_1 + x_2 = 8$, is -1 and the slope of the second binding constraint, $x_1 + 3x_2 = 19$, is -2/3.

Find the range of values for c_1 (with c_2 staying 7) such that the objective function line slope lies between that of the two binding constraints:

$$-1 \leq -c_1/7 \leq -2/3$$

Multiplying through by -7 (and reversing the inequalities):

$$14/3 \leq c_1 \leq 7$$

Example 1

- Range of Optimality for c_1

  Would a change in c_1 from 5 to 7 (with c_2 unchanged) cause a change in the optimal solution?

  The answer is 'no' because when $c_1 = 7$, the condition $14/3 \leq c_1 \leq 7$ is satisfied.

Example 1

- Range of Optimality for c_2

  Find the range of values for c_2 (with c_1 staying 5) such that the objective function line slope lies between that of the two binding constraints:

$$-1 \leq -5/c_2 \leq -2/3$$

Multiplying by -1: $1 \geq 5/c_2 \geq 2/3$

Inverting, $1 \leq c_2/5 \leq 3/2$

Multiplying by 5: $5 \leq c_2 \leq 15/2$

Example 1

- Range of Optimality for c_2

  Would a change in c_2 from 7 to 6 (with c_1 unchanged) cause a change in the optimal solution?

  The answer is 'no' because when $c_2 = 6$, the condition $5 \leq c_2 \leq 15/2$ is satisfied.

Simultaneous Changes

- The range of optimality for objective function coefficients is only applicable for changes made to one coefficient at a time.
- All other coefficients are assumed to be fixed at their initial values.
- If two or more coefficients are changed simultaneously, further analysis is usually necessary.
- However, when solving two-variable problems graphically, the analysis is fairly easy.

Simultaneous Changes

- Simply compute the slope of the objective function

$(-C_{x_1}/C_{x_2})$ for the new coefficient values.

- If this ratio is $\geq$ the lower limit on the slope of the objective function and $\leq$ the upper limit, then the changes made will not cause a change in the optimal solution.

Example 1

- Simultaneous Changes in c_1 and c_2

  Would simultaneously changing c_1 from 5 to 7 and changing c_2 from 7 to 6 cause a change in the optimal solution? (Recall that these changes <u>individually</u> did not cause the optimal solution to change.)

Recall that the objective function line slope must lie between that of the two binding constraints:

$$-1 \leq -c_1/c_2 \leq -2/3$$

The answer is 'yes' the optimal solution changes because -7/6 does not satisfy the above condition.

Right-Hand Sides

- Let us consider how a change in the right-hand side for a constraint might affect the feasible region and perhaps cause a change in the optimal solution.
- The change in the value of the optimal solution per unit increase in the right-hand side is called the <u>dual value</u>.
- The <u>range of feasibility</u> is the range over which the dual value is applicable.
- As the RHS increases, other constraints will become binding and limit the change in the value of the objective function.

Dual Value

- Graphically, a dual value is determined by adding +1 to the right hand side value in question and then resolving for the optimal solution in terms of the same two binding constraints.
- The dual value is equal to the difference in the values of the objective functions between the new and original problems.
- The dual value for a nonbinding constraint is 0.
- A negative dual value indicates that the objective function will **not** improve if the RHS is increased.

Example 1

- Dual Values

  Constraint 1: Since $x_1 \leq 6$ is not a binding constraint, its dual price is 0.

  Constraint 2: Change the RHS value of the second constraint to 20 and resolve for the optimal point determined by the last two constraints:
  $2x_1 + 3x_2 = 20$ and $x_1 + x_2 = 8$.

  The solution is $x_1 = 4$, $x_2 = 4$, $z = 48$. Hence, the dual price $= z_{new} - z_{old} = 48 - 46 = 2$.

Example 1

- Dual Values

Constraint 3: Change the RHS value of the third constraint to 9 and resolve for the optimal point determined by the last two constraints: $2x_1 + 3x_2 =$ 19 and $x_1 + x_2 = 9$.

The solution is: $x_1 = 8$, $x_2 = 1$, $z = 47$.

The dual price is $z_{\text{new}} - z_{\text{old}} = 47 - 46 = 1$.

Range of Feasibility

- The <u>range of feasibility</u> for a change in the right hand side value is the range of values for this coefficient in which the original dual value remains constant.
- Graphically, the range of feasibility is determined by finding the values of a right hand side coefficient such that the same two lines that determined the original optimal solution continue to determine the optimal solution for the problem.

Sensitivity Analysis: Computer Solution

Software packages such as *Lingo* and *Microsoft Excel* provide the following LP information:

- Information about the objective function:
  - its optimal value
  - coefficient ranges (ranges of optimality)
- Information about the decision variables:
  - their optimal values
  - their reduced costs
- Information about the constraints:
  - the amount of slack or surplus
  - the dual prices
  - right-hand side ranges (ranges of feasibility)

Reduced Cost

- The reduced cost associated with a variable is equal to the dual value for the non-negativity constraint associated with the variable.
- In general, if a variable has a non-zero value in the optimal solution, then it will have a reduced cost equal to 0.

Relevant Cost and Sunk Cost

- A resource cost is a relevant cost if the amount paid for it is dependent upon the amount of the resource used by the decision variables.
- Relevant costs **are** reflected in the objective function coefficients.
- A resource cost is a sunk cost if it must be paid regardless of the amount of the resource actually used by the decision variables.
- Sunk resource costs are **not** reflected in the objective function coefficients.

Cautionary Note on the Interpretation of Dual Values

- **Resource cost is sunk**

  The dual value is the maximum amount you should be willing to pay for one additional unit of the resource.

- **Resource cost is relevant**

  The dual value is the maximum premium over the normal cost that you should be willing to pay for one unit of the resource.

Example 1 (Again)

- **LP Formulation**

$$
\begin{array}{ll}
\text{Max} & 5x_1 + 7x_2 \\
\text{s.t.} & \begin{aligned} x_1 \phantom{{}+3x_2} &\le 6 \\ 2x_1 + 3x_2 &\le 19 \\ x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}
\end{array}
$$

Example 1

- **Range of Optimality for c_1 and c_2**

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	X1	5.0	0.0	5	2	0.33333333
C8	X2	3.0	0.0	7	0.5	2

Constraints

Cell	Name	Final Value	Dual Value	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B13	#1	5	0	6	1E+30	1
B14	#2	19	2	19	5	1
B15	#3	8	1	8	0.33333333	1.66666667

Dual Values

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	X1	5.0	0.0	5	2	0.33333333
C8	X2	3.0	0.0	7	0.5	2

Constraints

Cell	Name	Final Value	Dual Value	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B13	#1	5	0	6	1E+30	1
B14	#2	19	2	19	5	1
B15	#3	8	1	8	0.33333333	1.66666667

- **Range of Feasibility**

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	X1	5.0	0.0	5	2	0.33333333
C8	X2	3.0	0.0	7	0.5	2

Constraints

Cell	Name	Final Value	Dual Value	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B13	#1	5	0	6	1E+30	1
B14	#2	19	2	19	5	1
B15	#3	8	1	8	0.33333333	1.66666667

Example 2: Olympic Bike Co.

Olympic Bike is introducing two new lightweight bicycle frames, the Deluxe and the Professional, to be made from special aluminum and steel alloys. The anticipated unit profits are \$10 for the Deluxe and \$15 for the Professional.

The number of pounds of each alloy needed per frame is summarized on the next slide.

Example 2: Olympic Bike Co.

A supplier delivers 100 pounds of the aluminum alloy and 80 pounds of the steel alloy weekly.

	Aluminum Alloy	Steel Alloy
Deluxe	2	3
Professional	4	2

How many Deluxe and Professional frames should Olympic produce each week?

Example 2: Olympic Bike Co.

- Model Formulation
  - Verbal Statement of the Objective Function

    Maximize total weekly profit.
  - Verbal Statement of the Constraints

    Total weekly usage of aluminum alloy $\leq$ 100 pounds.

    Total weekly usage of steel alloy $\leq$ 80 pounds.
  - Definition of the Decision Variables

    x_1 = number of Deluxe frames produced weekly.

    x_2 = number of Professional frames produced weekly.

Example 2: Olympic Bike Co.

- **Model Formulation (continued)**

Max	$10x_1 + 15x_2$	(Total Weekly Profit)
s.t.	$2x_1 + 4x_2 \leq 100$	(Aluminum Available)
	$3x_1 + 2x_2 \leq 80$	(Steel Available)
	$x_1, x_2 \geq 0$	

Example 2: Olympic Bike Co.

- **Partial Spreadsheet Showing Solution**

	A	B	C	D
6		Decision Variables		
7		Deluxe	Professional	
8	Bikes Made	15	17.500	
9				
10	Maximized Total Profit		412.500	
11				
12	Constraints	Amount Used		Amount Avail.
13	Aluminum	100	<=	100
14	Steel	80	<=	80

Example 2: Olympic Bike Co.

- Optimal Solution

According to the output:

x_1 (Deluxe frames) = 15

x_2 (Professional frames) = 17.5

Objective function value = $412.50

Example 2: Olympic Bike Co.

- Range of Optimality

  Question:

  Suppose the profit on deluxe frames is increased to $20. Is the above solution still optimal? What is the value of the objective function when this unit profit is increased to $20?

Example 2: Olympic Bike Co.

- Sensitivity Report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	Deluxe	15	0	10	12.5	2.5
C8	Profess.	17.500	0.000	15	5	8.333333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B13	Aluminum	100	3.125	100	60	46.66666667
B14	Steel	80	1.25	80	70	30

Example 2: Olympic Bike Co.

- **Range of Optimality**

Answer:

The output states that the solution remains optimal as long as the objective function coefficient of x_1 is between 7.5 and 22.5. Because 20 is within this range, the optimal solution will not change. The optimal profit will change: $20x_1 + 15x_2 = 20(15) + 15(17.5) = \$562.50$.

Example 2: Olympic Bike Co.

- **Range of Optimality**

  Question:

  If the unit profit on deluxe frames were \$6 instead of \$10, would the optimal solution change?

Example 2: Olympic Bike Co.

- **Range of Optimality**

Adjustable Cells						
Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	Deluxe	15	0	10	12.5	2.5
C8	Profess.	17.500	0.000	15	5	8.333333333

Constraints						
Cell	Name	Final Value	Dual Value	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B13	Aluminum	100	3.125	100	60	46.66666667
B14	Steel	80	1.25	80	70	30

Example 2: Olympic Bike Co.

- **Range of Optimality**

Answer:

The output states that the solution remains optimal as long as the objective function coefficient of x_1 is between 7.5 and 22.5. Because 6 is outside this range, the optimal solution would change.

Example 2: Olympic Bike Co.

- **Range of Feasibility and Sunk Costs**

  Question:

  Given that aluminum is a sunk cost, what is the maximum amount the company should pay for 50 extra pounds of aluminum?

Example 2: Olympic Bike Co.

- **Range of Feasibility and Sunk Costs**

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	Deluxe	15	0	10	12.5	2.5
C8	Profess.	17.500	0.000	15	5	8.333333333

Constraints

Cell	Name	Final Value	Dual Value	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B13	Aluminum	100	3.125	100	60	46.6666667
B14	Steel	80	1.25	80	70	30

Example 2: Olympic Bike Co.

- **Range of Feasibility and Sunk Costs**

Answer:

Because the cost for aluminum is a sunk cost, the shadow price provides the value of extra aluminum. The shadow price for aluminum is the same as its dual value (for a maximization problem). The shadow price for aluminum is \$3.125 per pound and the maximum allowable increase is 60 pounds. Because 50 is in this range, the \$3.125 is valid. Thus, the value of 50 additional pounds is = 50(\$3.125) = \$156.25.

Example 2: Olympic Bike Co.

- **Range of Feasibility and Relevant Costs**

  Question:

  If aluminum were a relevant cost, what is the maximum amount the company should pay for 50 extra pounds of aluminum?

Example 2: Olympic Bike Co.

- **Range of Feasibility and Relevant Costs**

  Answer:

  If aluminum were a relevant cost, the shadow price would be the amount above the normal price of aluminum the company would be willing to pay. Thus if initially aluminum cost \$4 per pound, then additional units in the range of feasibility would be worth \$4 + \$3.125 = \$7.125 per pound.

Limitations of Classical Sensitivity Analysis

- **Classical sensitivity analysis provided by most computer packages does have its limitations.**
- **It is rarely the case that one solves a model once and makes recommendations.**
- **More often, a series of models is solved using a variety of input data sets before a final plan is adopted.**

Limitations of Classical Sensitivity Analysis

- **Simultaneous Changes**

  The range analysis for objective function coefficients and the constraint right-hand sides is only applicable for changes in a single coefficient.

  We should resolve the model with the new coefficient values.

Limitations of Classical Sensitivity Analysis

- **Changes in Constraint Coefficients**

  Classical sensitivity analysis provides no information about changes resulting from a change in a coefficient of a variable in a constraint.

  We must simply change the coefficient and rerun the model.

Limitations of Classical Sensitivity Analysis

- **Non-intuitive Dual Values**

  Constraints with variables naturally on both the left-hand and right-hand sides often lead to dual values that have a non-intuitive explanation.

  This is often the case with constraints that involve percentages.

  The model should be resolved with the new percentages.

Example 3

- Consider the following linear program:

$$
\begin{array}{lrcl}
\text{Min} & 6x_1 + 9x_2 & & (\$ \text{ cost}) \\
\\
\text{s.t.} & x_1 + 2x_2 & \leq & 8 \\
 & 10x_1 + 7.5x_2 & \geq & 30 \\
 & x_2 & \geq & 2 \\
\\
 & x_1, x_2 & \geq & 0
\end{array}
$$

Example 3

- **Computer Output**

OBJECTIVE FUNCTION VALUE = 27.000

Variable	Value	Reduced Cost
x_1	1.500	0.000
x_2	2.000	0.000

Constraint	Slack/Surplus	Dual Value
1	2.500	0.000
2	0.000	-0.600
3	0.000	-4.500

Example 3

- **Computer Output (continued)**

OBJECTIVE COEFFICIENT RANGES

Variable	Lower Limit	Current Value	Upper Limit
x_1	0.000	6.000	12.000
x_2	4.500	9.000	No Limit

RIGHTHAND SIDE RANGES

Constraint	Lower Limit	Current Value	Upper Limit
1	5.500	8.000	No Limit
2	15.000	30.000	55.000
3	0.000	2.000	4.000

Example 3

- **Optimal Solution**

According to the output:

$x_1 = 1.5$

$x_2 = 2.0$

Objective function value $= 27.00$

Example 3

- **Range of Optimality**

Question:

Suppose the unit cost of x_1 is decreased to \$4. Is the current solution still optimal? What is the value of the objective function when this unit cost is decreased to \$4?

Example 3

- Computer Output

OBJECTIVE COEFFICIENT RANGES

Variable	Lower Limit	Current Value	Upper Limit
x_1	0.000	6.000	12.000
x_2	4.500	9.000	No Limit

RIGHTHAND SIDE RANGES

Constraint	Lower Limit	Current Value	Upper Limit
1	5.500	8.000	No Limit
2	15.000	30.000	55.000
3	0.000	2.000	4.000

Example 3

- **Range of Optimality**

Answer:

The output states that the solution remains optimal as long as the objective function coefficient of x_1 is between 0 and 12. Because 4 is within this range, the optimal solution will not change. However, the optimal total cost will be affected: $6x_1 + 9x_2 = 4(1.5) + 9(2.0) = \$24.00$.

Example 3

- **Range of Optimality**

Question:

How much can the unit cost of x_2 be decreased without concern for the optimal solution changing?

Example 3

- Computer Output

OBJECTIVE COEFFICIENT RANGES

Variable	Lower Limit	Current Value	Upper Limit
x_1	0.000	6.000	12.000
x_2	4.500	9.000	No Limit

RIGHTHAND SIDE RANGES

Constraint	Lower Limit	Current Value	Upper Limit
1	5.500	8.000	No Limit
2	15.000	30.000	55.000
3	0.000	2.000	4.000

Example 3

- **Range of Optimality**

  Answer:

  The output states that the solution remains optimal as long as the objective function coefficient of x_2 does not fall below 4.5.

Example 3

- **Range of Feasibility**

  Question:

  If the right-hand side of constraint 3 is increased by 1, what will be the effect on the optimal solution?

Example 3

- **Computer Output**

OBJECTIVE COEFFICIENT RANGES

Variable	Lower Limit	Current Value	Upper Limit
x_1	0.000	6.000	12.000
x_2	4.500	9.000	No Limit

RIGHTHAND SIDE RANGES

Constraint	Lower Limit	Current Value	Upper Limit
1	5.500	8.000	No Limit
2	15.000	30.000	55.000
3	0.000	2.000	4.000

Example 3

- **Range of Feasibility**

Answer:

A dual value represents the improvement in the objective function value per unit increase in the right-hand side. A negative dual value indicates a deterioration (negative improvement) in the objective, which in this problem means an increase in total cost because we're minimizing. Since the right-hand side remains within the range of feasibility, there is no change in the optimal solution. However, the objective function value increases by $4.50.