Exercise 1 – ANOVA Decomposition

Given

Let
$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$
with $(x_1, x_2) \sim \text{Uniform}([0,1]^2)$.

We perform the ANOVA decomposition:
$$f(x_1, x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2)$$

calculate f_0:

$$f_0 = \mathbb{E}[f(x_1, x_2)] = \int_0^1 \int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\, dx_1 dx_2 = \frac{15}{2}$$

$f_1(x_1)$ and $f_2(x_2)$

$$f_1(x_1) = \mathbb{E}_{x_2}[f(x_1, x_2)] - f_0 = (12x_1 + 3) - \frac{15}{2} = 9x_1 - \frac{9}{2}$$

$$f_2(x_2) = \mathbb{E}_{x_1}[f(x_1, x_2)] - f_0 = (6x_2 + 3) - \frac{15}{2} = 3x_2 - \frac{3}{2}$$

compute $f_{12}(x_1, x_2)$

$$\begin{aligned} f_{12}(x_1, x_2) &= f(x_1, x_2) - f_0 - f_1(x_1) - f_2(x_2) \\ &= 12x_1 + 6x_2 - 6x_1x_2 - \frac{15}{2} - (9x_1 - \frac{9}{2}) - (3x_2 - \frac{3}{2}) \\ &= 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2} \end{aligned}$$

Determinate Variance Terms

Following the definition from the textbook:
$$\sigma_u^2 = \int (f_u(x_u) - \mathbb{E}[f_u])^2\, dx_u$$

Univariate terms

$$\mathbb{E}[f_1] = \int_0^1 \left(9x_1 - \frac{9}{2}\right) dx_1 = 0 \Rightarrow \sigma_1^2 = \int_0^1 \left(9x_1 - \frac{9}{2}\right)^2 dx_1 = \frac{27}{4}$$

$$\mathbb{E}[f_2] = \int_0^1 \left(3x_2 - \frac{3}{2}\right) dx_2 = 0 \Rightarrow \sigma_2^2 = \int_0^1 \left(3x_2 - \frac{3}{2}\right)^2 dx_2 = \frac{3}{4}$$

Interaction term

$$f_{12}(x_1, x_2) = 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}, \quad \mathbb{E}[f_{12}] = 0$$

Compute:

$$\sigma_{12}^2 = \iint_{[0,1]^2} \left(3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}\right)^2 dx_1 dx_2 = \frac{1}{4}$$

Final Results

$$f(x_1, x_2) = \frac{15}{2} + (9x_1 - \frac{9}{2}) + (3x_2 - \frac{3}{2}) + (3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2})$$

$$\sigma_1^2 = \frac{27}{4}, \quad \sigma_2^2 = \frac{3}{4}, \quad \sigma_{12}^2 = \frac{1}{4}$$

$$\mathrm{Var}(f) = \sigma_1^2 + \sigma_2^2 + \sigma_{12}^2 = \boxed{8}$$

Monte Carlo Estimation

2. Monte Carlo Estimation (6 Points)

Let $X \sim \mathcal{N}(1, 3)$, and $f(x) = 1 + 2x + x^2$.

(a) Analytical Computation

Expectation of $f(X)$:

$$\begin{aligned}\mathbb{E}[f(X)] &= \mathbb{E}[1 + 2X + X^2] \\ &= 1 + 2\mathbb{E}[X] + \mathbb{E}[X^2].\end{aligned}$$

Given $X \sim \mathcal{N}(1, 3)$:

- $\mathbb{E}[X] = 1$
- $\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = 3 + 1^2 = 4$

Substituting these values:

$$\mathbb{E}[f(X)] = 1 + 2(1) + 4 = 7.$$

Variance of $f(X)$:

Rewrite $f(X)$ as:

$$f(X) = (X - 1)^2 + 4(X - 1) + 4.$$

Let $Y = X - 1$, so $Y \sim \mathcal{N}(0, 3)$. Then:

$$f(X) = Y^2 + 4Y + 4.$$

The variance is unaffected by the constant term:

$$\mathrm{Var}(f(X)) = \mathrm{Var}(Y^2 + 4Y).$$

Since Y is a zero-mean Gaussian, Y^2 and Y are uncorrelated ($\mathrm{Cov}(Y^2, Y) = 0$):

$$\mathrm{Var}(Y^2 + 4Y) = \mathrm{Var}(Y^2) + \mathrm{Var}(4Y).$$

- $\mathrm{Var}(4Y) = 16\mathrm{Var}(Y) = 16 \times 3 = 48$
- For $Y \sim \mathcal{N}(0, \sigma^2)$, $\mathrm{Var}(Y^2) = 2\sigma^4$. Here, $\sigma^2 = 3$, so $\mathrm{Var}(Y^2) = 2(3^2) = 18$

Adding the variances:

$$\mathrm{Var}(f(X)) = 18 + 48 = 66.$$

Final Answer

(a)

$$\mathbb{E}[f(X)] = \boxed{7}, \quad \mathrm{Var}[f(X)] = \boxed{66}.$$

Question 3:.

a] using quadrature nodes: $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$

we aim to get a quadrature rule of the form:-

$$\int_0^1 f(x)\,dx = \sum_{i=1}^{3} w_i f(x_i)$$

$$= w_1 f(0) + w_2 f(\tfrac{1}{2}) + w_3 f(1)$$

to exactly integrate all Polynomials of degree ≤ 2

The weights must Correctly Compute:-

1) $f(x) = 1$ [degree 0]

$$\int_0^1 1\,dx = 1 = w_1 + w_2 + w_3$$

$\therefore$ $w_1 + w_2 + w_3 = 1$ (1)

2) $f(x) = x$ [degree 1]

$$\int_0^1 x\,dx = \left.\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$= w_1 * 0 + w_2 * \frac{1}{2} + w_3 * 1 =$$

$\therefore \frac{1}{2} w_2 + w_3 = \frac{1}{2}$ $\times 2$ $\Rightarrow$ $w_2 + 2w_3 = 1$ (2)

3) $f(x) = x^2$, [degree 2]

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$= w_1 * 0^2 + w_2 * \frac{1}{2}^2 + w_3 * 1^2$$

$\frac{1}{4} w_2 + w_3 = \frac{1}{3}$ $\times 12$ $\Rightarrow$ $3w_2 + 12w_3 = 4$ (3)

a) Cont.

By Solving the equations ①, ②, ③ together

From ②: $w_2 = 1 - 2w_3$

Substitute in ③: $3[1 - 2w_3] + 12w_3 = 4$

$\therefore 3 - 6w_3 + 12w_3 = 4$

$$w_3 = \frac{4-3}{6} = \frac{1}{6}$$

$$w_2 = 1 - 2 \times \frac{1}{6} = \frac{2}{3}$$

From ①: $w_1 = 1 - \frac{2}{3} - \frac{1}{6} = \frac{6-4-1}{6} = \frac{1}{6}$

$\therefore w_1 = \frac{1}{6}, \quad w_2 = \frac{2}{3}, \quad w_3 = \frac{1}{6}$

Part b) to show the rule integrate x^3 exactly:

Integral: $\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$

Quadrature aproximation:

$$\int_0^1 x^3 dx \approx \sum_{i=1}^{3} w_i f(x_i)$$

$$\approx \frac{1}{6} \times 0^3 + \frac{2}{3} \times \left(\frac{1}{2}\right)^3 + \frac{1}{6} \times 1^3$$

$$= \frac{2}{3} \times \frac{1}{8_4} + \frac{1}{6} = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}$$

Hence, the Quadrature approximation matches the True Integral. then, Confirming exactness of Cubic Polynomials (order 4)