Problem 4 Invariant-sets

a] using the Polar Coordinates $x = r\cos\theta$, $y = r\cos\theta$

$$\because\; x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\left[\cos^2\theta + \sin^2\theta\right] = \underline{\underline{r^2}}$$

differintiate with dt

$$r^2 = x^2 + y^2$$

$$2r\cdot\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} \longrightarrow (1)$$

L.H.S. $\quad x\dot{x} + y\dot{y} = x\left[\lambda x - y + (\alpha x - \beta y)(x^2 + y^2)\right]$
$\qquad y\left[x + \lambda y + (\beta x + \alpha y)(x^2 + y^2)\right]$

expand the multiplication & $x^2 + y^2 = r^2$

$$x\dot{x} + y\dot{y} = \lambda x^2 - \cancel{xy} + \alpha x r^2 - \cancel{xy\,\beta r^2} + \cancel{yx} + \lambda y^2 + \cancel{\beta\, xy\cdot r^2} + \alpha y^2\cdot r^2$$

$$= \lambda x^2 + \lambda y^2 + \alpha x^2 r^2 + \alpha y^2 r^2 \qquad [x^2 + y^2 = r^2]$$

$$= \lambda(x^2 + y^2) + \alpha r^2\cdot(x^2 + y^2)$$

$$= \lambda r^2 + \alpha r^4$$

subsitute in (1)

$$\boxed{\dot{r} = \lambda r + \alpha r^3}$$

Now, Let's analyse $\dot{r}$

$\dot{r} = \lambda r + \alpha r^3 \Rightarrow$

for $\dot{r} = 0$

$$\lambda r^* + \alpha r^{*3} = 0$$

$r^* = 0$ or $r^* = \sqrt{\frac{-\cancel{\lambda}}{\alpha}}$

I) from this for $\dot{r} = 0 \Rightarrow$ the origin is at Equilibrium.

II) For $\lambda < 0$:-

for $r \approx 0 \Rightarrow \dot{r} \approx \lambda r < 0$
then for $r \approx 0 \Rightarrow$ the function is decreasing towards 0,

III) for $1 > r > 0 \Rightarrow$ the term αr^* is so small
& λr would be the dominant part of $\dot{r}$
& since $\lambda < 0 \Rightarrow \dot{r}$ is also negative
which means the function is also decreasing towards 0

From I, II & III :-
the origin is at equilibrium
& the origin attracts the nearing points
for any inintial $r^* \in [0, 1]$

b] for $\lambda > 0$, $\alpha < 0$, $\dot{r} = \lambda r + \alpha r^3$

I) Solving for equilibrium Points. $\dot{r} = 0$

$r(\lambda + \alpha r^2) = 0 \Rightarrow r^* = 0$, $r^* = \sqrt{\frac{-\lambda}{\alpha}}$

Since $\lambda > 0, \alpha < 0 \Rightarrow$

then $r^* = \sqrt{\frac{-\lambda}{\alpha}}$ exists and $r^* > 0$

the $\left(\sqrt{\frac{-\lambda}{\alpha}}\right)$ is an equilibrium Point.

II) for $0 < r < \sqrt{\frac{-\lambda}{\alpha}}$

which means r is small

the in $\dot{r} = \lambda r + \alpha r^3$

λr dominates & $\lambda > 0$

then $\dot{r} > 0$, which mean $r(t)$ increases.

III) for $r > r^*$

which mean r is large

Then in, $\dot{r} = \lambda r + \alpha r^3$

αr^3 dominates & $\alpha < 0$

then $\dot{r} < 0$, which means $r(t)$ decreases

from I, II, III,

r^* is at equilibrium

for $r < r^*$ the function increases

& for $r > r^*$ the function decreases

$\therefore$ r^* is an attractor,

$\therefore$ $r^* = \sqrt{\frac{-\lambda}{\alpha}}$ is a stable limit cycle