Problem 1 Solution

Problem Statement

Consider the two-dimensional Gaussian PDF $n(\mathbf{z}; \bar{\mathbf{z}}, \mathbf{P})$, where $\mathbf{z} = (x_1, x_2)$ with mean $\bar{\mathbf{z}} = (\bar{x}_1, \bar{x}_2)$ and symmetric, positive definite covariance matrix

$$\mathbf{P} = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix}$$

with $\sigma_{12} = \sigma_{21}$.

We are tasked with finding constants $\bar{x}_c$ and σ_c^2 such that

$$n(\mathbf{z}; \bar{\mathbf{z}}, \mathbf{P}) = \frac{1}{2\pi\sigma_c^2} \exp\left(-\frac{1}{2} \left(\mathbf{z} - \bar{\mathbf{z}}\right)^T \mathbf{P}^{-1} \left(\mathbf{z} - \bar{\mathbf{z}}\right) \right)$$

can be factorized into the product of two univariate Gaussians:

$$n(x_1, x_2) = \frac{1}{2\pi\sigma_c^2} \exp\left(-\frac{1}{2\sigma_c^2} (x_1 - \bar{x}_c)^T \mathbf{P}^{-1} (x_1 - \bar{x}_c) \right) \times \frac{1}{2\pi\sigma_{22}^2} \exp\left(-\frac{(x_2 - \bar{x}_2)^2}{2\sigma_{22}^2} \right)$$

The goal is to factorize the bivariate Gaussian into the product of two univariate Gaussians, one depending on x_1 and the other on x_2.

Step 1: Factorization of the PDF

To begin, we define the vectors $\mathbf{z} = (x_1, x_2)$ and $\bar{\mathbf{z}} = (\bar{x}_1, \bar{x}_2)$, and the covariance matrix $\mathbf{P}$ as given in the problem statement.

We need to factorize the PDF into two univariate Gaussians. To do this, we first expand the quadratic form $(\mathbf{z} - \bar{\mathbf{z}})^T \mathbf{P}^{-1} (\mathbf{z} - \bar{\mathbf{z}})$.

Step 2: Expanding the quadratic form

The quadratic form is given by:

$$(\mathbf{z} - \bar{\mathbf{z}})^T \mathbf{P}^{-1} (\mathbf{z} - \bar{\mathbf{z}}) = \begin{pmatrix} \delta x_1 & \delta x_2 \end{pmatrix} \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}$$

where $\delta x_1 = x_1 - \bar{x}_1$ and $\delta x_2 = x_2 - \bar{x}_2$. Expanding the quadratic form, we get:

$$\sigma_{22}^2 (\delta x_1)^2 - 2\sigma_{12}^2 \delta x_1 \delta x_2 + \sigma_{11}^2 (\delta x_2)^2.$$

Step 3: Completing the square for δx_1

To separate δx_1 and δx_2, we complete the square on δx_1:

$$\sigma_{22}^2 (\delta x_1 - \frac{\sigma_{12}^2}{\sigma_{22}^2} \delta x_2)^2 = \sigma_{22}^2 (\delta x_1)^2 - 2\sigma_{12}^2 \delta x_1 \delta x_2 + \frac{\sigma_{12}^4}{\sigma_{22}^2} (\delta x_2)^2.$$

Thus, the quadratic form becomes:

$$\sigma_{22}^2(\delta x_1)^2 - 2\sigma_{12}^2\delta x_1 \delta x_2 + \sigma_{11}^2(\delta x_2)^2 = \sigma_{22}^2\left(\delta x_1 - \frac{\sigma_{12}^2}{\sigma_{22}^2}\delta x_2\right)^2 + \left(\sigma_{11}^2 - \frac{\sigma_{12}^4}{\sigma_{22}^2}\right)(\delta x_2)^2.$$

Step 4: Writing the exponential term

Now, the exponential term in the PDF is of the form:

$$\exp\left(-\frac{1}{2\det(\mathbf{P})}\left[\sigma_{22}^2(\delta x_1 - \frac{\sigma_{12}^2}{\sigma_{22}^2}\delta x_2)^2 + \left(\sigma_{11}^2 - \frac{\sigma_{12}^4}{\sigma_{22}^2}\right)(\delta x_2)^2\right]\right).$$

Comparing this with the desired form in the problem statement:

$$\exp\left(-\frac{(x_1 - \bar{x}_c)^2}{2\sigma_c^2}\right) \times \exp\left(-\frac{(x_2 - \bar{x}_2)^2}{2\sigma_{22}^2}\right),$$

we obtain the following:

Step 5: Comparing terms

By comparing the terms:

1. $\sigma_{22}^2\left(\delta x_1 - \frac{\sigma_{12}^2}{\sigma_{22}^2}\delta x_2\right)^2$ matches $(x_1 - \bar{x}_c)^2$.

Thus, the shifted mean is:

$$\bar{x}_c = \bar{x}_1 + \frac{\sigma_{12}^2}{\sigma_{22}^2}(x_2 - \bar{x}_2).$$

2. The effective variance for x_1 is:

$$\sigma_c^2 = \frac{\sigma_{22}^2}{\det(\mathbf{P})} = \frac{\sigma_{22}^2}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^4}.$$

Final Results

Shifted mean:

$$\bar{x}_c = \bar{x}_1 + \frac{\sigma_{12}^2}{\sigma_{22}^2}(x_2 - \bar{x}_2).$$

Effective variance:

$$\sigma_c^2 = \frac{\sigma_{22}^2}{\det(\mathbf{P})} = \frac{\sigma_{22}^2}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^4}.$$

Problem 1b solution

We have the two-dimensional Gaussian density

$$n(z;\bar{z},P) = \frac{1}{2\pi\sqrt{\det P}} \exp\Bigl(-\tfrac{1}{2}\,(z-\bar{z})^T P^{-1}(z-\bar{z})\Bigr),$$

with

$$z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \bar{z} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \quad P = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix}.$$

1. Marginal PDF of X_1.

A standard result for multivariate normals is that the marginal of a subset is again normal with the corresponding variance. Hence

$$\pi_{X_1}(x_1) = \int_{-\infty}^{\infty} n\bigl((x_1,x_2)^T;\bar{z},P\bigr)\,dx_2 = \mathcal{N}\bigl(x_1;\,\bar{x}_1,\,\sigma_{11}^2\bigr),$$

i.e.

$$\boxed{\pi_{X_1}(x_1) = \frac{1}{\sqrt{2\pi\,\sigma_{11}^2}} \exp\Bigl(-\frac{(x_1-\bar{x}_1)^2}{2\,\sigma_{11}^2}\Bigr).}$$

2. Conditional PDF of $X_2 \mid X_1$.

By definition,

$$\pi_{X_2}(x_2 \mid x_1) = \frac{n\bigl((x_1,x_2)^T;\bar{z},P\bigr)}{\pi_{X_1}(x_1)}.$$

Completing the square (or using the standard formula) shows this is again Gaussian with

$$\mu_{2|1} = \bar{x}_2 + \frac{\sigma_{12}^2}{\sigma_{11}^2}\,(x_1-\bar{x}_1), \qquad \sigma_{2|1}^2 = \sigma_{22}^2 - \frac{(\sigma_{12}^2)^2}{\sigma_{11}^2}.$$

Thus

$$\boxed{\pi_{X_2}(x_2 \mid x_1) = \frac{1}{\sqrt{2\pi\,\bigl(\sigma_{22}^2 - \frac{(\sigma_{12}^2)^2}{\sigma_{11}^2}\bigr)}} \exp\Biggl(-\frac{\bigl(x_2-\bar{x}_2-\frac{\sigma_{12}^2}{\sigma_{11}^2}(x_1-\bar{x}_1)\bigr)^2}{2\,\bigl(\sigma_{22}^2 - \frac{(\sigma_{12}^2)^2}{\sigma_{11}^2}\bigr)}\Biggr).}$$

Summary of results:

$$\pi_{X_1}(x_1) = \frac{1}{\sqrt{2\pi\,\sigma_{11}^2}} \exp\Bigl(-\frac{(x_1-\bar{x}_1)^2}{2\,\sigma_{11}^2}\Bigr),$$

$$\pi_{X_2}(x_2 \mid x_1) = \frac{1}{\sqrt{2\pi\,\bigl(\sigma_{22}^2 - \frac{(\sigma_{12}^2)^2}{\sigma_{11}^2}\bigr)}} \exp\Biggl(-\frac{\bigl(x_2-\bar{x}_2-\frac{\sigma_{12}^2}{\sigma_{11}^2}(x_1-\bar{x}_1)\bigr)^2}{2\,\bigl(\sigma_{22}^2 - \frac{(\sigma_{12}^2)^2}{\sigma_{11}^2}\bigr)}\Biggr).$$

Problem 2

$X_1 \quad X_2$ -> random var

joint PDF :

$$\pi_{x_1 x_2}(x_1, x_2) = \frac{1}{Z} \exp(-x_1^2 - x_2^2 - x_1^2 x_2^2)$$

Z -> normalization constant

Evaluate $E[X_1 X_2^2 \mid X_1 = a]$ $\quad a \in \mathbb{R}$ is a constant

-> Marginal density $\pi_{x_1}(x_1)$

$$\pi_{x_1}(x_1) = \int \pi_{x_1 x_2}(x_1, x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{Z} \exp(-x_1^2 - x_2^2 - x_1^2 x_2^2)\, dx_2$$

$$= \frac{\exp(-x_1^2)}{Z} \int_{-\infty}^{\infty} \exp(-x_2^2(1 + x_1^2))\, dx_2$$

$1 + x_1^2 > 0$ -> convert integrand to prob. density function of a Gaussian

$$\sigma := \frac{1}{\sqrt{2(x_1^2+1)}}$$

because $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x_2^2}{2\sigma^2}\right) dx_2 = 1 \Rightarrow$

$$\Rightarrow \quad = \frac{1}{Z} \sqrt{\frac{\pi}{1 + x_1^2}} \exp(-x_1^2)$$

-> Conditional density $\pi_{x_2|x_1}(x_2 | a)$

$$\pi_{x_2|x_1}(x_2 | a) = \frac{\pi_{x_1 x_2}(a, x_2)}{\pi_{x_1}(a)}$$

$$= \sqrt{\frac{1 + a^2}{\pi}} \exp(-x_2^2(1 + a^2))$$

$\rightarrow \quad E[X_1 X_2^2 \mid X_1 = a] = a \, E[X_2^2 \mid X_1 = a]$

since $X_1 = a$ is fixed.

we know that : $\pi_{X_2|X_1}(x_2 \mid a) = \sqrt{\frac{1+a^2}{\pi}} \exp(-x_2^2(1+a^2))$

This corresponds to the Gaussian distribution with variance $\sigma^2 = \frac{1}{2(1+a^2)}$ (Step 1)

in a Gaussian distribution $X \sim \mathcal{N}(0, \sigma^2)$

$E[x^2] = \sigma^2$

Therefore : $E[X_2^2 \mid X_1 = a] = \frac{1}{2(1+a^2)}$

Finally $E[X_1 X_2^2 \mid X_1 = a] = \frac{a}{2(1+a^2)}$

Problem 3:

$$d_{Hell}(P,q) = \frac{1}{2}\int_{R_N}\left(\sqrt{P(x)} - \sqrt{q(x)}\right)^2 . dx$$

multiply R.H.S by $\frac{P(x)}{P(x)} = \frac{(\sqrt{P(x)})^2}{P(x)} = \frac{P(x)}{(\sqrt{P(x)})^2}$

$$d_{Hell}(P,q)^2 = \frac{1}{2}\int_{R_N} P(x) . \left[\frac{\sqrt{P(x)}}{\sqrt{P(x)}} - \sqrt{\frac{q(x)}{P(x)}}\right]^2 . dx$$

$$= \frac{1}{2}\int_{R_N} P(x) . \left[1 - \sqrt{\frac{q(x)}{P(x)}}\right]^2 . dx$$

expand the square.

$$= \frac{1}{2}\int_{R_N} P(x) . \left[1 + \frac{q(x)}{P(x)} - 2\sqrt{\frac{q(x)}{P(x)}}\right] . dx$$

$$= \frac{1}{2}\int_{R_N}\left(P(x) + q(x) + P(x) \times -2\sqrt{\frac{q(x)}{P(x)}}\right) dx$$

$$= \frac{1}{2}\left[\int P(x) . dx + \int_{R_N} q(x) dx - 2\int_{R_N} P(x) . \sqrt{\frac{q(x)}{P(x)}} . dx\right]$$

$= \frac{1}{2}$ since $\int_{R_N} P(x) . dx = \int_{R_N} q(x) dx = 1$ [P.d.f Property Rule of full Prob.]

$$\therefore d_{Hell}(P,q)^2 = \frac{1}{2}\left[1 + 1 - 2\int_{R_N} P(x)\sqrt{\frac{q(x)}{P(x)}} dx\right]$$

$$= 1 - \int_{R_N} P(x) . \sqrt{\frac{q(x)}{P(x)}} . dx \quad \left[\int_{R_N} P(x) dx = 1\right]$$

$$= \int_{R_N} P(x) - P(x) . \sqrt{\frac{q(x)}{P(x)}} . dx$$

$$= \int_{R_N} P(x) . \left[1 - \sqrt{\frac{q(x)}{P(x)}}\right] dx \longrightarrow (1)$$

assuming $P(x) > 0$, $q(x) \geq 0$

applying the Inequality $1 - \sqrt{x} \leq \frac{-1}{2}\log x$ for $x > 0$

with $x = \sqrt{\frac{q(x)}{P(x)}}$

$$\therefore 1 - \sqrt{\frac{q(x)}{P(x)}} \leq \frac{-1}{2}\log\left(\frac{q(x)}{P(x)}\right)$$

$\because \quad \log\left(\frac{q(x)}{P(x)}\right) = -\log\left(\frac{P(x)}{q(x)}\right)$ Property of log

$\therefore \quad 1 - \sqrt{\frac{q(x)}{P(x)}} \leqslant \frac{1}{2} \log \frac{P(x)}{q(x)}$

$\because$ $P(x) > 0$ $\Rightarrow$ multiplying by $P(x)$ doesn't change Inequality.

$\therefore \quad P(x)\left[1 - \sqrt{\frac{q(x)}{P(x)}}\right] \leqslant \frac{1}{2} P(x) \log \frac{P(x)}{q(x)}$

apply Integral ~~over~~ $\int_{R_N}$ over R_N for both Sides.

$$\underbrace{\int_{R_N} P(x)\left[1 - \sqrt{\frac{q(x)}{P(x)}}\right] dx}_{\text{from } ①} \leqslant \frac{1}{2} \underbrace{\int_{R_N} P(x) \log \frac{P(x)}{q(x)}}$$

from ① $\quad \& \quad D_{KL}(P\|q) = \int_{R_N} P(x) \cdot \log \frac{P(x)}{q(x)} \cdot dx$

$\therefore \quad d_{Hell}(P,q)^2 \leqslant \frac{1}{2} D_{KL}(P\|q)$ #