Problem 3:-

a] we are required to show that :-

$$\mathbb{E}[S_{CrPs}(F_Y, Y_{obs})] = \int_{-\infty}^{\infty} S_{CrPs}(F_Y, y_{obs}) \, \pi_{Y_{obs}}(y_{obs}) \, dy_{obs} \quad [4.44]$$

is equivalent to:-

$$\mathbb{E}[S_{CrPs}(F_Y, Y_{obs})] = \int_{-\infty}^{\infty} (F_Y(y) - F_{Y_{obs}}(y))^2 dy + \int_{-\infty}^{\infty} F_{Y_{obs}}(y)(1 - F_{Y_{obs}}(y)) \, dy$$

First :- show that $\int_{-\infty}^{\infty} F_Y(y) F_{y_{obs}}(y) \, \pi_{Y_{obs}}(y_{obs}) \, dy_{obs} = F_Y(y) F_{Y_{obs}}(y)$

$$\mathbb{E}[S_{CrPs}(F_Y, Y_{obs})] = \int_{-\infty}^{\infty} S_{CrPs}(F_Y, y_{obs}) \, \pi_{Y_{obs}}(y_{obs}) \, dy_{obs}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} (F_Y(y) - F_{y_{obs}}(y))^2 dy \right] \pi_{Y_{obs}}(y_{obs}) \, dy_{obs} \quad [\text{swap the integral}$$

$$= \int_{-\infty}^{\infty} \left[\underbrace{\int_{-\infty}^{\infty} (F_Y(y) - F_{y_{obs}}(y))^2 \pi_{y_{obs}} \, dy_{obs}}_{} \right] dy$$

expand the square

$$(F_Y(y) - F_{Y_{obs}}(y))^2 = F_Y(y)^2 - 2 F_Y(y) F_{y_{obs}}(y) + F_{y_{obs}}(y)^2$$

$$E[S_{CrPs}(F_Y, Y_{obs})] = \int_{-\infty}^{\infty} \left[F_Y(y)^2 - 2 F_Y(y) \underbrace{\int_{-\infty}^{\infty} F_{y_{obs}}(y) \pi_{Y_{obs}}(y_{obs}) \, dy_{obs}}_{A} + \underbrace{\int_{-\infty}^{\infty} F_{y_{obs}}(y)^2 \pi_{y_{obs}}(y_{obs}) \, dy_{obs}}_{B} \right] dy$$

A: the definition of CDF : $F_{Y_{obs}}(y)$ & the hint

B: Since $F_{y_{obs}}(y) \in \{0, 1\} \Rightarrow F_{y_{obs}}(y)^2 = F_{y_{obs}}(y)$

then B : $F_{y_{obs}}(y)$

Subsititue A & B ∴

$$E[S_{crps}(F_Y, Y_{obs})] = \int_{-\infty}^{\infty} \left[F_Y(y)^2 - 2F_Y(y)\, F_{y_{obs}}(y) + F_{y_{obs}}(y) \right] dy$$

$$= \int_{-\infty}^{\infty} (F_Y(y) - F_{Y_{obs}}(y))^2\, dy + \int_{-\infty}^{\infty} F_{Y_{obs}}(y)(1 - F_{Y_{obs}}(y))\, dy \equiv 4.45.$$

b) first Calculate $E[S_{crps}(F_{Y_{obs}}, Y_{obs})]$ by 4.45

$$E[S_{crps}(F_{Y_{obs}}, Y_{obs})] = \underbrace{\int_{-\infty}^{\infty} (F_{y_{obs}}(y) - F_{Y_{obs}}(y))^2\, dy}_{\text{zero}} + \underbrace{\int_{-\infty}^{\infty} F_{Y_{obs}}(y)(1 - F_{y_{obs}}(y))\, dy}_{\text{constant for any distribution.}} .$$

since the second term doesn't depend on the distibuton of F_Y we let second term $= C$ ∴

* Calculate the difference.

$$E[S_{crps}(F_Y, Y_{obs})] - E[S_{crps}(F_{Y_{obs}}, Y_{obs})] =$$

$$\int_{-\infty}^{\infty} (F_Y(y) - F_{Y_{obs}}(y))^2\, dy + \cancel{C} - \cancel{C} = \int_{-\infty}^{\infty} (F_Y(y) - F_{Y_{obs}}(y))^2\, dy$$

$$\therefore E[S_{crps}(F_Y, Y_{obs})] - E[S_{crps}(F_{y_{obs}}, Y_{obs})] \geq 0$$

and can only be zero iff $F_Y(y) = F_{Y_{obs}}(y) \quad \forall y$