1. Bayesian Inference from Noisy Observation

(a) Finding w such that $[X = x, Y = y]$ is equivalent to $[X = x, W = w]$

We are given:

$$Y = X^2 + W, \quad W \sim \mathcal{N}(0, 1)$$

We want to find the value of w such that the event $[X = x, Y = y]$ is equivalent to $[X = x, W = w]$.

Step 1: Substitute $X = x$ into the equation:

$$Y = x^2 + W \Rightarrow W = Y - x^2$$

Step 2: Now substitute $Y = y$:

$$W = y - x^2$$

Conclusion: The event $[X = x, Y = y]$ is equivalent to $[X = x, W = y - x^2]$.

$$\boxed{w = y - x^2}$$

(b) Deriving the unnormalized posterior $\pi_{X|Y=y}(x)$

We are to derive the unnormalized posterior using Bayes' Theorem:

$$\pi_{X|Y=y}(x) \propto \pi_X(x) \cdot p_{Y|X}(y \mid x)$$

Step 1: Prior of X: Since $X \sim \mathcal{N}(1, 1)$,

$$\pi_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right)$$

Step 2: Likelihood $p_{Y|X}(y \mid x)$:
Given $Y = X^2 + W$ and $W \sim \mathcal{N}(0, 1)$, then conditional on $X = x$,

$$Y \sim \mathcal{N}(x^2, 1) \Rightarrow p_{Y|X}(y \mid x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-x^2)^2}{2}\right)$$

Step 3: Combine prior and likelihood:

$$\pi_{X|Y=y}(x) \propto \exp\left(-\frac{(x-1)^2}{2}\right) \cdot \exp\left(-\frac{(y-x^2)^2}{2}\right)$$

$$\Rightarrow \boxed{\pi_{X|Y=y}(x) \propto \exp\left(-\frac{(x-1)^2 + (y-x^2)^2}{2}\right)}$$

Exercise 2 :-

Part a)
$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

to show P is a Markov Chain:-

$$\forall i \quad \sum_{j=1}^{M} \tilde{P}_{ij} = 1$$

we split the sum into 2 Part $i = j$ & $i \neq j$

$$\sum_{j=1}^{M} \tilde{P}_{ij} = \sum_{j=1}^{M} \left[(1 - C_j)\delta_{ij} + \alpha_{ij} P_{ij} \right]$$

$$= \sum_{i=1}^{M} \alpha_{ij} P_{ij} + (1 - C_j) + \alpha_{ii} P_{ii} \quad \rightarrow ①$$

Since $\alpha_{ii} = 1 \wedge \left(\frac{P_i^t}{P_i} \right) = 1 \wedge 1 = 1 \Rightarrow \alpha_{ii} * P_{ii} = P_{ii}$

& $C_i = \sum_{K=1}^{M} \alpha_{Ki} \cdot P_{Ki}$ (substitute in ①)

$$\forall i, \sum_{j=1}^{M} \tilde{P}_{ij} = \sum_{j=1}^{M} (1 - C_j) + \sum_{i \neq i} \alpha_{ij} P_{ij} + P_{ii}$$

$$= 1 - C_i + 1 - \cancel{P_{ii}} + P_{ii}$$

$$= 2 - C_i \quad \rightarrow ②$$

$$C_i := \sum_{i=1}^{M} \alpha_{ij} P_{ij} \Rightarrow \sum_{j=1}^{M} \tilde{P}_{ij} = (1 - C_i) + C_i = 1$$

$$\therefore \forall i, \sum_{j=1}^{M} \tilde{P}_{ij} = 1 \quad \#$$

hence, $\tilde{P}$ is ~~MC~~ a valid Markov Chain.

Part b)

Case 1: $i \neq j \Rightarrow \delta_{ij} = 0$

$$\tilde{P}_{ij} = \alpha_{ij} P_{ij} \quad \& \quad \tilde{P}_{ji} = \alpha_{ji} \cdot P_{ji}$$

So: $\tilde{P}_{ij} P_j^* = \alpha_{ij} P_{ij} P_j^* \quad \& \quad \tilde{P}_{ji} \cdot P_i^* = \alpha_{ji} \cdot P_{ji} \cdot P_i^*$

Since P is symetric $\Rightarrow P_{ij} = P_{ji}$

$$\alpha_{ij} = \min\left(1, \frac{P_i^*}{P_j^*}\right)$$

$$\alpha_{ji} = \min\left(1, \frac{P_j^*}{P_i^*}\right)$$

$\boxed{1}$ if $\frac{P_i^*}{P_j^*} \leq 1 \Rightarrow \alpha_{ij} = \frac{P_i^*}{P_j^*}, \ \alpha_{ji} = 1$

$$\therefore \tilde{P}_{ij} P_j^* = \frac{P_i^*}{\cancel{P_j^*}} P_{ij} \cdot \cancel{P_j^*} = P_{ij} \cdot P_i^*$$

$$\tilde{P}_{ji} \cdot P_i^* = 1 \cdot \underline{\underline{P_{ji}}} \cdot P_i^* = P_{ij} \cdot P_i^* \quad \checkmark$$

$\boxed{2}$ if $\frac{P_i^*}{P_j^*} > 1 \Rightarrow \alpha_{ij} = 1, \ \alpha_{ji} = \frac{P_j^*}{P_i^*}$

$$\therefore \tilde{P}_{ij} P_j^* = 1 . P_{ij} \cdot P_j^* = P_{ij} \cdot P_j^*$$

$$P_{ji} \cdot P_i^* = \frac{P_j^*}{\cancel{P_i^*}} \cdot P_{ji} \cdot \cancel{P_i^*} = P_{ij} \cdot P_j^* \quad \boxed{\checkmark}$$

in $\boxed{1}$ & $\boxed{2}$ $\Rightarrow \forall_{i \neq j} \ \tilde{P}_{ij} P_j^* = \tilde{P}_{ji} \cdot P_i^*$

Case 2: $i = j \Rightarrow \delta_{ij} = 1 \Rightarrow \tilde{P}_{ii} = 1 - C_i + \alpha_{ii} P_{ii} = P_{ii} P_i$

trivially ∴ $\tilde{P}$ satisfies detailed balance $\forall i,j$

Part C:

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & \frac{1}{2} \end{bmatrix}, \quad P^* = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

First compute α:

$$\alpha = \begin{pmatrix} 1 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \end{pmatrix}$$

Second compute $C_j = \sum_i \alpha_{ij} \cdot P_{ij}$

$j = 1$: $C_1 = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = 1$

$j = 2$: $C_2 = 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 1$

$j = 3$: $C_3 = 0.5 \cdot \frac{1}{4} + 0.5 \times \frac{1}{4} + 1 \cdot \frac{1}{2} = \frac{1+1+4}{8} = \frac{6}{8} = \frac{3}{4}$

Compute P_{ij}: $\tilde{P}_{ij} = (1 - C_j)\,\delta_{ij} + \alpha_{ij} P_{ij}$

for $i = j$: $\tilde{P}_{11} = (1-1) \cdot 1 + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$\tilde{P}_{22} = (1 - C_2) + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$\tilde{P}_{33} = (1 - C_3) + \alpha_{33} \cdot P_{33} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

for $i \neq j$: $P_{ij} = \alpha_{ij} \cdot P_{ij}$

$$\tilde{P} = \begin{pmatrix} \frac{1}{2} & 1/4 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{4} \end{pmatrix} \begin{matrix} = 1 \\ = 1 \\ = 1 \end{matrix}$$

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