Question 3:..

a] using quadrature nodes: $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$

we aim to get a quadrature rule of the form:-

$$\int_0^1 f(x)\,dx = \sum_{i=1}^{3} w_i f(x_i)$$

$$= w_1 f(0) + w_2 f(\tfrac{1}{2}) + w_3 f(1)$$

to exactly integrate all Polynomials of degree ≤ 2

The weights must Correctly Compute:

1) $f(x) = 1$ [degree 0]

$$\int_0^1 1\,dx = 1 = w_1 + w_2 + w_3$$

$\therefore$ $w_1 + w_2 + w_3 = 1$ (1)

2) $f(x) = x$ [degree 1]

$$\int_0^1 x\,dx = \left.\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$= w_1 * 0 + w_2 * \frac{1}{2} + w_3 * 1 =$$

$\therefore$ $\frac{1}{2}w_2 + w_3 = \frac{1}{2}$ $\times$(2) $\Rightarrow$ $w_2 + 2w_3 = 1$ (2)

3) $f(x) = x^2$, [degree 2]

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$= w_1 * 0^2 + w_2 * \frac{1}{2}^2 + w_3 * 1^2$$

$\frac{1}{4}w_2 + w_3 = \frac{1}{3}$ $\times 12$ $\Rightarrow$ $3w_2 + 12w_3 = 4$ (3)

a) Cont.

By Solving the equations ①, ②, ③ together

From ②: $w_2 = 1 - 2w_3$

Substitute in ③: $3[1 - 2w_3] + 12w_3 = 4$

$\therefore\ 3 - 6w_3 + 12w_3 = 4$

$$w_3 = \frac{4-3}{6} = \frac{1}{6}$$

$$w_2 = 1 - 2 \times \frac{1}{6} = \frac{2}{3}$$

From ① $w_1 = 1 - \frac{2}{3} - \frac{1}{6} = \frac{6-4-1}{6} = \frac{1}{6}$

$\therefore\ w_1 = \frac{1}{6},\ w_2 = \frac{2}{3},\ w_3 = \frac{1}{6}$

Part b) to show the rule integrate x^3 exactly:-

Integral: $\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$

Quadrature aproximation:-

$$\int_0^1 x^3 dx \approx \sum_{i=1}^{3} w_i f(x_i)$$

$$\approx \frac{1}{6} \times 0^3 + \frac{2}{3} \times \left(\frac{1}{2}\right)^3 + \frac{1}{6} \times 1^3$$

$$= \frac{2}{3} \times \frac{1}{8_4} + \frac{1}{6} = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}$$

Hence, the Quadrature approximation matches the True Integral then, Confirming exactness of Cubic Polynomials (order 4)