Filtering and Smoothing Exercise

Problem 1: Filtering and Smoothing

Given the state-space model:

$$\begin{aligned} X_{n+1} &= \frac{1}{2}X_n + 1 + \Xi_n, \quad \Xi_n \sim \mathcal{N}(0,1) \\ Y_n &= X_n + \sqrt{2}\Sigma_n, \quad \Sigma_n \sim \mathcal{N}(0,1) \\ X_0 &\sim \mathcal{N}(-1,2) \end{aligned}$$

1. Prediction step: Distribution of X_1

The state transition is:

$$X_1 = \frac{1}{2}X_0 + 1 + \Xi_0$$

With:

$$\mathbb{E}[X_1] = \frac{1}{2}(-1) + 1 = \frac{1}{2}, \quad \mathrm{Var}(X_1) = \left(\frac{1}{2}\right)^2 \cdot 2 + 1 = \frac{3}{2}$$

$$\boxed{X_1 \sim \mathcal{N}\left(\frac{1}{2}, \frac{3}{2}\right)}$$

2. Filtering step: Distribution of $X_1 \mid Y_1 = 2$

Using Kalman filter update:

$$K = \frac{\frac{3}{2}}{\frac{3}{2}+2} = \frac{3}{7}, \quad \mathbb{E}[X_1 \mid Y_1 = 2] = \frac{1}{2} + \frac{3}{7}\left(2 - \frac{1}{2}\right) = \frac{8}{7}$$

$$\mathrm{Var}(X_1 \mid Y_1 = 2) = \left(1 - \frac{3}{7}\right) \cdot \frac{3}{2} = \frac{6}{7}$$

$$\boxed{X_1 \mid Y_1 = 2 \sim \mathcal{N}\left(\frac{8}{7}, \frac{6}{7}\right)}$$

3. Smoothing step: Distribution of $X_0 \mid Y_1 = 2$

The observation model is:

$$Y_1 = \frac{1}{2}X_0 + 1 + \underbrace{\Xi_0 + \sqrt{2}\Sigma_1}_{N(0,3)}$$

Conditional distribution parameters:

$$\mathrm{Var}(X_0 \mid Y_1 = 2) = \left(\frac{1}{2} + \frac{(1/2)^2}{3}\right)^{-1} = \frac{12}{7}$$

$$\mathbb{E}[X_0 \mid Y_1 = 2] = \frac{12}{7}\left(\frac{-1}{2} + \frac{(1/2)(2-1)}{3}\right) = -\frac{4}{7}$$

$$\boxed{X_0 \mid Y_1 = 2 \sim \mathcal{N}\left(-\frac{4}{7}, \frac{12}{7}\right)}$$

4. Pseudo Code for General Model

Model parameters:

$$\begin{aligned} X_{n+1} &= \alpha X_n + \beta \Xi_n, \quad \Xi_n \sim \mathcal{N}(0,1) \\ X_0 &\sim \mathcal{N}(m,1) \\ Y_n &= X_n + \sqrt{2}\Sigma_n, \quad \Sigma_n \sim \mathcal{N}(0,1) \\ \text{Input: } & \alpha, \beta, m, y \ (Y_1 = y) \end{aligned}$$

Algorithm 1 Compute Distributions

function COMPUTEDISTRIBUTIONS(α, β, m, y)
    // **Prediction:** X_1
    pred_mean $\leftarrow \alpha \cdot m$
    pred_var $\leftarrow \alpha^2 \cdot 1 + \beta^2$ ▷ Note: initial variance = 1
    // **Filtering:** $X_1 \mid Y_1 = y$
    $K \leftarrow$ pred_var/(pred_var + 2) ▷ Observation noise variance = 2
    filter_mean $\leftarrow$ pred_mean + $K \cdot (y -$ pred_mean)
    filter_var $\leftarrow (1 - K) \cdot$ pred_var
    // **Smoothing:** $X_0 \mid Y_1 = y$
    denom $\leftarrow \alpha^2 \cdot 1 + \beta^2 + 2$ ▷ $\alpha^2\text{Var}(X_0) + \beta^2$ + obs noise
    $K_s \leftarrow \alpha/$denom
    smooth_mean $\leftarrow m + K_s \cdot (y - \alpha \cdot m)$
    smooth_var $\leftarrow 1 - \alpha \cdot K_s$ ▷ $\text{Var}(X_0)$ - $\alpha K_s \text{Var}(X_0)$
    return (pred_mean, pred_var), (filter_mean, filter_var), (smooth_mean, smooth_var)
end function

Output:

- (i) X_1: $\mathcal{N}$(pred_mean, pred_var)
- (ii) $X_1 \mid Y_1 = y$: $\mathcal{N}$(filter_mean, filter_var)
- (iii) $X_0 \mid Y_1 = y$: $\mathcal{N}$(smooth_mean, smooth_var)

Note: This pseudo code assumes no constant term in the state transition equation. For models with constants (like the +1 in the original problem), additional adjustments would be needed.