ex 3 sheet 7

SDE: $dx_t = \sqrt{2\beta_0}\, \underbrace{dW_t}_{\text{Wiener process (BROWNIAN MOTION)}}$, $x_0 \sim N(0, I)$

$\sqrt{2\beta_0}$: DIFFUSION CONSTANT

2) $X_t = X_0 + \sqrt{2\beta_0}\, W_t$

$W_t \sim N(0, tI)$

$X_0 \sim N(0, I)$

$\sqrt{2\beta_0}\, W_t \sim N(0, 2\beta_0 t I)$

$X_t \sim N(0, I + 2\beta_0 t I)$

- REVERSE TIME PROCESS:

$X_s = X_{T-s}$

the reverse-time SDE of a pure diffusion process is:

$$dx_s = [-2\beta_0 \underbrace{\nabla_x \log \underbrace{p_s(x)}_{\text{PROB. DENSITY of } X_s}}_{\text{SCORE FUNCTION}}]ds + \sqrt{2\beta_0}\, \underbrace{d\bar{W}_s}_{\text{REVERSE TIME WIENER}}$$

- now $s = T - t$, so:

$$X_s \sim N(0, I + 2\beta_0 (T-s) I)$$

- then $p_s(x) = N(x; 0, \Sigma_s)$ then:

$$\log p_s(x) = -\frac{1}{2} x^T \Sigma_s^{-1} x + \text{const}$$

$$\nabla_x \log p_s(x) = -\Sigma_s^{-1} x \rightarrow \text{it's LINEAR!}$$

- Then $\mu_s = 0$

$$\Sigma_s = I + 2\beta_0 (T-s) I$$

b) The reverse SDE is:

$$dx_s = 2\beta_0 \Sigma_s^{-1} x_s \, ds + \sqrt{2\beta_0}\, d\bar{W}_s$$

Now discretize time: $s_k = k\Delta s$; $x_k \sim x_{s_k}$

Then:

$$X_{k+1} = X_k + 2\beta_0 \Sigma_{s_k}^{-1} X_k \Delta s + \sqrt{2\beta_0 \Delta s} \cdot \underbrace{\epsilon_k}_{\sim N(0,1)}$$

EULER MARUYAMA STEP $\Longleftarrow$

$$\boxed{\Sigma_{s_k}^{-1} = \frac{1}{1+2\beta_0(T-s_k)} I}$$