1. Bayesian Inference from Noisy Observation

(a) Finding w such that $[X = x, Y = y]$ is equivalent to $[X = x, W = w]$

We are given:

$$Y = X^2 + W, \quad W \sim \mathcal{N}(0, 1)$$

We want to find the value of w such that the event $[X = x, Y = y]$ is equivalent to $[X = x, W = w]$.

Step 1: Substitute $X = x$ into the equation:

$$Y = x^2 + W \Rightarrow W = Y - x^2$$

Step 2: Now substitute $Y = y$:

$$W = y - x^2$$

Conclusion: The event $[X = x, Y = y]$ is equivalent to $[X = x, W = y - x^2]$.

$$\boxed{w = y - x^2}$$

(b) Deriving the unnormalized posterior $\pi_{X|Y=y}(x)$

We are to derive the unnormalized posterior using Bayes' Theorem:

$$\pi_{X|Y=y}(x) \propto \pi_X(x) \cdot p_{Y|X}(y \mid x)$$

Step 1: Prior of X: Since $X \sim \mathcal{N}(1, 1)$,

$$\pi_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right)$$

Step 2: Likelihood $p_{Y|X}(y \mid x)$:
Given $Y = X^2 + W$ and $W \sim \mathcal{N}(0, 1)$, then conditional on $X = x$,

$$Y \sim \mathcal{N}(x^2, 1) \Rightarrow p_{Y|X}(y \mid x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - x^2)^2}{2}\right)$$

Step 3: Combine prior and likelihood:

$$\pi_{X|Y=y}(x) \propto \exp\left(-\frac{(x-1)^2}{2}\right) \cdot \exp\left(-\frac{(y - x^2)^2}{2}\right)$$

$$\Rightarrow \boxed{\pi_{X|Y=y}(x) \propto \exp\left(-\frac{(x-1)^2 + (y - x^2)^2}{2}\right)}$$