Exercise 1 Solution - Least Squares Fit

Exercise 1

Given the data points $(1, 2)$, $(2, 3)$, and $(3, 5)$, find the least squares line $y = a + bx$ that best fits these points. Compute the values of a and b.

Solution

Step 1: Set Up the System

For the line $y = Mx + B$, the equations for the three points are:

$$\begin{cases} 2 = M(1) + B \\ 3 = M(2) + B \\ 5 = M(3) + B \end{cases}$$

This can be written in matrix form $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} M \\ B \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

Step 2: Solve Using Normal Equations

The normal equations are $A^\top A\mathbf{x} = A^\top \mathbf{b}$.

Compute $A^\top A$:

$$A^\top A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

Compute $A^\top \mathbf{b}$:

$$A^\top \mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 23 \\ 10 \end{pmatrix}$$

Solve the System:

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} M \\ B \end{pmatrix} = \begin{pmatrix} 23 \\ 10 \end{pmatrix}$$

Expand to:

$$\begin{cases} 14M + 6B = 23 \\ 6M + 3B = 10 \end{cases}$$

1. Divide the second equation by 3:

$$2M + B = \frac{10}{3}$$

2. Substitute $B = \frac{10}{3} - 2M$ into the first equation:

$$14M + 6\left(\frac{10}{3} - 2M\right) = 23$$

Simplify:

$$14M + 20 - 12M = 23 \implies 2M = 3 \implies M = \frac{3}{2}$$

3. Solve for B:

$$B = \frac{10}{3} - 2\left(\frac{3}{2}\right) = \frac{10}{3} - 3 = \frac{1}{3}$$

Final Answer

The best-fit line is:

$$y = \boxed{\frac{3}{2}}x + \boxed{\frac{1}{3}}$$