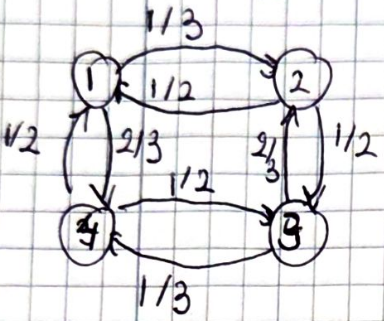
Markov chain

$S = \{1, 2, 3, 4\}$

$$[P_{ij}] = \begin{bmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$ From state j to i



a) $X_0 = 1 \rightarrow p_0 = [1 \quad 0 \quad 0 \quad 0]^T$

$$p_1 = [P_{ij}] \cdot p_0 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$p_2 = [P_{ij}]^2 \cdot p_0 = \begin{bmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

b) Convergence of X_n as $n \rightarrow \infty$

Key properties of $[P_{ij}]$

1. All states communicate
2. Periodic with period 2 $\rightarrow$ cycles of even lengths dominate

$$p_n = \begin{cases} [0 \quad 1/2 \quad 0 \quad 1/2]^T & \text{for } n \text{ odd} \\ [1/2 \quad 0 \quad 1/2 \quad 0]^T & \text{for } n \text{ even} \end{cases}$$

$$p_{n+1}(1) + p_{n+1}(3) = p_{n+1}(2) + p_{n+1}(4)$$

$$\Rightarrow \quad p_0(1) + p_0(3) \neq p_0(2) + p_0(4)$$

The distribution of X_n does **not** converge to a single limit.

Problem 2 :- given Π_0 [initial distribution]

$$\Pi_n = P^n \Pi_0 \quad , \quad \hat{\Pi}_n = \underbrace{(S_M P)^n}_{\text{Sampling of } P} \underbrace{S_M \Pi_0}_{\text{sample of } \Pi_0}$$

I) Identify Π_{n+1} & $\hat{\Pi}_{n+1}$

$$\Pi_{n+1} = P \cdot \Pi_n \quad , \quad \hat{\Pi}_{n+1} = S_M \cdot P \cdot \hat{\Pi}_n$$

II) The distance $d(\Pi_{n+1}, \hat{\Pi}_{n+1})$ can be identified as:-

$$d(\Pi_{n+1}, \hat{\Pi}_{n+1}) = d(P\Pi_n, S_M \cdot P\hat{\Pi}_n)$$

using triangular Inquality with $P\hat{\Pi}_n$ as intermidiate.

$$d(\Pi_{n+1}, \hat{\Pi}_{n+1}) = d(P\Pi_n, S_M P\hat{\Pi}_n) \leq d(P\Pi_n, P\hat{\Pi}_n) + d(P\hat{\Pi}_n, S_M P\hat{\Pi}_n)$$

III) break down of R.H.S of Inquality :-

1) $d(P\Pi_n, P\hat{\Pi}_n) \leq d(\Pi_n, \hat{\Pi}_n)$ Property of $\underline{P}$

2) $d(P\hat{\Pi}_n, S_M P\hat{\Pi}_n) \leq \frac{1}{M}$ Property of S_M Sampling

+ Subsitute in the inquality :-

$$\therefore d(\Pi_{n+1}, \hat{\Pi}_{n+1}) \leq d(P\Pi_n, P\hat{\Pi}_n) + d(P\hat{\Pi}_n, S_M P \cdot \hat{\Pi}_n)$$

$$\leq d(\Pi_n, \hat{\Pi}_n) + \frac{1}{M} \quad \#$$

ex 3 sheet 7

SDE: $dx_t = \sqrt{2\beta_0}\, dW_t$, $x_0 \sim N(0, I)$

$\sqrt{2\beta_0}$ → DIFFUSION CONSTANT

dW_t → Wiener process (BROWNIAN MOTION)

2) $X_t = X_0 + \sqrt{2\beta_0}\, W_t$

$$\begin{cases} W_t \sim N(0, tI) \\ X_0 \sim N(0, I) \\ \sqrt{2\beta_0}\, W_t \sim N(0, 2\beta_0 t I) \end{cases}$$

$X_t \sim N(0, I + 2\beta_0 t I)$ ←

- REVERSE TIME PROCESS:

$X_s = X_{T-s}$ the reverse-time SDE of a pure diffusion process is:

$$dx_s = [-2\beta_0 \underbrace{\nabla_x \log \underbrace{p_s(x)}_{\text{PROB. DENSITY of } X_s}}_{\text{SCORE FUNCTION}}]\,ds + \sqrt{2\beta_0}\, \underbrace{d\bar{W}_s}_{\text{REVERSE TIME WIENER}}$$

- now $s = T - t$, so:

$X_s \sim N(0, I + 2\beta_0 (T-s) I)$

- then $p_s(x) = N(x; 0, \Sigma_s)$ then:

$$\log p_s(x) = -\frac{1}{2} x^T \Sigma_s^{-1} x + \text{const}$$

↓

$$\nabla_x \log p_s(x) = -\Sigma_s^{-1} x \rightarrow \text{it's LINEAR!}$$

- Then $\mu_s = 0$

$\Sigma_s = I + 2\beta_0 (T - s) I$

b) The reverse SDE is:

$$dx_s = 2\beta_0 \Sigma_s^{-1} x_s ds + \sqrt{2\beta_0}\, d\bar{W}_s$$

Now discretize time: $s_k = k\Delta s$; $x_k \sim x_{s_k}$

Then: $$X_{k+1} = X_k + 2\beta_0 \Sigma_{s_k}^{-1} X_k \Delta s + \sqrt{2\beta_0 \Delta s} \cdot \underbrace{\epsilon_k}_{\sim \mathcal{N}(0,1)}$$

EULER MARUYAMA STEP $\Longleftarrow$ $$\boxed{\Sigma_{s_k}^{-1} = \frac{1}{1+2\beta_0(T-s_k)} I}$$