Monte Carlo Estimation

2. Monte Carlo Estimation (6 Points)

Let $X \sim \mathcal{N}(1, 3)$, and $f(x) = 1 + 2x + x^2$.

(a) Analytical Computation

Expectation of $f(X)$:

$$\begin{aligned} \mathbb{E}[f(X)] &= \mathbb{E}[1 + 2X + X^2] \\ &= 1 + 2\mathbb{E}[X] + \mathbb{E}[X^2]. \end{aligned}$$

Given $X \sim \mathcal{N}(1, 3)$:

- $\mathbb{E}[X] = 1$
- $\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = 3 + 1^2 = 4$

Substituting these values:

$$\mathbb{E}[f(X)] = 1 + 2(1) + 4 = 7.$$

Variance of $f(X)$:
Rewrite $f(X)$ as:

$$f(X) = (X - 1)^2 + 4(X - 1) + 4.$$

Let $Y = X - 1$, so $Y \sim \mathcal{N}(0, 3)$. Then:

$$f(X) = Y^2 + 4Y + 4.$$

The variance is unaffected by the constant term:

$$\mathrm{Var}(f(X)) = \mathrm{Var}(Y^2 + 4Y).$$

Since Y is a zero-mean Gaussian, Y^2 and Y are uncorrelated ($\mathrm{Cov}(Y^2, Y) = 0$):

$$\mathrm{Var}(Y^2 + 4Y) = \mathrm{Var}(Y^2) + \mathrm{Var}(4Y).$$

- $\mathrm{Var}(4Y) = 16\mathrm{Var}(Y) = 16 \times 3 = 48$
- For $Y \sim \mathcal{N}(0, \sigma^2)$, $\mathrm{Var}(Y^2) = 2\sigma^4$. Here, $\sigma^2 = 3$, so $\mathrm{Var}(Y^2) = 2(3^2) = 18$

Adding the variances:

$$\mathrm{Var}(f(X)) = 18 + 48 = 66.$$

Final Answer

(a)

$$\mathbb{E}[f(X)] = \boxed{7}, \quad \mathrm{Var}[f(X)] = \boxed{66}.$$