Exercise 1 – ANOVA Decomposition

Given

Let

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$

with $(x_1, x_2) \sim \text{Uniform}([0,1]^2)$.

We perform the ANOVA decomposition:

$$f(x_1, x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2)$$

calculate f_0:

$$f_0 = \mathbb{E}[f(x_1, x_2)] = \int_0^1 \int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\, dx_1 dx_2 = \frac{15}{2}$$

$f_1(x_1)$ and $f_2(x_2)$

$$f_1(x_1) = \mathbb{E}_{x_2}[f(x_1, x_2)] - f_0 = (12x_1 + 3) - \frac{15}{2} = 9x_1 - \frac{9}{2}$$

$$f_2(x_2) = \mathbb{E}_{x_1}[f(x_1, x_2)] - f_0 = (6x_2 + 3) - \frac{15}{2} = 3x_2 - \frac{3}{2}$$

compute $f_{12}(x_1, x_2)$

$$\begin{aligned} f_{12}(x_1, x_2) &= f(x_1, x_2) - f_0 - f_1(x_1) - f_2(x_2) \\ &= 12x_1 + 6x_2 - 6x_1x_2 - \frac{15}{2} - (9x_1 - \frac{9}{2}) - (3x_2 - \frac{3}{2}) \\ &= 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2} \end{aligned}$$

Determinate Variance Terms

Following the definition from the textbook:

$$\sigma_u^2 = \int (f_u(x_u) - \mathbb{E}[f_u])^2\, dx_u$$

Univariate terms

$$\mathbb{E}[f_1] = \int_0^1 \left(9x_1 - \frac{9}{2}\right) dx_1 = 0 \Rightarrow \sigma_1^2 = \int_0^1 \left(9x_1 - \frac{9}{2}\right)^2 dx_1 = \frac{27}{4}$$

$$\mathbb{E}[f_2] = \int_0^1 \left(3x_2 - \frac{3}{2}\right) dx_2 = 0 \Rightarrow \sigma_2^2 = \int_0^1 \left(3x_2 - \frac{3}{2}\right)^2 dx_2 = \frac{3}{4}$$

Interaction term

$$f_{12}(x_1, x_2) = 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}, \quad \mathbb{E}[f_{12}] = 0$$

Compute:

$$\sigma_{12}^2 = \iint_{[0,1]^2} \left(3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}\right)^2 dx_1 dx_2 = \frac{1}{4}$$

Final Results

$$f(x_1, x_2) = \frac{15}{2} + (9x_1 - \frac{9}{2}) + (3x_2 - \frac{3}{2}) + (3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2})$$

$$\sigma_1^2 = \frac{27}{4}, \quad \sigma_2^2 = \frac{3}{4}, \quad \sigma_{12}^2 = \frac{1}{4}$$

$$\mathrm{Var}(f) = \sigma_1^2 + \sigma_2^2 + \sigma_{12}^2 = \boxed{8}$$