1 Algorithm basis

Consider a HMM where $(X_t)_t$ represents the hidden variables and $(Y_t)_t$ the observed ones. Let's consider a sequence $(\mathbf{x}, \mathbf{y})$ of length T of this model. The joint probability is: $P(\mathbf{x}, \mathbf{y}) = P(x_1) \prod_{t=2}^{T} P(x_t|x_{t-1}) \prod_{t=1}^{T} P(y_t|x_t)$.

We will see that all what the information we need to solve our problem is encapsulated in the functions defined below.

$\forall t \in [1, T-1] : f_{t,t+1}(s, s') = max_{\{\mathbf{x}|(x_t, x_{t+1})=(s,s')\}} P(\mathbf{x}, \mathbf{y})$

To compute these functions we use the Max Product algorithm, it is based on the computation of two functions sequences α_t and β_t defined recursively by:

$\alpha_t(s) = P(Y_t = y_t|X_t = s) max_{s'}\{(P(X_t = s|X_{t-1} = s')\alpha_{t-1}(s'))\} \forall t \in [2, T]$

$\beta_t(s) = max_{s'}\{P(X_{t+1} = s'|X_t = s)P(Y_{t+1} = y_{t+1}|X_{t+1} = s)\beta_{t+1}(s')\} \forall t \in [1, T-1]$

$\alpha_0(s) = P(Y_0 = y_0|X_0 = s)P(X_0 = s)$

$\beta_T(s) = 1$