Classic HMM tutorial – see class website:
*L. R. Rabiner, "A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition," Proc. of the IEEE, Vol.77, No.2, pp.257--286, 1989.

Time series, HMMs, Kalman Filters

Machine Learning – 10701/15781
Carlos Guestrin
Carnegie Mellon University

March 28th, 2005

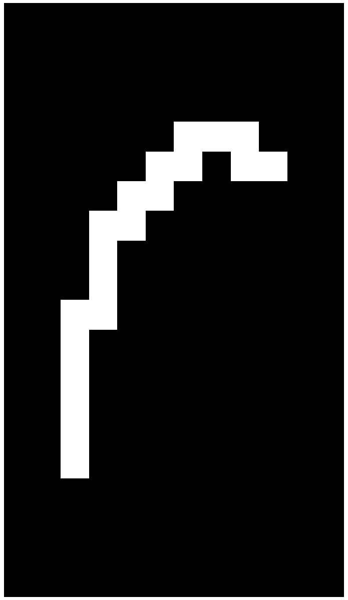
Adventures of our BN hero

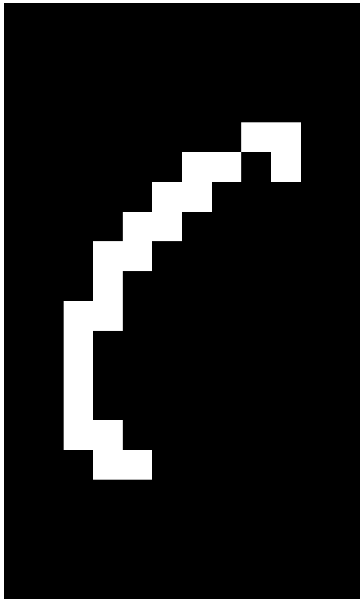
- Compact representation for probability distributions
- Fast inference
- Fast learning
- But… Who are the most popular kids?

1. Naïve Bayes

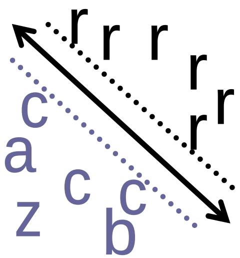
2 and 3.
Hidden Markov models (HMMs)
Kalman Filters

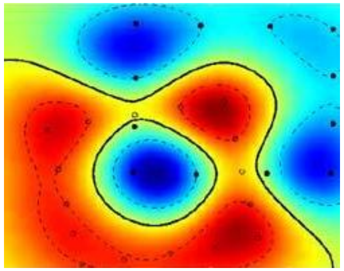
Handwriting recognition



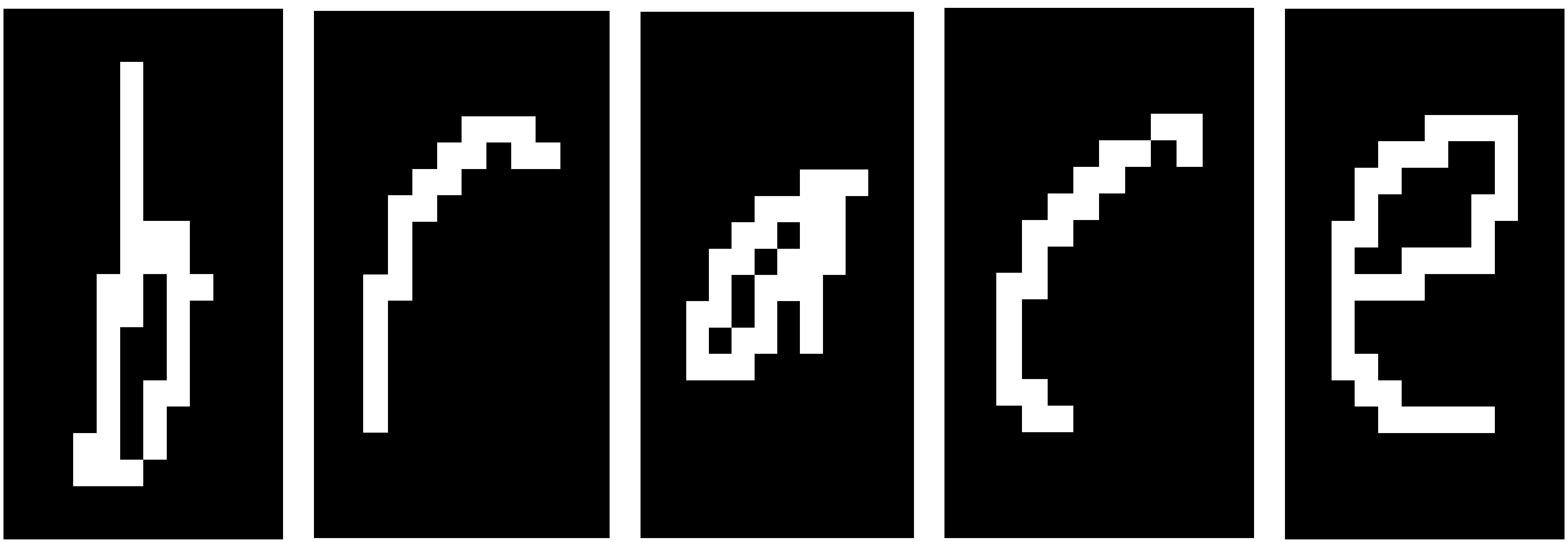


Character recognition, e.g., kernel SVMs

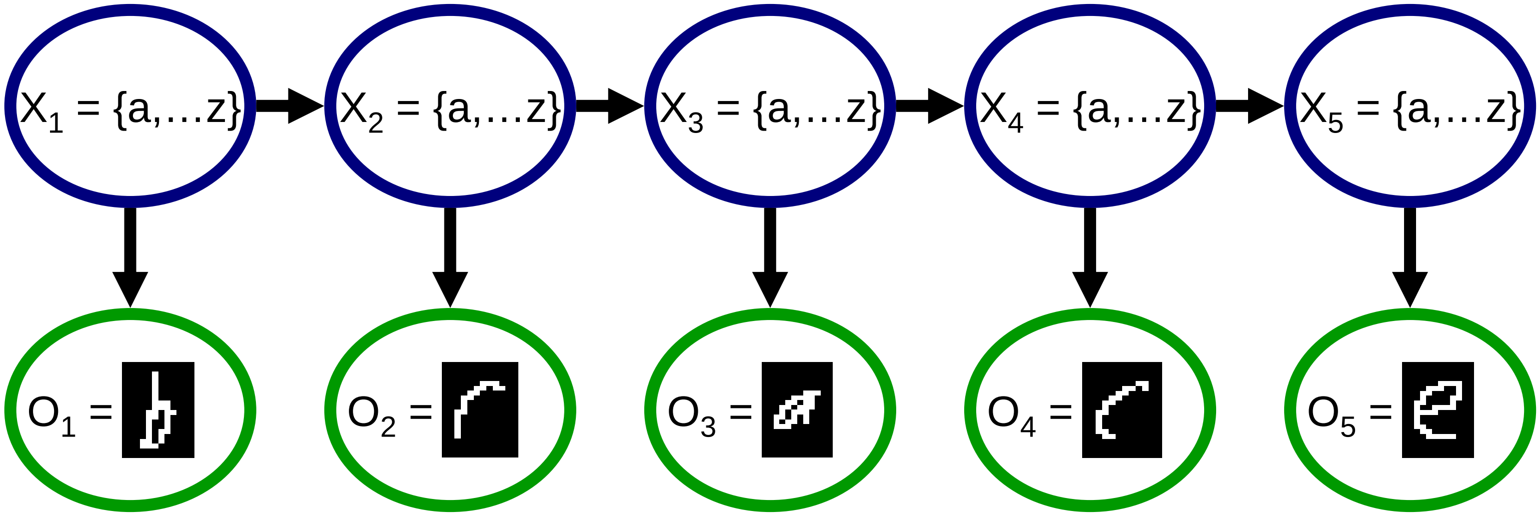




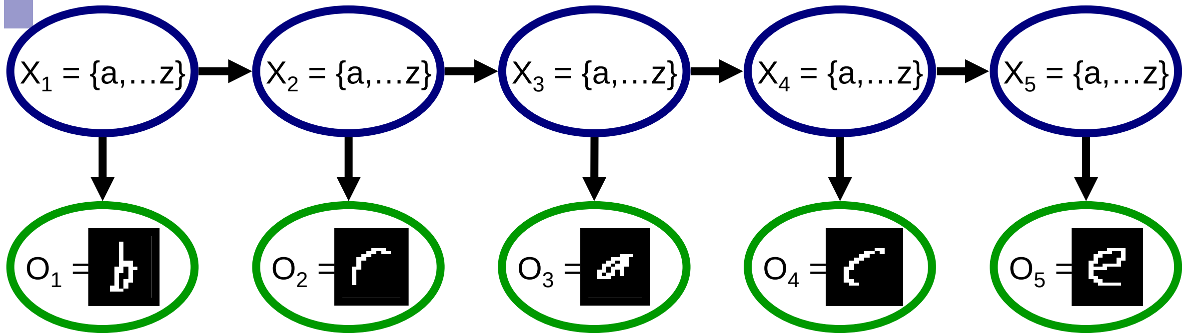
Example of a hidden Markov model (HMM)



Understanding the HMM Semantics



HMMs semantics: Details



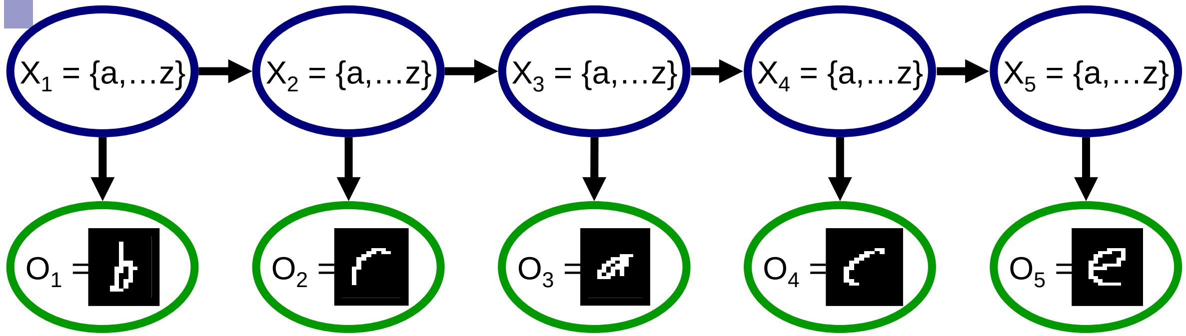
Just 3 distributions:

$$P(X_1)$$

$$P(X_i \mid X_{i-1})$$

$$P(O_i \mid X_i)$$

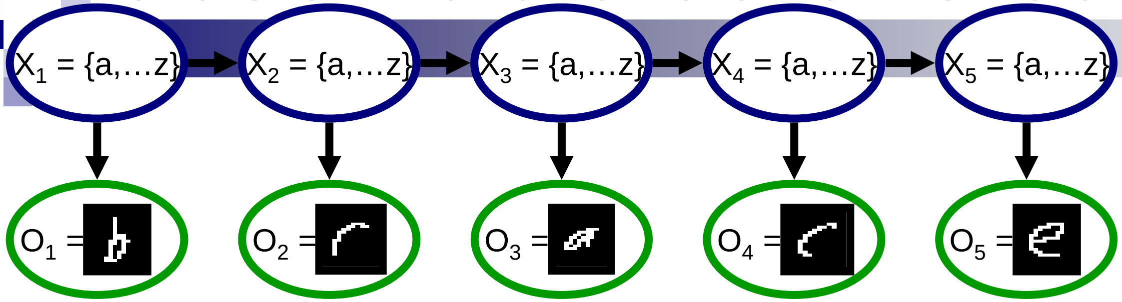
HMMs semantics: Joint distribution



$$P(X_1)$$
$$P(X_i \mid X_{i-1})$$
$$P(O_i \mid X_i)$$

$$P(X_1, \ldots, X_n \mid o_1, \ldots, o_n) = P(X_{1:n} \mid o_{1:n})$$
$$\propto P(X_1)P(o_1 \mid X_1) \prod_{i=2}^{n} P(X_i \mid X_{i-1})P(o_i \mid X_i)$$

Learning HMMs from fully observable data is easy



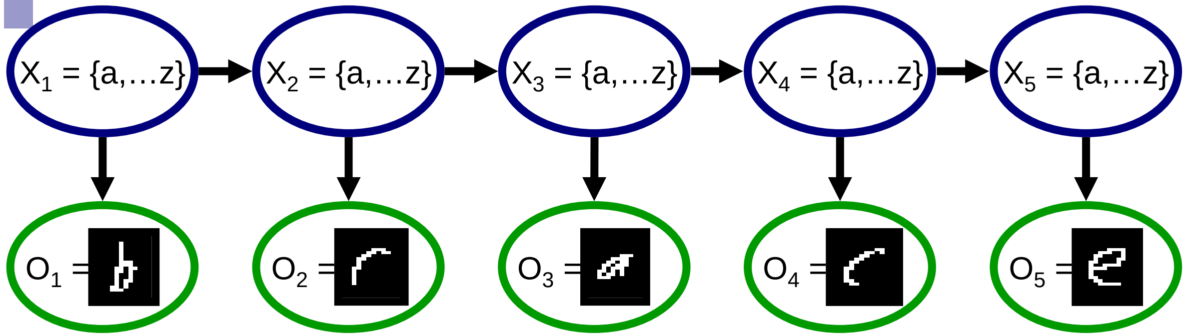
Learn 3 distributions:

$$P(X_1)$$

$$P(O_i \mid X_i)$$

$$P(X_i \mid X_{i-1})$$

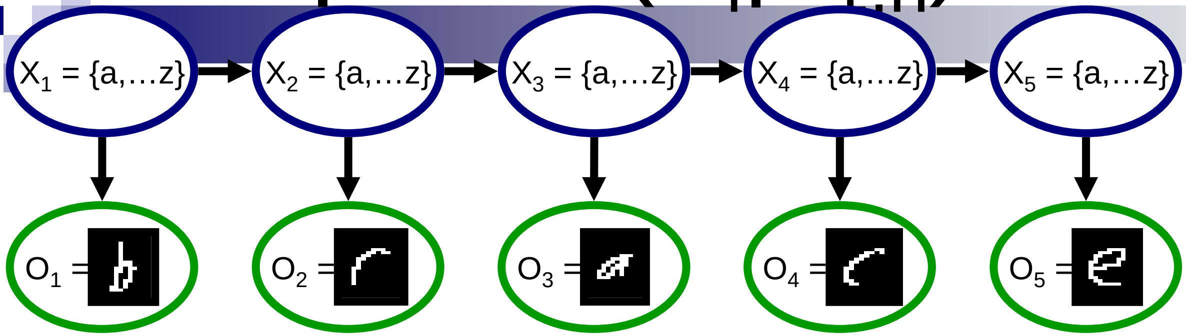
Possible inference tasks in an HMM



Marginal probability of a hidden variable:

Viterbi decoding – most likely trajectory for hidden vars:

Using variable elimination to compute $P(X_i|o_{1:n})$



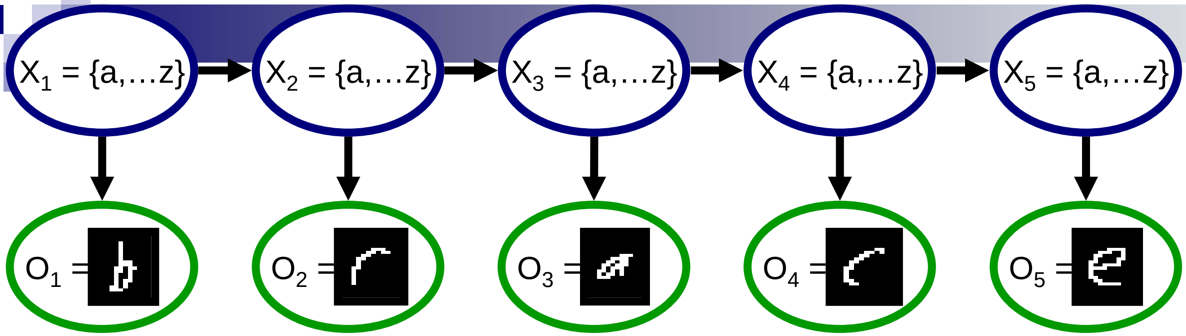
Compute:

$$P(X_i \mid o_{1..n})$$

Variable elimination order?

Example:

What if I want to compute $P(X_i|o_{1:n})$ for each i?



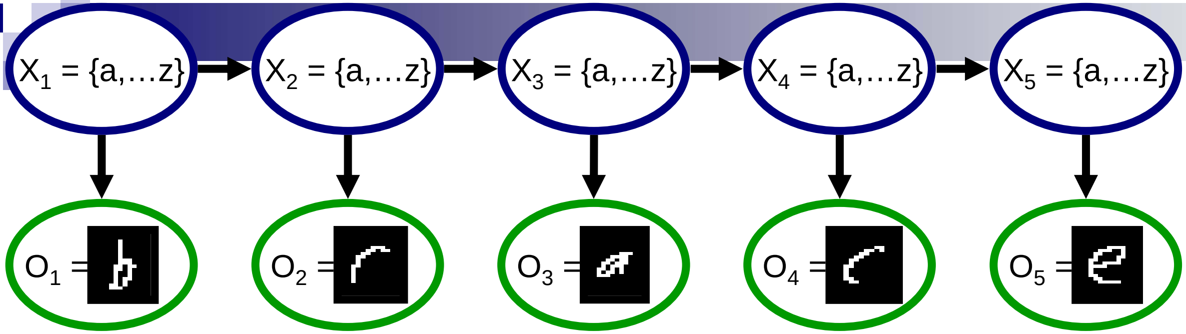
Compute:

$$P(X_i \mid o_{1..n})$$

Variable elimination for each i?

Variable elimination for each i, what's the complexity?

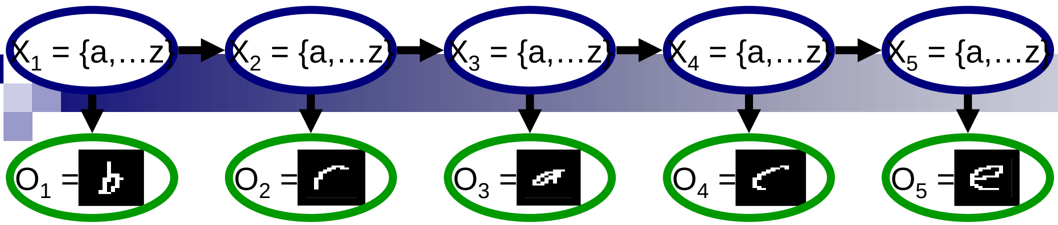
Reusing computation



Compute:

$$P(X_i \mid o_{1..n})$$

The forwards-backwards algorithm



$$P(X_i \mid o_{1..n})$$

- Initialization: $\alpha_1(X_1) = P(X_1)P(o_1 \mid X_1)$
- For i = 2 to n
  - Generate a forwards factor by eliminating X_{i-1}

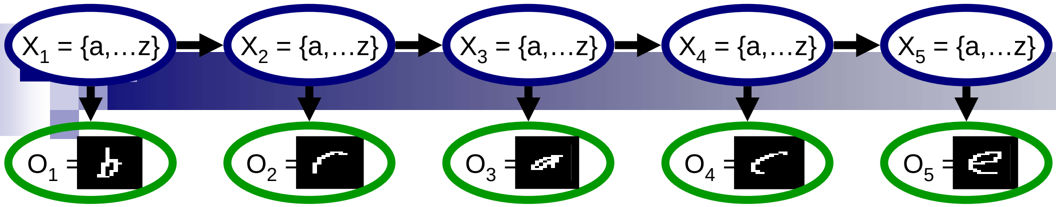
$$\alpha_i(X_i) = \sum_{x_{i-1}} P(o_i \mid X_i) P(X_i \mid X_{i-1} = x_{i-1}) \alpha_{i-1}(x_{i-1})$$

- Initialization: $\beta_n(X_n) = 1$
- For i = n-1 to 1
  - Generate a backwards factor by eliminating X_{i+1}

$$\beta_i(X_i) = \sum_{x_{i+1}} P(o_{i+1} \mid x_{i+1}) P(x_{i+1} \mid X_i) \beta_{i+1}(x_{i+1})$$

- ∀ i, probability is: $\boxed{P(X_i \mid o_{1..n}) = \alpha_i(X_i)\beta_i(X_i)}$

Most likely explanation

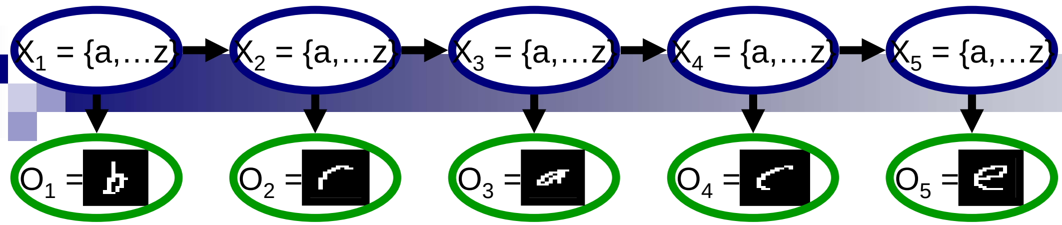


Compute:

Variable elimination order?

Example:

The Viterbi algorithm



- Initialization: $\alpha_1(X_1) = P(X_1)P(o_1 \mid X_1)$
- For i = 2 to n
  - Generate a forwards factor by eliminating X_{i-1}

$$\alpha_i(X_i) = \max_{x_{i-1}} P(o_i \mid X_i)P(X_i \mid X_{i-1} = x_{i-1})\alpha_{i-1}(x_{i-1})$$

- Computing best explanation: $x_n^* = \operatorname{argmax}_{x_n} \alpha_n(x_n)$
- For i = n-1 to 1
  - Use argmax to get explanation:

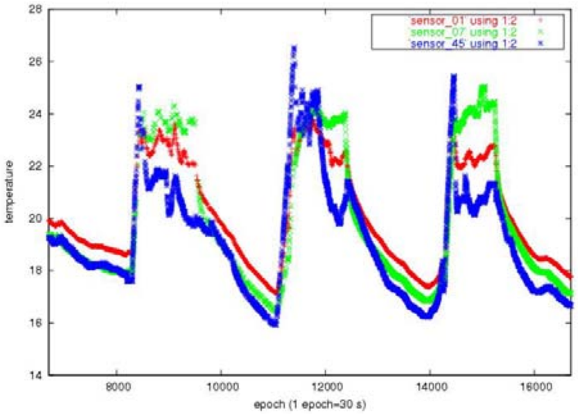
$$x_i^* = \operatorname{argmax}_{x_i} P(x_{i+1}^* \mid x_i)\alpha_i(x_i)$$

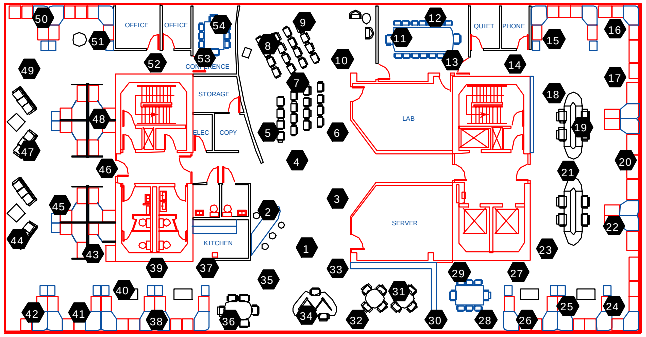
What about continuous variables?

- In general, very hard!
  - Must represent complex distributions

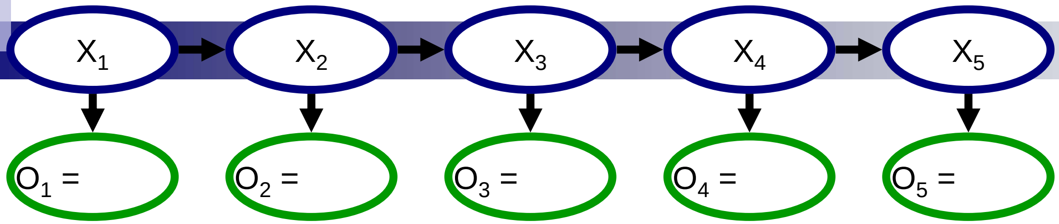
- A special case is very doable
  - When everything is Gaussian
  - Called a Kalman filter
  - One of the most used algorithms in the history of probabilities!

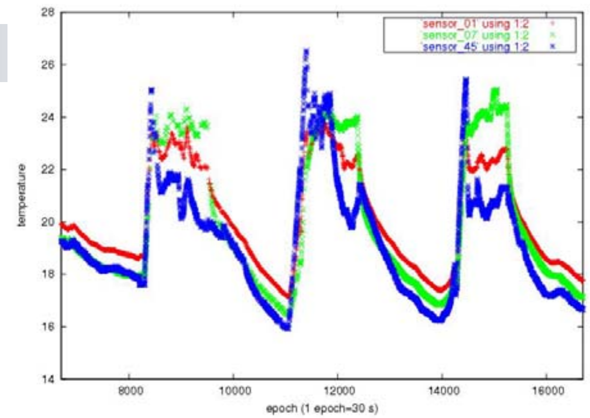
Time series data example: Temperatures from sensor network





Operations in Kalman filter

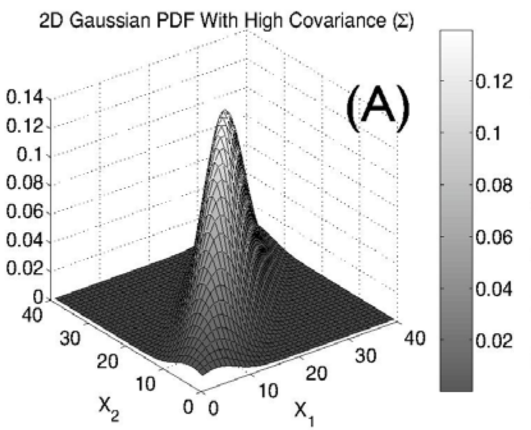


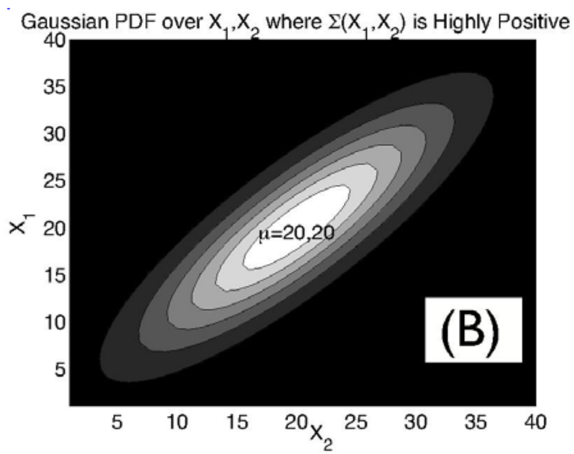


- Compute $p(X_t \mid O_{1:t} = o_{1:t})$
- Start with $p(X_0)$
- At each time step *t*:
  - **Condition** on observation
    $$p(X_t \mid o_{1:t}) \propto p(X_t \mid o_{1:t-1}) p(o_t \mid X_t)$$
  - **Roll-up** (marginalize previous time step)

$$p(X_{t+1} \mid o_{1:t}) = \int_{X_t} P(X_{t+1} \mid x_t) p(x_t \mid o_{1:t}) dx_t$$

Detour: Understanding Multivariate Gaussians

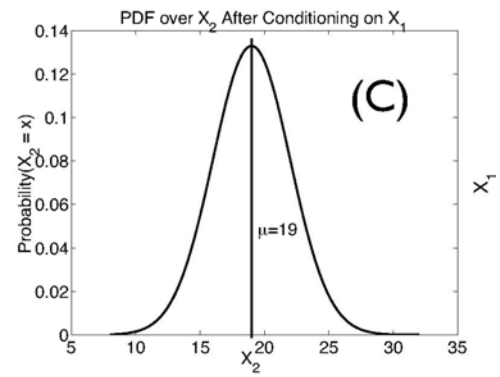




Observe attributes

Example: Observe X_1=18

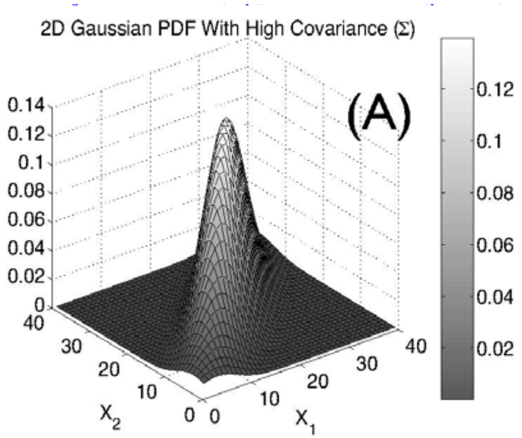
$P(X_2|X_1=18)$



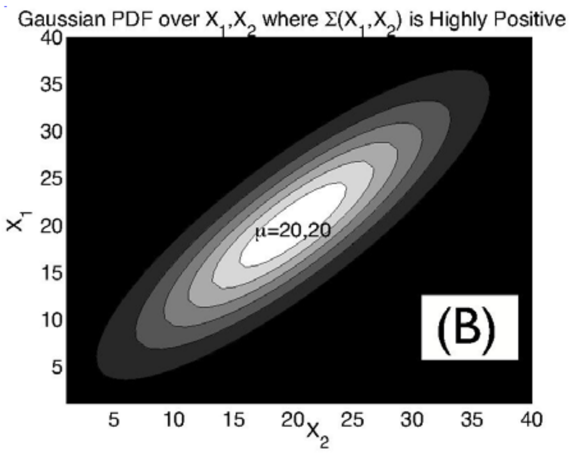
Characterizing a multivariate Gaussian

$$p(X_1, \ldots, X_n) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right\}$$

Mean vector:

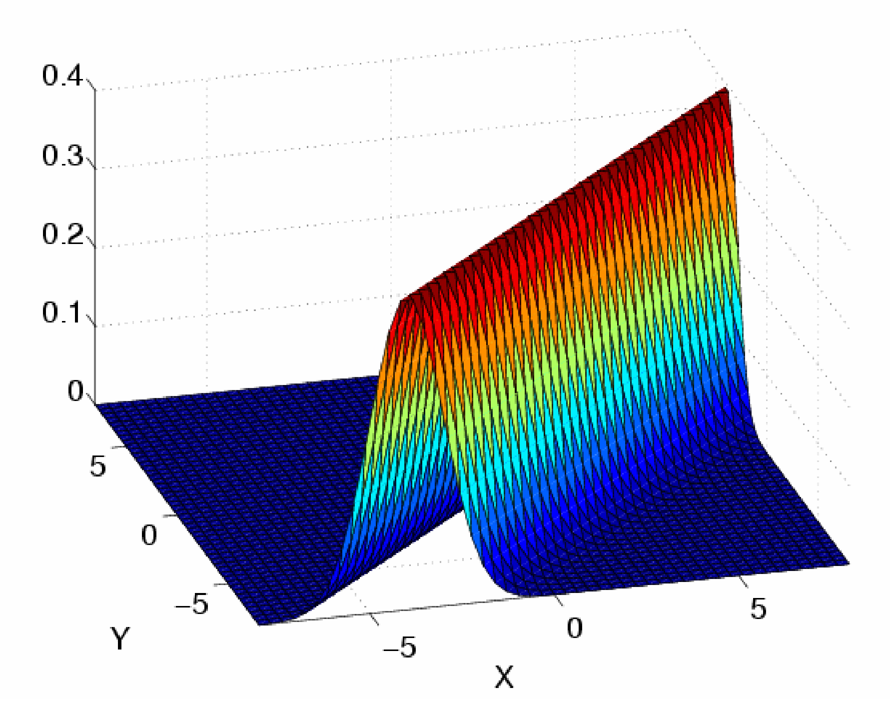


Covariance matrix:

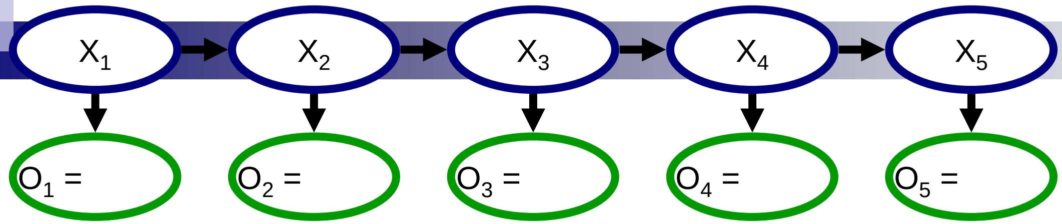


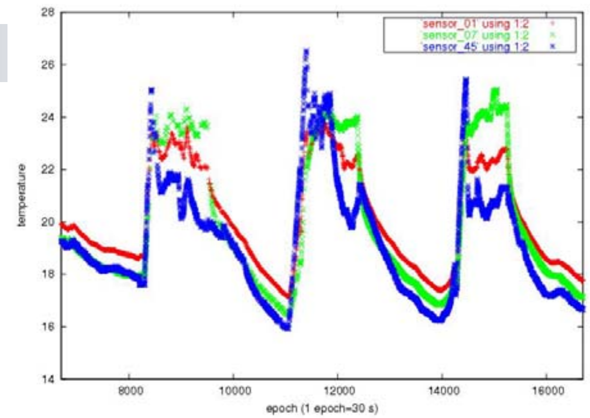
Conditional Gaussians

- Conditional probabilities
  - P(Y|X)



Kalman filter with Gaussians





$$P(X_1)$$

$$P(O_i \mid X_i)$$

$$P(X_i \mid X_{i-1})$$

- Equivalent to a linear system

Detour2: Canonical form

$$
\begin{aligned}
p(X_1, \ldots, X_n) &= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right\} \\
&= K \exp\left\{\eta^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\Lambda^{-1}\mathbf{x}\right\}
\end{aligned}
$$

- Standard form and canonical forms are related:

$$
\begin{aligned}
\mu &= \Lambda^{-1}\eta \\
\Sigma &= \Lambda^{-1}
\end{aligned}
$$

- Conditioning is easy in canonical form
- Marginalization easy in standard form

Conditioning in canonical form

$$p(X_t \mid o_{1:t}) \propto p(X_t \mid o_{1:t-1})p(o_t \mid X_t)$$

- First multiply: $p(A,B) = p(A)p(B \mid A)$

$p(A): \quad \eta_1,\ \Lambda_1$

$p(B \mid A): \quad \eta_2,\ \Lambda_2$

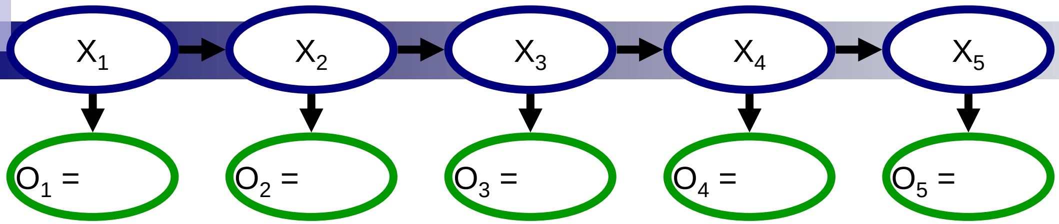
$p(A,B): \quad \eta_3 = \eta_1 + \eta_2,\ \Lambda_3 = \Lambda_1 + \Lambda_2$

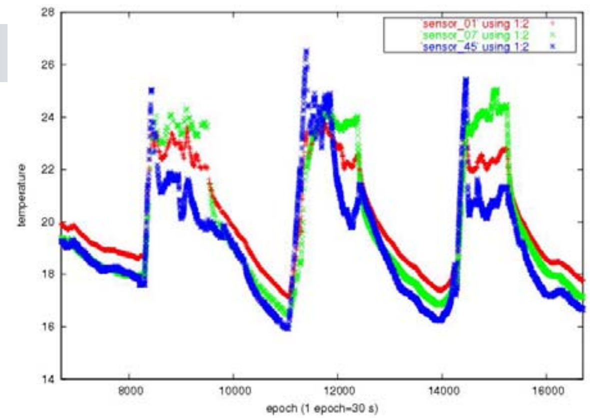
- Then, condition on value B = y $\quad p(A \mid B = y)$

$$\eta_{A|B=y} = \eta_A - \Lambda_{AB} . y$$

$$\Lambda_{AA|B=y} = \Lambda_{AA}$$

Operations in Kalman filter





- Compute $p(X_t \mid O_{1:t} = o_{1:t})$
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$$p(X_{t+1} \mid o_{1:t}) = \int_{X_t} P(X_{t+1} \mid x_t) p(x_t \mid o_{1:t}) dx_t$$

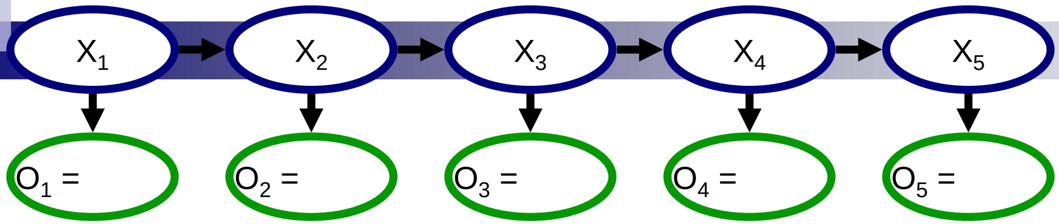
Roll-up in canonical form

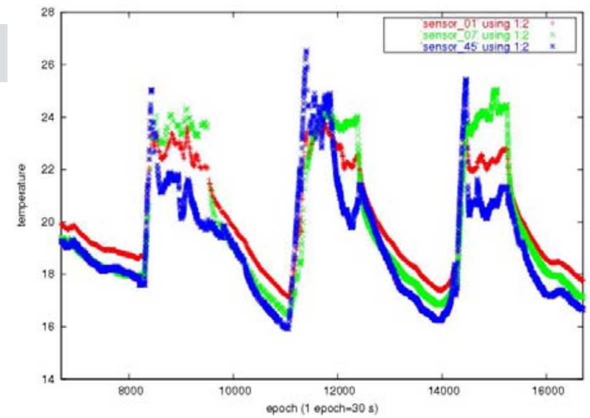
$$p(X_{t+1} \mid o_{1:t}) = \int_{X_t} P(X_{t+1} \mid x_t) p(x_t \mid o_{1:t}) dx_t$$

- First multiply: $p(A, B) = p(A)p(B \mid A)$
- Then, marginalize X_t: $p(A) = \int_B P(A, b) db$

$$\begin{aligned} \eta_A^m &= \eta_A - \Lambda_{AB}\Lambda_{BB}^{-1}\eta_B \\ \Lambda_{AA}^m &= \Lambda_{AA} - \Lambda_{AB}\Lambda_{BB}^{-1}\Lambda_{BA} \end{aligned}$$

Operations in Kalman filter





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Learning a Kalman filter

- Must learn: $P(X_1)$

$$P(O_i \mid X_i) = \frac{P(O_i, X_i)}{P(O_i)}$$

$$P(X_i \mid X_{i-1}) = \frac{P(X_i, X_{i-1})}{P(X_{i-1})}$$

- Learn joint, and use division rule:

$p(A): \quad \eta_1, \ \Lambda_1$

$p(A, B): \quad \eta_2, \ \Lambda_2$

$$p(B \mid A) = \frac{p(A, B)}{p(A)}: \quad \eta_3 = \eta_2 - \eta_1, \ \Lambda_3 = \Lambda_2 - \Lambda_1$$

Maximum likelihood learning of a multivariate Gaussian

$$\mu = \Lambda^{-1}\eta$$
$$\Sigma = \Lambda^{-1}$$

- Data: $< x_1^{(j)}, \ldots, x_n^{(j)} >$

- Means are just empirical means:

$$\widehat{\mu}_i = \frac{\sum_{j=1}^{m} x_i^{(j)}}{m}$$

- Empirical covariances:

$$\widehat{\Sigma}_{ik} = \frac{\sum_{j=1}^{m} (x_i^{(j)} - \widehat{\mu}_i)(x_k^{(j)} - \widehat{\mu}_k)}{m}$$

What you need to know

- Hidden Markov models (HMMs)
  - Very useful, very powerful!
  - Speech, OCR,…
  - Parameter sharing, only learn 3 distributions
  - Trick reduces inference from $O(n^2)$ to $O(n)$
  - Special case of BN
- Kalman filter
  - Continuous vars version of HMMs
  - Assumes Gaussian distributions
  - Equivalent to linear system
  - Simple matrix operations for computations