ETH zürich

Optimization in Energy Systems

Economic Dispatch, Power Flow Modeling

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Learning Objectives

- After this lecture, you should be able
  - to write down and solve an Economic Dispatch problem given cost functions for generators and the total load
  - to set up the AC and DC power flow equations for a system if given the grid parameters

Optimization Problems

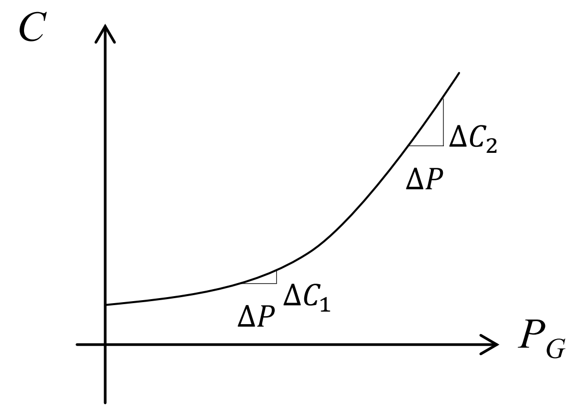
- Economic Dispatch
  - Find the generation dispatch which minimizes overall generation cost while supplying all loads

  $\Rightarrow$ Neglect grid

- Optimal Power Flow
  - Find the settings for the controllable variables, e.g. generation output, which minimizes the objective function, e.g. overall supply cost, taking into account the power flow equations and operational constraints.

  $\Rightarrow$ Take grid into account

Economic Dispatch



- Generation Cost (CHF/hr)

$$C_i(P_i) = a_i P_i^2 + b_i P_i + c_i$$

- Objective Function

$$C_T = \sum_{i=1}^{N} C_i = C_1(P_1) + C_2(P_2) + \ldots + C_N(P_N)$$

- Constraint

$$P_1 + P_2 + \ldots + P_N = P_T = P_L$$

Economic Dispatch

- Example
  - Two Generators:

$$C_1(P_{G1}) = 900 + 45P_{G1} + 0.01P_{G1}^2$$

$$C_2(P_{G2}) = 2500 + 43P_{G2} + 0.003P_{G2}^2$$

  - Load:

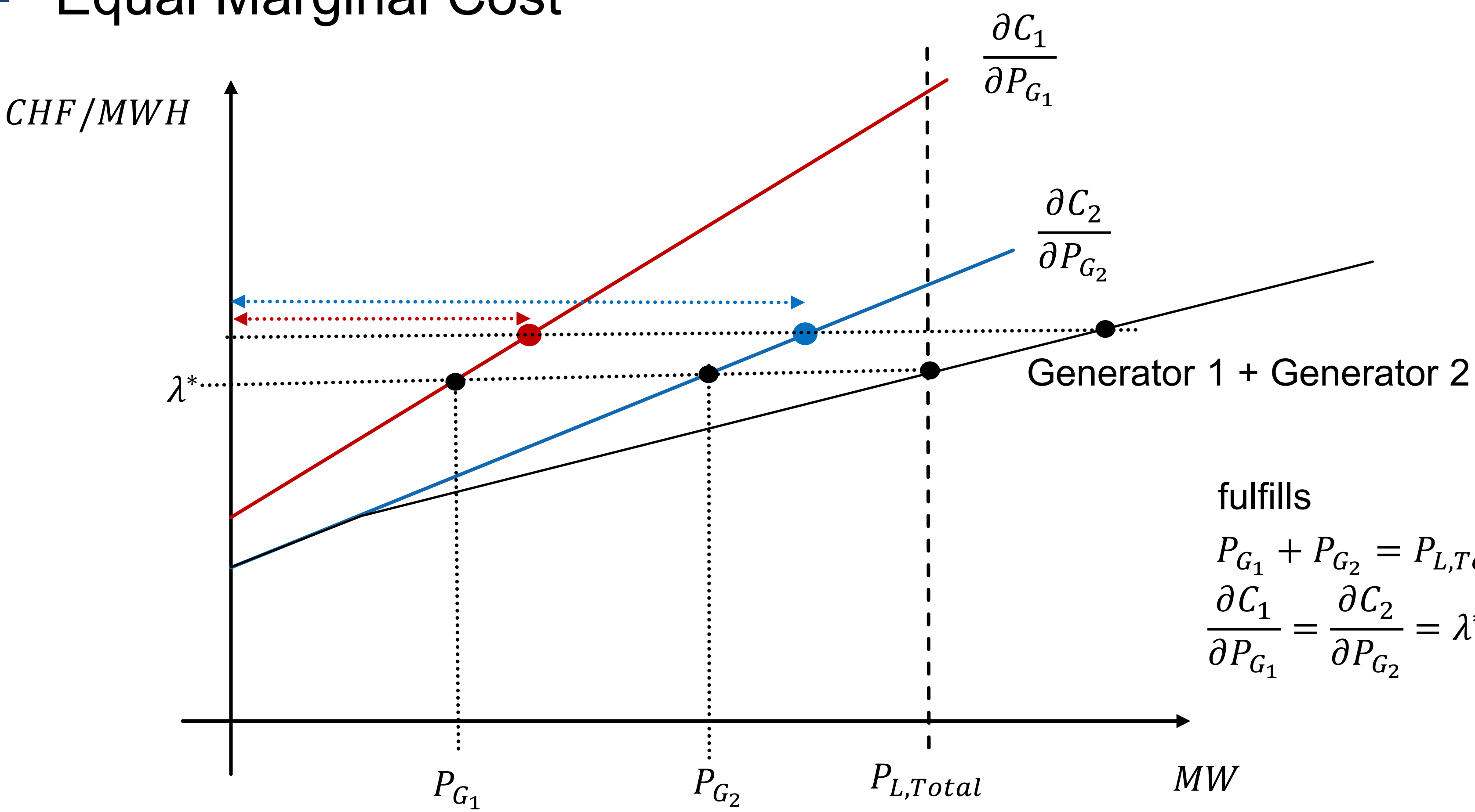
$$P_L = 700\ MW$$

$\Rightarrow$Solution (no limits on generation): $\frac{dC_1}{dP_1} = \frac{dC_2}{dP_2} = \ldots = \frac{dC_N}{dP_N} = \lambda$

$\Rightarrow$Lagrange Multiplier corresponds to price!

Economic Dispatch

- Equal Marginal Cost



fulfills

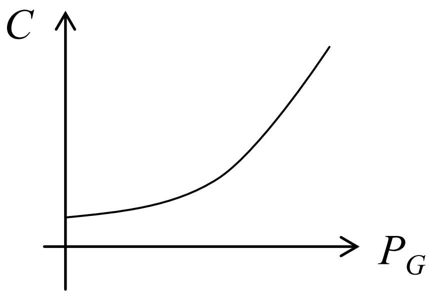
$$P_{G_1} + P_{G_2} = P_{L,Total}$$

$$\frac{\partial C_1}{\partial P_{G_1}} = \frac{\partial C_2}{\partial P_{G_2}} = \lambda^*$$

Economic Dispatch

- Generation Cost (CHF/hr)

$$C_i(P_i) = a_i P_i^2 + b_i P_i + c_i$$



- Objective Function

$$C_T = \sum_{i=1}^{N} C_i = C_1(P_1) + C_2(P_2) + \ldots + C_N(P_N)$$

- Constraints

$$P_1 + P_2 + \ldots + P_N = P_T$$

$$P_{i,min} < P_i < P_{i,max}$$

Economic Dispatch

- Example
  - Two Generators:

$$C_1(P_{G1}) = 900 + 45P_{G1} + 0.01P_{G1}^2 \qquad 0MW \leq P_{G1} \leq 200MW$$

$$C_2(P_{G2}) = 2500 + 43P_{G2} + 0.003P_{G2}^2 \qquad 0MW \leq P_{G2} \leq 600MW$$

  - Load:

$$P_L = 700\ MW$$

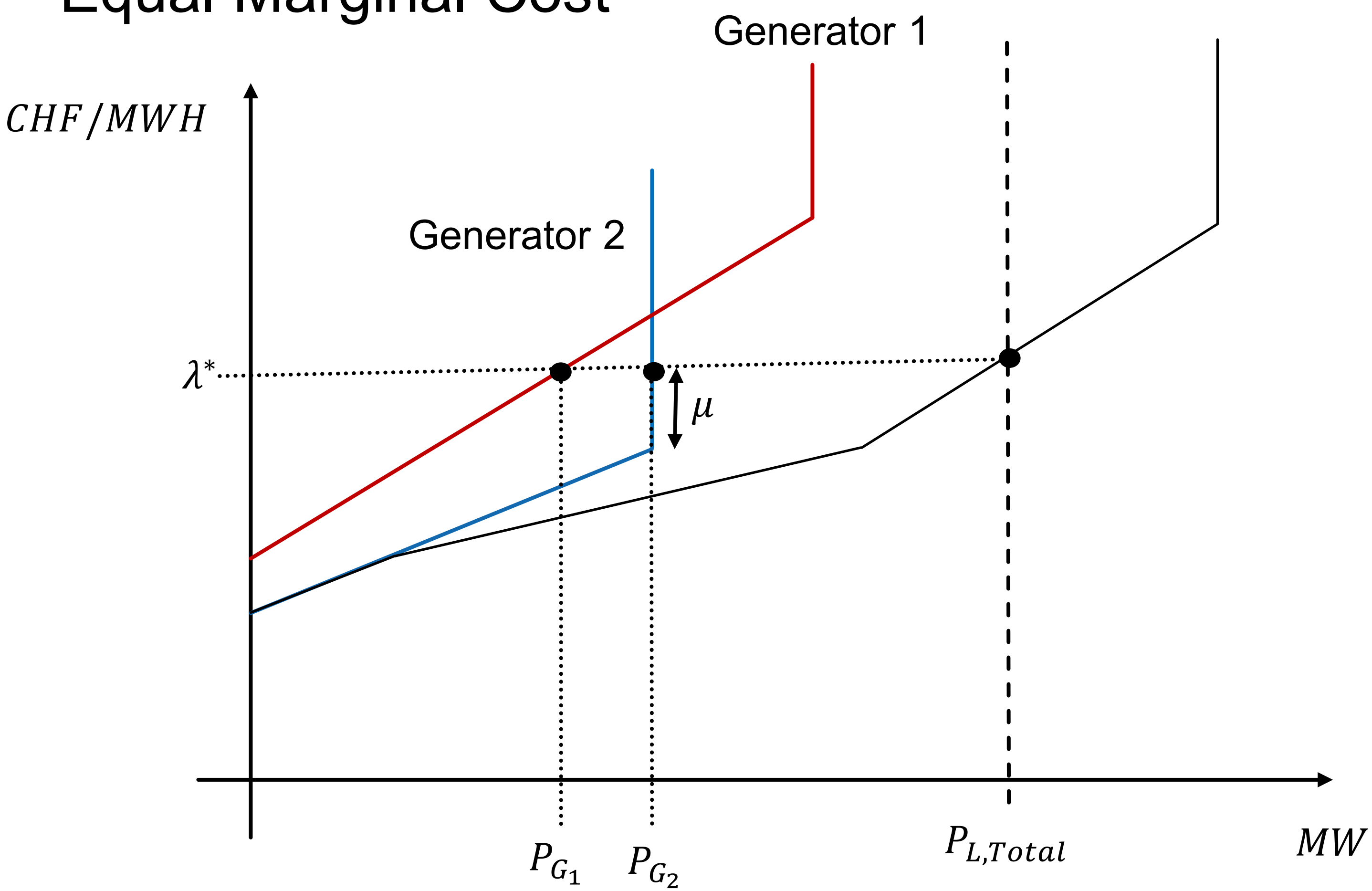
$\Rightarrow$ Solution (for generators i not at the limit): $\frac{\partial C_i}{\partial P_{G_i}} = \lambda$

$\Rightarrow$ Lagrange Multiplier for power balance corresponds to price!

$\Rightarrow$ Lagrange Multipliers for inequality constraint correspond to offset in price

Economic Dispatch

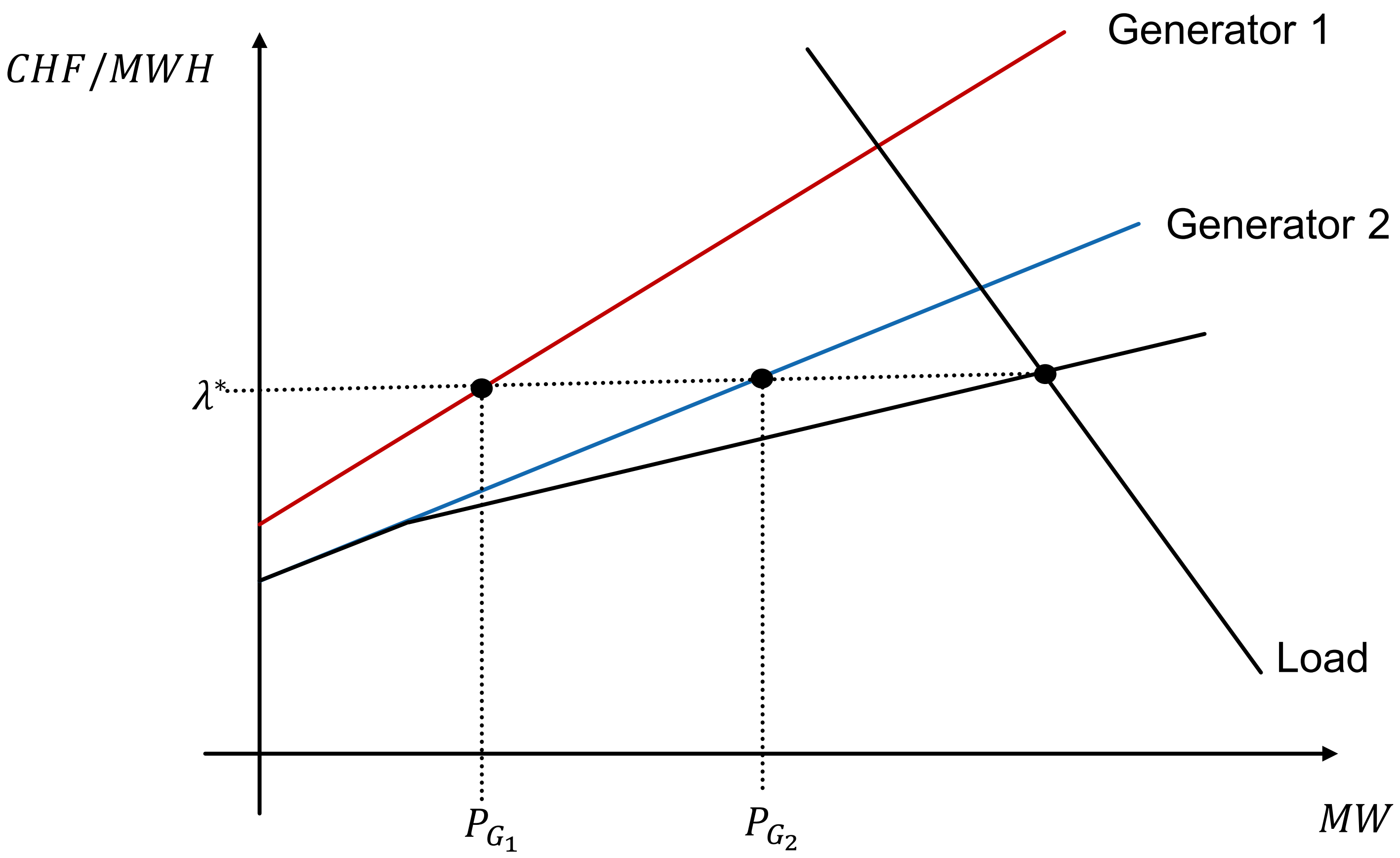
- Equal Marginal Cost



=> Solution: $\frac{\partial C_i}{\partial P_i} = \lambda$, for generator i not at the limit

Economic Dispatch

- Flexible Load



Economic Dispatch

- Generation Cost (CHF/hr)

$$C_i(P_{G_i}) = a_{G_i}P_{G_i}^2 + b_{G_i}P_{G_i} + c_{G_i}$$

$$a_{G_i}, b_{G_i} \geq 0$$

- Demand Curve

$$D_i(P_i) = a_{D_i}P_{D_i}^2 + b_{D_i}P_{D_i} + c_{D_i}$$

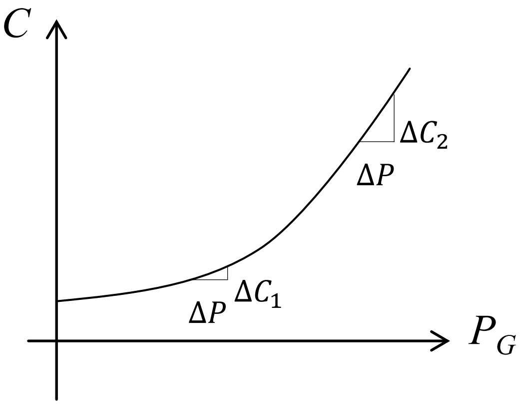
$$a_{D_i} \leq 0, b_{D_i} \geq 0$$

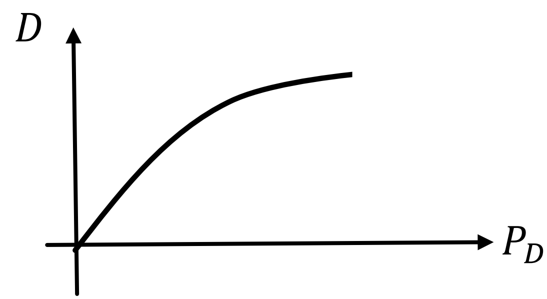
- Objective Function

$$-SW = \sum_{i=1}^{N_G} C_i(P_{G_i}) - \sum_{i=1}^{N_D} D_i(P_{D_i})$$

- Constraint

$$\sum_{i=1}^{N_G} P_{G_i} = \sum_{i=1}^{N_D} P_{D_i}$$



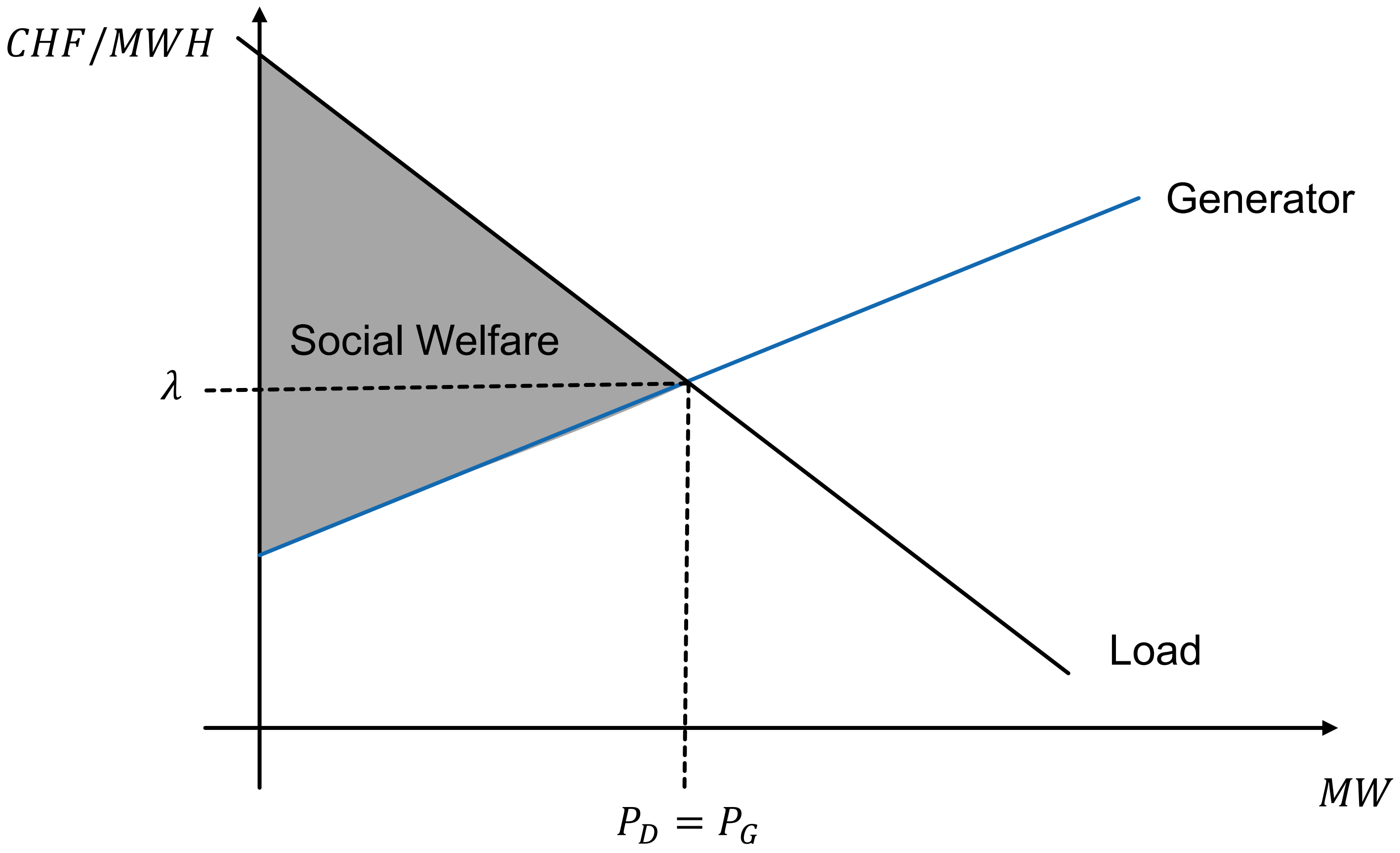


Maximize social welfare

Social Welfare = what load is willing to pay minus generation cost

Economic Dispatch

- Social Welfare

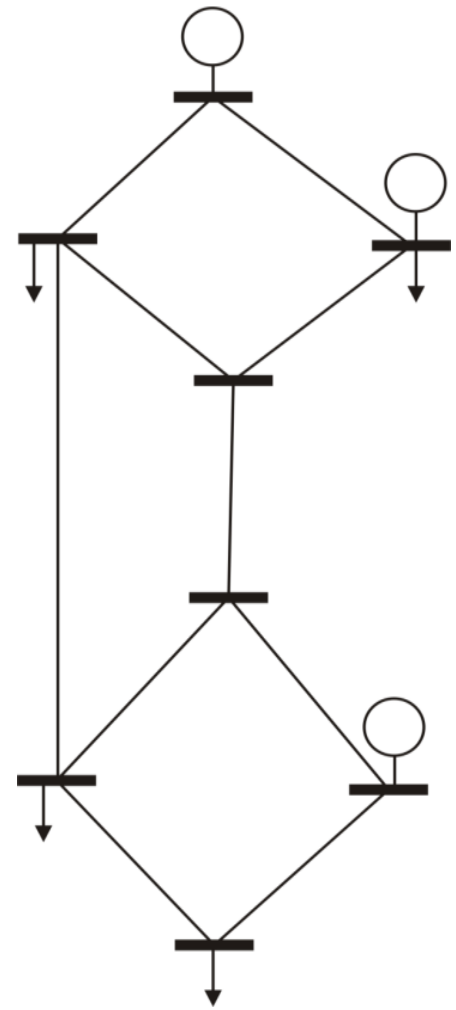


Optimal Power Flow

- Grid Model
  - Variables: Voltage magnitudes and angles at buses
  - Constraints: Power balances at nodes

$$S = P + jQ = Ue^{j\theta_U} \cdot \left(Ie^{j\theta_I}\right)^*$$

➡ get currents as function of voltages

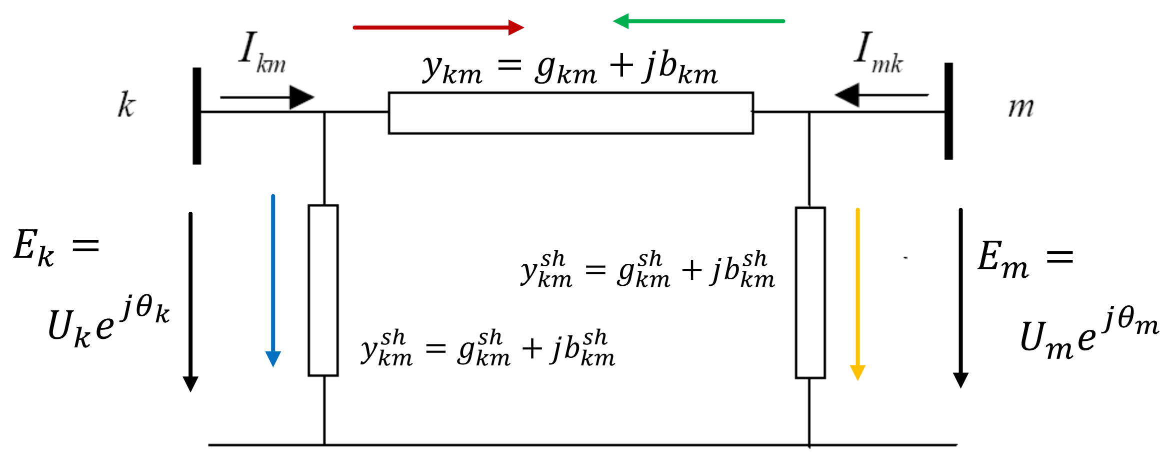


AC Power Flow

- Transmission Line - Derivation

$$I_{km} = y_{km}(E_k - E_m) + y_{km}^{sh} E_k \qquad = (y_{km} + y_{km}^{sh})E_k - y_{km}E_m$$

$$I_{mk} = y_{km}(E_m - E_k) + y_{km}^{sh} E_m \qquad = (y_{km} + y_{km}^{sh})E_m - y_{km}E_k$$



AC Power Flow

- Transmission Line - Derivation

$$S_{km} = E_k \cdot I_{km}^* \qquad = U_k e^{j\theta_k} \cdot \left(y_{km}^* \left(U_k e^{-j\theta_k} - U_m e^{-j\theta_m} \right) - j b_{km}^{sh} U_k e^{-j\theta_k} \right)$$

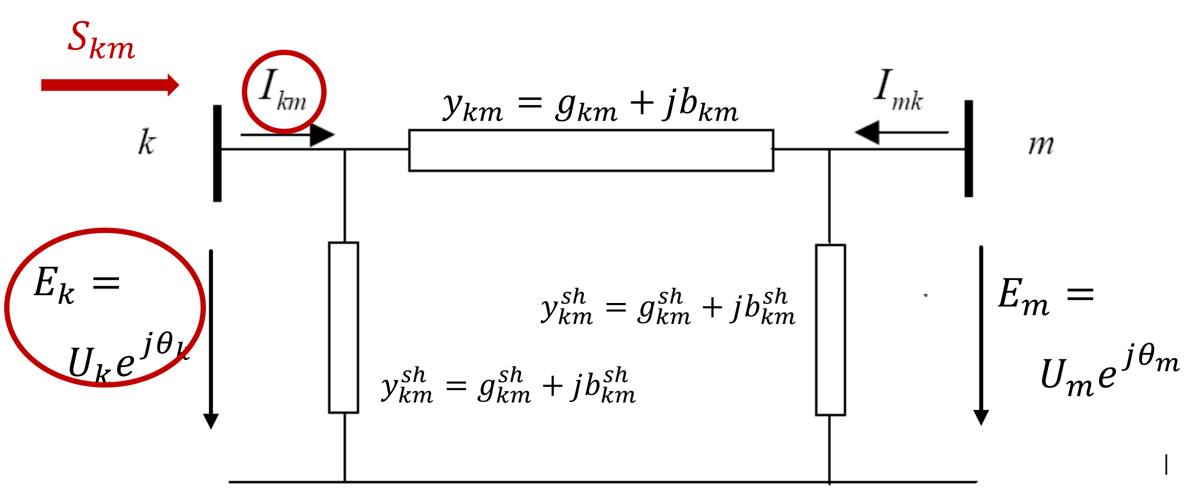
$$= U_k e^{j\theta_k} \cdot \left((g_{km} - j b_{km}) \left(U_k e^{-j\theta_k} - U_m e^{-j\theta_m} \right) - j b_{km}^{sh} U_k e^{-j\theta_k} \right)$$

$$= (g_{km} - j b_{km})(U_k^2 - U_k U_m e^{j(\theta_k - \theta_m)}) - j b_{km}^{sh} U_k^2$$

$$= (g_{km} - j b_{km})(U_k^2 - U_k U_m (\cos(\theta_k - \theta_m) + j \sin(\theta_k - \theta_m))) - j b_{km}^{sh} U_k^2$$

$$= P_{km} + j Q_{km}$$

$$I_{km} = y_{km}(E_k - E_m) + y_{km}^{sh} E_k \qquad = y_{km}\left(U_k e^{j\theta_k} - U_m e^{j\theta_m} \right) + j b_{km}^{sh} U_k e^{j\theta_k}$$

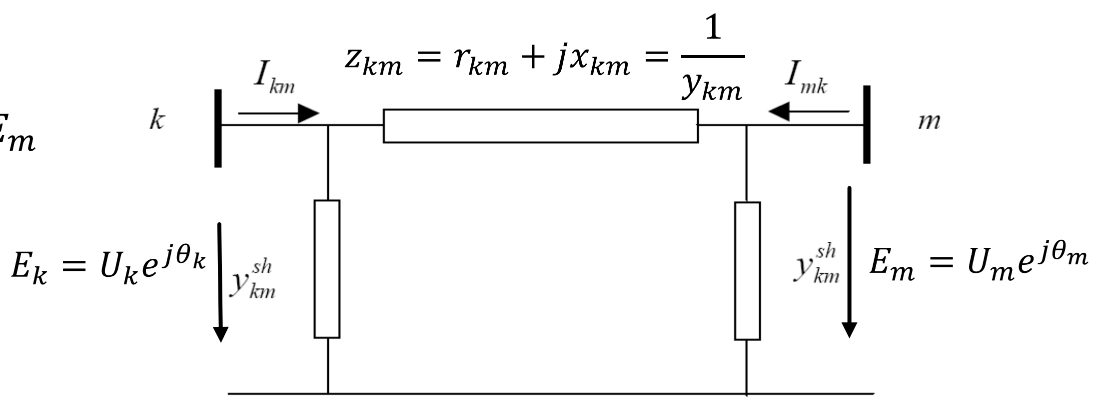


AC Power Flow

- Transmission Line

$$I_{km} = (y_{km} + y_{km}^{sh}) \cdot E_k - y_{km} \cdot E_m$$

$$y_{km} = g_{km} + jb_{km}$$
$$y_{km}^{sh} = jb_{km}^{sh}$$

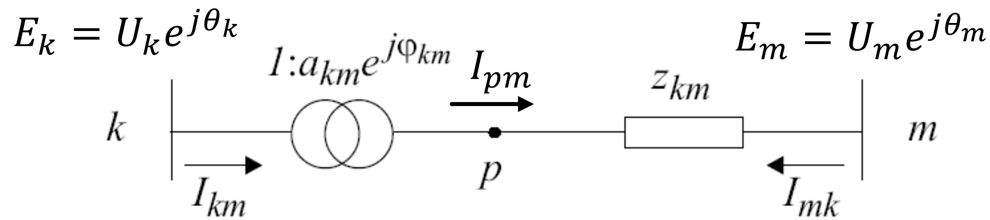


$$\boxed{\begin{aligned} P_{km} &= U_k^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} - U_k U_m b_{km} \sin\theta_{km} \\ Q_{km} &= -U_k^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos\theta_{km} - U_k U_m g_{km} \sin\theta_{km} \end{aligned}}$$

=> for other direction, i.e. m to k, switch indices

AC Power Flow

- Transformer - Derivation



$$E_p = t_{km} E_k \qquad I_{km} = t_{km}^* I_{pm}$$

$$t_{km} = a_{km} e^{j\phi_{km}}$$

$$y_{km} = \frac{1}{z_{km}} = g_{km} + jb_{km}$$

$$I_{pm} = (E_p - E_m) y_{km} = (t_{km} E_k - E_m) y_{km} = -I_{mk}$$

$$I_{km} = t_{km}^* (t_{km} E_k - E_m) y_{km} = a_{km}^2 y_{km} E_k - t_{km}^* y_{km} E_m$$

$$S_{km} = E_k \cdot I_{km}^*$$

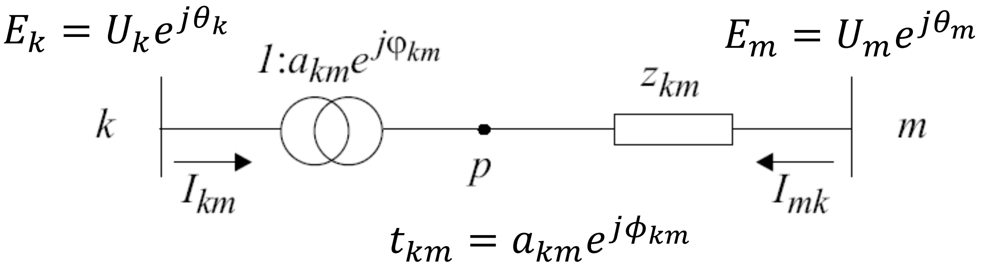
$$S_{mk} = E_m \cdot I_{mk}^*$$

AC Power Flow

- Transformer

$$I_{km} = a_{km}^2 y_{km} E_k - t_{km}^* y_{km} E_m$$

$$I_{mk} = -t_{km} y_{km} E_k + y_{km} E_m$$



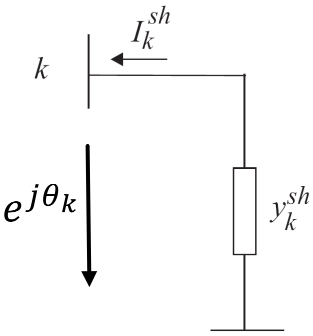
$$
\boxed{
\begin{aligned}
P_{km} &= (a_{km}U_k)^2 g_{km} - a_{km}U_kU_m g_{km}\cos(\theta_{km}+\varphi_{km}) - a_{km}U_kU_m b_{km}\sin(\theta_{km}+\varphi_{km})\\
Q_{km} &= -(a_{km}U_k)^2 b_{km} + a_{km}U_kU_m b_{km}\cos(\theta_{km}+\varphi_{km}) - a_{km}U_kU_m g_{km}\sin(\theta_{km}+\varphi_{km})\\
\\
P_{mk} &= {U_m}^2 g_{km} - a_{km}U_kU_m g_{km}\cos(\theta_{mk}-\varphi_{km}) - a_{km}U_kU_m b_{km}\sin(\theta_{mk}-\varphi_{km})\\
Q_{mk} &= -{U_m}^2 b_{km} + a_{km}U_kU_m b_{km}\cos(\theta_{mk}-\varphi_{km}) - a_{km}U_kU_m g_{km}\sin(\theta_{mk}-\varphi_{km})
\end{aligned}
}
$$

AC Power Flow

- Shunt Element

$$I_k^{sh} = -y_k^{sh} E_k$$

$$E_k = U_k e^{j\theta_k}$$

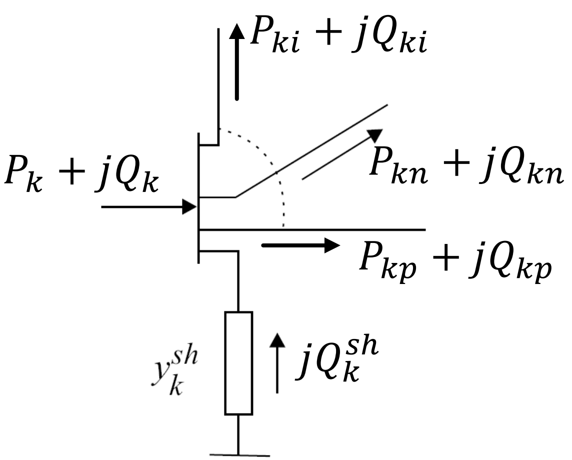


$$S_k^{sh} = P_k^{sh} + jQ_k^{sh} = -\left(y_k^{sh}\right)^* |E_k|^2 = -\left(y_k^{sh}\right)^* U_k^2$$

- Power Balance

$$P_k = \sum_{m \in \Omega_k} P_{km}$$

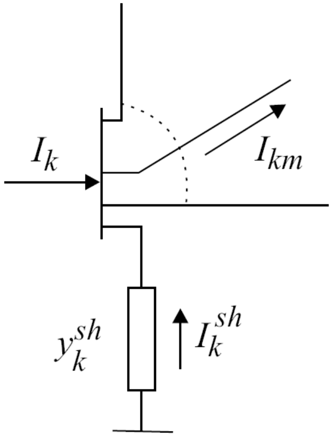
$$Q_k = -Q_k^{sh} + \sum_{m \in \Omega_k} Q_{km}$$



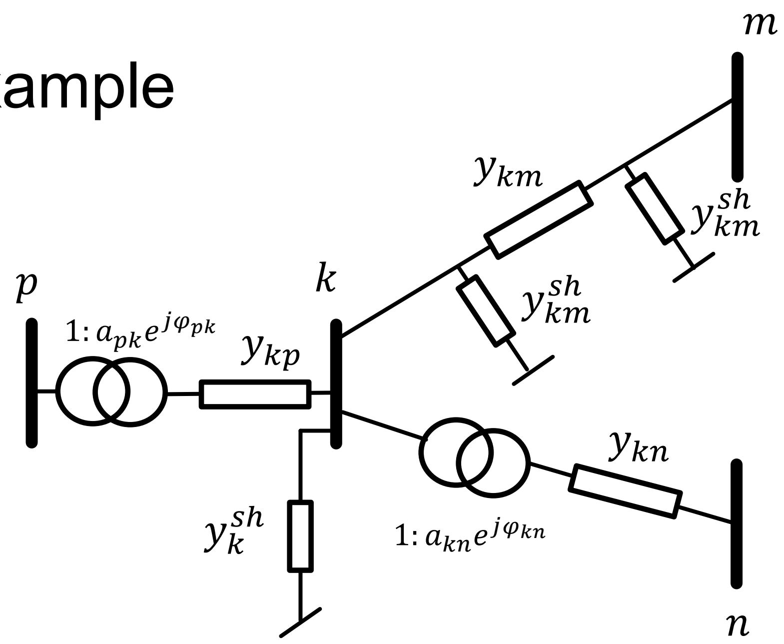
Nodal Equation

- Kirchhoff's Current Law

$$I_k = -I_k^{sh} + \sum_{m \in \Omega_k} I_{km}$$

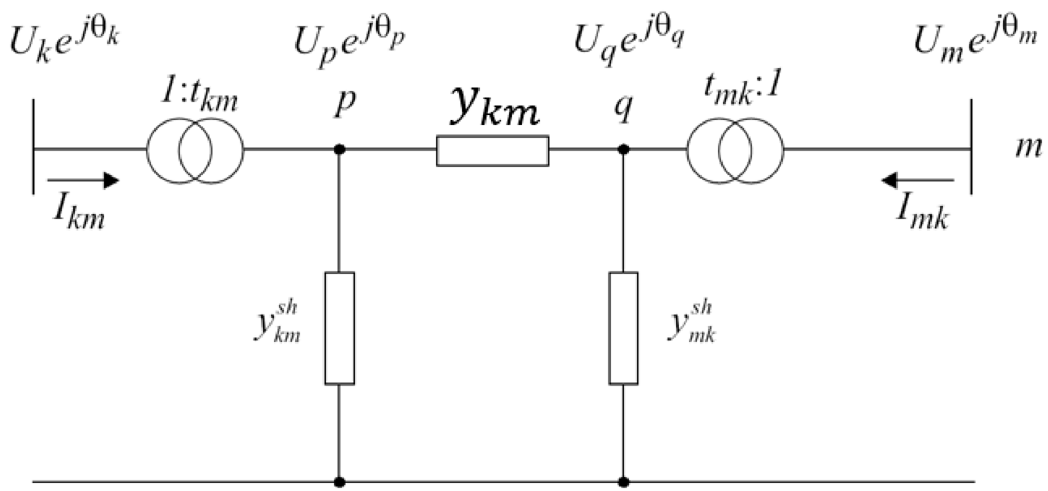


- Example



AC Power Flow

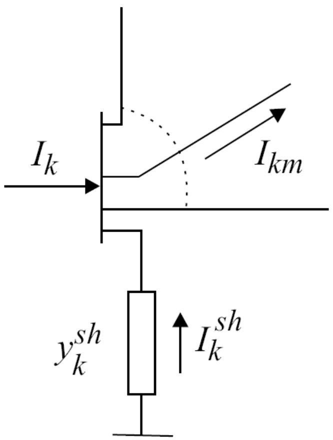
- Unified Model
  - captures all components in one circuit



Nodal Equation

- Kirchhoff's Current Law

$$I_k = -I_k^{sh} + \sum_{m \in \Omega_k} I_{km}$$



- Admittance Matrix

$$\mathbf{I} = \mathbf{YE}$$

  - $\mathbf{I}$: injection vector with elements $I_k, k = 1, \ldots, N$
  - $\mathbf{E}$: nodal voltage vector with elements $E_k = U_k e^{j\theta_k}$
  - $\mathbf{Y} = \mathbf{G} + \mathbf{jB}$: nodal admittance matrix with matrix elements

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & \ldots & Y_{n1} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \ldots & Y_{nn} \end{bmatrix} = \begin{bmatrix} G_{11} + jB_{11} & \ldots & G_{n1} + jB_{n1} \\ \vdots & \ddots & \vdots \\ G_{n1} + jB_{n1} & \ldots & G_{nn} + jB_{nn} \end{bmatrix}$$

$$Y_{km} = -t_{km}^* t_{mk} y_{km}$$

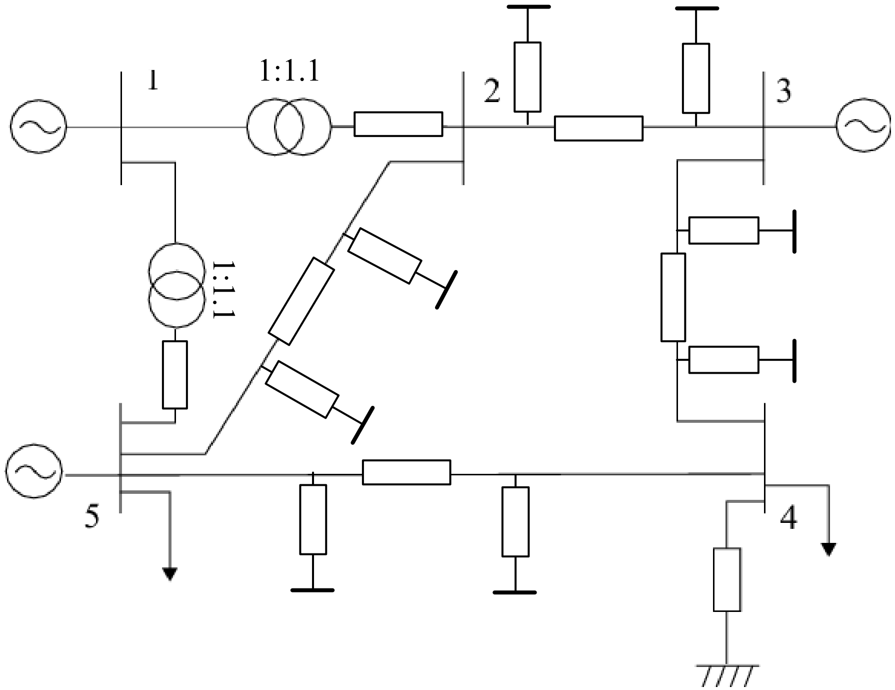
$$Y_{kk} = y_k^{sh} + \sum_{m \in \Omega_k} a_{km}^2 (y_{km}^{sh} + y_{km})$$

Admittance Matrix

$$Y_{km} = -t_{km}^* t_{mk} y_{km}$$

$$Y_{kk} = y_k^{sh} + \sum_{m \in \Omega_k} a_{km}^2 (y_{km}^{sh} + y_{km})$$

- Example



Line Parameters:

$$r_{km} + jx_{km} = 0.03 + 0.3j$$
$$jb_{km}^{sh} = 0.3j$$

Transformer Parameters:

$$jx_{km} = 0.8j$$

Shunt Parameter:

$$jb_k^{sh} = 1.2j$$

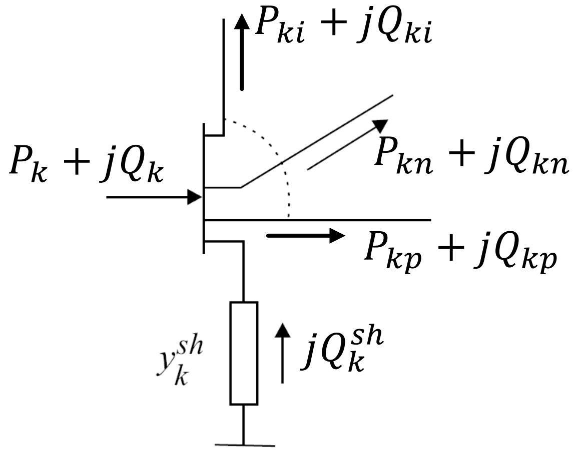
$$Y = \begin{bmatrix} -3.025j & 1.375j & 0 & 0 & 1.375j \\ 1.375j & 0.66 - 7.25j & -0.33 + 3.3j & 0 & -0.33 + 3.3j \\ 0 & -0.33 + 3.3j & 0.66 - 6j & -0.33 + 3.3j & 0 \\ 0 & 0 & -0.33 + 3.3j & 0.66 - 4.8j & -0.33 + 3.3j \\ 1.375j & -0.33 + 3.3j & 0 & -0.33 + 3.3j & 0.66 - 7.25j \end{bmatrix}$$

AC Power Flow

- Power Balance

$$P_k = \sum_{m \in \Omega_k} P_{km}$$

$$Q_k = -Q_k^{sh} + \sum_{m \in \Omega_k} Q_{km}$$



$$I_k = \sum_{m \in K} Y_{km} E_m = \sum_{m \in K} (G_{km} + jB_{km}) U_m e^{j\theta_m}$$

$$I_k^* = \sum_{m \in K} (G_{km} - jB_{km}) U_m e^{-j\theta_m}$$

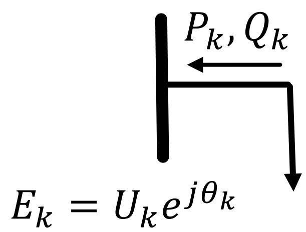
$$S_k = E_k I_k^* = \sum_{m \in K} (G_{km} - jB_{km}) U_k U_m e^{j(\theta_k - \theta_m)} = \sum_{m \in K} (G_{km} - jB_{km}) U_k U_m (\cos\theta_{km} + j\sin\theta_{km})$$

$$\boxed{\begin{aligned} P_k &= \sum_{m \in K} U_k U_m (G_{km} \cos\theta_{km} + B_{km} \sin\theta_{km}) \\ Q_k &= \sum_{m \in K} U_k U_m (G_{km} \sin\theta_{km} - B_{km} \cos\theta_{km}) \end{aligned}}$$

AC Power Flow

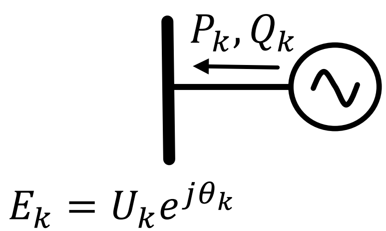
- Load

$$P_k = -P_L$$
$$Q_k = -Q_L$$



- Generator

$$P_k = P_G$$
$$U_k = U_L$$



$$\Rightarrow \quad S_k = P_k + jQ_k = E_k I_k^* = E_k \sum_{m \in \Omega_k} I_{km}$$

Power injected by generators minus power consumed by loads

$$P_k = U_k \sum_{m \in K} U_m (G_{km} cos\theta_{km} + B_{km} sin\theta_{km})$$

$$Q_k = U_k \sum_{m \in K} U_m (G_{km} sin\theta_{km} - B_{km} cos\theta_{km})$$

Power flowing into lines and shunts

AC Power Flow

- Procedure
  - Determine admittance matrix using

$$Y_{km} = -t_{km}^{*} t_{mk} y_{km}$$

$$Y_{kk} = y_k^{sh} + \sum_{m \in \Omega_k} a_{km}^2 (y_{km}^{sh} + y_{km})$$

→ real and imaginary entries in this matrix correspond to $G_{km} + jB_{km}$

  - Formulate the expressions for active and reactive power flowing out of the nodes

$$P_k = U_k \sum_{m \in K} U_m (G_{km} cos\theta_{km} + B_{km} sin\theta_{km})$$

$$Q_k = U_k \sum_{m \in K} U_m (G_{km} sin\theta_{km} - B_{km} cos\theta_{km})$$

  - Set them equal to the power injected by generators minus power consumed by the loads

Bus Types

- Load Bus (PQ Bus)
  - Active and reactive power consumption is constant and fixed
  - No generators connected to this type of bus
- Generator Bus (PU Bus)
  - Generator connected to this bus
  - Active power injection is controlled by generators to constant value
  - Voltage is controlled by generator to constant value
- Slack Bus
  - Only one bus of this type
  - Generator connected to this bus
  - Serves as reference bus for voltage angles, i.e. $\theta = 0$
  - Voltage is controlled by generator to constant value

Power Flow Equations

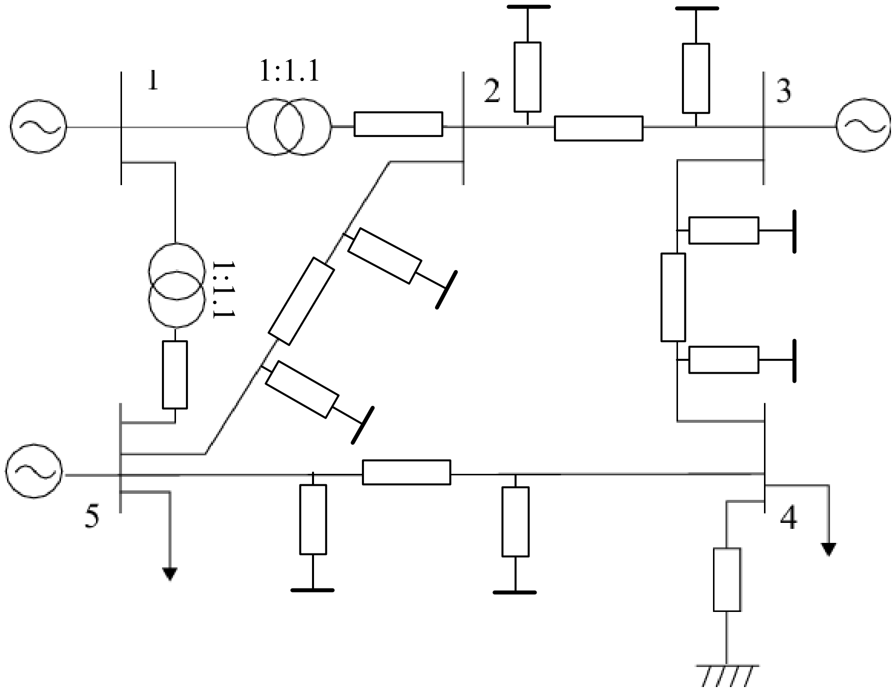
- Equations depending on bus type: Two variables per bus => we need two equation per bus
  - PQ bus
    $$-P_{L_k} = \sum_{m\in\Omega_k} P_{km}(\cdot) = U_k \sum_{m\in K} U_m(G_{km}cos\theta_{km} + B_{km}sin\theta_{km})$$
    $$-Q_{L_k} = -Q_{sh_k}(U_k) + \sum_{m\in\Omega_k} Q_{km}(\cdot) = U_k \sum_{m\in K} U_m(G_{km}sin\theta_{km} - B_{km}cos\theta_{km})$$
  - PU bus
    $$P_{G_k} - P_{L_k} = \sum_{m\in\Omega_k} P_{km}(\cdot) = U_k \sum_{m\in K} U_m(G_{km}cos\theta_{km} + B_{km}sin\theta_{km})$$
    $$U_k = U_{G_k}$$
  - slack bus
    $$U_k = U_{G_k}$$
    $$\theta_k = 0$$

Admittance Matrix

$$Y_{km} = -t_{km}^* t_{mk} y_{km}$$

$$Y_{kk} = y_k^{sh} + \sum_{m \in \Omega_k} a_{km}^2 (y_{km}^{sh} + y_{km})$$

- Example



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DC Load Flow

$$g_{km} = \frac{r_{km}}{r_{km}^2 + x_{km}^2}$$

$$b_{km} = -\frac{x_{km}}{r_{km}^2 + x_{km}^2}$$

- Linear Model
  - neglect loss terms (set $r_{km} = 0$ => $g_{km} = 0$)
  - set $U_k \approx U_m \approx 1pu$

    $\sin\theta_{km} \approx \theta_{km}, \ \cos\theta_{km} \approx 1$

$$P_{km} = \cancel{U_k^2 g_{km}} - \cancel{U_k U_m g_{km} \cos\theta_{km}} - \cancel{U_k U_m} b_{km} \underbrace{\sin\theta_{km}}_{=\theta_{km}}$$

$$= \frac{x_{km}}{\cancel{r_{km}^2} + x_{km}^2}(\theta_k - \theta_m) \quad = \frac{\theta_k - \theta_m}{x_{km}}$$

$$Q_{km} = -\cancel{U_k^2}(b_{km} + b_{km}^{sh}) + \cancel{U_k U_m} b_{km} \underbrace{\cos\theta_{km}}_{=1} - \cancel{U_k U_m g_{km} \sin\theta_{km}}$$

$$= -b_{km}^{sh}$$

DC Power Flow

- Line Model
  - neglect loss terms (set $g_{km} = 0$)
  - set $U_k \approx U_m \approx 1 \text{ p.u.}$ => need to work with p.u. values!

$$\sin\theta_{km} \approx \theta_{km}$$

$$\Longrightarrow \quad P_{km} = \frac{\theta_{km}}{x_{km}} = \frac{\theta_k - \theta_m}{x_{km}}$$

models active power flows but cannot be used to model reactive power

- System Equations
  - Power balance at all buses except slack bus

$$P_k = P_{G_k} - P_{L_k} = \sum_{m \in \Omega_k} P_{km} \quad \Longrightarrow \quad P = B\theta$$

vector of net injections (except slack)

vector of angles (except slack)

$$B_{km} = -\frac{1}{x_{km}}$$

$$B_{kk} = \sum_{m \in \Omega_k} \frac{1}{x_{km}}$$

  - Slack bus

$$\theta_k = 0$$

Per Unit System

- Definition

$$\text{quantity in p.u.} = \frac{\text{actual value}}{\text{base value}}$$

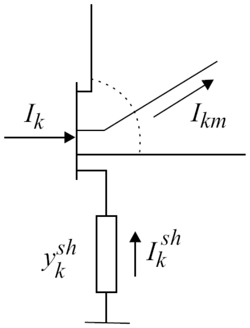
- Relation for Base Values: $S_B = U_B \cdot I_B = \frac{U_B^2}{Z_B}$

- Choice of Base Values
  - Power: one value for the entire system and equal to most frequently occurring power rating
  - Voltage: one value for all nodes between two transformers, most often equal to the nominal voltage of this part of the system
  - Current: follows from power and voltage
  - Impedance: follows from power and voltage

Summary

- Economic Dispatch
  - Find generation dispatch that minimizes cost of supply
  - Constraints include generation limits
  - Lagrange Multiplier of power balance constraint equal to marginal price

- Power Flow Modeling
  - Transmission lines, transformers, shunt elements
  - Nodal equations using admittance matrix



$$\mathbf{Y} = \begin{bmatrix} Y_{11} & \dots & Y_{n1} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \dots & Y_{nn} \end{bmatrix} = \begin{bmatrix} G_{11} + jB_{11} & \dots & G_{n1} + jB_{n1} \\ \vdots & \ddots & \vdots \\ G_{n1} + jB_{n1} & \dots & G_{nn} + jB_{nn} \end{bmatrix}$$

  - DC Power Flow

$$P_{km} = \frac{\theta_{km}}{x_{km}} = \frac{\theta_k - \theta_m}{x_{km}}$$