It is $\forall X \in M_{n,p}(\mathbb{R})$, choose $v_0 \in \mathbb{R}^p$, $\forall k=1,..T$, $u_k = \frac{X v_{k-1}}{\|X v_{k-1}\|}$, $v_k = \frac{X^T u_k}{\|X^T u_k\|}$ } The output: u_T, v_T

Goal of the proof (we will use in both cases geometric convergence)

1) Let's show that under good assumptions, $(v_k)_{k\geq 1}$ converges (up to a factor) to the leading right singular vector of X, ie let's show that under good assumptions, $(v_k)_{k\geq 1}$ converges (up to a factor) to an eigen vector of $A = X^TX$ associated to the biggest (in module) eigen value of $A = X^TX$.

2) Let's show 〃 〃 〃 $(u_k)_{k\geq 1}$ 〃 〃 leading left 〃 〃 〃 〃, ie 〃 〃 〃 〃 $(u_k)_{k\geq 1}$ 〃 〃 〃 〃 of $A^T = XX^T$ 〃 〃 〃 〃 $A^T = XX^T$

Proof number 1

- Our initial $X \in M_{n,p}(\mathbb{R})$, so $A = X^TX \in M_{p,p}(\mathbb{R})$ and is symetric (because $A^T = (X^TX)^T = X^T(X^T)^T = X^TX = A$)
$\Longrightarrow$ A is diagonalizable, so let $Sp(A) = \{\lambda_1, .. \lambda_p\}$ s.t $|\lambda_1| \leqslant ... \leqslant |\lambda_{p-1}| < |\lambda_p|$ (it's our assumption number 1) and $\exists (v_1, ... v_p)$ an orthonormal basis of $\mathbb{R}^p$ by definition (and $Av_i = \lambda_i v_i, \forall i=1,..p$)
So our $v_0 \in \mathbb{R}^p$ which is given can be written as $v_0 = \sum_{i=1}^{p} c_i v_i$ with $c_1, .. c_p \in \mathbb{R}$ and $c_p \neq 0$ (assumption 2)

So $Av_0 = A \sum_{i=1}^{p} c_i v_i = \sum_{i=1}^{p} c_i Av_i = \sum_{i=1}^{p} c_i \lambda_i v_i$
$A^2 v_0 = A \sum_{i=1}^{p} c_i \lambda_i v_i = \sum_{i=1}^{p} c_i \lambda_i^2 v_i$

So by direct reccursion, $\forall k=1..T$, $A^k v_0 = \sum_{i=1}^{p} c_i \lambda_i^k v_i$
so $A^k v_0 = \lambda_p^k \left(c_p v_p + \sum_{i=1}^{p-1} c_i \left(\frac{\lambda_i}{\lambda_p}\right)^k v_i\right)$
so $A^k v_0 \underset{k\to+\infty}{\simeq} \lambda_p^k c_p v_p$ because $|\lambda_p| > |\lambda_i|, \forall i=1,..p-1$

- Now, let's note that $\forall k=1,..T$, $v_k = \frac{X^T u_k}{\|X^T u_k\|} = \frac{X^TX v_{k-1} \|X v_{k-1}\|}{\|X v_{k-1}\| \|X^TX v_{k-1}\|} = \frac{A v_{k-1}}{\|A v_{k-1}\|}$, so by direct recursion, we have that $\forall k=1..T$, $v_k = \frac{A^k v_0}{\|A^k v_0\|}$

- Finally, let's show that $\left\| v_k - \frac{\lambda_p^k c_p v_p}{\|A^k v_0\|} \right\| \xrightarrow{k\to+\infty} 0$.

We have $\left\| v_k - \frac{\lambda_p^k c_p v_p}{\|A^k v_0\|} \right\| = \left\| \frac{A^k v_0}{\|A^k v_0\|} - \frac{\lambda_p^k c_p v_p}{\|A^k v_0\|} \right\| = \frac{1}{\|A^k v_0\|} \left\| \sum_{i=1}^{p-1} c_i \lambda_i^k v_i \right\|$

$$= \frac{|\lambda_p|^k}{\|A^k v_0\|} \left\| \sum_{i=1}^{p-1} c_i \left(\frac{\lambda_i}{\lambda_p}\right)^k v_i \right\|$$

triangular inequality and $\forall i=1,.p-1$, $|\lambda_i| \leqslant |\lambda_{p-1}|$ and $\|v_i\| = 1$

$$\leqslant \frac{|\lambda_p|^k}{\|A^k v_0\|} \left|\frac{\lambda_{p-1}}{\lambda_p}\right|^k \sum_{i=1}^{p-1} |c_i| \leqslant C \left|\frac{\lambda_{p-1}}{\lambda_p}\right|^k \sum_{i=1}^{p-1} |c_i| \xrightarrow[\text{geometrically}]{k\to+\infty} 0$$

because $|\lambda_{p-1}| < |\lambda_p|$

we proved before that $A^k v_0 \simeq \lambda_p^k c_p v_p$ so there exists a constant C such that $\frac{|\lambda_p|^k}{\|A^k v_0\|} \leqslant C$

So we just proved, that $v_k \xrightarrow[k\to+\infty]{\|\ \|} v_p = v^*$ (up to a factor) ☺

Proof number 2 : Same method!

We see that $\forall k \geq 2$, $u_k = \frac{(A^T)^k u_1}{\|(A^T)^k u_1\|}$ and A^T is symetric so diagonalizable with $(u_1, .. u_p)$ it's orthonormal basis, $Sp(A^T) = Sp(A)$ and we do the same method as before to show that $\left\| u_k - \frac{\lambda_1^k c_1 v_1}{\|(A^T)^k v_1\|} \right\| \xrightarrow{k\to+\infty} 0$ (we just adapt the proof number 1)

$\square$