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THIRD EDITION

PROBABILITY, STATISTICS AND RANDOM PROCESSES

T VEERARAJAN

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Preface to the Third Edition

I am extremely gratified by the overwhelming response shown for the first two editions of this book.

Though the previous editions catered to the requirements of engineering students as per the Anna University syllabus, a recent survey of the syllabi followed by many universities across the country made me revise the book. The two topics 'Reliability Engineering' and 'Statistical Quality Control' have been removed, as the mechanical engineering students study them along with other subjects. The topic 'Statistical Averages' has been thoroughly revised—more concepts about basics of statistical measures have been included and more number of worked examples and exercise problems have been added. Various chapters have been rearranged.

The Online Learning Centre of the book can be accessed at http://www.mhhe.com./veerarajan/psrp3e and contains the following material for students:

- useful web links for further reading
- Chapters on
  —Reliability Engineering
  —Statistical Quality Control

I am extremely grateful to the Management Chockalingapuram Devangar Varthaga Sangam, Aruppukottai, which has sponsored Sree Sowdambika College of Engineering, Aruppukottai, in which I am presently working, for the support extended to me in this project.

I wish to express my thanks to Prof. M Jegan Mohan, Principal, Sree Sowdambika College of Engineering for the keen interest shown and constant encouragement given to me while preparing this revised edition of the book.

While preparing this edition of the book, care has been taken to completely eliminate a few errors which were present in the previous edition.

Critical evaluation and suggestions for further improvement of the book are most welcome.

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Preface to the First Edition

This book is intended for a one-semester course in Probability, Statistics and Random Processes for undergraduate level Electrical/Electronics/Computer Engineering and Information Technology students. The various chapters also cover the syllabus content of 'Probability and Statistics' introduced recently for the BE (CSE and IT) course by the Madurai Kamaraj University of Tamil Nadu.

Most engineering students, who are used to a deterministic outlook of Physics and Engineering problems, find the theory of probability unreliable, vague and difficult. This is due to inadequate understanding of the basic concepts of probability theory and the wrong impression that the subject is an advanced branch of Mathematics.

The book is written in such a manner that beginners may develop an interest in the subject and may find it useful to pursue their studies. Basic concepts and proofs of theorems are explained in as lucid a manner as possible. Although the theory of probability is developed rigorously based on measure theory, it is developed in this book by a simple set-theory approach.

As engineering students find it easier to generalize specific results and examples than to specialize general results, considerable attention is devoted to working of problems. Nearly 300 problems including those with applications to communication theory are worked out in various chapters. Unless the students become personally involved in solving exercises, they cannot really develop an understanding and appreciation of the ideas and a familiarity with the pertinent techniques. Hence, in addition to a large number of short-answer questions under Part-A, over 350 problems have been given under Part-B of the Exercises in various chapters. Answers are provided at the end of every chapter.

Though chapters 7 and 8 are meant for Electrical/Electronics Engineering students, the other chapters that deal with probability theory, random variables, probability distributions and statistics may be useful to students of any discipline, such as those doing MCA and M.Sc courses who have Mathematical Statistics as a subject of their study.

I am sure that the students and the faculty will find this book very useful.

Critical evaluation and suggestions for improvement of the book will be highly appreciated and gratefully acknowledged.

I am extremely grateful to Dr K V Kuppusamy, Chairman, and Mr K Senthil Ganesh, Managing Trustee, RVS Educational Trust, Dindigul, for the support extended to me in this project.

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I wish to express my thanks to Dr K M Karuppannan, Principal, RVS College of Engineering and Technology, Dindigul, for the appreciative interest shown and constant encouragement given to me while writing this book.

I am thankful to my publishers, Tata McGraw-Hill Publishing Company Limited, New Delhi, for their painstaking efforts and cooperation in bringing out this book in a short span of time.

I would also like to thank Dr A Rangan, Professor, Department of Mathematics, IIT Madras; Dr S Leela Devi, Professor and Head, Department of Mathematics J J College of Engineering and Technology, Tiruchirapalli; and Mr Sitharselvan and Mr Muthuraman of Bannari Amman Institute of Technology for reviewing and providing valuable suggestions during the developmental stages of the book.

I have great pleasure in dedicating this book to my beloved students, past and present.

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1 Probability Theory

Probability theory had its origin in the analysis of certain games of chance that were popular in the seventeenth century. It has since found applications in many branches of Science and Engineering and this extensive application makes it an important branch of study. Probability theory, as a matter of fact, is a study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

Random Experiment

An experiment whose outcome or result can be predicted with certainty is called a deterministic experiment. For example, if the potential difference E between the two ends of a conductor and the resistance R are known, the current I flowing in the conductor is uniquely determined by Ohm's law, $I = \frac{E}{R}$.

Although all possible outcomes of an experiment may be known in advance, the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a random experiment.

Whenever a fair 6-faced cubic dice is rolled, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be, when the dice is rolled at a point of time.

Although the number of telephone calls received in a board in a 5-min. interval is a non-negative integer, we cannot predict exactly the number of calls received in the next 5-min. In such situations we talk of the chance or the probability of occurrence of a particular outcome, which is taken as a quantitative measure of the likelihood of the occurrence of the outcome.

Mathematical or Apriori Definition of Probability

Let S be the sample space (the set of all possible outcomes which are assumed equally likely) and A be an event (a sub-set of S consisting of possible outcomes) associated with a random experiment. Let $n(S)$ and $n(A)$ be the number of elements

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of S and A. Then the probability of event A occurring, denoted as $P(A)$, is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

For example, the probability of getting an even number in the die tossing experiment is 0.5, as $S = \{1, 2, 3, 4, 5, 6\}$, $E = \{2, 4, 6\}$, $n(S) = 6$ and $n(E) = 3$.

Statistical or Aposteriori Definition of Probability

Let a random experiment be repeated n times and let an event A occur n_A times out of the n trials. The ratio $\frac{n_A}{n}$ is called the relative frequency of the event A.

As n increases, $\frac{n_A}{n}$ shows a tendency to stabilise and to approach a constant value. This value, denoted by $P(A)$, is called the probability of the event A, i.e., $P(A) = \lim_{n \to \infty} \frac{n_A}{n}$.

For example, if we want to find the probability that a spare part produced by a machine is defective, we study the record of defective items produced by the machine over a considerable period of time. If, out of 10,000 items produced, 500 are defective, it is assumed that the probability of a defective item is 0.05.

Note *From both the definitions, it is obvious that $0 \le P(A) \le 1$. If A is an impossible event, $P(A) = 0$. Conversely, if $P(A) = 0$, then A can occur in a very small percentage of times in the long run. On the other hand, if A is a certain event, $P(A) = 1$. Conversely, if $P(A) = 1$, then A may fail to occur in a very small percentage of times in the long run.*

Axiomatic Definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by $P(A)$, is defined as a real number satisfying the following axioms.

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and

(iv) If $A_1, A_2, \ldots, A_n, \ldots$ are a set of mutually exclusive events, $P(A_1 \cup A_2 \cup \ldots \cup A_n \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$

The term *mutually exclusive* used in the above definition can be explained as follows. A set of events is said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the others. Two events A and B are mutually exclusive if A occurs and B does not occur and vice versa. In other words, A and B cannot occur simultaneously, i.e., $P(A \cap B) = 0$.

In the development of the probability theory, all results are derived directly or indirectly using only the axioms of probability, as can be seen from the following theorems.

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Theorem 1 The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, $P(\phi) = 0$.

Proof The certain event S and the impossible event ϕ are mutually exclusive.

Hence $\quad P(S \cup \phi) = P(S) + P(\phi)$ [Axiom (iii)].

But $\quad S \cup \phi = S$.

$\therefore \quad P(S) = P(S) + P(\phi)$

$\therefore \quad P(\phi) = 0$

Theorem 2 If $\bar{A}$ is the complementary event of A, $P(\bar{A}) = 1 - P(A) \leq 1$.

Proof A and $\bar{A}$ are mutually exclusive events, such that $A \cup \bar{A} = S$.

$$\therefore \quad P(A \cup \bar{A}) = P(S)$$
$$= 1 \text{ [Axiom (ii)]}$$

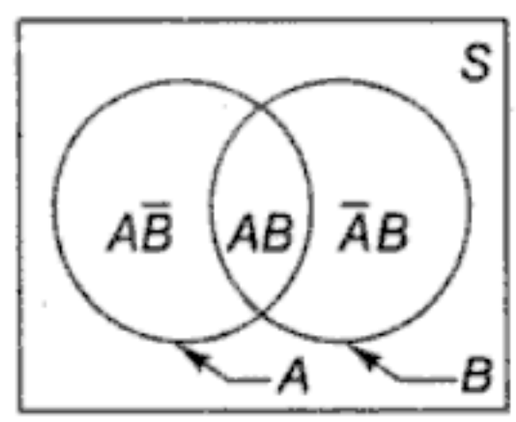
i.e., $\quad P(A) + P(\bar{A}) = 1$ [Axiom (iii)]

$$\therefore \quad P(\bar{A}) = 1 - P(A)$$

Since $P(A) \geq 0$, it follows that $P(\bar{A}) \leq 1$.

Theorem 3 If A and B are any 2 events, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

Proof A is the union of the mutually exclusive events $A\bar{B}$ and AB and B is the union of the mutually exclusive events $\bar{A}B$ and AB.



$$\therefore \quad P(A) = P(A\bar{B}) + P(AB) \text{ [Axiom (iii)]}$$

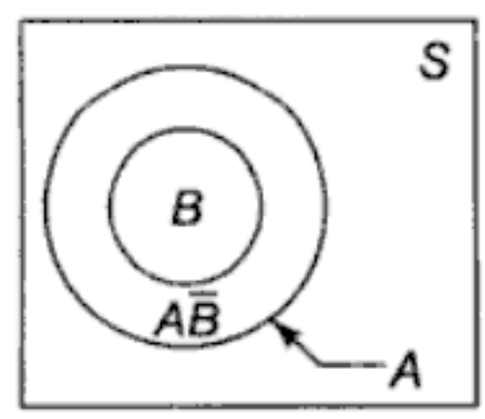
and $\quad P(B) = P(\bar{A}B) + P(AB)$ [Axiom (iii)]

$$\therefore \quad P(A) + P(B) = [P(A\bar{B}) + P(AB) + P(\bar{A}B) + P(AB)]$$
$$= P(A \cup B) + P(A \cap B)$$

The result follows. Clearly, $P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$]

Theorem 4 If $B \subset A$, $P(B) \leq P(A)$.

Proof B and $A\bar{B}$ are mutually exclusive events such that $B \cup A\bar{B} = A$.



$$\therefore \quad P(B \cup A\bar{B}) = P(A)$$

i.e. $\quad P(B) + P(A\bar{B}) = P(A)$ [Axiom (iii)]

$$\therefore \quad P(B) \leq P(A)$$

Note *In probability theory developed using the classical definition of probability, theorem 3 above is termed as Addition theorem of probability as applied to any 2 events. The theorem can be extended to any 3 events A, B and C as follows:*

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$$P(A \cup B \cup C) = P(\text{at least one of } A, B \text{ and } C \text{ occurs})$$
$$= P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

In the classical approach, probability axiom (iii) is termed as addition theorem of probability as applied to 2 mutually exclusive events, which is proved in the following way.

Let the total number of cases (outcomes) be n, of which n_A are favourable to the event A and n_B are favourable to the event B.

Therefore the number of cases favourable to A or B, i.e., $A \cup B$ is $(n_A + n_B)$, since the events A and B are disjoint.

$$\therefore \qquad P(A \cup B) = \frac{n_A + n_B}{n} = \frac{n_A}{n} + \frac{n_B}{n} = P(A) + P(B)$$

Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by $P(B/A)$ and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

For example, when a fair dice is tossed, the conditional probability of getting '1', given that an odd number has been obtained, is equal to 1/3 as explained below:

$$S = \{1, 2, 3, 4, 5, 6\}; A = \{1, 3, 5\}; B = \{1\}$$

$$\therefore \qquad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3}$$

As per the definition given above, $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$.

Rewriting the definition of conditional probability, we get $P(A \cap B) = P(A) \times P(B/A)$. This is sometimes referred to as *Product theorem of probability*, which is proved as follows.

Let n_A, n_{AB} be the number of cases favourable to the events A and $A \cap B$, out of the total number n of cases.

$$\therefore \qquad P(A \cap B) = \frac{n_{AB}}{n} = \frac{n_A}{n} \times \frac{n_{AB}}{n_A} = P(A) \times P(B/A)$$

The product theorem can be extended to 3 events A, B and C as follows:

$$P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/A \text{ and } B)$$

The following properties are easily deduced from the definition of conditional probability:

1. If $A \subset B$, $P(B/A) = 1$, since $A \cap B = A$
2. If $B \subset A$, $P(B/A) \geq P(B)$, since $A \cap B = B$, and $\dfrac{P(B)}{P(A)} \geq P(B)$, as $P(A) \leq P(S) = 1$

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3. If A and B are mutually exclusive events, $P(B/A) = 0$, since $P(A \cap B) = 0$
4. If $P(A) > P(B)$, $P(A/B) > P(B/A)$ (MKU — Apr. 96)
5. If $A_1 \subset A_2$, $P(A_1/B) \leq P(A_2/B)$ (BU — Apr. 96))

Independent Events

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

When 2 events A and B are independent, it is obvious from the definition that $P(B/A) = P(B)$. If the events A and B are independent, the product theorem takes the form $P(A \cap B) = P(A) \times P(B)$. Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events A and B are said to be independent (pairwise independent). The product theorem can be extended to any number of independent events: If $A_1, A_2, \ldots, A_n$ are n independent events.

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \times P(A_2) \times \ldots \times P(A_n)$$

When this condition is satisfied, the events $A_1, A_2, \ldots, A_n$ are also said to be *totally independent*. A set of events $A_1, A_2, \ldots, A_n$ is said to be *mutually independent* if the events are totally independent when considered in sets of 2, 3, ..., n events.

In the case of more than 2 events, the term 'independence' is taken as 'total independence' unless specified otherwise.

Theorem 1 If the events A and B are independent, the events $\overline{A}$ and B (and similarly A and $\overline{B}$) are also independent.

Proof The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive such that $(A \cap B) \cup (\overline{A} \cap B) = B$.

$\therefore \quad P(A \cap B) + P(\overline{A} \cap B) = P(B)$ (by addition theorem)

$$\begin{aligned} \therefore \quad P(\overline{A} \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)\,P(B) \text{ (by product theorem)} \\ &= P(B)\,[1 - P(A)] \\ &= P(\overline{A})\,P(B) \end{aligned}$$

Theorem 2 If the events A and B are independent, then so are $\overline{A}$ and $\overline{B}$.

(MU — Apr. 96)

Proof

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= P(\overline{A \cup B}) = 1 - P(A \cup B) \qquad (1) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \text{ (by addition theorem)} \\ &= 1 - P(A) - P(B) + P(A) \times P(B) \\ &\qquad \text{(since } A \text{ and } B \text{ are independent)} \\ &= [1 - P(A)] - P(B)\,[1 - P(A)] \\ &= P(\overline{A}) \times P(\overline{B}) \qquad (2) \end{aligned}$$

Note *From (1) and (2), if follows that when the events A and B are independent,* $P(A \cup B) = 1 - P(\overline{A}) \times P(\overline{B})$.

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Worked Examples 1(A)

Example 1 *A fair coin is tossed 4 times. Define the sample space corresponding to this random experiment. Also give the subsets corresponding to the following events and find the respective probabilities:*

(a) More heads than tails are obtained.

(b) Tails occur on the even numbered tosses.

Solution S = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

(a) Let A be the event in which more heads occur than tails.

Then A = {HHHH, HHHT, HHTH, HTHH, THHH}

(b) Let B be the event in which tails occur in the second and fourth tosses.

Then B = {HTHT, HTTT, TTHT, TTTT}

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{16};\ P(B) = \frac{n(B)}{n(S)} = \frac{1}{4}$$

Example 2 *There are 4 letters and 4 addressed envelopes. If the letters are placed in the envelopes at random, find the probability that (i) none of the letters is in the correct envelope and (ii) at least 1 letter is in the correct envelope, by explicitly writing the sample space and the event spaces.*

Solution Let the envelopes be denoted by A, B, C and D and the corresponding letters by a, b, c and d.

S = {(Aa, Bb, Cc, Dd), (Aa, Bb, Cd, Dc), (Aa, Bc, Cb, Dd), (Aa, Bc, Cd, Db), (Aa, Bd, Cb, Dc), (Aa, Bd, Cc, Db), (Ab, Ba, Cc, Dd), (Ab, Ba, Cd, Dc), (Ab, Bc, Ca, Dd), (Ab, Bc, Cd, Da), (Ab, Bd, Ca, Dc), (Ab, Bd, Cc, Da), (Ac, Ba, Cb Dd), (Ac, Ba, Cd, Db), (Ac, Bb, Ca, Dd), (Ac, Bb, Cd, Da), (Ac, Bd, Ca, Db), (Ac, Bd, Cb, Da), (Ad, Ba, Cb, Dc), (Ad, Ba, Cc, Db), (Ad, Bb, Ca, Dc), (Ad, Bb, Cc, Da), (Ad, Bc, Ca, Db), (Ad, Bc, Cb, Da)}

where 'Aa' means that the letter 'a' is placed in the envelope A.

Let E_1 denote the event in which none of the letters is in the correct envelope.

Then E_1 = {(Ab, Ba, Cd, Dc), (Ab, Bc, Cd, Da), (Ab, Bd, Ca, Dc), (Ac, Ba, Cd, Db), (Ac, Bd, Ca, Db), (Ac, Bd, Cb, Da), (Ad, Ba, Cb, Dc), (Ad, Bc, Ca, Db), (Ad, Bc, Cb, Da)}

Let E_2 denote the event in which at least one of the letters is in the correct envelope.

We note that E_2 is the complement of E_1. Therefore E_2 consists of all the elements of S except those in E_1.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{24} = \frac{3}{8} \text{ and } P(E_2) = 1 - P(E_1) = \frac{5}{8}$$

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Example 3 *A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v) exactly 1 is good, (vi) neither has major defects and (vii) neither is good.*

Solution Although the articles may be drawn one after the other, we can consider that both articles are drawn simultaneously, as they are drawn without replacement.

(i) $P(\text{both are good}) = \dfrac{\text{No. of ways drawing 2 good articles}}{\text{Total no. of ways of drawing 2 articles}}$

$$= \frac{10C_2}{16C_2} = \frac{3}{8}$$

(ii) $P(\text{both have major defects})$

$$= \frac{\text{No. of ways of drawing 2 articles with major defects}}{\text{Total no. of ways}}$$

$$= \frac{2C_2}{16C_2} = \frac{1}{120}$$

(iii) $P(\text{at least 1 is good}) = P(\text{exactly 1 is good or both are good})$

$= P(\text{exactly 1 is good and 1 is bad or both are good})$

$$= \frac{10C_1 \times 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iv) $P(\text{atmost 1 is good}) = P(\text{none is good or 1 is good and 1 is bad})$

$$= \frac{10C_0 \times 6C_2 + 10C_1 \times 6C_1}{16C_2} = \frac{5}{8}$$

(v) $P(\text{exactly 1 is good}) = P(\text{1 is good and 1 is bad})$

$$= \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$$

(vi) $P(\text{neither has major defects})$

$= P(\text{both are non-major defective articles})$

$$= \frac{14C_2}{16C_2} = \frac{91}{120}$$

(vii) $P(\text{neither is good}) = P(\text{both are defective})$

$$= \frac{6C_2}{16C_2} = \frac{1}{8}$$

Example 4 *From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is positive?*

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Solution If the product is to be positive, all the 4 numbers must be positive or all the 4 must be negative or 2 of them must be positive and the other 2 must be negative.

No. of ways of choosing 4 positive numbers = $6C_4 = 15$.

No. of ways of choosing 4 negative numbers = $8C_4 = 70$.

No. of ways of choosing 2 positive and 2 negative numbers

$$= 6C_2 \times 8C_2 = 420.$$

Total no. of ways of choosing 4 numbers from all the 14 numbers

$$= 14C_4 = 1001.$$

P(the product is positive)

$$= \frac{\text{No. of ways by which the product is positve}}{\text{Total no. of ways}}$$

$$= \frac{15 + 70 + 420}{1001} = \frac{505}{1001}$$

Example 5 *A box contains tags marked 1, 2, ..., n. Two tags are chosen at random without replacement. Find the probability that the numbers on the tags will be consecutive integers.*

Solution If the numbers on the tags are to be consecutive integers, they must be chosen as a pair from the following pairs.

$$(1, 2); (2, 3); (3, 4); \ldots; (n-1, n)$$

No. of ways of choosing any one pair from the above $(n - 1)$ pairs = $(n-1)\,C_1 = n - 1$.

Total No. of ways of choosing 2 tags from the n tags = nC_2.

$$\therefore \quad \text{Required probability} = \frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}$$

Example 6 *If n biscuits are distributed at random among m children, what is the probability that a particular child receives r biscuits, where r < n?*

(MKU — Nov. 96)

Solution The first biscuit can be given to any one of the m children, i.e., in m ways.

Similarly the second biscuit can be given in m ways.

Therefore 2 biscuits can be given in m^2 ways.

Extending, n biscuits can be distributed in m^n ways. The r biscuits received by the particular child can be chosen from the n biscuits in nC_r ways. If this child has got r biscuits, the remaining $(n - r)$ biscuits can be distributed among the remaining $(m - 1)$ children in $(m-1)^{n-r}$ ways.

$\therefore$ No. of ways of distributing in the required manner

$$= nC_r(m-1)^{n-r}$$

$$\therefore \quad \text{Required probability} = \frac{nC_r\,(m-1)^{n-r}}{m^n}$$

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Example 7 *If $P(A) = P(B) = P(AB)$, show that $P(A\overline{B} + \overline{A}B) = 0$ [$AB \equiv A \cap B$].*

Solution By addition theorem,

$$P(A \cup B) = P(A) + P(B) - P(AB) \quad (1)$$

From the Venn diagram on page (3), it is clear that

$$A \cup B = A\overline{B} + \overline{A}B + AB$$

$$\therefore \quad P(A \cup B) = P(A\overline{B}) + P(\overline{A}B) + P(AB) \text{ (by probability axiom)} \quad (2)$$

Using the given condition in (1),

$$P(A \cup B) = P(AB) \quad (3)$$

From (2) and (3), $P(A\overline{B}) + P(\overline{A}B) = 0$

Example 8 *If A, B and C are any 3 events such that $P(A) = P(B) = P(C) = 1/4$, $P(A \cap B) = P(B \cap C) = 0$; $P(C \cap A) = 1/8$. Find the probability that at least 1 of the events A, B and C occurs.*

Solution P(at least one of A, B and C occurs) $= P(A \cup B \cup C)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \quad (1)$$

Since $P(A \cap B) = P(B \cap C) = 0$, $P(A \cap B \cap C) = 0$. Equation (1) becomes

$$P(A \cup B \cup C) = \frac{3}{4} - 0 - 0 - \frac{1}{8} = \frac{5}{8}$$

Example 9 *Solve Example 5, if the tags are chosen at random with replacement.*

Solution If the tag with '1' is chosen in the first draw, the tag with '2' must be chosen in the second draw. Probability for each $= 1/n$.

$\therefore$ P('1' in the first draw and '2' in the second draw) $= 1/n^2$ (Product theorem)

Similarly, P('n' in the first draw and '$n - 1$' in the second draw $= 1/n^2$.

If the number drawn first is '2', the number drawn second may be '1' or '3'.

Probability of drawing consecutive numbers in this case

$$= \frac{1}{n} \times \frac{2}{n} = \frac{2}{n^2}$$

Similarly, when the first number drawn is '3', '4', ..., '$(n - 1)$' probability of drawing consecutive numbers will be $2/n^2$.

All the above possibilities are mutually exclusive.

$$\therefore \quad \text{Required probability} = \frac{1}{n^2} + \frac{1}{n^2} + (n-2) \times \frac{2}{n^2} = \frac{2(n-1)}{n^2}$$

Example 10 *A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?*

Solution Let A = one of the tubes drawn is good and B = the other tube is good.

$$P(A \cap B) = P(\text{both tubes drawn are good})$$

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$$= \frac{6C_2}{10C_2} = \frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., $P(B/A)$ is required.

By definition,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$$

Example 11 *Two defective tubes get mixed up with 2 good ones. The tubes are tested, one by one, until both defectives are found. What is the probability that the last defective tube is obtained on (i) the second test, (ii) the third test and (iii) the fourth test?*

Solution Let D represent defective and N represent non-defective tube.

(i) P(Second D in the II test) = P(D in the I test and D in the II test)

$$= P(D_1 \cap D_2), \text{ say}$$
$$= P(D_1) \times P(D_2) \text{ (by independence)}$$
$$= \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$$

(ii) P(second D in the III test) = $P(D_1 \cap N_2 \cap D_3 \text{ or } N_1 \cap D_2 \cap D_3)$

$$= P(D_1 \cap N_2 \cap D_3) + P(N_1 \cap D_2 \cap D_3)$$
$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2}$$
$$= \frac{1}{3}$$

(iii) P(second D in the IV test) = $P(D_1 \cap N_2 \cap N_3 \cap D_4) + P(N_1 \cap D_2 \cap N_3 \cap D_4) + P(N_1 \cap N_2 \cap D_3 \cap D_4)$

$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1$$
$$= \frac{1}{2}$$

Example 12 *In a shooting test, the probability of hitting the target is 1/2 for A, 2/3 for B and 3/4 for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.*

Solution Let $A \equiv$ Event of A hitting the target, and so on.

$$P(\overline{A}) = \frac{1}{2}, P(\overline{B}) = \frac{1}{3}, P(\overline{C}) = \frac{1}{4}$$

$$P(\overline{A} \cap \overline{B} \cap \overline{C}) = P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \text{ (by independence)}$$

ie., P(none hits the target) $= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$

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P(at least one hits the target)

$$= 1 - P(\text{none hits the target})$$

$$= 1 - \frac{1}{24} = \frac{23}{24}$$

Example 13 *A and B alternately throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.* (BU — Apr. 96)

Solution Throwing 6 with 2 dice ≡ Getting 6 as the sum of the numbers shown on the upper faces of the 2 dice.

$$P(\text{throwing 6 with 2 dice}) = \frac{5}{36}$$

$$P(\text{throwing 7 with 2 dice}) = \frac{1}{6}$$

Let $A \equiv$ Event of A throwing 6.

Let $B \equiv$ Event of B throwing 7.

A plays in the first, third, fifth, …, trials.

Therefore A will win, if he throws 6 in the first trial or third trial or in subsequent (odd) trials.

$$\therefore \quad P(A \text{ wins}) = P(A \text{ or } \overline{A}\ \overline{B}\ A \text{ or } \overline{A}\ \overline{B}\ \overline{A}\ \overline{B}\ A \text{ or } \ldots)$$

$$= P(A) + P(\overline{A}\ \overline{B}\ A) + P(\overline{A}\ \overline{B}\ \overline{A}\ \overline{B}\ A) + \ldots \text{ (Addition theorem)}$$

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \times \frac{5}{36} + \ldots \text{ upto } \infty$$

$$= \frac{5/36}{1 - (155/216)} \text{ (since the series is an infinite geometric series)}$$

$$= \frac{30}{61}$$

Example 14 *Show that $2^n - (n + 1)$ equations are needed to establish the mutual independence of n events.*

Solution n events are mutually independent, if they are totally independent when considered in sets of 2, 3, …, n events.

Sets of r events can be chosen from the n events in nC_r ways.

To establish total independence of r events, say, $A_1, A_2, \ldots, A_r$ chosen in any one of the nC_r ways, we need one equation, namely, $P(A_1, A_2, \ldots, A_r) = P(A_1) \times P(A_2) \ldots \times P(A_r)$.

Therefore to establish total independence of all the nC_r sets, each of r events, we need nC_r equations.

Therefore the number of equations required to establish mutual independence

$$= \sum_{r=2}^{n} nC_r$$

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$$= (nC_0 + nC_1 + nC_2 + \ldots + nC_n) - (1 + n)$$
$$= (1 + 1)^n - (n + 1)$$
$$= 2^n - (n + 1)$$

Example 15 *Two fair dice are thrown independently. Three events A, B and C are defined as follows.*

(a) Odd face with the first die

(b) Odd face with the second die

(c) Sum of the numbers in the 2 dice is odd. Are the events A, B and C mutually independent? (MKU — Apr. 97)

Solution $P(A) = \frac{3}{6} = \frac{1}{2}$; $P(B) = \frac{3}{6} = \frac{1}{2}$

The outcomes favourable to the event C are (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5) and so on.

$$\therefore \quad P(C) = \frac{1}{2}$$

$$P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4}$$

$\therefore \quad P(A \cap B) = P(A)\,P(B)$, and so on

But $P(A \cap B \cap C) = 0$, since C cannot happen when A and B occur. Therefore $P(A \cap B \cap C) \neq P(A) \times P(B) \times P(C)$.

Therefore the events are pairwise independent, but not mutually independent.

Example 16 *If A, B and C are random subsets (events) in a sample space and if they are pairwise independent and A is independent of $(B \cup C)$, prove that A, B and C are mutually independent.* (MU — Nov. 96)

Solution Given: $P(AB) = P(A) \times P(B)$ (1)

$P(BC) = P(B) \times P(C)$ (2)

$P(CA) = P(C) \times P(A)$ (3)

$P[A\,(B \cup C)] = P(A) \times P(B \cup C)$ (4)

Consider $P[A(B \cup C)] = P(AB \cup AC)$

$= P(AB) + P(AC) - P(AB \cap AC)$ (by addition theorem)

$= P(A) \times P(B) + P(A) \times P(C) - P(ABC)$ [by (1) and (3)] (5)

Therefore from (4) and (5), we get

$$P(ABC) = P(A) \times P(B) + P(A) \times P(C) - P(A) \times P(B \cup C)$$
$$= P(A) \times [P(B) + P(C) - P(B \cup C)]$$
$$= P(A) \times P(B \cap C) \text{ (by addition theorem)}$$
$$= P(A) \times P(B) \times P(C) \text{ [by (2)]} \quad (6)$$

From (1), (2), (3) and (6), the required result follows.

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Exercise 1(A)

Part A (Short answer questions)

1. What is a random experiment? Give an example.
2. Give the apriori definition of probability with an example.
3. Give the aposteriori definition of probability with an example.
4. Give the relative frequency definition of probability with an example.
5. Define the sample space and an event associated with a random experiment with an example.
6. Give the axiomatic definition of probability.
7. State the axioms of probability.
8. What do you infer from the statements $P(A) = 0$ and $P(A) = 1$?
9. Define mutually exclusive events with an example.
   (**Example:** Getting an odd number and getting an even number when a 6-faced die is tossed are 2 mutually exclusive events.)
10. From a bag containing 3 red and 2 black balls, 2 balls are drawn at random. Find the probability that they are of the same colour.
11. When 2 cards are drawn from a well-shuffled pack of playing cards, what is the probability that they are of the same suit?
12. When A and B are 2 mutually exclusive events such that $P(A) = 1/2$ and $P(B) = 1/3$, find $P(A \cup B)$ and $P(A \cap B)$.
13. If $P(A) = 0.29$, $P(B) = 0.43$, find $P(A \cap \overline{B})$, if A and B are mutually exclusive.
14. When A and B are 2 mutually exclusive events, are the values $P(A) = 0.6$ and $P(A \cap \overline{B}) = 0.5$ consistent? Why?
15. Prove that the probability of an impossible event is zero (or prove that $P(\phi) = 0$.
16. Prove that $P(\overline{A}) = 1 - P(A)$, where $\overline{A}$ is the complement of A.
17. State addition theorem as applied to any 2 events. Extend it to any 3 events.
18. If $P(A) = 3/4$, $P(B) = 5/8$, prove that $P(A \cap B) \geq 3/8$.
19. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?
20. The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will get an electric contract is 4/9. If the probability of getting at least one contract is 4/5, what is the probability that he will get both?
21. If $P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, find $P(\overline{A} \cap \overline{B})$.
22. If $P(A) = 0.35$, $P(B) = 0.75$ and $P(A \cup B) = 0.95$, find $P(\overline{A} \cup \overline{B})$.
23. Prove that $P(A \cup B) \leq P(A) + P(B)$. When does the equality hold good?
24. If $B \subset A$, prove that $P(B) \leq P(A)$.
25. Give the definitions of joint and conditional probabilities with examples.
26. Give the definition of conditional probability and deduce the product theorem of probability.

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27. If $A \subset B$, prove that $P(B/A) = 1$.
28. If $B \subset A$, prove that $P(B/A) \geq P(B)$.
29. If A and B are mutually exclusive events, prove that $P(B/A) = 0$
30. If $P(A) > P(B)$, prove that $P(A/B) > P(B/A)$.
31. If $A \subset B$, prove that $P(A/C) \leq P(B/C)$.
32. If $P(A) = 1/3$, $P(B) = 3/4$ and $P(A \cup B) = 11/12$, find $P(A/B)$ and $P(B/A)$.
33. When are 2 events said to be independent? Give an example for 2 independent events.
34. What is the probability of getting atleast 1 head when 2 coins are tossed?
35. When 2 dice are tossed, what is the probability of getting 4 as the sum of the face numbers?
36. If the probability that A solves a problem is 1/2 and that for B is 3/4 and if they aim at solving a problem independently, what is the probability that the problem is solved?
37. If $P(A) = 0.65$, $P(B) = 0.4$ and $P(A \cap B) = 0.24$, can A and B be independent events?
38. 15% of a firm's employees are BE degree holders, 25% are MBA degree holders and 5% have both the degrees. Find the probability of selecting a BE degree holder, if the selection is confined to MBAs.
39. In a random experiment, $P(A) = 1/12$, $P(B) = 5/12$ and $P(B/A) = 1/15$, find $P(A \cup B)$.
40. What is the difference between total independence and mutual independence?
41. Can 2 events be simultaneously independent and mutually exclusive? Explain.
42. If A and B are independent events, prove that $\overline{A}$ and B are also independent.
43. If A and B are independent events, prove that A and $\overline{B}$ are also independent.
44. If $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.15$, find $P(A/\overline{B})$.
45. If A and B are independent events, prove that $\overline{A}$ and $\overline{B}$ are also independent.
46. If A and B are independent events, prove that
$$P(A \cup B) = 1 - P(\overline{A}) \times P(\overline{B}).$$
47. A and B toss a fair coin alternately with the understanding that the one who obtains the head first wins. If A starts, what is his chance of winning?

Part B

48. Write the sample space associated with the experiment of tossing 3 coins at a time and the event of getting heads from the first 2 coins. Also find the corresponding probability.

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49. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition listed. This is continued until 2 consecutive defectives are produced or 4 items have been checked, whichever occurs first. Describe a sample space for this experiment.
50. An urn contains 2 white and 4 black balls. Two balls are drawn one by one without replacement. Write the sample space corresponding to this experiment and the subsets corresponding to the following events.
    (a) The first ball drawn is white.
    (b) Both the balls drawn are black.
    Also find the probabilities of the above events.
51. A box contains three 10-Ω resistors labelled R_1, R_2 and R_3 and two 50-Ω resistors labelled R_4 and R_5. Two resistors are drawn from this box without replacement. List all the outcomes of this random experiment as pairs of resistors. Also list the outcomes associated with the following events and hence find the corresponding probabilities.
    (a) Both the resistors drawn are 10-Ω resistors.
    (b) One 10-Ω resistor and one 50-Ω resistor are drawn.
    (c) One 10-Ω resistor is drawn in the first draw and one 50-Ω resistor is drawn in the second draw.
52. A box contains 3 white balls and 2 black balls. We remove at random 2 balls in succession. What is the probability that the first removed ball is white and the second is black? (BDU — Apr. 96)
53. An urn contains 3 white balls, 4 red balls and 5 black balls. Two balls are drawn from the urn at random. Find the probability that (i) both of them are of the same colour and (ii) they are of different colours.
54. One integer is chosen at random from the numbers 1, 2, 3, …, 100. What is the probability that the chosen number is divisible by (i) 6 or 8 and (ii) 6 or 8 or both?
55. If there are 4 persons A, B, C and D and if A tossed with B, then C tossed with D and then the winners tossed. This process continues till the prize is won. What are the probabilities of each of the 4 to win?
    (MKU — Nov. 96)
56. Ten chips numbered 1 through 10 are mixed in a bowl. Two chips are drawn from the bowl successively and without replacement. What is the probability that their sum is 10?
57. A bag contains 10 tickets numbered 1, 2, …, 10. Three tickets are drawn at random and arranged in ascending order of magnitude. What is the probability that the middle number is 5?
58. Two fair dice are thrown independently. Four events A, B, C and D are defined as follows:
    A: Even face with the first dice.
    B: Even face with the second dice.
    C: Sum of the points on the 2 dice is odd.

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D: Product of the points on the 2 dice exceeds 20.
Find the probabilities of the 4 events.

59. A box contains 4 white, 5 red and 6 black balls. Four balls are drawn at random from the box. Find the probability that among the balls drawn, there is at least 1 ball of each colour.
60. Four persons are chosen at random from a group consisting of 4 men, 3 women and 2 children. Find the chance that the selected group contains at least 1 child.
61. A committee of 6 is to be formed from 5 lecturers and 3 professors. If the members of the committee are chosen at random, what is the probability that there will be a majority of lecturers in the committee?
62. Twelve balls are placed at random in 3 boxes. What is the probability that the first box will contain 3 balls?
63. If A and B are any 2 events, show that $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$. (MU — Apr. 96)
64. A and B are 2 events associated with an experiment. If $P(A) = 0.4$ and $P(A \cup B) = 0.7$, find $P(B)$ if (i) A and B are mutually exclusive (ii) A and B are independent.
65. If $P(A + B) = 5/6$, $P(AB) = 1/3$ and $P(\overline{B}) = 1/2$, prove that the events A and B are independent.
66. If $A \subset B$, $P(A) = 1/4$ and $P(B) = 1/3$, find $P(A/B)$ and $P(B/A)$.
67. m objects are selected from n objects ($m < n$). What is the probability that the selection contains a particular object that was present in the n given objects?
68. What is the probability that there will be 53 sundays in (i) a leap year and (ii) a non-leap year?
69. If the probability that a communication system has high selectivity is 0.54 and the probability that it will have high fidelity is 0.81 and the probability that it will have both is 0.18, find the probability that (i) a system with high fidelity will also have high selectivity and (ii) a system with high selectivity will also have high fidelity.
70. An electronic assembly consists of two subsystems A and B. From previous testing procedures, the following probabilities are assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ fail}) = 0.15$ and $P(B \text{ fails alone}) = 0.15$. Evaluate (i) $P(A \text{ fails alone})$ and (ii) $P(A \text{ fails}/B \text{ has failed})$.
71. A consignment of 15 tubes contains 4 defectives. The tubes are selected at random, one by one, and examined. Assuming that the tubes tested are not put back, what is the probability that the ninth one examined is the last defective? (BDU — Apr. 97)
72. A card is drawn from a 52-card deck, and without replacing it, a second card is drawn. The first and second cards are not replaced and a third card is drawn.
    (a) If the first card is a heart, what is the probability of second card being a heart?

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Clearly B_1, B_2 and B_3 are exhaustive and mutually exclusive events.

Let A = event of drawing a white ball from the second urn after transfer.

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{15}{26};\ P(B_2) = \frac{3C_2}{13C_2} = \frac{1}{26};\ P(B_3) = \frac{10 \times 3}{13C_2} = \frac{10}{26},$$

$$\begin{aligned} P(A/B_1) &= P(\text{drawing a white ball/2 white balls have been transferred}) \\ &= P(\text{drawing a white ball/urn II contains 5 white and 5 black balls}) \\ &= \frac{5}{10} \end{aligned}$$

Similarly, $P(A/B_2) = \frac{3}{10}$ and $P(A/B_3) = \frac{4}{10}$.

By theorem of total probability,

$$\begin{aligned} P(A) &= P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3) \\ &= \frac{15}{26} \times \frac{5}{10} + \frac{1}{26} \times \frac{3}{10} + \frac{10}{26} \times \frac{4}{10} \\ &= \frac{59}{130} \end{aligned}$$

Example 3 *In a coin tossing experiment, if the coin shows head, 1 dice is thrown and the result is recorded. But if the coin shows tail, 2 dice are thrown and their sum is recorded. What is the probability that the recorded number will be 2?*

(BDU — Apr. 96)

Solution When a single dice is thrown, $P(2) = \frac{1}{6}$.

When 2 dice are thrown, the sum will be 2, only if each die shows 1.

$\therefore$ $P(\text{getting 2 as sum with 2 dice}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ (since independence)

By theorem of total probability,

$$\begin{aligned} P(2) &= P(H) \times P(2/H) + P(T) \times P(2/T) \\ &= \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{36} \\ &= \frac{7}{72} \end{aligned}$$

Example 4 *A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white?*

Solution Since 2 white balls have been drawn out, the bag must have contained 2, 3, 4 or 5 white balls.

Let B_1 = Event of the bag containing 2 white balls, B_2 = Event of the bag containing 3 white balls, B_3 = Event of the bag containing 4 white balls and B_4 = Event of the bag containing 5 white balls.

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where $y = \frac{1}{\sqrt{2\pi npq}} e^{-(x-np)^2/2npq}$, which is the density of a normal distribution with mean np and variance npq.

As the reader is familiar with normal distribution, it can be easily seen that

$$\sum_{r=r_1}^{r_2} P_n(r) = \int_{t_1}^{t_2} \phi(t)dt$$

where $\phi(t)$ is the standard normal density and $t_1 = \frac{r_1 - np - 1/2}{\sqrt{npq}}$ and $t_2 = \frac{r_2 - np + 1/2}{\sqrt{npq}}$. Now $\int_{t_1}^{t_2} \phi(t)dt$ can be computed using the Table of areas under normal curve.

Generalisation of Bernoulli's Theorem Multinomial Distribution

If $A_1, A_2, \ldots, A_k$ are exhaustive and mutually exclusive events associated with a random experiment such that $P(A_i \text{ occurs}) = p_i$, where $p_1 + p_2 + \ldots + p_k = 1$, and if the experiment is repeated n times, then the probability that A_1 occurs r_1 times, A_2 occurs r_2 times, ..., A_k occurs r_k times is given by

$$P_n(r_1, r_2, \ldots, r_k) = \frac{n!}{r_1!\,r_2!\ldots r_k!} p_1^{r_1} \times p_2^{r_2} \ldots p_k^{r_k}$$

where $r_1 + r_2 + \ldots + r_k = n$.

Proof The r_1 trials in which the event A_1 occurs can be chosen from the n trials in nC_{r_1} ways. The remaining $(n - r_1)$ trials are left over for the other events.

The r_2 trials in which the event A_2 occurs can be chosen from the $(n - r_1)$ trials in $(n - r_1)C_{r_2}$ ways.

The r_3 trials in which the event A_3 occurs can be chosen from the $(n - r_1 - r_2)$ trials in $(n - r_1 - r_2)C_{r_3}$ ways, and so on.

Therefore the number of ways in which the events $A_1, A_2, \ldots, A_k$ can happen $= nC_{r_1} \times (n - r_1)C_{r_2} \times (n - r_1 - r_2)C_{r_3} \times \ldots \times (n - r_1 - r_2 \ldots - r_{k-1})C_{r_k}$

$$= \frac{n!}{r_1!\,r_2!\ldots r_k!}.$$

Consider any one of the above ways in which the events $A_1, A_2, \ldots, A_k$ occur. Since the n trials are independent, $r_1, r_2, \ldots, r_k$ trials are also independent.

$\therefore$ $P(A_1 \text{ occurs } r_1 \text{ times}) = p_1^{r1}$

$P(A_2 \text{ occurs } r_2 \text{ times}) = p_2^{r_2}$, and so on.

$\therefore$ $P(A_1 \text{ occurs } r_1 \text{ times}, A_2 \text{ occurs } r_2 \text{ times}, \ldots, A_k \text{ occurs } r_k \text{ times}) = p_1^{r1} \times p_2^{r2} \times \ldots \times p_k^{rk}$

Since the ways in which the events happen are mutually exclusive, the required probability is given by

$$P_n(r_1, r_2, \ldots, r_k) = \frac{n!}{r_1!\,r_2!\ldots r_k!} p_1^{r1} \times p_2^{r2} \times \ldots \times p_k^{rk}$$

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Worked Examples 1(C)

Example 1 *A coin is tossed an infinite number of times. If the probability of a head in a single toss is p, show that the probability that kth head is obtained at the nth tossing, but not earlier is* $(n-1)C_{k-1}\,p^k q^{n-k}$, *where* $q = 1 - p$.

Solution k heads should be obtained at the nth tossing, but not earlier.

Therefore $(k-1)$ heads must be obtained in the first $(n-1)$ tosses and 1 head must be obtained at the nth toss.

$$\begin{aligned}\therefore \quad \text{Required probability} &= P[k-1 \text{ heads in } (n-1) \text{ tosses}] \\ &\quad \times P(1 \text{ head in 1 toss}) \\ &= (n-1)C_{k-1}p^{k-1}q^{n-k} \times p \\ &= (n-1)C_{k-1}p^k q^{n-k}\end{aligned}$$

Example 2 *Each of two persons A and B tosses 3 fair coins. What is the probability that they obtain the same number of heads?*

Solution $P(A$ and B get the same no. of heads)

$= P$(they get no head each or 1 head each or 2 heads each or 3 heads each)

$= P$(each gets 0 head) + P(each gets 1 head) + P(each gets 2 heads) + P(each gets 3 heads) (since the events are mutually exclusive)

$= P(A$ gets 0 head$) \times P(B$ gets 0 head$)$ + … (since A and B toss independently)

$$= \left[3C_0\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_1\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_2\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_3\left(\frac{1}{2}\right)^3\right]^2$$

$$= \frac{1}{64}(1+9+9+1) = \frac{5}{16}$$

Example 3 *A coin with P(head) = p = 1 − q is tossed n times. Show that the probability that the number of heads obtained is even is* $0.5[1 + (q-p)^n]$.

Solution P(even no. of heads are obtained)

$$= P(0 \text{ head or } 2 \text{ heads or } 4 \text{ heads or } \ldots)$$

$$= nC_0q^np^0 + nC_2q^{n-2}p^2 + nC_4q^{n-4}p^4 + \ldots \quad (1)$$

Consider

$$1 = (q+p)^n = nC_0q^n p^0 + nC_1q^{n-1}p^1 + nC_2q^{n-2}p^2 + \ldots \quad (2)$$

$$(q-p)^n = nC_0q^np^0 - nC_1q^{n-1}p^1 + nC_2q^{n-2}p^2 - \ldots \quad (3)$$

Adding (2) and (3), we get

$$1 + (q-p)^n = 2[nC_0q^np^0 + nC_2q^{n-2}p^2 + nC_4q^{n-4}p^4 + \ldots] \quad (4)$$

Using (4) in (1), the required probability $= 0.5[1 + (q-p)^n]$.

Example 4 *If at least 1 child in a family with 2 children is a boy, what is the probability that both children are boys?* (MKU — Apr. 96)

Solution p = Probability that a child is a boy $= \dfrac{1}{2}$

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$\therefore \qquad q = \frac{1}{2}$ and $n = 2$

P(at least one boy) = p(exactly 1 boy) + p(exactly 2 boys)

$$= 2C_1\left(\frac{1}{2}\right)^2 + 2C_2\left(\frac{1}{2}\right)^2 \qquad \text{(by Bernoulli's theorem)}$$

$$= \frac{3}{4}$$

P(both are boys/at least one is a boy)

$$= \frac{P(\text{both are boys} \cap \text{at least one is a boy})}{P(\text{at least one is a boy})}$$

$$= \frac{P(\text{both are boys})}{P(\text{at least one is a boy})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 5 *Find the probability of getting at least 60 heads, when 100 fair coins are tossed.*

Solution Since the coins are fair, $p = q = \frac{1}{2}$.

$$n = 100$$

$\therefore \qquad np = 50$ and $\sqrt{npq} = 5$

Required probability $P = \sum_{r=60}^{100} 100C_r \left(\frac{1}{2}\right)^{100}$

$$= \int_{1.9}^{10.1} \phi(t)\mathrm{d}t \simeq \int_{1.9}^{\infty} \phi(t)dt \text{ (by Demoivre–Laplace approximation)}$$

$$= 0.5 - \int_{0}^{1.9} \phi(t)dt = 0.5 - 0.4719 \text{ (from the Normal Table)}$$

$$= 0.0281$$

Example 6 *A fair dice is rolled 5 times. Find the probability that 1 shows twice, 3 shows twice and 6 shows once.*

Solution This problem is a direct application of Bernoulli's generalisation.

Here $n = 5$, $A_1 \equiv$ (getting 1), $A_3 \equiv$ (getting 3)

$A_6 =$ (getting 6), $k_1 = 2$, $k_3 = 2$ and $k_6 = 1$; $p_1 = p_3 = p_6 = 1/6$

$$\therefore \quad \text{Required probability} = \frac{5!}{2!2!1!}\left(\frac{1}{6}\right)^2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{1}{6}\right)^1$$

$$= 0.0039$$

(It is to be noted that $k_2 = k_4 = k_5 = 0$, though $p_2 = p_4 = p_5 = \frac{1}{6}$.)

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Exercise 1(C)

Part A (Short answer questions)

1. State Bernouilli's theorem on independent trials.
2. A fair coin is tossed 4 times. What is the probability of getting more heads than tails?
3. When 12 coins are tossed 256 times, how many times may one expect 8 heads and 4 tails?
4. If war breaks out on the average once in 25 years, find the probability that in 50 years at a stretch, there will be no war.
5. State the generalised form of Bernoulli's theorem on independent trials.
6. State De Moivre–Laplace theorem.

Part B

7. A binary number (composed only of the digits '0' and '1') is made up of n digits. If the probability of an incorrect digit appearing is p and that errors in different digits are independent of one another, find the probability of forming an incorrect number.
8. Suppose that twice as many items are produced (per day) by machine 1 as by machine 2. However 4% of the items from machine 1 are defective while machine 2 produces only about 2% defectives. Suppose that the daily output of the 2 machines is combined. A random sample of 10 items is taken from the combined output. What is the probability that this sample contains 2 defectives?
9. Binary digits are transmitted over a noisy communication channel in blocks of 16 binary digits. The probability that a received binary digit is in error because of channel noise is 0.1. If errors occur in various digit positions within a block independently, find the probability that the number of errors per block is greater than or equal to 5.
10. A company is trying to market a digital transmission system (modem) that has a bit error probability of 10^{-4}, and the bit errors are independent. The buyer will test the modem by sending a known message of 10^4 digits and checking the received message. If more than 2 errors occur, the modem will be rejected. Find the probability that the customer will buy the company's modem.
11. A fair coin is tossed 10,000 times. Find the probability that the number of heads obtained is between 4900 and 5100, using DeMoivre-Laplace approximation.
12. Over a period of 12h, 180 calls are made at random. What is the 'probability' that in a 4 h interval the number of calls is between 50 and 70? Use DeMoivre-Laplace approximation.

$\left[\textbf{Hint: } P(\text{a particular call occurs in the 4-h interval}) = p = \frac{4}{12} = \frac{1}{3}\right].$

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13. A random experiment can terminate in one of 3 events A, B and C with probabilities 1/2, 1/4 and 1/4 respectively. The experiment is repeated 6 times. Find the probability that the events A, B and C occur once, twice and thrice respectively.
14. A throws 3 fair coins and B throws 4 fair coins. Find the chance that A will throw more number of heads than would B.
15. In a large consignment of electric bulbs 10% are defective. A random sample of 20 bulbs is taken for inspection. Find the probability that (i) exactly 3 of them are defective, (ii) at most 3 of then are defective and (iii) at least 3 of them are defective.
16. A lot contains 1% defective items. What should be the number of items in a random sample so that the probability of finding at least 1 defective in it is at least 0.95?
17. In a precision bombing attack there is a 50% chance that any one bomb will hit the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% or more chance of completely destroying the target?

Answers

Exercise 1(A)

10. P(both balls are of the same colour)
$= P$(both balls are red or both are black)
$= P$(both are red) + P(both are black)
$$= \frac{3C_2}{5C_2} + \frac{2C_2}{5C_2} = \frac{2}{5}$$
11. Required probability = P(both are spades) + P(both are clubs) + P(both are hearts) + P(both are diamonds)
$$= 4 \times \frac{13C_2}{52C_2} = \frac{4}{17}$$
12. $P(A \cup B) = P(A) + P(B) = 5/6$; $P(A \cap B) = 0$, by definition.
13. When A and B are mutually exclusive, $P(A \cap \overline{B}) = P(A) = 0.29$
14. $P(A) = P(A \cap B) + P(A \cap \overline{B}) = 0 + P(A \cap \overline{B})$ i.e., $0.6 = 0.5$, which is absurd. Hence the given values are inconsistent.
18. P(any event) ≤ 1; $P(A \cup B) \leq 1$; $P(A) + P(B) - P(A \cap B) \leq 1$
$$\therefore P(A \cap B) \geq \frac{3}{4} + \frac{5}{8} - 1 \left(= \frac{3}{8}\right)$$
19. $P(S \cup A) = P(S) + P(A) - P(S \cap A)$
$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$$
20. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

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$$\therefore P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$$

21. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - [0.4 + 0.7 - 0.3] = 0.2$

22. $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$
$= 1 - [P(A) + P(B) - P(A \cup B)] = 0.85$

23. Equality holds good, when A and B are mutually exclusive events.

25. The probability for the simultaneous occurrence of two events A and B is called the joint probability of A and B. Probability of getting 2 heads, when 2 coins are tossed, is an example of joint probability.

32. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; $P(A \cap B) = \frac{1}{3} + \frac{3}{4} - \frac{11}{12} = \frac{1}{6}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{9}; P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{2}$$

34. Required probability $= 1 - P$(both tails)

$$= 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

35. Required probability $= P$ (1 from I dice and 3 from II dice or 2 from 1 and 2 from II or 3 from I and 1 from II)

$$= \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{12}$$

36. Required probability $= 1 - P$ (the problem is not solved)

$$= 1 - \frac{1}{2} \times \frac{1}{4} = \frac{7}{8}$$

37. No, since $P(A \cap B) \neq P(A) \times P(B)$

38. $P(BE/MBA) = \frac{P(BE \cap MBA)}{P(MBA)} = \frac{0.05}{0.25} = 0.2$

39. $P(A \cap B) = P(A) \times P(B/A) = \frac{1}{12} \times \frac{1}{15} = \frac{1}{180}$

$P(A \cup \mathrm{B}) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{12} + \frac{5}{12} - \frac{1}{180} = \frac{89}{180}$$

41. A and B are independent, if $P(A \cap B) = P(A) \times P(B)$. They are mutually exclusive, if $P(A \cap B) = 0$.
They are both independent and mutually exclusive if $P(A) \times PB) = 0$, i.e., if $P(A) = 0$ or $P(B) = 0$ or $P(A) = 0$ and $P(B) = 0$. The third case is trivial. Hence A and B can be both independent and mutually exclusive, provided either of the events is an impossible event.

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values of X may be assumed as $x_1, x_2, \ldots, x_n, \ldots$. In the finite case, the list of values terminates and in the countably infinite case, the list goes upto infinity.

For example, the number shown when a die is thrown and the number of alpha particles emitted by a radioactive source are discrete RVs.

Probability Function

If X is a discrete RV which can take the values $x_1, x_2, x_3, \ldots$ such that $P(X = x_i) = p_i$, then p_i is called the *probability function or probability mass function or point probability function*, provided p_i $(i = 1, 2, 3, \ldots)$ satisfy the following conditions:

(*i*) $p_i \geq 0$, for all i, and

(*ii*) $\sum_i p_i = 1$

The collection of pairs $\{x_i, p_i\}$, $i = 1, 2, 3, \ldots$, is called *the probability distribution of the* RV X, which is sometimes displayed in the form of a table as given below:

$X = x_i$	$P(X = x_i)$
x_1	p_1
x_2	p_2
$\vdots$	$\vdots$
x_r	p_r
$\vdots$	$\vdots$

Continuous Random Variable

If X is an RV which can take all values (i.e., *infinite number* of values) in an interval, then X is called a *continuous* RV.

For example, the length of time during which a vacuum tube installed in a circuit functions is a continuous RV.

Probability Density Function

If X is a continuous RV such that

$$P\left\{x - \frac{1}{2}\mathrm{d}x \leq X \leq x + \frac{1}{2}\mathrm{d}x\right\} = f(x)\mathrm{d}x$$

then $f(x)$ is called the *probability density function* (shortly denoted as pdf) of X, provided $f(x)$ satisfies the following conditions:

(i) $f(x) \geq 0$, for all $x \in R_x$, and

(ii) $\int_{R_X} f(x)\mathrm{d}x = 1$

Moreover, $P(a \leq X \leq b)$ or $P(a < X < b)$ is defined as

$$P(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x.$$

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12. If the pdf of a continuous RV X is $f(x) = \frac{\alpha}{\pi} \times \frac{1}{x^2 + \alpha^2}$, $\alpha > 0, -\infty < x < \infty$, then X follows a *Cauchy distribution* with parameter a.

Worked Examples 2(A)

Example 1 *From a lot containing 25 items, 5 of which are defective, 4 items are chosen at random. If X is the number of defectives found, obtain the probability distribution of X, when the items are chosen (i) without replacement and (ii) with replacement.*

Solution Since only 4 items are chosen, X can take the values 0, 1, 2, 3 and 4. The lot contains 20 non-defective and 5 defective items.

Case (i): When the items are chosen without replacement, we can assume that all the 4 items are chosen simultaneously.

$$\therefore \quad P(X = r) = P(\text{choosing exactly } r \text{ defective items})$$
$$= P(\text{choosing } r \text{ defective and } (4 - r) \text{ good items})$$
$$= \frac{5C_r \times 20C_{4-r}}{25C_4} \quad (r = 0, 1, \ldots, 4)$$

Case (ii): When the items are chosen with replacement, we note that the probability of an item being defective remains the same in each draw.

i.e., $$p = \frac{5}{25} = \frac{1}{5}, q = \frac{4}{5} \text{ and } n = 4$$

The problem is one of performing 4 Bernoulli's trials and finding the probability of exactly r successes.

$$\therefore \quad P(X = r) = 4C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{4-r} \quad (r = 0, 1, \ldots, 4)$$

Example 2 *A shipment of 6 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If X is the number of defective sets purchased by the hotel, find the probability distribution of X. (MU — Apr. 96)*

Solution All the 3 sets are purchased simultaneously. Since there are only 2 defective sets in the lot, X can take the values 0, 1 and 2.

$$\therefore \quad P(X = r) = P(\text{choosing exactly } r \text{ defective sets})$$
$$= P[\text{choosing } r \text{ defective and } (3 - r) \text{ good sets}]$$
$$= \frac{2C_r \times 4C_{3-r}}{6C_3} \quad (r = 0, 1, 2)$$

The required probability distribution is represented in the form of the following table.

$X = r$	p_r
0	1/5
1	3/5
2	1/5
Total	1

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Example 3 *A random variable X may assume 4 values with probabilities (1 + 3x)/4, (1 – x)/4, (1 + 2x)/4 and (1 – 4x)/4. Find the condition on x so that these values represent the probability function of X?*

Solution $P(X = x_1) = p_1 = (1 + 3x)/4$; $p_2 = (1 - x)/4$;
$p_3 = (1 + 2x)/4$; $p_4 = (1 - 4x)/4$

If the given probabilities represent a probability function, each $p_i \geq 0$ and $\sum_i p_i = 1$.

In this problem, $p_1 + p_2 + p_3 + p_4 = 1$, for any x.

But $p_1 \geq 0$, if $x \geq -1/3$; $p_2 \geq 0$, if $x \leq 1$; $p_3 \geq 0$, if $x \geq -1/2$ and $p_4 \geq 0$, if $x \leq 1/4$.

Therefore, the values of x for which a probability function is defined lie in the range $-1/3 \leq x \leq 1/4$.

Example 4 *If the random variable X takes the values 1, 2, 3 and 4 such that 2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4), find the probability distribution and cumulative distribution function of X.*

Solution Let $P(X = 3) = 30K$. Since $2P(X = 1) = 30K$, $P(X = 1) = 15K$.

Similarly $P(X = 2) = 10K$ and $P(X = 4) = 6K$.

Since S $p_i = 1$, $15K + 10K + 30K + 6K = 1$.

$$\therefore \quad K = \frac{1}{61}$$

The probability distribution of X is given in the following table:

$X = i$	1	2	3	4
p_i	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

The cdf $F(x)$ is defined as $F(x) = P(X \leq x)$. Accordingly the cdf for the above distribution is found out as follows:

When $x < 1$, $F(x) = 0$

When $1 \leq x < 2$, $F(x) = P(X = 1) = \frac{15}{61}$

When $2 \leq x < 3$, $F(x) = P(X = 1) + P(X = 2) = \frac{25}{61}$

When $3 \leq x < 4$, $F(x) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{55}{61}$

When $x \geq 4$, $F(x) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = 1$.

Example 5 *A random variable X has the following probability distribution.*

x:	*– 2*	*– 1*	*0*	*1*	*2*	*3*
p(x):	*0.1*	*K*	*0.2*	*2K*	*0.3*	*3K*

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(a) Find K, (b) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$, (c) find the cdf of X and (d) evaluate the mean of X. (BU — Apr. 96)

Solution (a) Since S $P(x) = 1$, $6K + 0.6 = 1$

$$\therefore \qquad K = \frac{1}{15}$$

$\therefore$ the probability distribution becomes

x	:	-2	-1	0	1	2	3
$p(x)$	:	1/10	1/15	1/5	2/15	3/10	1/5

(b) $P(X < 2) = P(X = -2, -1, 0 \text{ or } 1)$

$$= P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$$

[since the events $(X = -2)$, $(X = -1)$ etc. are mutually exclusive]

$$= \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{1}{2}$$

$$P(-2 < X < 2) = P(X = -1, 0 \text{ or } 1)$$

$$= P(X = -1) + P(X = 0) + P(X = 1)$$

$$= \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{2}{5}$$

(c) $F(x) = 0$, when $x < -2$

$$= \frac{1}{10}, \text{ when } -2 \le x < -1$$

$$= \frac{1}{6}, \text{ when } -1 \le x < 0$$

$$= \frac{11}{30}, \text{ when } 0 \le x < 1$$

$$= \frac{1}{2}, \text{ when } 1 \le x < 2$$

$$= \frac{4}{5}, \text{ when } 2 \le x < 3$$

$$= 1, \text{ when } 3 \le x$$

(d) The mean of X is defined as $E(X) = \Sigma x p(x)$ (refer to Chapter 4)

$$\therefore \qquad \text{Mean of } X = \left(-2 \times \frac{1}{10}\right) + \left(-1 \times \frac{1}{15}\right) + \left(0 \times \frac{1}{5}\right) + \left(1 \times \frac{2}{15}\right) + \left(2 \times \frac{3}{10}\right) + \left(3 \times \frac{1}{5}\right)$$

$$= -\frac{1}{5} - \frac{1}{15} + \frac{2}{15} + \frac{3}{5} + \frac{3}{5} = \frac{16}{15}$$

Example 6 *The probability function of an infinite discrete distribution is given by $P(X = j) = 1/2^j$ $(j = 1, 2, \ldots, \infty)$. Verify that the total probability is 1 and find the mean and variance of the distribution. Find also $P(X \text{ is even})$, $P(X \le 5)$ and $P(X \text{ is divisible by } 3)$.*

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Solution Let $P(X = j) = p_j$

$$\sum_{j=1}^{\infty} p_j = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \infty, \text{ that is a geometric series.}$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

The mean of X is defined as $E(X) = \sum_{j=1}^{\infty} jp_j$ (refer to Chapter 4).

$$\therefore \qquad E(X) = a + 2a^2 + 3a^3 + \ldots \infty, \text{ where } a = \frac{1}{2}$$

$$= a(1 + 2a + 3a^2 + \ldots \infty)$$

$$= a\,(1 - a)^{-2} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$$

The variance of X is defined as $V(X) = E(X^2) - [E(X)]^2$,

where $E(X^2) = \sum_{j=1}^{\infty} j^2 p^j$ (refer to Chapter 4).

$$E(X^2) = \sum_{j=1}^{\infty} j^2 a^j, \text{ where } a = \frac{1}{2}$$

$$= \sum_{j=1}^{\infty} [j(j+1) - j]a^j = \sum_{j=1}^{\infty} j(j+1)a^j - \sum_{j=1}^{\infty} ja^j$$

$$= a\,(1.2 + 2.3a + 3.4a^2 + \ldots \infty) - a(1 + 2a + 3a^2 + \ldots \infty)$$

$$= a \times 2\,(1 - a)^3 - a \times (1 - a)^{-2}$$

$$= \frac{2\,a}{(1-a)^3} - \frac{a}{(1-a)^2} = 8 - 2 = 6$$

$$\therefore \qquad V(X) = E(X^2) - \{E(X)\}^2 = 6 - 4 = 2$$

$P(X \text{ is even}) = P(X = 2 \text{ or } X = 4 \text{ or } X = 6 \text{ or etc.})$

$P\,(X = 2) + P(X = 4) + \ldots + \infty$

(since the events are mutually exclusive)

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$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots + \infty$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

$P(X \geq 5) = P(X = 5 \text{ or } X = 6 \text{ or } X = 7 \text{ or etc.})$

$= P(X = 5) + P(X = 6) + \ldots + \infty$

$$= \frac{\frac{1}{2^5}}{1-\frac{1}{2}} = \frac{1}{16}$$

$P(X \text{ is divisible by } 3) = P(X = 3 \text{ or } X = 6 \text{ or } X = 9 \text{ etc.})$

$= P(X = 3) + P(X = 6) + \ldots + \infty$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \ldots + \infty$$

$$= \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{7}$$

Example 7 *A random variable X has the following probability distribution.*

x	:	0	1	2	3	4	5	6	7
$p(x)$	:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find (i) the value of K, (ii) $P(1.5 < X < 4.5/X > 2)$ and (iii) the smallest value of λ for which $P(X \leq \lambda) > 1/2$.

Solution $\Sigma\, p(x) = 1$

$\therefore \quad 10K^2 + 9K = 1$

i.e., $(10K - 1)(K + 1) = 0$

$\therefore \quad K = \frac{1}{10}$ or -1.

The value $K = -1$ makes some values of $p(x)$ negative, which is meaningless.

$\therefore \quad K = \frac{1}{10}$

The actual distribution is given below:

x	:	0	1	2	3	4	5	6	7
$p(x)$	:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

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(i) $P(1.5 < X < 4.5 / X > 2) = P(A/B)$, say

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P[(1\cdot5 < X < 4\cdot5) \cap (X > 2)]}{P(X > 2)}$$

$$= \frac{P(X=3) + P(X=4)}{\sum_{r=3}^{7}(X=r)} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(ii) By trials, $P(X \le 0) = 0$; $P(X \le 1) = \frac{1}{10}$; $P(X \le 2) = \frac{3}{10}$

$$P(X \le 3) = \frac{5}{10};\ P(X \le 4) = \frac{8}{10}$$

Therefore, the smallest value of λ satisfying the condition $P(X \le l) > 1/2$ is 4.

Example 8 *If* $p(x) = \begin{cases} x\,e^{-x^2/2} & x \ge 0 \\ 0 & x < 0 \end{cases}$

(a) show that p(x) is a pdf (of a continuous RV X.)
(b) find its distribution function P(x). (BU — Nov. 96)

Solution (a) If $p(x)$ is to be a pdf, $p(x) \ge 0$ and

$$\int_{R_X} p(x)\,dx = 1$$

Obviously, $p(x) = xe^{-x^2/2}$ 0, when $x \ge 0$

Now $\int_0^\infty p(x)dx = \int_0^\infty xe^{-x^2/2}\,dx = \int_0^\infty e^{-t}\,dt$ (putting $t = x^2/2$) = 1

$\therefore$ $p(x)$ is a legitimate pdf of a RV X.

$$F(x) = P(X \le x) = \int_0^x f(x)dx$$

$\therefore$ $F(x) = 0$, when $x < 0$

and $F(x) = \int_0^X xe^{-x^2/2}\,dx = 1 - e^{-x^2/2}$, when $x \ge 0$.

Example 9 *If the density function of a continuous RV X is given by*

$$\begin{aligned} f(x) &= ax, && 0 \le x \le 1 \\ &= a, && 1 \le x \le 2 \\ &= 3a - ax, && 2 \le x \le 3 \\ &= 0, && \text{elsewhere} \end{aligned}$$

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Example 10 *A continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$.*

(MU — Apr. 96)

Solution By the property of pdf,

$\int_{R_x} f(x)\,dx = 1$. X takes values between 2 and 5.

$$\therefore \qquad \int_2^5 k(1+x)dx = 1$$

$$\text{i.e.,} \qquad \frac{27}{2}k = 1$$

$$\therefore \qquad k = \frac{2}{27}$$

Now $p(X < 4) = p(2 < X < 4) = \int_2^4 k(1+x)dx = \frac{16}{27}$

Example 11 *A continuous RV X has a pdf $f(x) = kx^2e^{-x}$; $x \geq 0$. Find k, mean and variance.* (MKU — Apr. 97)

Solution By the property of pdf,

$$\int_0^\infty kx^2e^{-x}dx = 1$$

$$\text{i.e.,} \qquad 2k = 1$$

$$\therefore \qquad k = \frac{1}{2}$$

Mean of X is defined as

$$E(X) = \int_{R_x} xf(x)dx$$

(refer to Chapter 4)

Variance of X is defined as

$$V(X) = E(X^2) - \{E(X)\}^2,$$

where $E(X^2) = \int_{R_x} x^2f(x)dx$ (refer to Chapter 4)

$$\therefore \qquad E(X) = \frac{1}{2}\int_0^\infty x^3e^{-x}dx$$

$$= \frac{1}{2}[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})]_0^\infty$$

$$= 3$$

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$$= \frac{F(x) - F(t)}{1 - F(t)} \text{ for } x > t$$

$$= 0 \text{ for } x < t$$

Therefore, the conditional density function $f(x/X > t)$ is given by

$$f(x/X > t) = \frac{d}{dx} F(x/X > t)$$

$$= \frac{f(x)}{1 - F(t)}, x > t$$

For the exponential distribution with parameter λ,

$$f(x) = \lambda e^{-\lambda x}, x > 0, \text{ and } F(x) = \int_0^X \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

$$\therefore \quad f(x/X > t) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)} = f(x - t)$$

Example 19 *If $f(t)$ is the unconditional density of the time of failure of a system and $h(t)$ is the hazard rate (or conditional failure rate or conditional density of T, given $T > t$) find $f(t)$ in terms of $h(t)$. Deduce that T follows a Rayleigh distribution, when $h(t) = t$.*

Solution The conditional density of T, given $T > t$ or the hazard rate, is given by $h(t) = f(t/T > t)$

$$= \frac{f(t)}{1 - F(t)} = \frac{F'(t)}{1 - F(t)}$$

$$\therefore \quad \int_0^t h(t)dt = \int_0^t \frac{F'(t)}{1 - F(t)} dt$$

$$= [-\log \{1 - F(t)\}]_0^t$$

$= -\log \{1 - F(t)\}$ [since $F(0) = P(T \leq 0) = 0$ as the system was put into operation at $t = 0$]

$$\therefore \quad F(t) = 1 - e^{-\int_0^t h(t)dt}$$

$$\therefore \quad f(t) = h(t) \times e^{-\int_0^t h(t)dt}$$

When $h(t) = t$,

$$f(t) = t \times e^{-\int_0^t t dt}$$

$$= te^{-t^2/2}, \text{ which is the pdf of a Raleigh distribution}$$

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57. If the cdf of a continuous RV X is given by $F(x) = \frac{1}{2}e^{kx}$, $x \le 0$, and $F(x) = 1 - \frac{1}{2}e^{-kx}$, $x > 0$, find $P(|x| \le 1/k)$. Prove that the density function of X is $f(x) = \frac{k}{2}e^{-k|x|}$ $-\infty < x < \infty$, given that $k > 0$.
58. If the distribution function of a continuous RV X is given by $F(x) = 0$, when $x < 0$; $= x$, when $0 \le x < 1$ and $= 1$, when $1 \le x$, find the pdf of X. Also find $P(1/3 < X < 1/2)$ and $P(1/2 < X < 2)$ using the cdf of X.
59. A point is chosen on a line of length a at random. What is the probability that the ratio of the shorter to the longer segment is less than 1/4?
60. If the RV k is uniformly distributed over (1, 7) what is the probability that the roots of the equation $x^2 + 2kx + (2k + 3) = 0$ are real?
61. If $f(t)$ is the unconditional density of time to failure T of a system and $h(t)$ is the conditional density of T, given $T > t$, find $h(t)$ when (i) $f(t) = \lambda e^{-\lambda t}$ (ii) $f(t) = l^2 te^{-\lambda t}$, $t > 0$. Prove also that $h(t)$ is not a density function.
62. If the continuous RV X follows $N(1000, 20)$, find
(i) $P(X < 1024)$, (ii) $P(X < 1024 / X > 961)$ and
(iii) $P(31 < \sqrt{X} \le 32)$.

Two-Dimensional Random Variables

So far we have considered only the one-dimensional RV, i.e., we have considered such random experiments, the outcome of which had only one characteristic and hence was assigned a single real value. In many situations, we will be interested in recording 2 or more characteristics (numerically) of the outcome of a random experiment. For example, both voltage and current might be of interest in a certain experiment.

Definitions: Let S be the sample space associated with a random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes s Î S. Then (X, Y) is called a *two-dimensional random variable.*

If the possible values of (X, Y) are finite or countably infinite, (X, Y) is called *a two-dimensional discrete* RV. When (X, Y) is a two-dimensional discrete RV the possible values of (X, Y) may be represented as (x_i, y_j), i = 1, 2, K, m, K; j = 1, 2, …, n, ….

If (X, Y) can assume all values in a specified region R in the xy-plane, (X, Y) is called a *two-dimensional continuous* RV.

Probability Function of (X, Y)

If (X, Y) is a two-dimensional discrete RV such that $P(x = x_i, y = y_j) = p_{ij}$, then p_{ij} is called the *probability mass function* or simply the *probability function* of (X, Y) provided the following conditions are satisfied.

(i) $p_{ij} \ge 0$, for all i and j (ii) $\sum_j \sum_i p_{ij} = 1$

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Worked Examples 2(B)

Example 1 *Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X, Y).*

Solution As there are only 2 white balls in the box, X can take the values 0, 1 and 2 and Y can take the values 0, 1, 2 and 3.

$P(X = 0, Y = 0)$ = P(drawing 3 balls none of which is white or red)

= P(all the 3 balls drawn are black)

$$= {}^4C_3/{}^9C_3 = \frac{1}{21}$$

$P(X = 0, Y = 1)$ = P(drawing 1 red and 2 black balls)

$$= \frac{{}^3C_1 \times {}^4C_2}{{}^9C_3} = \frac{3}{14}$$

Similarly, $P(X = 0, Y = 2) = \dfrac{{}^3C_2 \times {}^4C_1}{{}^9C_3} = \dfrac{1}{7}$; $P(X = 0, Y = 3) = \dfrac{1}{84}$

$P(X = 1, Y = 0) = \dfrac{1}{7}$; $P(X = 1, Y = 1) = \dfrac{2}{7}$; $P(X = 1, Y = 2) = \dfrac{1}{14}$;

$P(X = 1, Y = 3) = 0$ (since only 3 balls are drawn)

$P(X = 2, Y = 0) = \dfrac{1}{21}$; $P(X = 2, Y = 1) = \dfrac{1}{28}$; $P(X = 2, Y = 2) = 0$;

$P(X = 2, Y = 3) = 0$

The joint probability distribution of (X, Y) may be represented in the form of a table as given below:

X	Y			
	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Note *Sum of all the cell probabilities = 1.*

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$P_{0*} = 0.1 + 0.04 + 0.06 = 0.2$; $P_{1*} = 0.4$; $P_{2*} = 0.4$

$P_{*0} = 0.5$; $P_{*1} = 0.2$; $P_{*2} = 0.3$

Now $P_{0*} \times P_{*0} = 0.2 \times 0.5 = 0.1 = P_{00}$

$P_{0*} \times P_{*1} = 0.2 \times 0.2 = 0.04 = P_{01}$

$P_{0*} \times P_{*2} = 0.2 \times 0.3 = 0.06 = P_{02}$

Similarly we can verify that

$P_{1*} \times P_{*0} = P_{10}$; $P_{1*} \times P_{*1} = P_{11}$; $P_{1*} \times P_{*2} = P_{12}$;

$P_{2*} \times P_{*0} = P_{20}$; $P_{2*} \times P_{*1} = P_{21}$; $P_{2*} \times P_{*2} = P_{22}$

Hence the RVs X and Y are independent.

Example 5 *The joint pdf of a two-dimensional RV (X, Y) is given by* $f(x, y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$.

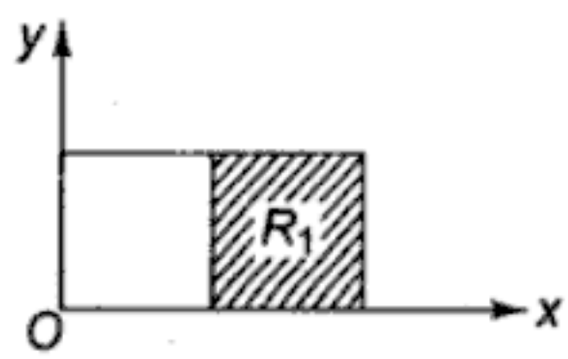
Compute $P(X > 1)$, $P(Y < \frac{1}{2})$, $P(X > 1/Y < 1/2)$

$P(Y < \frac{1}{2}/X > 1)$, $P(X < Y)$ and $P(X + Y \le 1)$.

Solution Here the rectangle defined by $0 \le x \le 2$, $0 \le y \le 1$ is the range space R. $R_1, R_2, \ldots$, are event spaces.

(i) $$P(X > 1) = \iint_{\substack{R_1 \\ (x>1)}} f(x, y)\,dx\,dy$$

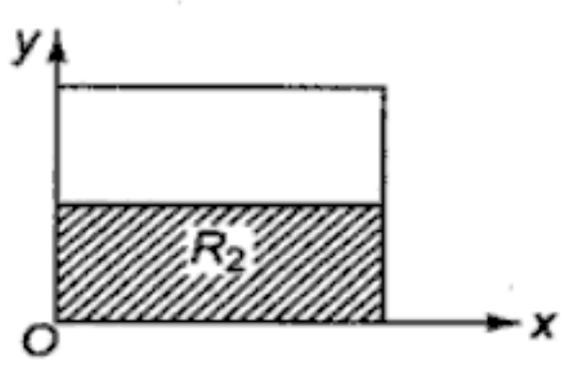
$$= \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{19}{24}$$



(ii) $$P(Y < 1/2) = \iint_{\substack{R_2 \\ \left(y<\frac{1}{2}\right)}} \left(xy^2 + \frac{x^2}{8} \right) dx\,dy$$

$$= \int_0^{1/2} \int_0^2 \left(xy^2 + \frac{x^2}{8} \right) dx\,dy$$

$$= \frac{1}{4}$$



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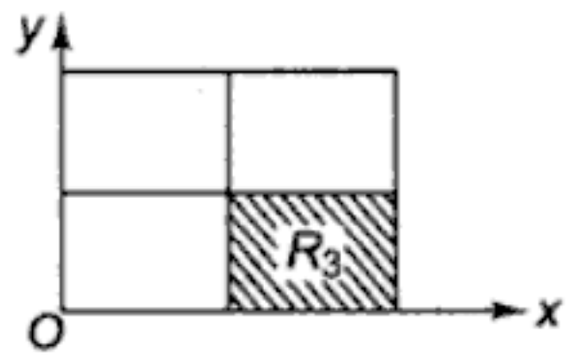
(iii) $P(X > 1, Y < 1/2) = \int\limits_{\substack{R_3 \\ \left(x>1 \,\&\, y<\frac{1}{2}\right)}} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^{1/2}\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{5}{24}$$

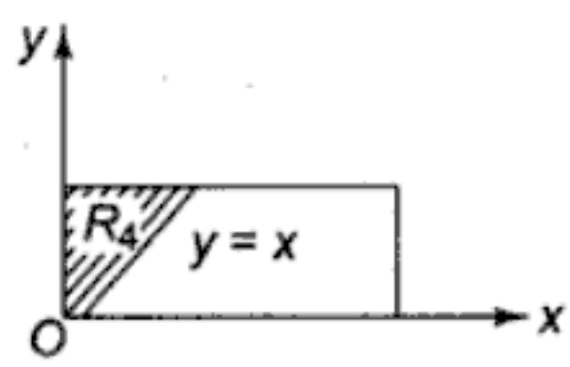


(iv) $P(X > 1/Y < \frac{1}{2}) = \dfrac{P\left(X>1, Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)} = \dfrac{5/24}{1/4} = \dfrac{5}{6}$

(v) $P(Y < \frac{1}{2}/X > 1) = \dfrac{P\left(X>1, Y<\frac{1}{2}\right)}{P(X>1)} = \dfrac{5/24}{19/24} = \dfrac{5}{19}$

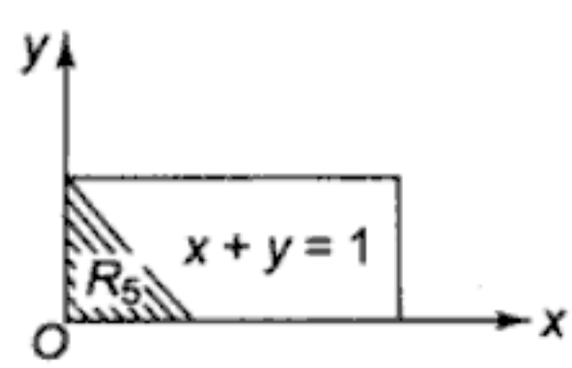
(vi) $P(X < Y) = \int\limits_{\substack{R_4 \\ (x<y)}} \int \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^1\int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{53}{480}$$



(vii) $P(X + Y \le 1) = \int\limits_{\substack{R_5 \\ (x+y\le 1)}} \int \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^1\int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{13}{480}$$



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Example 6 *If the joint pdf of the RV (X, Y) is given by f(x, y)*

$= \dfrac{1}{2\pi\sigma^2} \exp\{-(x^2 + y^2)/2\sigma^2\}, -\infty < x, y < \infty$, *find* $P(X^2 + Y^2 \le a^2)$.

Solution Here the entire xy-plane is the range space R and the event-space D is the interior of the circle $x^2 + y^2 = a^2$.

$$P(X^2 + Y^2 \le a^2) = \iint_{x^2+y^2 \le a^2} f(x, y)\, dx\, dy$$

Transform from cartesian system to polar system, i.e., put $x = r\cos\theta$ and $y = r\sin\theta$.

Then $dx\, dy = r d\, r d\, \theta$.

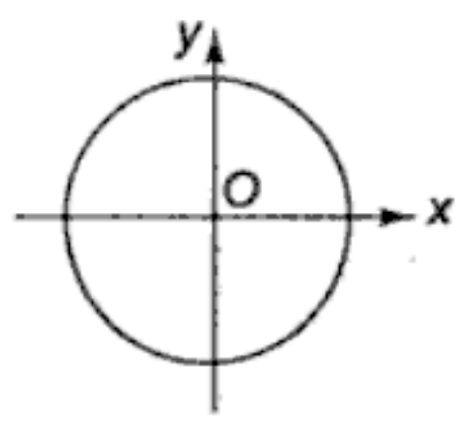
The domain of integration becomes $r \le a$.

$$\text{Then } P(X^2 + Y^2 \le a^2) = \int_0^{2\pi}\int_0^a \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, r dr d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(-e^{-r^2/2\sigma^2}\right)_0^a d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(1 - e^{-a^2/2\sigma^2}\right) d\theta$$

$$= 1 - e^{-a^2/2\sigma^2}$$



Example 7 *A gun is aimed at a certain point (origin of the co-ordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius R about the origin. Assume that the joint density of X and Y is constant in this circle given by*

$$f_{xy}(x, y) = c, \quad \text{for } x^2 + y^2 \le R^2$$
$$= 0, \quad \text{otherwise}$$

(i) Compute c and (ii) show that

$$f_x(x) = \frac{2}{\text{चR}}\sqrt{1 - \left(\frac{x}{R}\right)^2}, \text{for } -R \le x \le R$$

$$= 0, \quad \text{otherwise}$$

(BDU — Nov. 96)

Solution Here the range space is the interior of the circle $x^2 + y^2 = R^2$. By the property of joint pdf,

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$$\iint_{x^2+y^2 \le R^2} f(x, y)\mathrm{d}x\,\mathrm{d}y = 1$$

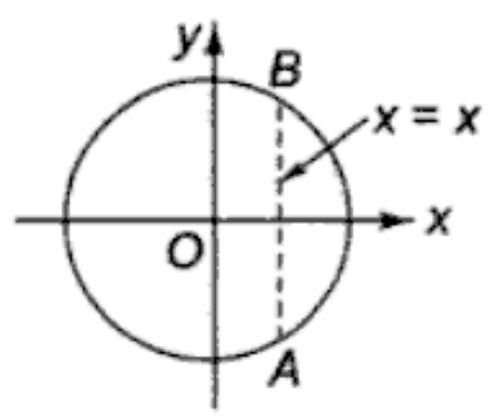
i.e.,
$$\iint_{x^2+y^2 \le R^2} c\,\mathrm{d}x\,\mathrm{d}y = 1$$

Changing over to polar co-ordinates, we have

$$\int_0^{2\pi}\int_0^{R} c\,r\,\mathrm{d}r\,\mathrm{d}\,\theta = 1$$

$$\therefore \qquad c = \frac{1}{\pi R^2}$$

Note *We have defined earlier that* $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\mathrm{d}y$. *This definition holds good if the range space is the entire xy-plane. If the range space is different from the entire xy-plane* $f_X(x)$ *is given by* $\int f(x, y)dy$, *for which the limits are fixed as follows: Draw an arbitrary line parallel to y-axis (since x is to be treated as a constant). The y co-ordinates of the ends of the segment of such a line that lies within the range space are the required limits. These limits will be either constants or functions of x.*



The point $\quad A = \left\{x, -\sqrt{R^2 - x^2}\right\}$

and the point $B = \left\{x, \sqrt{R^2 - x^2}\right\}$

Now
$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,\mathrm{d}y$$

$$= \frac{2}{\pi R^2}\sqrt{R^2 - x^2} = \frac{2}{\pi R}\sqrt{1 - \left(\frac{x}{R}\right)^2} \quad -R \le x \le R$$

Note *Whenever we are required to find the marginal and conditional density functions, the ranges of the concerned variables should also be specified.*

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(b) $f_x(x) = \int_{-x}^{x} \frac{1}{8} x\,(x-y)\mathrm{d}y$

$$= \frac{x^3}{4}, \text{ in } 0 < x < 2$$

(c) $f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{2x^2}\,(x-y), -x < y < x$

(d) $f_y(y) = \int_{-y}^{2} \frac{1}{8}\, x\,(x-y)\mathrm{d}x$, in $-2 \le y \le 0$

$$= \int_{y}^{2} \frac{1}{8}\, x\,(x-y)\mathrm{d}x, \text{ in } 0 \le y \le 2$$

i.e., $$f_y(y) = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5}{48} y^3, & \text{in } -2 \le y \le 0 \\ \frac{1}{3} - \frac{y}{4} + \frac{1}{48} y^3, & \text{in } 0 \le y \le 2 \end{cases}$$

Example 10 *Train X arrives at a station at random in the time interval (0, T) and stops for 'a'min. Train Y arrives independently in the same interval and stops for 'b' min.*

(i) Find the probability P_1 that X will arrive before Y.

(ii) Find the probability P_2 that the two trains meet.

(iii) Assuming that they meet, find the probability P_3 that X arrived before Y.

(MSU — Nov. 96)

Solution Let the trains X and Y arrive at the station at time instances X and Y respectively.

Then the lengths of the intervals $(0, X)$ and $(0, Y)$, namely X and Y are continuous RVs. Each of X and Y is uniformly distributed in $(0, T)$ (since the times of arrival are equally likely) with pdf $\frac{1}{T}$.

Since the 2 trains arrive independently, X and Y are independent RVs.

$\therefore$ The joint pdf of (X, Y) is given by

$$f(x, y) = f_x(x) \times f_y(y) = \frac{1}{T^2};\ 0 \le x, y \le T$$

The range space is the square defined by $0 \le x \le T$ and $0 \le y \le T$.

(i) $P_1 = P(X < Y) = \iint\limits_{\substack{x<y \\ (\Delta OBC)}} f(x, y)\mathrm{d}x\mathrm{d}y$

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Part B

22. If X denotes the number of aces and Y the number of queens obtained when 2 cards are drawn at random (without replacement) from a deck of cards, obtain the joint probability distribution of (X, Y).
23. The joint probability function of two discrete RVs X and Y is given by $f(x, y) = c(2x + y)$, where x and y can assume all integers such that $0 \le x \le 2$ and $0 \le y \le 3$, and $f(x, y) = 0$ otherwise. (a) find the value of c and (b) find $P(X \ge 1, Y \le 2)$. (MKU — Apr. 96)

Note *$f(x, y)$ should not be mistaken as pdf, It is used instead of $p(x_i, y_i)$*

24. The joint probability distribution of a two-dimensional discrete RV (X, Y) is given below:

Y	X					
	0	1	2	3	4	5
0	0	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05

(i) Find $P(X > Y)$ and $P\{\max(X, Y) = 3\}$ and
(ii) Find the probability distribution of the RV $Z = \min(X, Y)$.

25. The input to a binary communication system, denoted by a RV X, takes one of two values 0 or 1 with probabilities 3/4 and 1/4 respectively. Because of errors caused by noise in the system, the output Y differs from the input occasionally. The behaviour of the communication system is modeled by the conditional probabilities given below:

$$P(Y = 1/X = 1) = 3/4 \text{ and } P(Y = 0/X = 0) = 7/8$$

Find (i) $P(Y = 1)$, (ii) $P(Y = 0)$ and (iii) $P(X = 1/Y = 1)$.

26. The following table represents the joint probability distribution of the discrete RV (X, Y). Find all the marginal and conditional distributions.

Y	X		
	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

27. The joint distribution of X_1 and X_2 is given by $f(x_1, x_2) = \dfrac{x_1 + x_2}{21}$, $x_1 = 1, 2$ and 3; $x_2 = 1$ and 2. Find the marginal distributions of X_1 and X_2. (MU — Nov. 96).

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51. (i) $k = 6$;
(ii) $F(x) = 0$, when $x < 0$; $= 3x^2 - 2x^3$, when $0 \le x < 1$; $= 1$, when $1 \le x$;
(iii) the root of the equation $6a^3 - 9a^2 + 2 = 0$ that lies between 0 and 1;
(iv) $\frac{1}{2}$.

52. $k = \frac{1}{8}$; $F(x) = 0$, when $x < 0$; $= \frac{x^2}{16}$, when $0 \le x < 2$; $= \frac{1}{4}(x - 1)$, when $2 \le x < 4$; $= -\frac{1}{16}(20 - 12x + x^2)$, when $4 \le x < 6$; $= 1$, when $6 \le x$.

53. (i) $\frac{9}{16}$ (ii) $\frac{27}{64}$

54. $\frac{1}{3}$

55. $k = 5$; $P = \frac{1}{33}$

56. (a) $\frac{1}{4}$ (b) $\frac{4}{9}$ (c) 5

57. $P = 1 - \frac{1}{e}$; $f(x) = \frac{k}{2} e^{-k|x|}$

58. $f(x) = 1$ in $0 \le x \le 1$ and $= 0$, elsewhere; $\frac{1}{6}$; $\frac{1}{2}$.

59. $\frac{1}{5}$

60. $\frac{2}{3}$

61. (i) λ; (ii) $\frac{\lambda^2 t}{1 + \lambda t}$

62. (i) 0.8849 (ii) 0.8819 (iii) 0.8593

Exercise 2(B)

6. A

4. $f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}$

$$\exp\left[-\frac{1}{2(1-r^2)}\left\{\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2r(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right\}\right],$$

$-\infty < x, y < \infty$

12.

$(X + Y)$:	2	3	4
p:	0.1	0.5	0.4

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23. $c = \frac{1}{42}$; $P = \frac{4}{7}$

24. (i) 0.75, 0.21 (ii)

Z	0	1	2	3
P	0.28	0.30	0.25	0.17

25. (i) $\frac{9}{32}$ (ii) $\frac{23}{32}$ (iii) $\frac{2}{3}$

26. $\{i, p_{i*}\}$ $\{j, p_{*j}\}$ CPD of $X/Y = 1$ CPD of $X/Y = 2$

$x = i$	p_{i*}
1	5/36
2	19/36
3	1/3

$y = j$	p_{*j}
1	1/4
2	14/45
3	79/180

$x = i$	p_{i1}/p_{*1}
1	1/3
2	2/3
3	0

$x = i$	p_{i2}/p_{*2}
1	0
2	5/14
3	9/14

CPD of $X/Y = 3$ CPD of $Y/X = 1$ CPD of $Y/X = 2$ CPD of $Y/X = 3$

$x = i$	p_{i3}/p_{*3}
1	10/79
2	45/79
3	24/79

$y = j$	p_{1j}/p_{1*}
1	3/5
2	0
3	2/5

$x = j$	p_{2j}/p_{2*}
1	6/19
2	4/19
3	9/19

$y = j$	p_{3j}/p_{3*}
1	0
2	3/5
3	2/5

27.

M.D. of X_1

$X_1 = i$	$pi*$
1	5/21
2	7/21
3	9/21

M.D. of X_2

$X_2 = j$	$p*j$
1	9/21
2	12/21

28. (i) $\frac{5}{6}$ (ii) $\frac{7}{24}$ (iii) $\frac{5}{32}$

29. (i) 1/8 (ii) 3/8 (iii) 5/24 (iv) 3/5

30. $\frac{31}{256}$

31. (i) $\frac{e^4 - 5}{e^5 - 6}$ (ii) $f_x(x) = e^{-x}, x > 0; f_y(y) = y\, e^{-y}, y > 0.$

32. $f_x(x) = 2x$ in $0 < x < 1$; $f_y(y) = 2\,(1 - y)$ in $0 < y < 1$.

33. $k = 1/2$; $f_x(x) = \frac{1}{2}\,(2 - x)$, $0 \le x \le 2$; $f_y(y) = (1/2)\, y$, $0 \le y \le 2$; $f(x/y)$

$= 1/y$; $0 < x < y$; $f(y/x) = \frac{1}{2 - x}$, $x < y < 2$.

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34. $f(y/x) = 2y/x^2, 0 < y < x$

35. (i) $a = 1/24$

(ii) $f_x(x) = x/4, 1 \le x \le 3; f_y(y) = y/6, 2 \le y \le 4$

(iii) $f(x/y) = x/4, 1 < x < 3; f(y/x) = y/6, 2 < y < 4$

36. $f_{x1}(x_1) = e^{-x1}$; $f_{x2}(x_2) = e^{-x2}$; X_1 and X_2 are independent; $(1 - e^{-1})^2$; $(1 - 2e^{-1})$

37. $f_x(x) = e^{-x}, 0 < x < \infty$; $f_y(y) = (y+1)^{-2}, 0 < y < \infty$; X and Y are not independent.

38. $K = 1/16; f_x(x) = \frac{1}{8}(x^3 + 2x), 0 \le x \le 2; f_y(y) = \frac{1}{8}(y^3 + 2y), 0 \le y \le 2;$

$$f(x/y) = x/2\,\frac{(x^2 + y^2)}{(y^2 + 2)}\ 0 < x < 2; f(y/x) = y/2\,\frac{(x^2 + y^2)}{(x^2 + 2)}, 0 < y < 2$$

39. $f_x(x) = 3/4, \frac{2x+3}{(1+x)^4}\ x > 0$ and $f_y(y) = 3/4, \frac{2x+3}{(1+y)^4}\ y > 0;$

Not independent.

40. (i) 1/2 (ii) 0.3988 (iii) 0.4514

41. $f_y(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) - \infty < y < \infty$

$$f(x/y) = N\left(\frac{r\sigma_x}{\sigma_y}\cdot y, \sigma_x\sqrt{1-r^2}\right)$$

42. (i) 1/256

(ii) $f_x(x) = 4x(1 - x^2)$ in (0, 1); $f_y(y) = 4y^3$ in (0, 1); $f(y/x)$

$$= \left|\frac{2y}{1-x^2}\right| f(x/y) = \frac{2x}{y^2}$$

(iii) X and Y are not independent.

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3 Functions of Random Variables

In the analysis of electrical systems, we will be often interested in finding the properties of a signal after it has been subjected to certain processing operations by the system, such as integration, weighted averaging, etc. These signal processing operations may be viewed as transformations of a set of input variables to a set of output variables. If the input is a set of random variables (RVs), then the output will also be a set of RVs. In this chapter, we deal with techniques for obtaining the probability law (distribution) for the set of output RVs when the probability law for the set of input RVs and the nature of transformation are known.

Function of One Random Variable

Let X be an RV with the associated sample space S_x and a known probability distribution. Let g be a scalar function that maps each $x \in S_x$ into $y = g(x)$. The expression $Y = g(X)$ defines a new RV Y. For a given outcome, $X(s)$ is a number x and $g[X(s)]$ is another number specified by $g(x)$. This number is the value of the RV Y, i.e., $Y(s) = y = g(x)$. The sample space S_y of Y is the set

$$S_y = \{y = g(x): x \in S_x)\}$$

How to find $f_y(y)$, when $f_x(x)$ is known

Let us now derive a procedure to find $f_y(y)$, the pdf of Y, when $Y = g(X)$, where X is a continuous RV with pdf $f_x(x)$ and $g(x)$ is a strictly monotonic function of x.

Case (i): $g(x)$ is a strictly increasing function of x.

$$\begin{aligned} F_y(y) &= P(Y \le y), \text{ where } F_y(y) \text{ is the cdf of } Y \\ &= P[g(X) \le y] \\ &= P[X \le g^{-1}(y)] \\ &= F_x(g^{-1}(y)] \end{aligned}$$

Differentiating both sides with respect to y,

$$f_y(y) = f_x(x)\frac{dx}{dy}, \text{ where } x = g^{-1}(y) \tag{1}$$

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Case (ii): $g(x)$ is a strictly decreasing function of x.

$$\begin{aligned} F_y(y) &= P(Y \le y) \\ &= P(g(X) \le y] \\ &= P[X \ge g^{-1}(y)] \\ &= 1 - P[X \le g^{-1}(y)] \\ &= 1 - F_x[g^{-1}(y)] \end{aligned}$$

$$\therefore \qquad f_y(y) = -f_x(x)\frac{dx}{dy} \qquad (2)$$

Combining (1) and (2), we get

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$$

i.e.,
$$f_y(y) = \frac{f_x(x)}{|g'(x)|}$$

Note *The above formula for $f_y(y)$ can be used only when $x = g^{-1}(y)$ is single valued.*

When $x = g^{-1}(y)$ takes finitely many values $x_1, x_2, \cdots x_n$, i.e., when the roots of the equation $y = g(x)$ are $x_1, x_2, \cdots, x_n$, the following extended formula should be used for finding $f_y(y)$:

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + \frac{f_x(x_2)}{|g'(x_2)|} + \cdots + \frac{f_x(x_n)}{|g'(x_n)|} \text{ or}$$

$$f_y(y) = f_x(x_1)\left|\frac{dx_1}{dy}\right| + f_x(x_2)\left|\frac{dx_2}{dy}\right| + \ldots + f_x(x_n)\left|\frac{dx_n}{dy}\right|$$

One Function of Two Random Variables

If a RV Z is defined as $Z = g(X, Y)$ where X and Y are RVs, we proceed to find $f_z(z)$ in the following way.

If z is a given number, we can find a region D_z in the xy-plane such that all points in D_z satisfy the condition $g(x, y) \le z$.

i.e., $\qquad (Z \le z) = [g(X, Y) \le z] = [(X, Y) \in D_z)]$

Now $\qquad F_z(z) = P(Z \le z) = P[(X, Y) \in D_z] = \iint_{D_z} f(x, y)\, dx\, dy$

where $f(x, y)$ is the joint pdf of (X, Y). Thus, to find $F_z(z)$ it is sufficient to find the region D_z for every z and to evaluate the above integral. $f_z(z)$ is then found out as usual.

Theorem 1 If two RVs are independent, then the density function of their sum is given by the convolution of their density functions.

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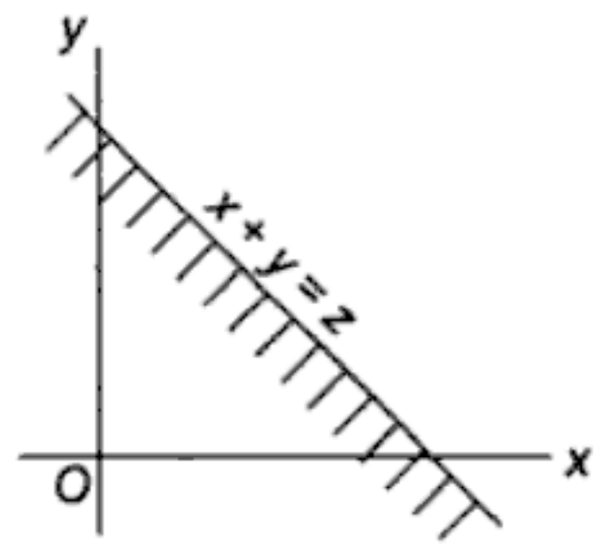


Proof Let the joint pdf of (X, Y) be $f(x, y)$

Let
$$Z = X + Y$$
$$F_z(z) = P(X + Y \le z)$$
$$= \iint_{(x+y\le z)} f(x, y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f(x, y)\, dx\, dy$$

Differentiating both sides with respect to z (Note that the upper limit for the inner integral is a function of z),

$$f_z(z) = \int_{-\infty}^{\infty} f(z-y, y)\, dy \tag{1}$$

Since X, Y are independent RVs

$$f(x, y) = f_x(x)\, f_Y(y)$$
$$\therefore \quad f(z-y, y) = f_x(z-y)\, f_y(y) \tag{2}$$

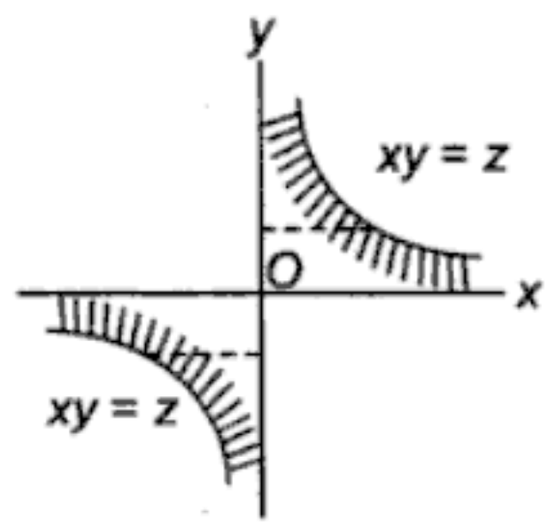
Using (2) in (1), we get

$$f_z(z) = \int_{-\infty}^{\infty} f_x(z-y)\, f_y(y)\, dy, \text{ which is the convolution of } f_x(x) \text{ and } f_y(y).$$

Corollary

If $f_x(x) = 0$, for $x < 0$, and $f_y(y) = 0$, for $y < 0$, then $f_x(z-y)\, f_y(y) \ne 0$, only when $0 < y < z$.

$$\therefore \quad f_z(z) = \int_0^z f_x(z-y)\, f_y(y)\, dy,\ z > 0$$



Theorem 2 If two RVs X and Y are independent, find the pdf of $Z = XY$ in terms of the density functions of X and Y.

Let the joint pdf of (X, Y) be $f(x, y)$

$$F_z(z) = \iint_{xy \le z} f(x, y)\, dx\, dy$$

Note *$xy = z$ is a rectangular hyperbola as shown in the figure.*

$$F_z(z) = \int_{-\infty}^{0} \int_{z/y}^{0} f(x, y)\, dx\, dy + \int_0^{\infty} \int_0^{z/y} f(x, y)\, dx\, dy$$

Differentiating both sides with respect to z,

$$f_z(z) = \int_{-\infty}^{0} -\frac{1}{y} f\left(\frac{z}{y}, y\right) dy + \int_0^{\infty} \frac{1}{y} f\left(\frac{z}{y}, y\right) dy$$

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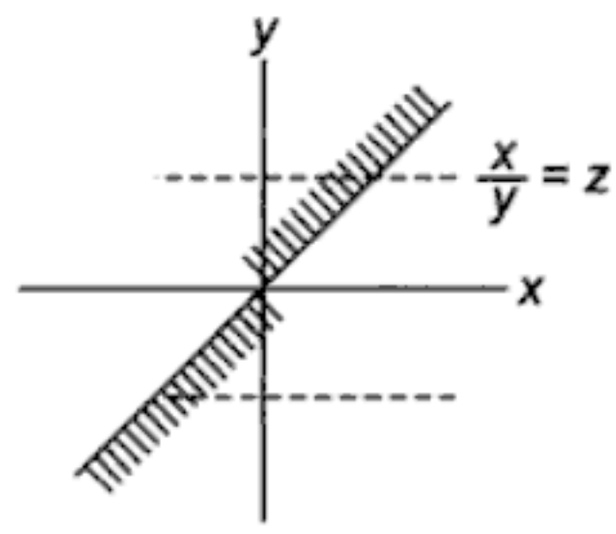
$$= \int_{-\infty}^{\infty} \left|\frac{1}{y}\right| f\left(\frac{z}{y}, y\right) dy$$

$$= \int_{-\infty}^{\infty} \left|\frac{1}{y}\right| f_x\left(\frac{z}{y}\right) f_y(y)\, dy \quad \text{(since } X \text{ and } Y \text{ are independent)}$$

Theorem 3 If two RVs X and Y are independent, find the pdf of $Z = \dfrac{X}{Y}$ in terms of the density functions of X and Y.



Let the joint pdf of (X, Y) be $f(x, y)$.

$$F_z(z) = \iint_{\frac{x}{y} \le z} f(x, y)\, dx\, dy$$

i.e.,
$$F_z(z) = \int_{-\infty}^{0} \int_{yz}^{\infty} f(x, y)\, dx\, dy + \int_{0}^{\infty} \int_{-\infty}^{yz} f(x, y)\, dx\, dy$$

Differentiating both sides with respect to z,

$$f_z(z) = \int_{-\infty}^{0} -yf(yz, y)\, dy + \int_{0}^{\infty} yf(yz, y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| f(yz, y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| f_x(yz) f_y(y)\, dy \quad \text{(since } X \text{ and } Y \text{ are independent)}$$

Two Functions of Two Random Variables

Theorem If (X, Y) is a two-dimensional RV with joint pdf $f_{xy}(x, y)$ and if $Z = g(X, Y)$ and $W = h(X, Y)$ are two other RVs, then the joint pdf of (Z, W) is given by

$$f_{zw}(z, w) = |J| f_{xy}(x, y), \text{ where } J = \frac{\partial(x, y)}{\partial(z, w)}$$

is called the Jacobian of the transformation and is given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix}$$

Note *This theorem holds good, only if the equations $z = g(x, y)$ and $w = h(x, y)$ when solved, give unique values of x and y in terms of z and w.*

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An alternative method to find the pdf of $Z = g(X, Y)$

Introduce a second RV $W = h(X, Y)$ and obtain the joint pdf of (Z, W), as suggested in the above theorem. Let it be $f_{zw}(z, w)$. The required pdf of Z is then obtained as the marginal pdf, i.e., $f_z(z)$ is obtained by simply integrating $f_{zw}(z, w)$ with respect to w.

i.e., $$f_z(z) = \int_{-\infty}^{\infty} f_{zw}(z, w)\, dw$$

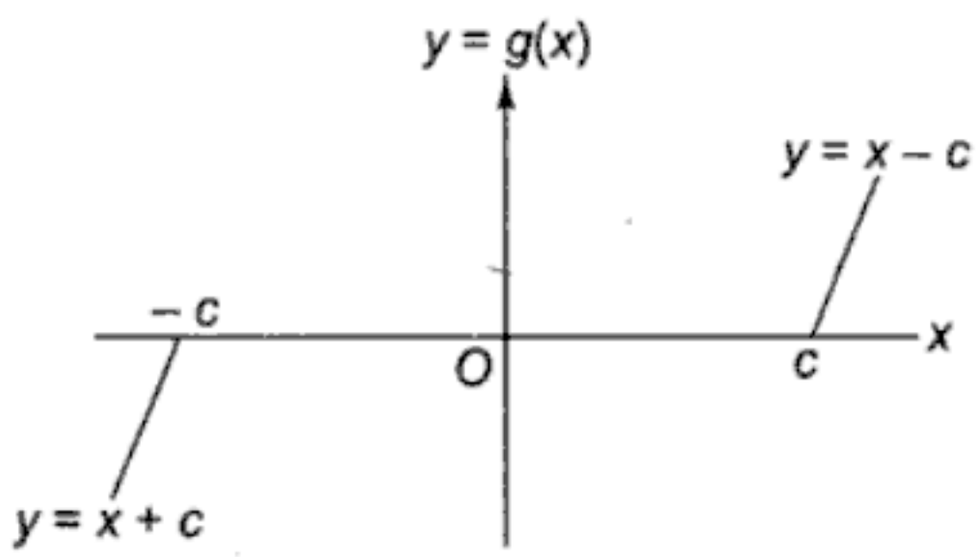
Worked Examples 3

Example 1 *Find the distribution function of the RV $Y = g(X)$, in terms of the distribution function of X, if it is given that*

$$g(x) = \begin{cases} x - c & \text{for } x > c \\ 0 & \text{for } |x| \le c \\ x + c & x < -c \end{cases}$$ (MSU — Apr. 96)

Solution If $y < 0$, $F_y(y) = P(Y \le y)$

$= P(X + c \le y)$



$= P(X \le y - c)$

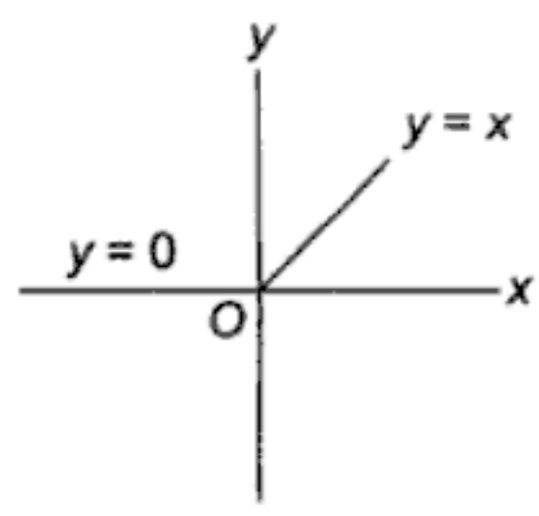
$= F_x(y - c)$

If $y \ge 0$, $F_y(y) = P(X - c \le y)$

$= F_x(y + c)$

Example 2 *The random variable Y is defined by $Y = \frac{1}{2}(X + |X|)$, where X is another RV. Determine the density and distribution function of Y in terms of those of X.*

(MSU — Nov. 96)



Solution When $X \ge 0$, $Y = X$

When $X < 0$, $Y = 0$

If $y < 0$, $F_y(y) = P(Y \le y) = 0$ (since there is no X, for which $Y \le y$)

If $y \ge 0$, $F_y(y) = P(Y \le y)$

$= P(X \le y / X \ge 0)$

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$$= P(0 \le X \le y)/P(X \ge 0)$$

$$= \frac{F_x(y) - F_x(0)}{1 - F_x(0)}$$

$\therefore$ When $y < 0, f_y(y) = 0$ and

when $y \ge 0, f_y(y) = f_x(y)/[1 - F_x(0)]$

Example 3 *(a) Find the density function of $Y = aX + b$ in terms of the density function of X.*

(b) Let X be a continuous RV with pdf

$$f(x) = \frac{x}{12}, \text{ in } 1 < x < 5$$

$$= 0, \text{ elsewhere}$$

find the probability density function of $Y = 2X - 3$.

(MU — Apr. 96 and Nov. 96)

Solution (a) (i) Let $a > 0$

$$F_y(y) = P(Y \le y) = P(aX + b \le y)$$

$$= P\left(X \le \frac{y-b}{a}\right) \quad \text{(since } a > 0)$$

$$= F_x\left(\frac{y-b}{a}\right) \tag{1}$$

(ii) Let $a < 0$

$$F_y(y) = P(Y \le y) = P(aX + b \le y)$$

$$= P(aX \le y - b)$$

$$= P\left(X \ge \frac{y-b}{a}\right) \quad \text{(Since } a < 0)$$

$$= 1 - F_x\left(\frac{y-b}{a}\right) \tag{2}$$

From (1), $$f_y(y) = \frac{1}{a} f_x\left(\frac{y-b}{a}\right) \tag{3}$$

From (2), $$f_y(y) = -\frac{1}{a} f_x\left(\frac{y-b}{a}\right) \tag{4}$$

Combining (3) and (4),

$$f_y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right)$$

(b) $y = 2x - 3$, since $Y = 2X - 3$

$\therefore$ $x = \frac{1}{2}(y + 3)$, i.e., x is a single valued function of y

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$$\therefore \qquad f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$$
$$= \frac{x}{12} \times \frac{1}{2}$$
$$= \frac{1}{48}(y+3), \text{ in } -1 < y < 7$$

Note *The range of y is obtained from that of x (given in the problem) using the relation between x and y.*

Example 4 *If X is a continuous RV with some distribution defined over (0, 1) such that*

$$P(X \le 0.29) = 0.75, \text{ determine } k \text{ so that}$$
$$P(Y \le k) = 0.25, \text{ where } Y = 1 - X$$

Solution $\quad P(Y \le k) = 0.25$

i.e., $\quad P(1 - X \le k) = 0.25$

i.e., $\quad P(X \ge 1 - k) = 0.25$

$\therefore \quad P(X \le 1 - k) = 0.75 \qquad (1)$

But it is given that $P(X \le 0.29) = 0.75 \qquad (2)$

Comparing (1) and (2), $k = 0.71$

Example 5 *If $Y = X^2$, where X is a Gaussian random variable with zero mean and variance σ^2, find the pdf of the random variable Y.* (BU — Nov. 96)

Solution $\quad F_y(y) = P(Y \le y) = P(X^2 \le y)$

$$= P(-\sqrt{y} \le X \le \sqrt{y}), \text{ if } y \ge 0$$
$$= F_x(\sqrt{y}) - F_x(-\sqrt{y}). \qquad (1)$$

and $F_y(y) = 0$, if $y < 0$ [since $X^2 = y$ has no roots, when $y < 0$]

Differentiating (1) with respect to y,

$$f_y(y) = \frac{1}{2\sqrt{y}}\left\{f_x(\sqrt{y}) + f_x(-\sqrt{y})\right\}, \text{ if } y \ge 0 \qquad (2)$$
$$= 0, \text{ if } y < 0$$

It is given that X follows $N(0, \sigma)$.

$$\therefore \qquad f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad -\infty < x < \infty$$

Using this value in (2), we get

$$f_y(y) = \frac{1}{\sigma\sqrt{2\pi y}} e^{-y/2\sigma^2} \qquad y > 0$$

Example 6 *If the continuous RV X has pdf $f_x(x) = \frac{2}{9}(x+1)$, in $-1 < x < 2$ and $= 0$, elsewhere, find the pdf of $Y = X^2$.*

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Example 13 *(i) If X is uniformly distributed in $(-\pi/2, \pi/2)$ find the pdf of $Y = \tan X$.*

(ii) If X has the Cauchy's distribution with parameter 1, find the pdf of $Y = \tan^{-1} X$.

Solution (i) X is $U(-\pi/2, \pi/2)$

$$\therefore \qquad f_x(x) = \frac{1}{\pi}$$

$$y = \tan x$$

Therefore, $x = \tan^{-1} y$, which is single valued in $(-\pi/2, \pi/2)$ i.e., for a given value of y, there exists only one value of $\tan^{-1} y$ in $(-\pi/2, \pi/2)$.

$$f_y(y) = f_x(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{\frac{1}{\pi}}{1 + y^2} \qquad -\infty < y < \infty$$

which is the pdf of a Cauchy's distribution.

(ii) $y = \tan^{-1} x$ is a monotonic increasing function

$$\therefore \qquad f_y(y) = f_x(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{\pi(1 + x^2)} \sec^2 y \quad (\text{since } x = \tan y)$$

$$= \frac{1}{\pi}, \quad \frac{-\pi}{2} < y < \frac{\pi}{2}$$

i.e., Y is uniformly distributed in $(-\pi/2, \pi/2)$.

Example 14 *(i) If X has an arbitrary distribution function $F_x(x)$, find $g(x)$ so that the random variable $Y = g(X)$ may be uniformly distributed in (0, 1).*

(ii) If X is uniformly distributed in (0, 1), find $g(x)$, so that the random variable $Y = g(X)$ may have an arbitrary distribution with cdf $F_y(y)$.

(iii) If X is uniformly distributed in (–1, 1), find $g(x)$, so that the random variable $Y = g(X)$ may have the density function $f_y(y) = 2e^{-2y}$, $y > 0$.

Solution (i) Y is to be uniform in (0, 1)

$$\therefore \qquad f_y(y) = 1 \quad \text{and} \quad F_y(y) = \int_0^y f_y(y)\, dy = y$$

$$\therefore \qquad F_y\{g(x)\} = g(x) \qquad (1)$$

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Example 18 *The current I and the resistance R in a circuit are independent continuous RVs with the following density functions.*

$$f_i(i) = 2i, \qquad 0 \le i \le 1$$
$$= 0 \qquad \text{elsewhere}$$
$$f_r(r) = r^2/9 \qquad 0 \le r \le 3$$
$$= 0 \qquad \text{elsewhere}$$

Find the pdf of the voltage E in the circuit,

where E = IR.

Solution By Theorem 2,

$$f_z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_x\left(\frac{z}{y}\right) f_y(y)\, dy$$

when $Z = XY$ and X and Y are independent RVs

Taking $x = i$, $y = r$ and $z = e$, we have

$$f_e(e) = \int_{-\infty}^{\infty} \frac{1}{|r|} f_i\left(\frac{e}{r}\right) \times f_r(r)\, dr$$

$$= \int_{e}^{3} \frac{1}{r} \times 2\frac{e}{r} \times \frac{r^2}{9}\, dr$$

Note $$f_i\left(\frac{e}{r}\right) = 2\frac{e}{r}, \text{ if } 0 \le \frac{e}{r} \le 1,$$
i.e., if $r \ge e$.

$$f_r(r) = \frac{r^2}{9}, \text{ if } 0 \le r \le 3$$

Hence the limits for r are taken as e and 3.

$$\therefore \qquad f_e(e) = \frac{2e}{9}(3 - e), \quad 0 \le e \le 3$$

Example 19 *If X and Y are independent RVs each following N (0, 2) prove that* $Z = \dfrac{X}{Y}$ *follows a Cauchy's distribution.*

Solution $f_x(x) = \dfrac{1}{2\sqrt{2\pi}} e^{-x^2/8}$ *and* $f_y(y) = \dfrac{1}{2\sqrt{2\pi}} e^{-y^2/8} \quad -\infty < x, \quad y < \infty$

By Theorem 3

$$f_z(z) = \int_{-\infty}^{\infty} |y| f_x(yz) f_y(y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| \frac{1}{2\sqrt{2\pi}} e^{-y^2z^2/8} \times \frac{1}{2\sqrt{2\pi}} e^{-y^2/8}\, dy$$

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$$= \frac{1}{4\pi}\int_0^\infty y\, e^{-(1+z^2)y^2/8}\, dy \quad \text{(since the integrand is an even function)}$$

$$= \frac{1}{\pi} \times \frac{1}{1+z^2} \quad -\infty < z < \infty$$

which is the pdf of Cauchy's distribution.

Example 20 *If X and Y are independent RVs each following N(0, 2), find the pdf of Z = 2X + 3Y.*

Solution Introduce the auxiliary RV $W = Y$.

$\therefore \quad z = 2x + 3y$ and $w = y$

solving, $x = \frac{1}{2}(z - 3w)$ and $y = w$

$$J = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{-3}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

Since X and Y are independent normal RVs $f_{xy}(x, y) = \frac{1}{8\pi} e^{-(x^2+y^2)/8}$, $-\infty < x, y < \infty$. The joint pdf of (Z, W) is given by

$$f_{zw}(z, w) = |J| f_{xy}(x, y)$$

$$= \frac{1}{2} \times \frac{1}{8\pi} e^{-\{(z-3w)^2 + 4w^2\}/32} \quad -\infty < z,\ w < \infty$$

The pdf of Z is the marginal pdf, obtained by integrating $f_{zw}(z, w)$ with respect to w over the range of w.

$$\therefore \quad f_z(z) = \frac{1}{16\pi}\int_{-\infty}^{\infty} e^{-(13w^2 - 6zw + z^2)/32}\, dw$$

$$= \frac{1}{16\pi} e^{-z^2/8 \times 13} \int_{-\infty}^{\infty} e^{-13\left(w - \frac{3z}{13}\right)^2/32}\, dw$$

$$= \frac{1}{(2\sqrt{13})\sqrt{2\pi}} e^{-z^2/2(2\sqrt{13})^2} \quad -\infty < z < \infty$$

which is $N(0, 2\sqrt{13})$.

Example 21 *If X and Y each follow an exponential distribution with parameter 1 and are independent, find the pdf of U = X – Y.*

$$f_x(x) = e^{-x},\ x > 0, \text{ and } f_y(y) = e^{-y},\ y > 0$$

Solution Since X and Y are independent.

$$f_{xy}(x, y) = e^{-(x+y)}; \quad x, y > 0$$

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Consider the auxiliary RV $V = Y$ along with

$$U = X - Y.$$

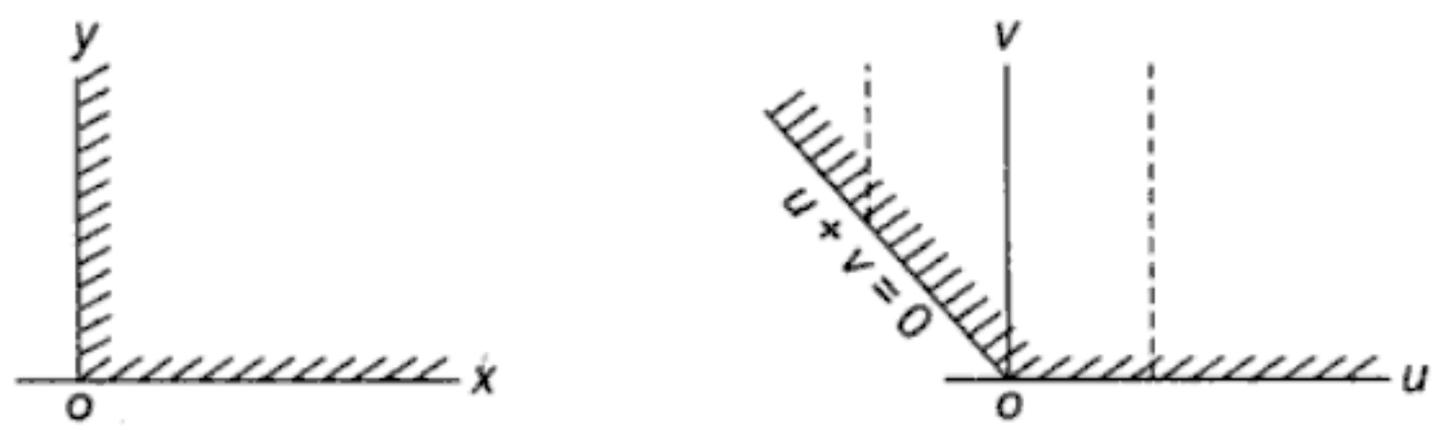
$$\therefore \qquad x = u + v \quad \text{and} \quad y = v$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The joint pdf of (U, V) is given by

$$f_{uv}(u, v) = |J| f_{xy}(x, y)$$
$$= e^{-(x+y)} = e^{-(u+2v)}$$

The range space of (U, V) is found out from the map of the range space of (X, Y) under the transformations $x = u + v$ and $y = v$.



Therefore, the range space of (U, V) is given by $v > -u$, when $u < 0$ and $v > 0$, when $u > 0$.

Now the pdf of U is given by

$$f_u(u) = \int_{-u}^{\infty} e^{-(u+2v)}\, dv, \quad \text{when } u < 0$$

and

$$= \int_{0}^{\infty} e^{-(u+2v)}\, dv, \quad \text{when } u > 0$$

$$\therefore \qquad f_u(u) = \frac{1}{2} e^{u}, \quad \text{when } u < 0.$$

and

$$= \frac{1}{2} e^{-u}, \quad \text{when } u > 0.$$

Example 22 *If the joint pdf of (X, Y) is given by $f_{xy}(x, y) = x + y$; $0 \le x, y \le 1$, find the pdf of $U = XY$.*

Solution Introduce the auxiliary RV $V = Y$.

$$\therefore \qquad x = \frac{u}{v} \quad \text{and} \quad y = v$$

$$J = \begin{vmatrix} \dfrac{1}{v} & -\dfrac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

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Part B

31. The RV X is of the continuous type with distribution function $F_x(x)$. If $g(x)$ is defined as

$$g(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x \le 0 \end{cases}$$

what is the distribution function of $Y = g(X)$? (MSU — Apr. 96)

32. If $g(x)$ is defined as

$$g(x) = \begin{cases} -b & \text{if} \quad x \le -b \\ x & \text{if} \quad -b < x < b \\ b & \text{if} \quad x \ge b \end{cases}$$

find the distribution function of $Y = g(X)$ in terms of that of X.

33. If X follows a normal distribution with mean zero and variance σ^2, find the density function of $Y = \frac{1}{2}(X + |X|)$.

34. If the density function of a continuous RV X is given by $f_x(x) = \frac{2}{9}(x+1)$, for $-1 < x < 2$, and $= 0$, otherwise, find the density function of $Y = \frac{1}{2}(X + |X|)$.

35. If X is a continuous RV with density function $f_x(x) = e^{-x}$, $x > 0$, find the density function of $Y = 2X + 1$. Hence or otherwise find $P(Y \ge 5)$.

36. If the density function of a continuous RV X is given by $f_x(x) = 2x$, for $0 < x < 1$, and $= 0$, elsewhere, find the density and distribution functions of $Y = 3X + 1$.

37. If $Y = aX^2$ $(a > 0)$, where X is a Gaussian RV with zero mean and variance σ^2, find the pdf of the RV Y. (MSU — Nov. 96)

38. (i) If the continuous RV X is uniformly distributed in $(-3, 3)$, find the density function of $Y = 2X^2 - 3$.
    (ii) If the continuous RV X is uniformly distributed in $(-2, 2)$, find the density function of $Y = 6 - X^2$.

39. (i) A fluctuating electric current I may be considered as a uniformly distributed RV over the interval (9, 11). If this current flows through a 2-Ω resistor, find the density function of the power $P = 2I^2$.
    (ii) If the voltage E across a resistor is a RV uniformly distributed between 5 and 10 V, find the density function of the power $W = \frac{E^2}{r}$, when $r = 1000\ \Omega$.

40. (i) If the continuous RV X has density function $f_x(x) = 2\,e^{-2x}$, $x > 0$, find the density function of $Y = X^2$.
    (ii) If the density function of a continuous RV X is given by $f_x(x) = e^{-x}$, $x > 0$, find the density function of $Y = 3/(X+1)^2$.

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27. $f_{uv}(u, v) = \dfrac{1}{|v|} f_{xy}(x, y)$

28. $f_{uv}(u, v) = |v| f_{xy}(x, y)$

29. $F_{uv}(u, v) = \dfrac{1}{2} f_{xy}(x, y)$

30. $f_{r\phi}(r, \theta) = |r| f_{xy}(x, y)$

31. $F_y(y) = 0$, if $y < -1$; $= F_x(0)$, if $-1 \le y < 1$ and $= 1$, if $1 \le y$

32. $F_y(y) = 0$, if $y < -b$; $= F_x(y)$, if $-b \le y < b$ and $= 1$, if $b \le y$

33. $f_y(y) = \dfrac{2}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2}$, $y \ge 0$ and 0, elsewhere

34. $f_y(y) = \dfrac{1}{4}(y + 1)$, $0 < y < 2$ and $= 0$, elsewhere

35. $f_y(y) = \dfrac{1}{2} e^{-(y-1)/2}$, $y > 1$; $1/e^2$

36. $f_y(y) = \dfrac{2}{9}(y - 1)$ for $1 < y < 4$ and $= 0$, elsewhere; $F_y(y) = (y - 1)^2/9$.

37. $f_y(y) = \dfrac{1}{\sigma\sqrt{2\pi ay}} e^{-y/2a\sigma^2}$, $y > 0$

38. (i) $f_y(y) = \dfrac{1}{6\sqrt{2}\sqrt{y+3}}$ for $-3 \le y \le 15$

(ii) $f_y(y) = \dfrac{1}{4\sqrt{6-y}}$ for $2 \le y \le 6$

39. (i) $f_p(p) = \dfrac{1}{8}\sqrt{\dfrac{2}{p}}$ for $162 \le p \le 242$

(ii) $f_w(w) = \sqrt{\dfrac{10}{w}}$ for $\dfrac{1}{40} \le w \le \dfrac{1}{10}$

40. (i) $f_y(y) = \dfrac{1}{\sqrt{y}} e^{-2\sqrt{y}}$ for $y > 0$

(ii) $f_y(y) = \dfrac{1}{6}\left(\dfrac{3}{y}\right)^{3/2} e^{1-\sqrt{3/y}}$ for $0 \le y \le 3$

41. $f_y(y) = \dfrac{1}{3} y^{-2/3} \quad e^{-y^{1/3}}$ for $y > 0$

42. (i) $f_y(y) = \dfrac{1}{12\pi} y^{-2/3}$ for $-8\pi^3 \le y \le 8\pi^3$

(ii) $f_y(y) = \dfrac{1}{8\pi} y^{-3/4}$ for $0 \le y \le 16\pi^4$

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43. (i) $f_s(s) = \frac{3}{4\pi}\left(1 - \sqrt{\frac{s}{4\pi}}\right)$ for $0 \le s \le 4\pi$

(ii) $f_v(v) = \frac{3}{2\text{च}}\left\{\left(\frac{3v}{4\text{च}}\right)^{-\frac{1}{3}} - 1\right\}$ for $0 \le v \le \frac{4\pi}{3}$

44. $f_y(y) = 1$ in $(0, 1)$

45. (i) $f_g(g) = \frac{1}{200g^2}$ for $\frac{1}{1100} \le g \le \frac{1}{900}$

(ii) $f_y(y) = \frac{1}{y^2}$ for $\frac{1}{2} \le y \le 1$

46. (i) $f_y(y) = \frac{1}{2y}$ for $e \le y \le e^3$

(ii) $f_y(y) = -\frac{2}{y}\log y$ for $\frac{1}{e} \le y \le 1$

47. (i) $f_y(y) = \frac{1}{\pi\sqrt{1-y^2}}$ for $-1 \le y \le 1$

(ii) $f_y(y) = \frac{2}{\pi\sqrt{1-y^2}}$ for $0 \le y \le 1$

49. $f_z(z) = \frac{\alpha\beta}{\beta - \alpha}(e^{-}\alpha^z - e^{-\beta z})$ if $\beta \ne \alpha$

$= \alpha^2 z \quad e^{-\alpha z}$ if $\beta = \alpha$

50. $f_z(z) = \frac{1}{4}(2 + z)$ if $-2 < y < 0$

$= \frac{1}{4}(2 - z)$ if $2 > y > 0$

51. $f_z(z) = 2[e^{-z} - e^{-2z}]\,U(z)$

52. $f_z(z) = \begin{cases} \frac{1}{6}(z - 2) & \text{when } 2 \le z \le 4 \\ \frac{1}{12}(8 - z) & \text{when } 4 \le z \le 8 \end{cases}$

54. $f_z(z) = \frac{2}{a^2}(a - z)$, for $0 < z < a$

55. (i) $f_{uv}(u, v) = \frac{1}{2}$ for $0 \le u + v \le 2$ and $0 \le u - v \le 2$

(ii) $f_u(u) = \begin{cases} u & \text{if } 0 \le u \le 1 \\ 2 - u & \text{if } 1 \le u \le 2 \end{cases}$

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(iii) $f_v(v) = \begin{cases} 1 + v & \text{if } -1 \le v \le 0 \\ 1 - v & \text{if } 0 \le v \le 1 \end{cases}$

56. If $Z = X + Y$, $F_z(z) = 0$, if $z < 0$

$$= \frac{1}{2}\left(\frac{z^2}{2} - e^{-z} - z + 1\right), \text{ if } 0 \le z < 2$$

$$= \frac{1}{2}\left[(1 - e^{-2} + (1 + e^2)(e^{-2} - e^{-z})\right], \text{ if } 2 \le z$$

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$$\mu_3 = \mu'_3 - 3\mu'_2\, d + 2d^3; \text{ and}$$
$$\mu_4 = \mu'_4 - 4\mu'_3\, d + 6\mu'_2\, d^2 - 3d^4, \quad \text{where } d = \mu - a.$$

Dispersion

Though a measure of central tendency may be taken as a representative value of the random variable, it is important to know how the values are clustered around or scattered away from the measure of central tendency while studying the distribution of the random variable. The property of the random variable (or its distribution) by which its values are clustered around or scttering away from the central value is called *dispersion*, *spread*, *scatter* or *variability*.

Following are the measures of dispersion which are commonly used.

(i) Quartile Deviation
(ii) Mean Deviation
(iii) Standard Deviation

Definitions

1. The rth quartile of a continuous random variable (or its distribution), denoted by Q_r, is defined by

$$\int_{-\infty}^{Q_r} f(x)\,dx = \frac{r}{4}\,;\, r = 1, 2, 3$$

Q_1 and Q_3 are called the first (lower) and the third (upper) *quartiles* respectively.

Q_2 is obviously the median (middle or second quartile).

Now $\dfrac{Q_3 - Q_1}{2}$, that is half of the difference between the upper and lower quartiles, is called *the semi-interquartile range* or *the quartile deviation* and denoted by Q

i.e. $$Q = \frac{Q_3 - Q_1}{2}$$

In the case of discrete random variable, the quartiles are defined in a manner similar to the definition of median.

2. Mean Deviation of a random variable X about the mean value μ is defined by

$$\text{M.D.} = E\{|X - \mu|\}$$

i.e. $$\text{M.D.} = \sum_i |x_i - \mu| p_i, \text{ when } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} |x - \mu| f(x)dx, \text{ when } X \text{ is continuous}$$

Mean deviations about the averages median and mode may also be required in statistical analysis.

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Mean deviation about the median is the minimum.

Let $\phi(a)$ be the mean deviation about a.

i.e. $$\phi(a) = E\{|X-a|\} = \int_{-\infty}^{\infty} |x-a| f(x)\mathrm{d}x$$

$$= \int_{-\infty}^{a} (a-x) f(x)\mathrm{d}x + \int_{a}^{\infty} (x-a) f(x)\mathrm{d}x$$

[$\because$ in $(-\infty, a)$, $x < a$ and in (a, ∞), $x > a$]

$$\therefore \quad \phi'(a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + (a-a) f(a)\,\frac{\mathrm{d}a}{\mathrm{d}a}$$

$$-\left[\int_{\infty}^{a} (-1)\, f(x)\,\mathrm{d}x + (a-a)\, f(a)\,\frac{\mathrm{d}a}{\mathrm{d}a}\right]$$

$$\left[\because \frac{\partial}{\partial a}\int_{k}^{a} g(x,a)\,\mathrm{d}x = \int_{k}^{a} \frac{\partial}{\partial a} g(x,a)\,\mathrm{d}x + g(a,a)\,\frac{\mathrm{d}a}{\mathrm{d}a}\right]$$

i.e. $$\phi'(a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}x - \int_{a}^{\infty} f(x)\,\mathrm{d}x$$

$$\phi''(a) = f(a) + f(a) = 2f(a) > 0 \quad [\because f(x) \text{ is the density function}]$$

$$\phi'(a) = 0, \text{ if } F(a) - \{1 - F(a)\} = 0$$

i.e. $2F(a) = 1$

i.e., $F(a) = \dfrac{1}{2}$

i.e. a is the median.

Therefore, mean deviation about a is minimum, when a is the median.

3. Standard Deviation (S.D.) of a random variable X or its distribution is defined as the positive square root of the second order central moment μ_2, (called *the variance* of the random variable) denoted by σ_x.

i.e. $$\sigma_x = \sqrt{\mu_2} \text{ or } \sigma_x^2 = E\{(X-\mu)^2\}$$

The variance of random variable X itself is denoted by Var (X). When it is required to compare the dispersion or variability of two random variables (or their distributions), we cannot use their standard deviations which have the dimensions of the random variables. We therefore use dimensionless ratios, known as coefficients of variation.

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The Coefficient of Variation

(C.V.) of a Random Variable X (or its distribution) is defined as

$$\text{C.V.} = \frac{\text{S.D.}}{\text{Mean}} \times 100$$

If $(\text{C.V.})_X > (\text{C.V.})_Y$, then the random variable X (or its distribution) is more variable (or less consistent) than the random variable Y (or its distribution).

Properties of variance

1. $\text{Var}(X) = E(X^2) - \{E(X)\}^2$

Proof From the relation between central and non-central moments, we have $\mu_2 = \mu_2' - d^2$, where $\mu_2' = E(X-a)^2$ and $d = \mu - a$ or $E(X) - a$

Taking $a = 0$, we get

$$\text{Var}(X) = \mu_2 = E(X^2) - \{E(X)\}^2$$

2. $\text{Var}\,(aX + b) = a^2\,\text{Var}(X)$

Proof $\text{Var}\,(aX+b) = E\{(aX+b) - E(aX+b)\}^2$, by definition

$$\begin{aligned} &= E\{(aX+b) - aE(X) - b\}^2 \\ &= E\{a[X - E(X)]\}^2 \\ &= a^2 E\{X - E(X)\}^2 \\ &= a^2\,\text{Var}\,(X) \end{aligned}$$

Skewness

Skewness, which means lack of symmetry, is the property of a random variable or its distribution by which we get an idea about the shape of the probability curve of the distribution. If the probability curve is not symmetrical but has a longer tail on one side than on the other, the distribution is said to be skewed. If a distribution is skewed, then the averages mean, median and mode will take different values and the quartiles will not be equidistant from the median.

The measures of skewness in common use are the following:

(i) Mean-Median,
(ii) Mean-Mode,
(iii) $(Q_3 - \text{Median}) - (\text{Median} - Q_1)$ or $Q_3 + Q_1 - 2\,\text{Median}$
(iv) the third order central moment $\mu_3 = E(X-\mu)^3$.

When the comparison of the skewness of the distributions of two random variables is to be made, we compute *coefficients of skewness* which are pure numbers, given below.

(i) Pearson's coefficient of skewness, S_k, defined by

$$S_k = \frac{\text{Mean} - \text{Median}}{\text{S.D.}} \text{ or equivalently}$$

$$S_k = \frac{3\,(\text{Mean} - \text{Mode})}{\text{S.D.}}$$

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Note *For a moderately asymmetrical distribution, it is known that Mean – Median = 3(Mean – Mode) and so either of the above definitions may be used.*

(ii) Bowley's quartile coefficient of skewness S_k, defined by

$$S_k = \frac{Q_3 + Q_1 - 2\,\text{Median}}{Q_3 - Q_1}$$

(iii) Moment coefficient of skewness v_1, defined by

$$v_1 = \frac{\mu_3}{\mu_2^{3/2}} \text{ or } \frac{\mu_3}{\sigma^3}$$

If any of the above coefficients of skewness is positive, the distribution will be skewed towards the right, otherwise it will be skewed towards the left.

Kurtosis

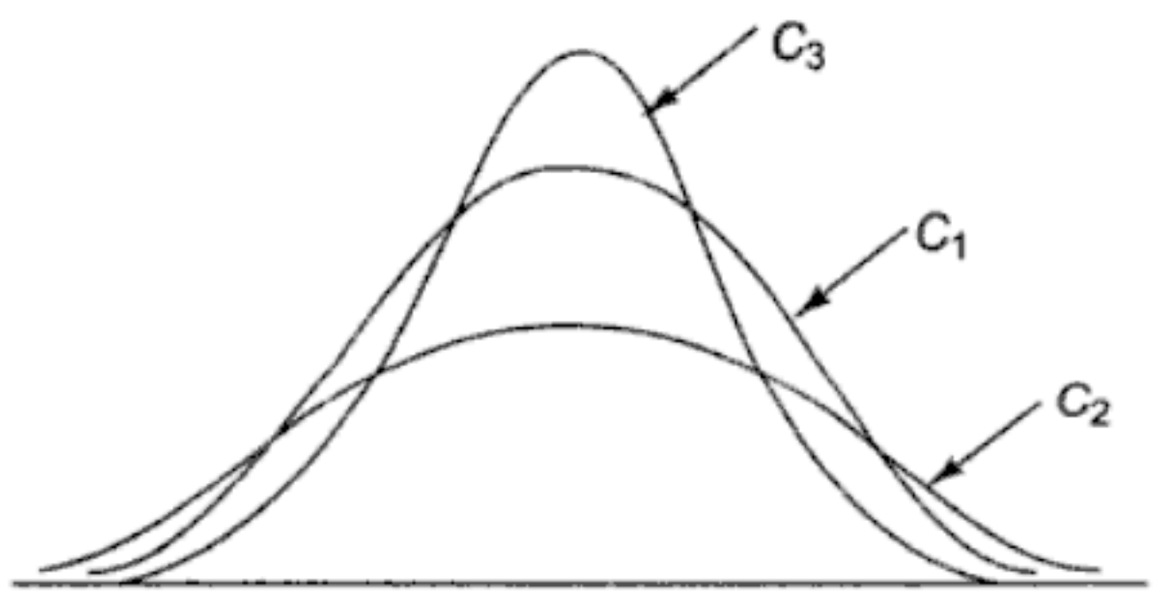
Even if we know the measures of central tendency, dispersion and skewness of a radom variable (or its distribution), we cannot get a complete idea about the distribution. In order to analyse the distribution completely, another characteristic *kurtosis* is also required. Kurtosis means the convexity of the probability curve of the distribution. Using the measure of coefficient of kurtosis, we can have an idea about the flatness or peakedness of the probability curve near its top.

The only measure of kurtosis used is the fourth order central moment μ_4, given by

$$\mu_4 = \text{E}(X - \mu)^4$$

The coefficient of kurtosis, denoted by β_2 is defined as

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad \text{or} \quad \frac{\mu_4}{\sigma^4}$$



Curve C_1 which is neither flat nor peaked is called a *mesokurtic* curve, for which $\beta_2 = 3$.

Curve C_2 which is flatter than the curve C_1 is called *platykurtic* curve, for which $\beta_2 < 3$.

Curve C_3 which is more peaked than C_1 is called *leptokurtic* curve, for which $\beta_2 > 3$.

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Pearson's Shape Coefficients

As $\beta_1 = v_1^2 = \dfrac{\mu_3^2}{\mu_2^3}$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$ determine the shape of the probability curve, they are called (Pearson's) shape coefficients.

Equivalently v_1 and $v_2 = \beta_2 - 3$ are also used as shape coefficients.

Expected Values of a Two-Dimensional RV

If (X, Y) is a two-dimensional discrete RV with joint probability mass function p_{ij}, then $E\{g(X, Y)\} = \sum_j \sum_i g(x_i, y_i\}p_{ij}$.

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X, Y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$$

Properties of Expected Values

We give below the proofs of the properties for continuous RVs. Students can prove the properties for discrete RVs.

(i) $E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$, where $f_X(x)$ is the marginal density of X.

Proof
$$\begin{aligned} E\{g(X)\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) f(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} g(x) \left[\int_{-\infty}^{\infty} f(x, y)\, dy \right] dx \\ &= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \end{aligned}$$

(ii) $E\{h(Y)\} = \int_{-\infty}^{\infty} h(y) f_Y(y)\, dy$

where $f_Y(y)$ is the marginal density of Y.

(Proof is left as an exercise to the student.)

(iii) $E(X + Y) = E(X) + E(Y)$

Proof
$$\begin{aligned} E(X + Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) f(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dx\, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) dx\, dy \\ &= E(X) + E(Y) \end{aligned}$$

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(iv) In general, $E(XY) \neq E(X) \times E(Y)$, but if X and Y are independent RVs, $E(XY) = E(X) \times E(Y)$.

Proof $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)\,\mathrm{d}x\,\mathrm{d}y$$

(since X and Y are independent)

$$= \int_{-\infty}^{\infty} xf_X(x)\mathrm{d}x \times \int_{-\infty}^{\infty} yf_Y(y)\mathrm{d}y$$

$$= E(X) \times E(Y)$$

In general, if X and Y are independent,

$$E\{g(X) \times h(Y)\} = E\{g(X)\} \times E\{h(Y)\}$$

Conditional Expected Values

If (X, Y) is a two-dimensional discrete RV with joint probability mass function p_{ij}, then the conditional expectations of $g(X, Y)$ are defined as follows:

$$E\{g(X, Y)/Y = Y_j\} = \sum_i g(x_i, y_j) \times P(X = x_i/Y = y_j)$$

$$= \sum_i g(x_i, y_j)\frac{p_{ij}}{p_{*j}}$$

and $$E\{g(X, Y)/X = x_i\} = \sum_j g(x_i, y_j)\, p_{ij}/p_{i*}$$

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X, Y)/Y\} = \int_{-\infty}^{\infty} g(x, y) \times f(x/y)\,\mathrm{d}x \text{ and}$$

$$E\{g(X, Y)/X\} = \int_{-\infty}^{\infty} g(x, y) \times f(y/x)\,\mathrm{d}y$$

In particular, the conditional means are defined as

$$\mu_{Y/X} = E(Y/X) = \int_{-\infty}^{\infty} yf(y/x)\,\mathrm{d}y \text{ and}$$

$$\mu_{X/Y} = E(X/Y) = \int_{-\infty}^{\infty} xf(x/y)\,\mathrm{d}x$$

The conditional variances are defined as

$$\sigma^2{}_{Y/X} = E\{(Y - \mu_{Y/X})^2\} = \int_{-\infty}^{\infty} (y - \mu_{Y/X})^2 f(y/x)\,\mathrm{d}y \text{ and}$$

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$$= 40 - 3 \times 20 \times 2 + 2 \times 2^3 = -64$$

A measure of skewness $= \mu_3 = -64$

$$\mu_4' = E(X-5)^4 = 50$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2' \cdot \mu_1'^2 - 3\mu_1'^4$$

$$= 50 - 4 \times 40 \times 2 + 6 \times 20 \times 4 - 3 \times 16 = 162$$

As $\mu_3 < 0$, the distribution is skewed to the left.

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{162}{256} < 3.$$

Therefore the distribution is platykurtic.

Example 4 *The first three moments of a distribution about the value 2 of the random variable X are 1, 16 and – 40. Find the coefficient of variation and skewness of the distribution. Also find the first three simple moments.*

Solution $\mu_1' = E(X-2) = 1$

$\therefore$ $E(X) = \text{Mean} = 3$

$$\mu_2' = E(X-2)^2 = 16$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 16 - 1 = 15$$

$$\mu_3' = E(X-2)^3 = -40$$

$$\mu_3 = \mu_3' - 4\mu_2'\,\mu_1' + 2\mu_1'^3$$

$$= -40 - 3 \times 16 \times 1 + 2 \times 1^3$$

$$= -86$$

$$\text{Coefficient of variation} = \frac{\text{S.D.}}{\text{Mean}} \times 100$$

$$= \frac{\sqrt{15}}{3} \times 100$$

$$= 129.1\%$$

$$\text{Coefficient of skewness} = \nu_1 = \frac{\mu_3}{\sigma^3}$$

$$= \frac{-86}{15\sqrt{15}} = -1.480$$

Now $E(X-0)^r = E\{(X-2)+2\}^r$

$$= E(X-2)^r + 2 \cdot rC_1\, E(X-2)^{r-1} + 4 \cdot rC_2\, E(X-2)^{r-2} + 8 \cdot rC_3\, E(X-2)^{r-3} + \ldots,$$

$\therefore$ $E(X) = E(X-2) + 2 = 1 + 2 = 3$

$$E(X^2) = E(X-2)^2 + 2 \times 2\, E(X-2) + 4 \times 1$$

$$= 16 + 4 + 4 = 24$$

$$E(X^3) = E(X-2)^3 + 2 \times 3 \cdot E(X-2)^2 + 4 \times 3 \cdot E(X-2) + 8 \times 1$$

$$= -40 + 96 + 12 + 8 = 76$$

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Example 7 *Find the mean and variance of the Pascal's (negative binomial distribution) distribution, given by* $P(X = k) = \binom{n+k-1}{k} p^n q^k$, $k = 0, 1, 2, \ldots$

Solution
$$E(X) = \sum_{k=0}^{\infty} k \binom{n+k-1}{k} p^n q^k$$
$$= p^n[1 \times nC_1 q^1 + 2(n+1)C_2 q^2 + 3(n+2)C_3 q^3 + \ldots]$$
$$= np^n q[1 + (n+1)C_1 q + (n+2)C_2 q^2 + \ldots]$$
$$= np^n q(1-q)^{-(n+1)} = \frac{nq}{p}$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \left(\frac{n+k-1}{k}\right) p^n q^k$$
$$= p^n[1^2 nC_1 q^1 + 2^2(n+1)\, C_2 q^2 + 3^2\,(n+2)C_3 q^3 + \ldots]$$
$$= p^n\left[nq + \frac{1\times 2+2}{2!}(n+1)\,nq^2 + \frac{(2\times 3+3)}{3!}(n+2)\,(n+1)\,nq^3 + \cdots\right]$$
$$= np^n q\left[\left\{1 + \frac{(n+1)}{1!}q + \frac{(n+1)\,(n+2)}{2!}q^2 + \cdots\right\} + (n+1)q \times \left\{1 + \frac{(n+2)}{1!}q + \frac{(n+2)(n+3)}{2!}q^2 + \cdots\right\}\right]$$
$$= np^n q[(1-q)^{-(n+1)} + (n+1)q(1-q)^{-(n+2)}]$$
$$= np^n q\left[\frac{1}{p^{n+1}} + \frac{(n+1)q}{p^{n+2}}\right]$$
$$= \frac{nq}{p} + \frac{n(n+1)q^2}{p^2}$$
$$\therefore \quad \mathrm{Var}(X) = E(X^2) - \{E(X)\}^2$$
$$= \frac{nq}{p} + \frac{n(n+1)q^2}{p^2} - \frac{n^2q^2}{p^2}$$
$$= \frac{nq}{p}\left(1 + \frac{q}{p}\right) = \frac{nq}{p^2}$$

Example 8 *If the continuous RV X has Rayleight density* $f(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} \times U(x)$, *find* $E(X^n)$ *and deduce the values of* $E(X)$ *and* $\mathrm{var}(X)$.

Solution By definition,
$$E(X^n) = \int_0^{\infty} x^n \times \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}\, dx$$

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Example 15 *A Box contains 2^n tickets of which nC_r tickets bear the number r ($r = 0, 1, 2\ldots, n$). Two tickets are drawn from the box. Find the expectation of the sum of their numbers.*

Solution Total number of tickets in the box.

$$\sum_{r=0}^{n} nC_r = nC_0 + nC_1 + \ldots + nC_n$$

$$= (1 + 1)^n = 2^n, \text{ as given.}$$

Let the RVs X and Y represent the numbers on the first and second tickets respectively.

Then $\quad E(X + Y) = E(X) + E(Y)$

X can take the values 0, 1, 2, ..., n with probabilities $\frac{nC_0}{2^n}, \frac{nC_1}{2^n}, \cdots \frac{nC_n}{2^n}$ respectively.

$$\therefore \quad E(X) = 1 \times \frac{nC_1}{2^n} + 2 \times \frac{nC_2}{2^n} + \ldots + n \times \frac{nC_n}{2^n}$$

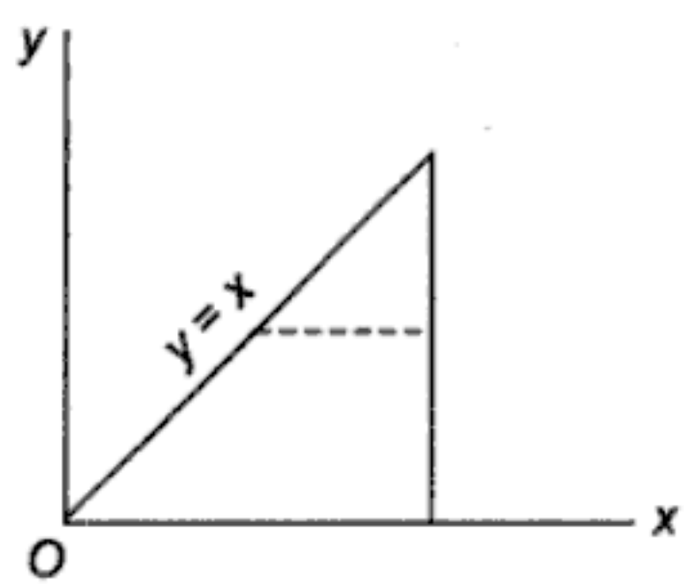
$$= \frac{n}{2^n}\{(n-1)C_0 + (n-1)C_1 + \ldots + (n-1)C_{n-1}\}$$

$$= \frac{n}{2^n}(1+1)^{n-1} = \frac{n}{2}$$

Similarly, $\quad E(Y) = \frac{n}{2}$

$$\therefore \quad E(X + Y) = n$$

Example 16 *If the joint pdf of (X, Y) is given by $f(x, y) = 24y\,(1 - x)$, $0 \le y \le x \le 1$, find $E(XY)$.*



Solution $\quad E(XY) = \int_0^1 \int_y^1 xyf(x, y)\,dx\,dy$

$$= 24 \int_0^1 \int_y^1 xy^2(1 - x)\,dx\,dy$$

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$$= 24 \int_0^1 y^2 \left(\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3} \right) dy$$

$$= \frac{4}{15}$$

Example 17 *If X and Y are two independent RVs with $f_X(x) = e^{-x}\, U(x)$ and $f_Y(y) = e^{-y}\, U(y)$ and $Z = (X - Y)\, U\,(X - Y)$, prove that $E(Z) = 1/2$.*

Solution $U(X - Y) = \begin{cases} 1 & \text{if } X > Y \\ 0 & \text{if } X < Y \end{cases}$

$\therefore$ $$Z = \begin{cases} X - Y & \text{if } X > Y \\ 0 & \text{if } X < Y \end{cases}$$

$$E(Z) = \int_0^\infty \int_0^\infty z\, e^{-(x+y)}\, dx\, dy$$

[since X and Y are independent, $f(x, y) = f_X(x) \times f_Y(y)$]

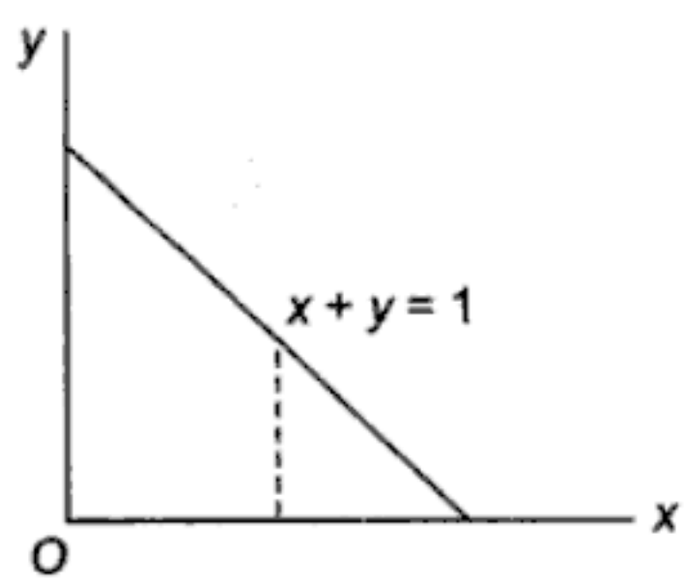
$$= \int_0^\infty \int_y^\infty (x - y)\, e^{-(x+y)}\, dx\, dy$$

$$= \int_0^\infty e^{-y}[(x - y)\,(-e^{-x}) - e^{-x}]_y^\infty\, dy$$

$$= \int_0^\infty e^{-2y}\, dy = \frac{1}{2}$$

Example 18 *The joint pdf of (X, Y) is given by $f(x, y) = 24xy$; $x > 0$, $y > 0$, $x + y \le 1$, and $f(x, y) = 0$, elsewhere, find the conditional mean and variance of Y, given X.*

Solution



$$f_X(x) = \int_0^{1-x} 24xy\, dy$$

$$= 12x\,(1 - x)^2,\ 0 < x < 1$$

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Now $$f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{2y}{(1-x)^2}, 0 < y < 1 - x$$

$$E(Y/X = x) = \int_0^{1-x} yf(y/x)\, dy$$

$$= \int_0^{1-x} \frac{2y^2}{(1-x)^2}\, dy = \frac{2}{3}(1-x)$$

$$E(Y^2/x) = \int_0^{1-x} y^2 \times f(y/x)\, dy = \frac{1}{2}(1-x)^2$$

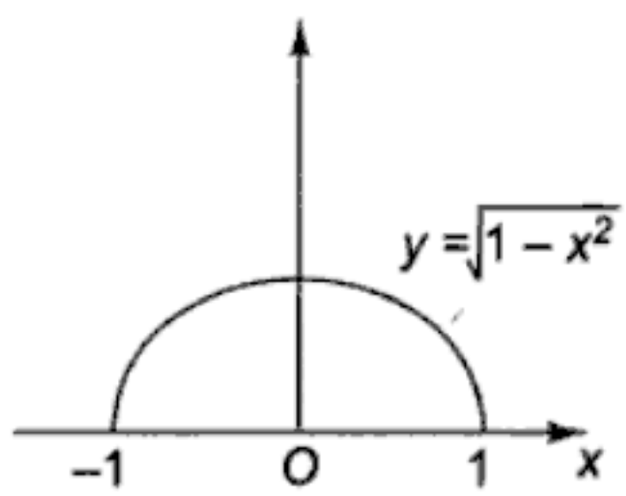
$$\text{Var}(Y^2/x) = E(Y^2/x) - \{E(Y/x)\}^2$$

$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2$$

$$= \frac{1}{18}(1-x)^2$$

Example 19 *If (X, Y) is uniformly distributed over the semicircle bounded by* $y = \sqrt{1-x^2}$ *and* $y = 0$, *find* $E(X/Y)$ *and* $E(Y/X)$. *Also verify the* $E\{E(X/Y)\} = E(X)$ *and* $E\{E(Y/X)\} = E(Y)$.

Solution



$$f(x, y) = k$$

$$\iint f(x, y)\, dy\, dx = 1$$

i.e., $$\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} k\, dy\, dx = 1$$

i.e., $$2k \int_0^1 \sqrt{1-x^2}\, dx = 1$$

$\therefore$ $$k = \frac{2}{\pi}$$

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{2}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2}, \; -1 \le x \le 1$$

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$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2\left(1-r^2\right)}\left(\frac{y}{\sigma_y}-\frac{rx}{\sigma_x}\right)^2-\frac{x^2}{2\sigma_x^2}\right\}$$

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-x^2/2\,\sigma^2{}_X\right)$$

[refer to the worked Example 12 in Chapter 2 on two-dimensional RVs]

$$\therefore \qquad f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{\sigma_y\sqrt{1-r^2}\sqrt{2\pi}}$$

$$\exp\left\{-\frac{1}{2(1-r^2)\sigma_y^2}\left(y-\frac{rx\sigma_y}{\sigma_x}\right)^2\right\}$$

which is a $N\left(\frac{rx\sigma_y}{\sigma_x}, \sigma_y\sqrt{1-r^2}\right)$

$$\therefore \qquad E(Y/X = x) = \int_{-\infty}^{\infty} yf(y/x)\,\mathrm{d}y = \frac{rx\sigma_y}{\sigma_y}$$

and $\qquad \mathrm{Var}(Y/X = x) = \sigma^2{}_y(1-r^2)$

$$\mathrm{Var}(Y/X) = E(Y^2/X) - \{E(Y/X)\}^2$$

$$\therefore \qquad E(Y^2/X) = \sigma^2{}_y(1-r^2) + r^2x^2\sigma_y^2/\sigma_x^2$$

By Property 3 of conditional expected values,

$$E(XY) = E\{XE(Y/X)\}$$

$$= E\left\{r\frac{\sigma_y}{\sigma_x}X^2\right\} = r\frac{\sigma_y}{\sigma_x}\times\sigma^2{}_x = r\sigma_x\sigma_y$$

Again, by the same property

$$E(X^2Y^2) = E\{X^2\times E(Y^2/X)\}$$

$$= E\left[\sigma_y^2\left(1-r^2\right)X^2+\frac{r^2\sigma_y^2}{\sigma_x^2}X^4\right]$$

$$= \sigma_y^2(1-r^2)E(X^2)+\frac{r^2\sigma_y^2}{\sigma_x^2}E(X^4)$$

$$= \sigma_x^2\sigma_y^2(1-r^2)+\frac{r^2\sigma_y^2}{\sigma_x^2}\times 3\sigma_x^4$$

$$= (1+2r^2)\sigma_x^2\sigma_y^2$$

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Exercise 4(A)

Part A (Short answer questions)

1. What do you mean by central tendency and dispersion of a random variable?
2. What is meant by skewness and kurtosis with respect to the probability distribution of a RV?
3. Define β_1 and β_2 coefficients. What are their uses?
4. Define mathematical expectation of $g(X)$ when X is a (i) discrete RV, and (ii) continuous RV.
5. If X is and RV, prove that Var $(X) = E(X^2) - E^2(X)$ and hence prove that $E(X^2) \geq [E(X)]^2$.
6. If μ_X and σ_X are the mean and SD of the RVX, find μ_Y and σ_Y, where $Y = (X - \mu_X)/\sigma_X$.
7. Define the raw and central moments of an RV and state the relation between them.
8. For a RVX, $E(X) = 10$ and Var $(X) = 25$, find the positive values of a and b such that $E(X) = 0$ and Var $(Y) = 1$, where $Y = aX - b$.
9. A fair coin is tossed until a head appears. What is the expected value of the number of tosses required?
10. On the average, how many times must a die be thrown until one gets a 6?
11. The probability distribution of an RVX is given by

X	:	0	1	2	3
p_X	:	0.1	0.3	0.4	0.2

Find $E(Y)$ where $Y = X^2 + X$.

12. Find $E(X)$ and Var (X), when X is a discrete RV taking the values 1, 2, 3,… , n with equal probability.
13. Find the mean of the RVX, if its p.m.f. is given by $P(X = j) = (1 - a)\, a^j$; $j = 0, 1, 2, \ldots \infty$.
14. Find $E(X)$, if $P(X = x_k) = 2^{-k}$, where $x_k = \dfrac{(-1)^{k-1}2^k}{k}$; $k = 1, 2, 3, \ldots \infty$.
15. Find $E(X)$, if X is a RV such that $P(X = r) = pq^r$; $r = 0, 1, 2, \ldots \infty$, where $p + q = 1$.
16. Find $E(X)$, if X is a discrete RV such that $P(X = r) = n\mathrm{C}_r p^r q^{n-r}$; $r = 0, 1, 2, \ldots, n$, where $p + q = 1$.
17. Find $E(X)$, if X is a discrete RV such that $P(X = r) = e^{-\lambda}\, \lambda^r/r!$; $r = 0, 1, 2, \ldots \infty$.
18. Find the mean and mode of the RVX, given that $P(X = r) = 2^{-r}$; $r = 1, 2, 3, \ldots \infty$.
19. Find the mean and variance of the RVX, whose pdf is given by $f(x) = me^{-mx}$; $x > 0$.
20. Find the mean and median of the RVX, whose pdf is given by $f(x) = 2x$; $0 \leq x \leq 1$.

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21. Find the median and mode of the RVX, whose pdf is given by $f(x) = \sin x$; $0 \le x \le \frac{\pi}{2}$.
22. For a probability distribution, SD = 2, coefficient of variation = 5% and Pearson's coefficient of skewness = 0.5. Find the mean and mode of the distribution.
23. If the first three simple moments of an RVX are $\frac{1}{2}(n+1)$, $\frac{1}{6}(n+1)(2n+1)$ and $\frac{1}{4}n(n+1)^2$, find whether the distribution of X is symmetrical or skewed.
24. Show that for a discrete probability distribution, the β_2-coefficient is always greater than 1.
25. Define the expected value of $g(X, Y)$, where (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$.
26. If (X, Y) is a two-dimensional continuous RV, express $E\{g(X)\}$ and $E\{h(Y)\}$ in terms of the marginal densities of X and Y.
27. If X and Y are independent RVs, prove that $E(X\,Y) = E(X) \cdot E(Y)$.
28. If X and Y are independent RVs with means 2 and 3 and variances 1 and 2 respectively, find the mean and variance of $Z = 2X - 5Y$.
29. If (X, Y) is a two-dimensional RV, define $E\{g(X, Y)/X\}$ and $E\{g(X, Y)/Y\}$.
30. If (X, Y) is a two-dimensional continuous RV, define conditional mean and conditional variance of X, given Y.
31. If X and Y are independent RVs, prove that $E(Y/X) = E(Y)$.
32. If X and Y are independent RVs, prove that $E\{E(Y/X)\} = E(Y)$.
33. If X and Y are independent RVs, prove that $E(X\,Y) = E\,\{X \cdot E(Y/X)\}$
34. If the joint pdf of (X, Y) is given by $f(x, y) = 2 - x - y$, in $0 \le x, y \le 1$, find $E(X)$ and $E(Y)$.
35. If the joint pdf of (X, Y) is given by $f(x, y) = 2$, in $0 \le x < y \le 1$, find $E(X)$.

Part B

36. If t is any positive real number, show that the function defined by $p(x) = e^{-t}(1 - e^{-t})^{x-1}$; $x = 1, 2, 3, ...$ can represent the probability mass function of a RVx. Find the mean and variance of the distribution of X. [**Hind:** Put $e^{-t} = p$ and $1 - e^{-t} = q$]
37. A discrete RVX takes the values $0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1$ with equal probability. Find the mean and variance of the distribution of X.
38. A discrete RVX takes the values $a, a+1, a+2, ..., a+2n$ with equal probability, find the SD and mean deviation of X about its mean.
39. Find the mean and variance of $Y = (2X + 1)^2$, if the probability distribution of X is given by

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$X = i$	:	0	1	2	3
p_i	:	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{24}$	$\frac{1}{8}$

40. Find the mean, median, mode, QD, mean deviation about the mean and SD of the following probability distribution.

$X = x_i$	:	5	15	25	35	45	55	65	75
p_i	:	0.05	0.08	0.07	0.12	0.28	0.20	0.10	0.10

41. Compute β_1 and β_2 coefficients for the following probability distribution. Also comment on the skewness and kurtosis of the distribution.

$X = x_i$	:	1	2	3	4	5	6	7	8
p_i	:	0.008	0.032	0.142	0.216	0.240	0.206	0.143	0.013

42. The first four moments of a distribution about $X = 4$ are 1, 4, 10, 45 respectively. Find the mean, variance, β_1 and β_2.
43. The first three moments of a distribution about the value 3 of the RVX are 2, 10, –30 respectively. Show that the first three simple moments are 5, 31, 141. Find also μ_2 and μ_3.
44. The distribution of a RVX has mean 10, variance 16, $\upsilon_1 = 1$ and $\beta_2 = 4$. Obtain the first four simple moments of X.
45. What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success?
46. What is the expection of (i) the sum of the points on n dice, and (ii) the product of the points on n dice?
47. Three tickets are chosen at random without replacement from 100 tickets, numbered 1, 2, 3, ..., 100. Find the expectation of the sum of the numbers.
48. From an urn containing 3 red and 2 black balls, a man is to draw 2 balls at random, without replacement, being promised Rs 20/- for each red ball he draws and Rs. 10/- for each black ball. Find his expectation.
49. A person draws 3 balls from a bag containing 3 white, 4 red and 5 black balls. He is offered Rs 10/-, Rs 5/- and Rs. 2/-, if the draws 3 balls of the same colour, 2 balls of the same colour and one ball of different colour and 3 balls of different colours respectively. Find the expected value of the amount he gets.
50. Box A contains 5 cards numbered from 1 to 5 and Box B contains 4 cards numbered from 1 to 4. One card is drawn from each of these boxes. Find the mathematical expectation of the sum of the numbers on the cards drawn.
51. If the density function of a continuous RVX is given by $f(x) = \dfrac{kx}{(1+x)^3}$, $x \geq 0$, find the value of k, median and mode.
52. The density function of a continuous RVX is given by $f(x) = \dfrac{3}{4}\, x\,(2 - x)$, $0 \leq x \leq 2$. Find the mean, median, mode, harmonic mean, MD about mean, SD, β_1 and β_2 of the distribution of X.

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53. The density function of a continuous RVX is given by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, \text{in } -3 \le x \le -1 \\ \frac{1}{16}(6-2x^2), \text{in } -1 \le x \le 1 \\ \frac{1}{16}(3-x)^2, \text{in } 1 \le x \le 3 \end{cases}$$

Find the mean, SD and MD about the mean.

54. If the density function of a continuous RVX is given by $f(x) = k\, e^{-|x|}$, $-\infty < x < \infty$, show that $k = \frac{1}{2}$, mean = 0, MD about the mean = 1 and SD = $\sqrt{2}$.

55. If the density function of a continuous RVX is given by $f(x) = \lambda e^{-\lambda(x-a)}$, $a \le x < \infty$, show that [(mean – SD) of the distribution of X] = a, $\beta_1 = 4$ and $\beta_2 = 9$.

56. If the continuous RVX has the density function $f(x) = 2x\, e^{-x^2}$, $x \ge 0$ and if $Y = X^2$, find the mean and variance of Y.

57. If X and Y are independent RVs with density functions $f_X(x) = \frac{8}{x^3}$, $x > 2$ and $f_Y(y) = 2y$, $0 < y < 1$ respectively and $Z = XY$, find $E(Z)$.

58. If the joint pdf of (X, Y) is given by $f(x, y) = 2$, $0 \le x < y \le 1$, find the conditional mean and conditional variance of X, given that $Y = y$.

59. If the joint pdf of (X, Y) is given by $f(x, y) = 21\, x^2 y^3$, $0 \le x < y \le 1$, find the conditional mean and conditional variance of X, given that $Y = y$, $0 < y < 1$.

60. If the joint pdf of (X, Y) is given by $f(x, y) = 3xy\,(x + y)$, $0 \le x, y \le 1$, verify that $E\{E(Y/X)\} = E(Y) = \frac{17}{24}$.

Linear Correlation

In many situations, the outcome of a random experiment will have two measurable characteristics, viz., will result in two random variables X and Y. Often we will be interested in finding whether the two different RV's are related to each other. If they are related, we will try to determine the nature of relationship and degree of relationship (correlation). Assuming that there is some correlation between X and Y, we will then try to find a formula expressing the relationship and use this formula to predict the most likely value of one RV corresponding to any given value of the other RV.

To examine whether the two RV's are inter-related, we collect n pairs of values of X and Y corresponding to n repetitions of the random experiment. Let

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$$= E\{XY) - E(X) \times E(Y) \tag{2}$$

Also we know that $\quad E\{X - E(X)\}^2 = E(X^2) - \{E(X)\}^2 \quad (3)$

and $\quad E\{Y - E(Y)\}^2 = E\{Y^2\} - \{E(Y)\}^2 \quad (4)$

Using (2), (3) and (4) in (1), we get

$$r_{XY} = \frac{E(XY) - E(X) \cdot E(Y)}{\sqrt{\left\{E\left(X^2\right) - E^2(X)\right\}\left\{E\left(Y^2\right) - E^2(Y)\right\}}} \tag{5}$$

where $E^2(X)$ means $\{E(X)\}^2$.

We will mainly deal with linear correlation of discrete RV's X and Y. X will take the values $x_1, x_2, \ldots, x_n$ with frequency 1 each and Y will simultaneously take the values $y_1, y_2, \ldots, y_n$ with frequency 1 each. Hence $E(X) = \frac{1}{n}\sum x_i$; $E(X^2) = \frac{1}{n}\sum x_i^2$, $E(XY) = \frac{1}{n}\sum x_i y_i$ etc. Using these values in (5), the working formula for the compution of r_{XY} is got as

$$r_{XY} = \frac{\frac{1}{n}\Sigma x_i y_i - \frac{1}{n}\Sigma x_i \cdot \frac{1}{n}\Sigma y_i}{\sqrt{\left\{\frac{1}{n}\Sigma x_i^2 - \left(\frac{1}{n}\Sigma x_i\right)^2\right\}\left\{\frac{1}{n}\Sigma y_i^2 - \left(\frac{1}{n}\Sigma y_i\right)^2\right\}}} \tag{6}$$

or

$$r_{XY} = \frac{n\,\Sigma xy - \Sigma x \cdot \Sigma y}{\sqrt{\left\{n\Sigma x^2 - (\Sigma x)^2\right\}\left\{n\Sigma y^2 - (\Sigma y)^2\right\}}} \tag{7}$$

Properties of Correlation Coefficient

1. $-1 \le r_{XY} \le 1$ or $|\text{Cov}(X, Y)| \le \sigma_X \cdot \sigma_Y$.

   Let us consider

   $$E[a\{X - E(X)\} + \{Y - E(Y)\}]^2 = a^2\,\sigma_x^{\,2} + 2a\,C_{XY} + \sigma^2{}_Y \tag{1}$$

   The R.H.S. expression is a quadratic expression in a, that is a real quantity. It is positive, as it is the expected value of a perfect square.

   Hence, by the property of quadratic expressions, the discriminant of the R.H.S. ≤ 0

   i.e., $\quad 4C^2{}_{XY} - 4\sigma^2{}_X\,\sigma^2{}_Y \le 0$

   i.e., $\quad C^2{}_{XY} \le \sigma_X^{\,2} \cdot \sigma_Y^{\,2} \quad (2)$

   i.e., $\quad \dfrac{C_{XY}^2}{\sigma_X^2 \cdot \sigma_Y^2} \le 1$

   i.e., $\quad r^2{}_{XY} \le 1$

   $\therefore \quad |r_{XY}| \le 1$ or $-1 \le r_{XY} \le 1$

   From step (2), it is clear that $|C_{XY}| \le \sigma_X \cdot \sigma_Y$

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Note *When $0 < r_{XY} \le 1$, the correlation between X and Y is said to be positive or direct.*

When $-1 \le r_{XY} \le 0$, the correlation is said to be negative or inverse.
When $-1 \le r_{XY} \le -0.5$ or $0.5 \le r_{XY} \le 1$, the correlation is assumed to be high, otherwise the correlation is assumed to be poor.

2. Correlation coefficient is independent of change of origin and scale.

i.e., If $U = \dfrac{X-a}{h}$ and $V = \dfrac{Y-b}{k}$, where $h, k > 0$, then $r_{XY} = r_{UV}$.

By the transformations, $X = a + hU$ and $Y = b + kV$

$\therefore \quad E(X) = a + hE(U)$ and $E(Y) = b + kE(V)$

$\therefore \quad X - E(X) = h\{U - E(U)\}$ and $Y - E(Y) = k\{V - E(V)\}$

Then $\quad C_{XY} = E[h\{U - E(U)\} \cdot k\{V - E(V)\} = hk\, C_{UV}$

$\sigma^2{}_X = E[h^2\, \{U - E(U)\}^2] = h^2\, \sigma^2{}_U$

$\sigma^2{}_Y = E[k^2\, \{V - E(V)\}^2] = k^2\, \sigma^2{}_V$

$$\therefore \quad r_{XY} = \frac{\text{Cov}\,(X, Y)}{\sqrt{\sigma_X^2 \cdot \sigma_Y^2}} = \frac{hk\, C_{UV}}{\sqrt{h^2 \cdot \sigma_U^2 \cdot k^2\, \sigma_V^2}}$$

$$= \frac{C_{UV}}{\sigma_U \cdot \sigma_V} = r_{UV}$$

Note *If X and Y take considerably large values, computation of r_{XY} will become difficult. In such problems, we may introduce change of origin and scale and compute r using the above property.]*

3. Two independent RV's X and Y are uncorrelated, but two uncorrelated RV's need not be independent.

When X and Y are independent, $E(XY) = E(X) \cdot E(Y)$.

$\therefore \quad C_{XY} = 0$ and hence $r_{XY} = 0$

viz., X and Y are uncorrelated.

The converse is not true, since $E(XY) = E(X) \cdot E(Y)$, when $r_{XY} = 0$.

This does not imply that X and Y are independent, as X and Y are independent only when $f(x, y) = f_X(x) \cdot f_Y(y)$.

Note *When $E(XY) = 0$, X and Y are said to be orthogonal RV's.*

4. $$r_{XY} = \frac{\sigma_X^2 + \sigma_Y^2 - \sigma_{(X-Y)}^2}{2\sigma_X\, \sigma_Y}$$

Let $\quad Z = X - Y$. Then $E(Z) = E(X) - E(Y)$

$\therefore \quad Z - E(Z) = [X - E(X)] - [Y - E(Y)]$

$\therefore \quad \sigma^2{}_Z = E[Z - E(Z)]^2 = E[\{X - E(X)\} - \{Y - E(Y)\}]^2$

$= E\{X - E(X)\}^2 + E\{Y - E(Y)\}^2 - 2E\,[\{X - E(X)\}\ \{Y - E(Y)\}]$

$= \sigma^2{}_X + \sigma^2{}_Y - 2C_{XY}$

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$$\therefore \qquad C_{XY} = \frac{\sigma_X^2 + \sigma_Y^2 - \sigma_Z^2}{2}$$

$$\therefore \qquad r_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y} = \frac{\sigma_X^2 + \sigma_Y^2 - \sigma_{(X-Y)}^2}{2\sigma_X \sigma_Y}$$

Similarly we can prove that

$$\sigma^2_{(X+Y)} = \sigma^2_X + \sigma^2_Y + 2C_{XY}$$

and hence $r_{XY} = \dfrac{\sigma_{(X+Y)}^2 - \sigma_X^2 - \sigma_Y^2}{2\sigma_X \sigma_Y}$

Rank Correlation Coefficient

Sometimes the actual numerical values of X and Y may not be available, but the positions of the actual values arranged in order of merit (ranks) only may be available. The ranks of X and Y will in general, be different and hence may be considered as random variables. Let them be denoted by U and V. The correlation coefficient between U and V is called *the rank correlation coefficient* between (the ranks of) X, Y and denoted by ρ_{XY}.

Let us now derive a formula for ρ_{XY} or r_{UV}. Since U represents ranks of n values of X, U takes the values $1, 2, 3, \cdots, n$.

Similarly V takes the same values $1, 2, 3, \cdots, n$ in a different order.

$$E(U) = E(V) = \frac{1}{n}(1 + 2 + \cdots + n) = \frac{n+1}{2}$$

$$E(U^2) = E(V^2) = \frac{1}{n}(1^2 + 2^2 + \cdots + n^2) = \frac{(n+1)(2n+1)}{6}$$

$$\therefore \qquad \sigma_U^2 = \sigma_V^2 = E(U^2) - E^2(U)$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{(n+1)}{12}\{2(2n+1) - 3(n+1)\}$$

$$= \frac{n^2 - 1}{12}$$

Let $\qquad D = U - V \quad \therefore E(D) = 0$

and $\qquad \sigma^2_D = E(D^2)$

By property (4) given above,

$$\rho_{XY} = r_{UV} = \frac{\sigma_U^2 + \sigma_V^2 - \sigma_D^2}{2\sigma_U \sigma_V}, \text{ where } D = U - V$$

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$$= \frac{\left(\frac{n^2-1}{6}\right) - \sigma_D^2}{2\left(\frac{n^2-1}{12}\right)}$$

$$= 1 - \frac{6}{n^2-1}\sigma_D^2 \text{ or } 1 - \frac{6E\left(D^2\right)}{n^2-1}$$

$$= 1 - \frac{6\,\Sigma d^2}{n\left(n^2-1\right)} \quad \left[\because E\left(D^2\right) = \frac{1}{n}\Sigma d^2\right]$$

Note *The formula for the rank correlation coefficient is known as spearman's formula. The values of r_{XY} and ρ_{XY} (or r_{UV}) will be, in general, different.*

Worked Examples 4(B)

Example 1 *Compute the coefficient of correlation between X and Y, using the following data:*

X	:	1	3	5	7	8	10
Y	:	8	12	15	17	18	20

Solution

x_i	y_i	x_i^2	y_i^2	$x_i\,y_i$
1	8	1	64	8
3	12	9	144	36
5	15	25	225	75
7	17	49	289	119
8	18	64	324	144
10	20	100	400	200
34	90	248	1446	582

Thus $n = 6$

$\Sigma x_i = 34,\ \Sigma y_i = 90$

$\Sigma x_i^2 = 248,\ \Sigma y_i^2 = 1446$

$\Sigma x_i\,y_i = 582$

$$r_{XY} = \frac{n\,\Sigma xy - \Sigma x \cdot \Sigma y}{\sqrt{\left\{n\,\Sigma x^2 - (\Sigma x)^2\right\}\left\{n\,\Sigma y^2 - (\Sigma y)^2\right\}}}$$

$$= \frac{6 \times 582 - 34 \times 90}{\sqrt{\left\{6 \times 248 - (34)^2\right\}\left\{6 \times 1446 - (90)^2\right\}}}$$

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x	y	$u = \frac{x-15}{5}$	$v = y - 23$	u^2	v^2	uv
5	16	– 2	– 7	4	49	14
10	19	– 1	– 4	1	16	4
15	23	0	0	0	0	0
20	26	1	3	1	9	3
25	30	2	7	4	49	14
	Total	0	– 1	10	123	35

$$r_{XY} = r_{UV} = \frac{n\Sigma uv - \Sigma u \cdot \Sigma v}{\sqrt{\left\{n\Sigma u^2 - (\Sigma u)^2\right\}\left\{n\Sigma v^2 - (\Sigma v)^2\right\}}}$$

$$= \frac{5 \times 35 - 0 \times (-1)}{\sqrt{(5 \times 10 - 0)(5 \times 125 - 1)}}$$

$$= \frac{175}{\sqrt{50 \times 624}} = 0.9907$$

Example 4 *The following table gives the bivariate frequency distribution of marks in an intelligence test obtained by 100 students according to their age:*

Age (x) in yrs *Marks (y)*	*18*	*19*	*20*	*21*	*Total*
10–20	4	2	2	–	8
20–30	5	4	6	4	19
30–40	6	8	10	11	35
40–50	4	4	6	8	22
50–60	–	2	4	4	10
60–70	–	2	3	1	6
Total	19	22	31	28	100

Calculate the coefficient of correlation between age and intelligence.

Solution Since the frequencies of various values of x and y are not equal to 1 each the formula for the computation of r_{XY} is taken with a slight modification as given below:

$$r_{XY} = r_{UV} = \frac{N\,\Sigma f_{XY}\,uv - \Sigma f_X\,u \cdot \Sigma f_Y\,v}{\sqrt{\left\{N\,\Sigma f_X u^2 - (\Sigma f_X u)^2\right\}\left\{N\,\Sigma f_Y v^2 - (\Sigma f_Y\,v)^2\right\}}} \quad (1)$$

where $u = x - 20$, $v = \frac{y-35}{10}$, f_X represents frequencies of X-distribution, f_Y represents frequencies of Y-distribution and f_{XY} are the cell frequencies.

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i.e.,
$$E(X^2) + k\frac{\sigma_X}{\sigma_Y}E(Y^2) + \left(k + \frac{\sigma_X}{\sigma_Y}\right)E(XY)$$
$$-\left[E^2(X) + k\frac{\sigma_X}{\sigma_Y}E^2(Y) + \left(k + \frac{\sigma_X}{\sigma_Y}\right)E(X)\cdot E(Y)\right] = 0$$

i.e.,
$$\{E(X^2) - E^2(X)\} + \frac{k\sigma_X}{\sigma_Y}\{E(Y^2) - E^2(Y)\}$$
$$+\left(k + \frac{\sigma_X}{\sigma_Y}\right)\{E(XY) - E(X)\cdot E(Y)\} = 0$$

i.e.,
$$\sigma^2_X + k\frac{\sigma_X}{\sigma_Y}\cdot\sigma^2_Y + \left(k + \frac{\sigma_X}{\sigma_Y}\right)\text{Cov}(X, Y) = 0$$

Dividing throughout by σ_X,
$$(\sigma_X + k\sigma_Y) + (\sigma_X + k\sigma_Y)\cdot\frac{\text{cov}(X, Y)}{\sigma_X\,\sigma_Y} = 0$$

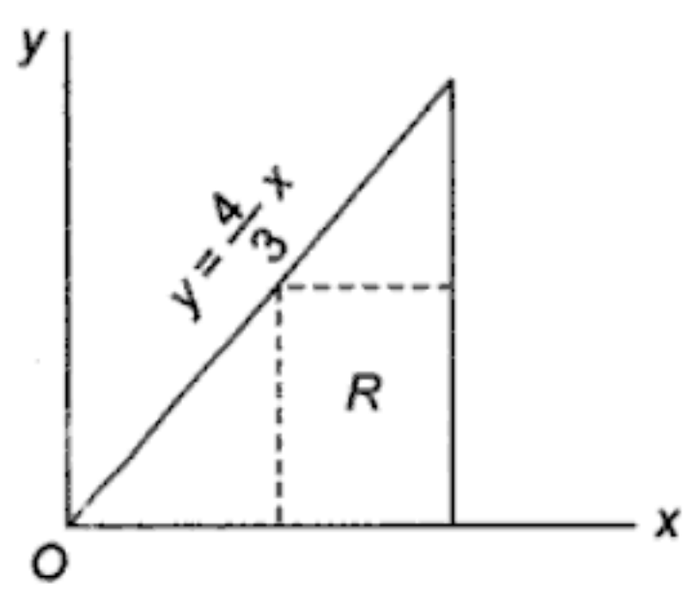
i.e., $(\sigma_X + k\sigma_Y)(1 + r_{XY}) = 0$

Assuming that $r_{XY} \neq -1$, we get
$$\sigma_X + k\sigma_Y = 0$$

$\therefore$
$$k = -\frac{\sigma_X}{\sigma_Y}$$

Example 9 *If (X, Y) is a two-dimensional RV uniformly distributed over the triangular region R bounded by y = 0, x = 3 and y = 4/3 x. Find $f_X(x)$, $f_Y(y)$, E(X), Var(X), E(Y), Var(Y) and ρ_{XY}.*

Solution



Since (X, Y) is uniformly distributed, $f(x, y)$ = a constant = k

Now
$$\iint f(x, y)\,dx\,dy = 1$$

i.e.,
$$\int_0^4\int_{3y/4}^3 k\,dx\,dy = 1$$

i.e.,
$$k\int_0^4\left(3 - \frac{3y}{4}\right)dy = 1$$

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i.e., $$6k = 1$$

$\therefore$ $$k = \frac{1}{6}$$

$$f_Y(y) = \int_{3y/4}^{3} \frac{1}{6} dx = \frac{1}{8}(4-y), 0 < y < 4$$

$$f_X(x) = \int_{3}^{4x/3} \frac{1}{6} dy = \frac{2}{9} x, 0 < x < 3$$

$$E(X) = \int x f_X(x)\, dx = \int_0^3 \frac{2}{9} x^2\, dx = 2$$

$$E(Y) = \int y f_Y(y)\, dy = \int_0^4 \frac{y}{8} \times (4-y)\, dy = \frac{4}{3}$$

$$E(X^2) = \int_0^3 \frac{2}{9} \times x^3\, dx = \frac{9}{2}$$

$$E(Y^2) = \int_0^4 \frac{y^2}{8} \times (4-y)\, dy = \frac{8}{3}$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{9}{2} - 4 = \frac{1}{2}$$

$$\text{Var}(Y) = E(Y^2) - \{E(Y)\}^2 = \frac{8}{3} - \frac{16}{9} = \frac{8}{9}$$

$$E(XY) = \int_0^4 \int_{3y/4}^{3} \frac{1}{6} xy\, dx\, dy$$

$$= \frac{3}{64} \int_0^4 (16 - y^2) y\, dy = 3$$

$$\rho_{XY} = \frac{E(XY) - E(X) \times E(Y)}{\sigma_x \sigma_y} = \frac{3 - 2 \times \frac{4}{3}}{\frac{1}{\sqrt{2}} \times 2 \frac{\sqrt{2}}{3}} = \frac{1}{2}$$

Example 10 *Find the correlation co-efficient between X and Y, which are jointly normally distributed with*

$$f(x, y) = \frac{1}{2\pi\, \sigma_x\, \sigma_y \sqrt{1 - r^2}} \exp\left\{-\frac{1}{2(1 - r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\, \sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\}$$

Solution $$\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x \sigma_y} + \frac{y^2}{\sigma_y^2} = \left(\frac{x}{\sigma_x} - \frac{ry}{\sigma_y}\right)^2 + (1 - r^2)\frac{y^2}{\sigma_y^2}$$

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$$= \frac{1}{\sigma_x^2}\left(x - \frac{ry\sigma x}{\sigma_y}\right)^2 + (1 - r^2)\frac{y^2}{\sigma_y^2}$$

$$E(XY) = \frac{1}{2\pi\sigma_x\,\sigma_y\sqrt{1-r^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp$$

$$\left\{-\frac{1}{2(1-r^2)}\left[\frac{1}{\sigma_x^2}\left(x - \frac{ry\sigma_x}{\sigma_y}\right)^2 + \frac{(1-r)^2\,y^2}{\sigma_y^2}\right]\right\} xy\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{\sigma_y\sqrt{2\pi}}\int_{-\infty}^{\infty} y\exp\left(\frac{-y^2}{2\sigma_y^2}\right)\int_{-\infty}^{\infty}\frac{x}{\left(\sigma_x\sqrt{1-r^2}\right)\sqrt{2\pi}}$$

$$\exp\left\{\frac{-\left(x - \frac{ry\sigma_x}{\sigma_y}\right)^2}{2(1-r^2)\sigma_x^2}\right\}\mathrm{d}x\,\mathrm{d}y \qquad (1)$$

The inner integral is the mean of the normal distribution with mean $\frac{ry\sigma_x}{\sigma_y}$ and variance $(1 - r^2)\,\sigma_X^2$.

$\therefore$ the inner integral $= \frac{ry\sigma_x}{\sigma_y}$

Using this value in (1),

$$E(XY) = \left(\frac{r\sigma_x}{\sigma_y}\right) \times \frac{1}{\sigma_y\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2\exp\left(-\frac{y^2}{2\sigma_y^2}\right)\mathrm{d}y$$

$$= \frac{r\sigma_x}{\sigma_y}\,E(Y^2) \text{ for } N(0, \sigma_Y)$$

$$= \frac{r\sigma_x}{\sigma_y} \times \sigma_Y^2$$

$$= r\sigma_X\sigma_Y$$

$$\therefore \qquad \rho_{XY} = \frac{E(XY) - E(X)E(y)}{\sigma_x\,\sigma_y} = r$$

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24. Find the rank correlation coefficient between the ranks of the variable X and Y:

X:	10	15	12	17	13	16	24	14	22
Y:	30	42	45	46	33	34	40	35	39

25. The competitors in a musical contest were ranked by the three judges A, B, C in the following order:

Rank by A:	1	6	5	10	3	2	4	9	7	8
Rank by B:	3	5	8	4	7	10	2	1	6	9
Rank by C:	6	4	9	8	1	2	3	10	5	7

Using rank correlation technique, find which pair of judges have more or less the same taste in music.

26. If X, Y, Z are uncorrelated RV's having the same variance, find the correlation coefficient between $(X + Y)$ and $(Y + Z)$.
27. If X and Y are two uncorrelated RV's with zero means, prove that $U = X \cos \alpha + Y \sin \alpha$ and $V = X \sin \alpha - Y \cos \alpha$ are also uncorrelated.
28. X and Y are independent RV's with means 5 and 10 and variances 4 and 9 respectively. Obtain the correlation coefficient between U and V, where $U = 3X + 4Y$ and $V = 3X - Y$.
29. If X_1, X_2, X_3 are three uncorrelated RV's having variances v_1, v_2, v_3 respectively, obtain the coefficient of correlation between $(X_1 + X_2)$ and $(X_2 + X_3)$.
30. Show that (i) $E\{aX + bY\} = aE(X) + bE(Y)$ and (ii) var $(aX + bY) = a^2$ var $(X) + b^2$ var $(Y) + 2abC\,(X, Y)$, where $C(X, Y)$ is the covariance of (X, Y).
31. It two RV's are uncorrelated, prove that the variance of their sum is equal to the sum of their variances.
32. If the joint density fucntion of (X, Y) is given by $f(x, y) = 2 - x - y$, $0 \le x$, $y \le 1$, find $E(X)$, $E(Y)$, var (X), var (Y) and r_{XY}.
33. If the two dimensional RV (X, Y) is uniformly distributed in $0 \le x < y \le 1$, find $E(X)$, $E(Y)$, var (X), var (Y) and r_{XY}.
34. If the two dimensional RV (X, Y) is uniformly distributed over R, where R is defined by $\{(x, y)/x^2 + y^2 \le 1, y \ge 0\}$, find r_{XY}.
35. If the joint pdf of (X, Y) is given by $f(x, y) = x + y$, $0 \le x, y \le 1$, find r_{XY}.
36. Let X be a RV with mean value = 3 and variance = 2. Find the second moment of X about the origin. Another RVY is defined by $Y = -6X + 22$. Find the mean value of Y and the correlation of X and Y.

Regression

When the random variables X and Y are linearly correlated, the points plotted on the scatter diagram, corresponding to n pairs of observed values of X and Y, will have a tendency to cluster round a straight line. This straight is called *the regression line*. The regression line can be taken as the best fitting straight line for the observed pairs of values of X and Y in the least square sense, with which the students are familiar.

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of Y, namely, y_i. Hence the difference between Y and Y_E is called *the error of estimate of Y*. This error will vary from one observed value to the other and a random variable. The standard deviation of this RV $(Y - Y_E)$ is called *the standard error of estimate of Y* and denoted by S_Y.

Now $E\{Y - Y_E\} = E\left[Y - \left\{\bar{y} + \frac{r_{XY}\,\sigma_Y}{\sigma_X}(X - \bar{x})\right]\right.$

$$= (\bar{y} - \bar{y}) - \frac{r_{XY}\,\sigma_Y}{\sigma_X}(\bar{x} - \bar{x})$$

$$= 0$$

$$\sigma^2_{(Y - Y_E)} = E\{(Y - Y_E)^2\} - E^2(Y - Y_E)$$

$$= E\left[Y - \left\{\bar{y} + \frac{r_{XY}\,\sigma_Y}{\sigma_X}(X - \bar{x})\right\}\right]^2$$

$$= E\left[(Y - \bar{y})^2 + \frac{r^2_{XY}\,\sigma^2_Y}{\sigma^2_X}(X - \bar{x})^2 - \frac{2r_{XY}\,\sigma_Y}{\sigma_X}(X - \bar{x})(Y - \bar{y})\right]$$

(i.e.) $$S^2{}_Y = \sigma^2_Y + \frac{r^2_{XY}\,\sigma^2_Y}{\sigma^2_X}\sigma^2_X - \frac{2r_{XY}\,\sigma_Y}{\sigma_X}\,\text{Cov}\,(X, Y)$$

$$= \sigma^2_Y + r^2{}_{XY} \cdot \sigma^2_Y - 2r^2_{XY}\,\sigma^2_Y$$

$[\because \text{Cov}\,(X, Y) = r_{XY}\,\sigma_X\,\sigma_Y]$

$$= (1 - r^2_{XY})\,\sigma^2{}_Y \text{ or } S_Y = \sqrt{1 - r^2_{XY}}\,\sigma_Y \tag{1}$$

Similarly, the standard error of estimate of X, denoted by S_X is given by

$$S^2_X = (1 - r^2_{XY})\,\sigma^2{}_X \text{ or } S_X = \sqrt{1 - r^2_{XY}}\,\sigma_X \tag{2}$$

Note *We may use (1) or (2) to prove that* $|r_{XY}| \le 1$.

From (1), $S_Y = \sqrt{1 - r^2_{XY}}\,\sigma_Y$

Since S_Y *and* σ_Y *are positive,* $1 - r^2_{XY} \ge 0$

$\therefore$ $r^2_{XY} \le 1$

i.e., $|r_{XY}| \le 1$ *or* $-1 \le r_{XY} \le 1]$

Worked Examples 4(C)

Example 1 *Obtain the equations of the lines of regression from the following data:*

X:	1	2	3	4	5	6	7
Y:	9	8	10	12	11	13	14

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Solution

X	Y	$U = X - 4$	$V = Y - 11$	U^2	V^2	UV
1	9	−3	−2	9	4	6
2	8	−2	−3	4	9	6
3	10	−1	−1	1	1	1
4	12	0	1	0	1	0
5	11	1	0	1	0	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
	Total	0	0	28	28	26

$$\bar{x} = E(X) = 4 + \frac{1}{n}\Sigma u = 4$$

$$\bar{y} = E(Y) = 11 + \frac{1}{n}\Sigma v = 11$$

$$\sigma^2_X = \frac{1}{n}\Sigma u^2 - \left(\frac{1}{n}\Sigma u\right)^2 = \frac{1}{7} \times 28 = 4$$

$$\sigma^2_Y = \frac{1}{n}\Sigma v^2 - \left(\frac{1}{n}\Sigma v\right)^2 = \frac{1}{7} \times 28 = 4$$

$$C_{XY} = \frac{1}{n}\Sigma uv - \left(\frac{1}{n}\Sigma u\right)\cdot\left(\frac{1}{n}\Sigma v\right) = \frac{1}{7} \times 26 = 3.7$$

The regression line of Y on X is

$$y - \bar{y} = \frac{p_{XY}}{\sigma^2_X}(x - \bar{x})$$

i.e., $$y - 11 = \frac{3.7}{4}(x - 4)$$

i.e., $3.7x - 4y + 29.2 = 0$

The regression line of X on Y is

$$x - \bar{x} = \frac{p_{XY}}{\sigma^2_X}(y - \bar{y})$$

i.e., $$x - 4 = \frac{3.7}{4}(y - 11)$$

i.e., $4x - 3.7y + 24.7 = 0$

Example 2 *Obtain the equations of the regression lines from the following data, using the method of least squares. Hence find the coefficient of correlation between X and Y. Also estimate the value of (i) Y, when X = 38 and (ii) X, when Y = 18.*

Solution

X:	22	26	29	30	31	31	34	35
Y:	20	20	21	29	27	24	27	31

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Put $U = X - 29$ and $V = Y - 27$.

Let the equation of the regression line of Y on X be $y = Ax + B$ or equivalently

$$v = au + b \tag{1}$$

The normal equations for finding a and b are

$$a\,\Sigma u + nb = \Sigma v \tag{2}$$

and

$$a\,\Sigma u^2 + b\Sigma u = \Sigma uv \tag{3}$$

x	y	$u = x - 29$	$v = y - 27$	u^2	v^2	uv
22	20	– 7	– 7	49	49	49
26	20	– 3	– 7	9	49	21
29	21	0	– 6	0	36	0
30	29	1	2	1	4	2
31	27	2	0	4	0	0
31	24	2	– 3	4	9	– 6
34	27	5	0	25	0	0
35	31	6	4	36	16	24
	Total	6	– 17	128	163	90

Using the relevant values from the table in (2) and (3), we have

$$6a + 8b = -17 \tag{2$'$}$$

$$128a + 6b = 90 \tag{3$'$}$$

Solving (2)′ and (3)′, we get

$$a = 0.83;\ b = -2.75$$

Hence the regression line of Y on X is

$$y - 27 = 0.83\,(x - 29) - 2.75$$

i.e.,

$$y = 0.83x + 0.18 \tag{4}$$

Let the equation of the regression line of X on Y be $x = Cy + D$ or equivalently

$$u = cv + d \tag{5}$$

The normal equations for finding c and d are

$$c\Sigma v + nd = \Sigma u \tag{6}$$

and

$$c\Sigma v^2 + d\Sigma v = \Sigma uv \tag{7}$$

Using the relevant values from the table in (6) and (7), we have

$$-17c + 8d = 6 \tag{6$'$}$$

$$163c - 17d = 90 \tag{7$'$}$$

Solving (6)′ and (7)′, we get

$$c = 0.81;\ d = 2.47$$

Hence the regression line of X on Y is

$$x - 29 = 0.81(y - 27) + 2.47$$

i.e.,

$$x = 0.81y + 9.60 \tag{8}$$

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Solution If μ is the common mean, the point (μ, μ) lies on $y = ax + b$ and $x = cy + d$

[$\because$ They intersect at $(\bar{x}, \bar{y})$]

$$\therefore \quad \mu = a\mu + b \tag{1}$$

$$\mu = c\mu + d \tag{2}$$

From (1), $$\mu = \frac{b}{1-a}$$

From (2), $$\mu = \frac{d}{1-c}$$

$$\therefore \quad \frac{b}{1-a} = \frac{d}{1-c}$$

$$\therefore \quad \frac{b}{d} = \frac{1-a}{1-c}$$

Now $$\frac{\sigma_Y^2}{\sigma_X^2} = \frac{b_{YX}}{b_{XY}} = \frac{a}{c};$$

$$\therefore \quad \frac{\sigma_Y}{\sigma_X} = \sqrt{\frac{a}{c}}$$

Example 8 *Find the standard error of estimate of Y on X and of X on Y from the following data:*

X:	1	2	3	4	5
Y:	2	5	9	13	14

Solution

x	y	x^2	y^2	xy
1	2	1	4	2
2	5	4	25	10
3	9	9	81	27
4	13	16	169	52
5	14	25	196	70
15	43	55	475	161

$$r_{XY} = \frac{n\Sigma xy - \Sigma x \cdot \Sigma y}{\sqrt{\left\{n\Sigma x^2 - (\Sigma x)^2\right\}\left\{n\Sigma y^2 - (\Sigma y)^2\right\}}}$$

$$= \frac{5 \times 161 - 15 \times 43}{\sqrt{\left\{5 \times 55 - (15)^2\right\}\left\{5 \times 475 - (43)^2\right\}}}$$

$$= \frac{160}{\sqrt{50 \times 526}} = 0.9866$$

$$\sigma^2{}_X = \frac{1}{n}\Sigma x^2 - \left(\frac{1}{n}\Sigma x\right)^2$$

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5. If the characteristic function of a continuous RV X with density function $f(x)$ is $\phi(\omega)$, then $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\omega)\, e^{-ix\omega}\, d\omega$.

$$\phi(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

RS can be identified as the Fourier transform of $f(x)$.
Therefore, by Fourier inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\omega)\, e^{-i\omega x}\, d\omega$$

6. If the density function of X is known, the density function of $Y = g(X)$ can be found from the CF of Y, provided $Y = g(X)$ is one-to-one.

Proof Let the density function of X be $f_X(x)$.

Then $$\phi_{g(X)}(\omega) = \int_{-\infty}^{\infty} e^{ig(x)\omega} f_X(x)\, dx$$

Put $$g(X) = Y \text{ and hence } g(x) = y$$

Then $$\phi_Y(\omega) = \int_{-\infty}^{\infty} e^{iy\omega} h(y)\, dy, \text{ say}$$

Therefore, $h(y)$ is the density function of Y.

Cumulant Generating Function (CGF)

If $M(t)$ is the MGF of a RV X, then $\log_e M(t)$ is called the cumulant generating function of X and denoted by $K(t)$.

The coefficient of $\frac{t^r}{r!}$ in the expansion of $K(t)$ in ascending powers of t is called the *rth order cumulant of X* and denoted by λ_r.

i.e., $$K(t) = \sum_{r=1}^{\infty} \frac{\lambda_r t^r}{\lfloor r}$$

Also $$\lambda_r = \left\{ \frac{d^r}{dt^r} K(t) \right\}_{t=0}$$

If $\phi(\omega)$ is the characteristic function of a RV X, then $\log_e \phi(\omega)$ is called the *second characteristic function of X* and denoted by $\psi(\omega)$.

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Solution Hence find the mean and variance.

$$\phi(\omega) = \sum_{r=0}^{\infty} e^{i\omega r} pq^r$$

$$= p \sum_{r=0}^{\infty} (qe^{i\omega})^r = p(1 - qe^{i\omega})^{-1}$$

$$\phi^{(1)}(\omega) = p(1 - qe^{i\omega})^{-2}\, i\, q\, e^{i\omega}$$

$$\phi^{(2)}(\omega) = i^2 pq\, [2(1 - q\, e^{i\omega})^{-3}\, qe^{i2\omega} + (1 - qe^{i\omega})^{-2}\, e^{i\omega}]$$

$$\therefore \qquad E(X) = \frac{1}{i}\, \phi^1(0) = \frac{q}{p} \text{ and } E(X^2) = \frac{1}{i^2}\, \phi^{(2)}(0) = \frac{q}{p^2}\, (1 + q)$$

$$\therefore \qquad \mu_X = \frac{q}{p} \text{ and } \sigma^2{}_X = \frac{q}{p^2}$$

Example 6 *Obtain the characteristic function of the normal distribution. Deduce the first four central moments.* (MKU — Apr. 96)

Solution Let X follow $N(\mu, \sigma)$

Then $Z = \dfrac{X - \mu}{\sigma}$ follows $N(0, 1)$, i.e., the standard normal distribution whose density function is $\phi(z) = \dfrac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$.

Now
$$\phi_Z(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, e^{i\omega z}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - i\omega)^2 + \frac{i^2\omega^2}{2}}\, dz$$

$$= e^{-\omega^2/2}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \cdot \sqrt{2}\, dt, \text{ on putting } \frac{z - i\omega}{\sqrt{2}} = t$$

$$= e^{-\omega^2/2} \left[\text{since} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right]$$

$$\therefore \qquad \phi_X(\omega) = \phi_{\sigma Z + \mu}(\omega)$$

$$= e^{i\mu\omega}\, \phi_Z(\sigma\omega) \quad \text{(by Property 3)}$$

$$= e^{i\mu\omega}\, e^{-\sigma^2\omega^2/2}$$

Now
$$\phi_X(\omega) = e^{i\omega(\mu + i\sigma^2\, \omega/2)}$$

$$= 1 + \frac{1}{\lfloor 1}\, i\omega\, (\mu + i\, \sigma^2\, \omega/2) + \frac{1}{\lfloor 2}\, i^2\, \omega^2\, (\mu + i\, \sigma^2\, \omega/2)^2 + \ldots$$

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$$= 1 + \frac{i\omega\mu}{\lfloor 1} + \frac{i^2\,\omega^2}{\lfloor 2}(\sigma^2 + \mu^2) + \frac{i^3\,\omega^3}{\lfloor 3}\,(3\,\mu\,\sigma^2 + \mu^3)$$

$$+ \frac{i^4\,\omega^4}{\lfloor 4}\,(3\,\sigma^4 + 6\,\mu^2\,\sigma^2 + \mu^4) + \ldots \qquad (1)$$

$$\therefore \qquad \mu'_1 = E(X) = \text{coefficient of } \frac{i\,\omega}{\lfloor 1} \text{ in } (1) = \mu$$

Similarly, $E(X^2) = \sigma^2 + \mu^2$;

$E(X^3) = 3\mu\sigma^2 + \mu^3$ and,

$E(X^4) = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$.

Using the relation $\mu_k = E\{(X - \mu)^k\}$, we get

$$\mu_1 = 0,\ \mu_2 = \sigma^2,\ \mu_3 = 0 \text{ and } \mu_4 = 3\sigma^4$$

Example 7 *Find the characteristic function of the Erlang distribution given by* $f(x) = \dfrac{\lambda^n}{\lfloor n-1}\, x^{n-1}\, e^{-\lambda x}\, U(x)$ *and hence find its mean and variance.*

Solution $$\phi(\omega) = \frac{\lambda^n}{\lfloor n-1} \int_0^\infty x^{n-1}\, e^{-(\lambda - i\omega)\,x}\, dx$$

$$= \frac{\lambda^n}{\lfloor n-1\,(\lambda - i\omega)^n} \int_0^\infty t^{n-1}\, e^{-t}\, dt, \text{ on putting } (\lambda - i\omega)\,x = t$$

$$= \left(\frac{\lambda}{\lambda - i\omega}\right)^n \frac{1}{\lfloor n-1}\,\Gamma(n) = \frac{1}{\left(1 - \dfrac{i\omega}{\lambda}\right)^n}$$

[since $\Gamma n = \lfloor n-1$ when n is a positive integer]

Now $$\phi(\omega) = \left(1 - \frac{i\omega}{\lambda}\right)^{-n}$$

$$= 1 + \frac{n}{\lfloor 1}\,\frac{i\omega}{\lambda} + \frac{n(n+1)}{\lfloor 2}\,\frac{i^2\,\omega^2}{\lambda^2} + \cdots + \infty$$

$$\therefore \qquad E(X) = \frac{n}{\lambda} \text{ and } E(X^2) = \frac{n(n+1)}{\lambda^2}$$

$$\therefore \qquad \text{Var}(X) = \frac{n}{\lambda^2}$$

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7. State the uniqueness theorem of characteristic functions.
8. If $Y = aX + b$, find the relation between the characteristic functions of X and Y.
9. If X and Y are 2 independent RVs prove that $\phi_{X+Y}(\omega) = \phi_X \times \phi_Y(\omega)$.
10. If the characteristic function of a continuous RV X is $\phi(\omega)$, express its density function $f(x)$ in terms of $\phi(\omega)$.
11. Define the cumulant generating function/the second characteristic function of a RV X and what is its use?
12. If the rth moment of a continuous RV X about the origin is $r!$, find the MGF of X.
13. If the MGF of a RV X is $\dfrac{2}{2-t}$ find the SD of X.
14. Find the MGF/CF of a uniform distribution in (a, b).
15. Find the MGF/CF of the binomial distribution.
16. Find the MGF/CF of the Poisson distribution.
17. State and prove the reproductive property of the Poisson distribution.
18. Find the CF/MGF of the geometric distribution.
19. Find the CF/MGF of the exponential distribution.
20. If the CF of the standard normal distribution $N(0, 1)$ is $e^{-\omega^2/2}$, find the CF of the general normal distribution $N(\mu, \sigma)$.
21. Find the CF of X whose pdf is given by $f(x) = \dfrac{1}{2} e^{-|x|}, -\infty < x < \infty$.

Part B

22. Find the MGF of a RV which is uniformly distributed over $(-1, 2)$ and hence find its mean and variance.
23. Find the characteristic function of the binomial distribution and hence find its mean and variance.
24. Obtain the MGF of the Poisson distribution; deduce the values of the first four central moments. (MKU — Apr. 96)
25. Find the characteristic function of the negative binomial distribution given by $P(X = r) = (n + r - 1)\, C_r\, q^r\, p^n$, $(r = 0, 1, 2, ..., \infty)$, $p + q = 1$, and hence find its mean and variance.
26. Find the MGF of the two-parameter exponential distribution whose density function is given by $f(x) = \lambda\, e^{-\lambda(x-a)}$, $x \geq a$. Hence find its mean and variance.
27. If the desnity function of a continuous RV X is given by $f(x) = \dfrac{1}{2} e^{-|x|}$, $-\infty < x < \infty$, find the MGF of X. Hence find its mean and variance.
28. Find the characteristic function of the Cauchy's distribution given by $f(x) = \dfrac{1}{\pi} \times \dfrac{1}{1+x^2} - \infty < x < \infty$. Comment about the first two moments.

    (**Hint**: Use contour integration.)

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By Tchebycheff's inequality

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\therefore \qquad P\{|X-\mu| \geq 3/2\ \sigma\} \leq \frac{4}{9} = 0.444,$$

which is a poor upper bound.

Example 3 *Can we find a RV X for which $P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = 0.6$?*

Solution $P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = P\{|X-\mu| \leq 2\sigma\}$

$$\geq 1 - \frac{1}{2^2} \text{ by Tchebycheff's inequality}$$

i.e., $$P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} \geq 0.75$$

Therefore, there does not exist a RV X satistying the given condition.

Example 4 *A discrete RV X takes the values – 1, 0, 1 with probabilities $\frac{1}{8}$, $\frac{3}{4}$, $\frac{1}{8}$ respectively. Evaluate $P\{|X-\mu| \geq 2\sigma\}$ and compare it with the upper bound given by Tchebycheff's inequality.*

Solution $E(X) = -1 \times \frac{1}{8} + 0 \times \frac{3}{4} + 1 \times \frac{1}{8} = 0$

$$E(X^2) = 1 \times \frac{1}{8} + 0 \times \frac{3}{4} + 1 \times \frac{1}{8} = \frac{1}{4};\ \text{Var}(X) = \frac{1}{4}$$

$$\therefore \qquad P\{|X-\mu| \geq 2\sigma\} = P\{|X| \geq 1\}$$

$$= P\{X = -1 \text{ or } X = 1\}$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \tag{1}$$

By Tchebycheff's inequality

$$P\{|X-\mu| \geq 2\sigma\} \leq \frac{1}{2^2} = \frac{1}{4} \tag{2}$$

The two values coincide.

Example 5 *If X denotes the sum of the numbers obtained when 2 dice are thrown, obtain an upper bound for $P\{|X-7| \geq 4\}$. Compare with the exact probability.*

Solution Let X_1, X_2 denote the outcomes of the first and second dice respectively.

$$E(X_1) = E(X_2) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{7}{2}$$

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17. A fair die is tossed 600 times. Use Tchebycheff's inequality to find a lower bound for the probability of getting 80 to 120 sixes.
18. An unbiased coin is tossed 100 times. Show that the probability that the number of heads will be between 30 and 70 is greater than 0.93.
19. If X is the number obtained in a throw of a fair die, show that the Tcheycheff's inequality gives $P\{|X - \mu| > 2.5\} < 0.47$, while the actual probability is zero.
20. Using Tchebycheff's inequality, find how many times a fair coin must be tossed in order that the probability that the ratio of the number of heads to the number of tosses will lie between 0.4 and 0.6 will be at least 0.9.
21. A random variable X has pdf $f(x) = e^{-x}, x \geq 0$. Use Tchebycheff's inequality to show that $P\{|X - 1| > 1\} < 1/4$ and show also that the actual probability is e^{-3}.
22. A random variable X has the pmf $P(X = 1) = 1/18$, $P(X = 2) = 16/18$, $P(X = 3) = 1/18$. Show that there is a value of c such that $P\{|X - \mu| \geq c\} = \dfrac{\sigma^2}{c^2}$, so that, in general, the bound given by Tchebycheff's inequality cannot be improved.

Convergence Concepts and Central Limit Theorem

Let us consider the sequence of RVs $X_1, X_2, X_3, \cdots X_n, \cdots$ or random sequence. The concept of convergence of random sequences is essential in the study of random signals. A few definitions and criteria that are used for determining the convergence of random sequences are given below.

(1) Convergence everywhere and almost everywhere

If $\{X_n\}$ is a sequence of RVs and X is a RV such that $\left[\lim_{n\to\infty}(X_n) = X\right]$ i.e., $X_n \to X$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X everywhere.

If $P\{X_n \to X\} = 1$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X almost everywhere.

(2) Convergence in probability or stochastic convergence

If $P\{|X_n - X| > \varepsilon\} \to 0$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X in probability or stochastically.

As a particular case of this kind of convergence we have the following result, known as *Bernoulli's law of large numbers*.

If X represents the number of successes out of n Bernoulli's trials with probability of success p (in each trial), then $\{X/n\}$ converges in probability to p.

i.e.,
$$P\left\{\left|\frac{X}{n} - p\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

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$$\therefore \qquad \lim_{n \to \infty} \phi_{yn}(\omega) = e^{\frac{-1}{2}\omega^2}$$

which is the characteristic function of $N(0, 1)$.

$$\therefore \qquad Y_n \to N(0, 1), \text{ as } n \to \infty$$

Therefore, CLT holds good for the sequence $\{X_k\}$.

Example 6 *Show that the central limit theorem holds good for the sequence $\{X_k\}$, if*

$$P\{X_k = \pm k^{\alpha}\} = \frac{1}{2} \times k^{-2\alpha},\ P\{X_k = 0\} = 1 - k^{-2\alpha},\ \alpha < \frac{1}{2}.$$ (BDU — Nov. 96)

Note *Liapounoff's form of CLT holds good for a sequence $\{X_k\}$, if*

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} E\left\{|X_k - \mu_k|^3\right\}}{\left\{\sum_{k=1}^{n} Var(X_k)\right\}^{\frac{3}{2}}} = 0 \quad \textit{Condition is assumed.}$$

Solution We have to verify whether this condition is satisfied by the given $\{X_k\}$.

Now $$\mu_k = E(X_k) = \frac{1}{2} \times k^{-\alpha} - \frac{1}{2} \times k^{-\alpha} = 0$$

$$E(X_k^2) = \frac{1}{2} \times k^{2\alpha} \times k^{-2\alpha} + \frac{1}{2} k^{2\alpha} \times k^{-2\alpha} = 1$$

$$\therefore \qquad \text{Var}(X_k) = 1$$

$$E\{|X_k - \mu_k|^3 = E\{|X_k|^3\}$$

$$= k^{3\alpha} \times \frac{1}{2} k^{-2\alpha} + k^{3\alpha} \times \frac{1}{2} k^{-2\alpha}$$

$$= k^{\alpha}$$

$$\therefore \qquad \sum_{k=1}^{n} E\{|X_k - \mu_k|^3\} = 1^{\alpha} + 2^{\alpha} + \cdots + n^{\alpha} < n \times n^{\alpha} \qquad \text{(since each term} \le n\alpha)$$

$$\sum_{k=1}^{n} \text{Var}(X_k) = n$$

$$\therefore \qquad \lim_{n \to \infty} \frac{\sum_{k=1}^{n} E\left\{|X_k - \mu_k|^3\right\}}{\left\{\sum_{k=1}^{n} \text{Var}(X_k)\right\}^{\frac{3}{2}}}$$

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19. $\bar{x} = \dfrac{a+bc}{1-bd}; \bar{y} = \dfrac{c+ad}{1-bd}; r = \sqrt{bd}$; No.

20. $\sqrt{\dfrac{a_1b_2}{a_2b_1}}; \left(\dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}\right)\sqrt{\dfrac{a_1a_2}{b_1b_2}}$

Exercise 4(D)

5. $M(t) = E(e^{tX})$, whereas $\phi(\omega) = E(e^{i}\omega^{X})$
7. The characteristic function of a RV uniquely determines its pdf. (or) The necessary and sufficient condition for two distributions with pdf's $f_1(x)$ and $f_2(x)$ to be identical is that their characteristic functions $\phi_1(\omega)$ and $\phi_2(\omega)$ are identical.
11. It generates cumulants.

12. $M(t) = \sum_{r=0}^{\infty} E(X^r)\, t^r/r! = \sum_{r=0}^{\infty} t^r = \dfrac{1}{1-t}$

13. $\dfrac{2}{2-t} = \left(1 - \dfrac{t}{2}\right)^{-1} = 1 + \dfrac{t}{2} + \dfrac{t^2}{4} + \cdots$

$E(X) = \dfrac{1}{2}; E(X^2) = \dfrac{1}{2}; \text{Var}(X) = \dfrac{1}{4}; \sigma_x = \dfrac{1}{2}$

14. $M(t) = E(e^{tX}) = \int_a^b \dfrac{1}{b-a} e^{tx}\, dx$

$$= \frac{1}{b-a}\left(\frac{e^{bt} - e^{at}}{t}\right)$$

$$\phi(\omega) = \frac{1}{b-a}\left(\frac{e^{ib\omega} - e^{ia\omega}}{i\omega}\right)$$

15. $M(t) = \sum_{r=0}^{n} nC_r\, p^r q^{n-r} e^{tr} = (q + pe^t)^n$

$\phi(\omega) = (q + pe^{i\omega})^n$

16. $M(t) = \sum_{r=0}^{\infty} \dfrac{e^{-\lambda}\lambda^r}{r!} e^{tr} = e^{-\lambda(1-e^t)}$

$\phi(\omega) = e^{-\lambda(1-e^{i\omega})}$

18. $\text{CF} = \dfrac{p}{1 - qe^{i\omega}}$

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(2) *The name 'binomial distribution' is given since the probabilities* $nC_r\, q^{n-r}\, p^r$ $(r = 0, 1, 2, \ldots, n)$ *are the successive terms in the expansion of the binomial expression* $(q + p)^n$.

(3) *If we assume that n trials constitute a set and if we consider N sets, the frequency function of the binomial distribution is given by* $f(r) = N\, p(r) = N \cdot nC_r\, q^{n-r}\, p^r$, $r = 0, 1, 2, \ldots, n$. *In other words, the number of sets in which we get exactly r successes (the occurrences of the event A)* $= N \cdot nC_r\, q^{n-r}\, p^r$; $r = 0, 1, 2, \ldots, n$.

Mean and Variance of the Binomial Distribution

We have already found out $E(X)$ and $\text{Var}(X)$ for the binomial distribution $B(n, p)$ using the moment generating function in Example 3 in Worked Example 4 (d). Here we shall find them directly using the definitions of $E(X)$ and $\text{Var}(X)$.

$$E(X) = \sum_r x_r p_r$$

$$= \sum_{r=0}^{n} r \cdot nC_r\, p^r\, q^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{n!}{r!(n-r)!}\, p^r\, q^{n-r} \tag{1}$$

$$= np \cdot \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\{(n-1)-(r-1)\}!}\, p^{r-1}\, q^{(n-1)-(r-1)}$$

$$= np \sum_{r=1}^{n} (n-1)\, C_{r-1} \cdot p^{r-1} \cdot q^{(n-1)-(r-1)}$$

$$= np\,(q+p)^{n-1}$$

$$= np \tag{2}$$

$$E(X^2) = \sum_r x_r^2\, p_r = \sum_0^n r^2\, p_r$$

$$= \sum_{r=0}^{n} \{r(r-1) + r\}\, \frac{n!}{r!(n-r)!}\, p^r\, q^{n-r}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} (n-2)C_{r-2}\, p^{r-2}\, q^{n-r} + np, \quad \text{[by (1) and(2)]}$$

$$= n(n-1)\, p^2\, (q+p)^{n-2} + np$$

$$= n(n-1)\, p^2 + np$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= n(n-1)\, p^2 + np - n^2p^2$$

$$= np\,(1-p)$$

$$= npq$$

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Using recurrence relation (2), we may compute moments of higher order, provided we know moment of lower order.

Putting $k = 1$ in (2), we get

$$\mu_2 = \lambda\left(\frac{d\mu_1}{d\lambda} + \mu_0\right)$$

$$= \lambda \ (\because \mu_0 = 1 \text{ and } \mu_1 = 0)$$

Putting $k = 2$ in (2), we get

$$\mu_3 = \lambda\left(\frac{d\mu_2}{d\lambda} + 2\mu_1\right) = \lambda.$$

Putting $k = 3$ in (2), we get

$$\mu_4 = \lambda\left(\frac{d\mu_3}{d\lambda} + 3\mu_2\right) = \lambda(3\lambda + 1)$$

Note *The interesting property of the Poisson distribution is the equality of its mean, variance and third-order central moment.*

3. Geometric distribution

Definition: Let the RV X denote the number of trials of a random experiment required to obtain the first success (occurrence of an event A). Obviously X can assume the values 1, 2, 3, ...

Now $X = r$, if and only if the first $(r - 1)$ trials result in failure (occurrence of $\overline{A}$) and the rth trial results in success (occurrence of A). Hence

$$P(X = r) = q^{r-1}p; \quad r = 1, 2, 3, \ldots, \infty$$

where

$$P(A) = p \text{ and } P\left(\overline{A}\right) = q.$$

If X is a discrete RV that can assume the values 1, 2, 3, ..., ∞ such that its probability mass function is given by

$$P(X = r) = q^{r-1}\, p; \ r = 1, 2, \ldots, \infty \ \text{ where } \ p + q = 1$$

then X is said to follow *a geometric distribution*.

Note *Geometric distribution is a legitimate probability distribution, since*

$$\sum_{r=1}^{\infty} P(X = r) = \sum_{r=1}^{\infty} q^{r-1}p$$

$$= p(1 + q + q^2 + \ldots \infty)$$

$$= \frac{p}{1-q} = 1$$

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Worked Examples 5(A)

Example 1 *Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls, (ii) at least 1 boy, (iii) at most 2 girls and (iv) children of both sexes. Assume equal probabilities for boys and girls.*

Considering each child as a trial, n = 4. Assuming that birth of a boy is a success, $p = \frac{1}{2}$ *and* $q = \frac{1}{2}$. *Let X denote the number of successes (boys).*

Solution (i) $P(2 \text{ boys and } 2 \text{ girls}) = P(X = 2)$

$$= 4C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{4-2}$$

$$= 6 \times \left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

$\therefore$ No. of families having 2 boys and 2 girls

$= N \cdot (P(X = 2)$ (where N is the total no. of families considered)

$$= 800 \times \frac{3}{8}$$

$$= 300.$$

(ii) $P(\text{at least 1 boy}) = P(X \geq 1)$

$$= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= 1 - P(X = 0)$$

$$= 1 - 4C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

$\therefore$ No. of families having at least 1 boy

$$= 800 \times \frac{15}{16} = 750.$$

(iii) $P(\text{at most 2 girls}) = P(\text{exactly 0 girl, 1 girl or 2 girls})$

$$= P(X = 4, X = 3 \text{ or } X = 2)$$

$$= 1 - \{P(X = 0) + P(X = 1)\}$$

$$= 1 - \left\{ 4C_0 \cdot \left(\frac{1}{2}\right)^4 + 4C_1 \cdot \left(\frac{1}{2}\right)^4 \right\}$$

$$= \frac{11}{16}$$

$\therefore$ No. of families having at most 2 girls

$$= 800 \times \frac{11}{16} = 550.$$

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x:	0	1	2	3	4	5	6	Total
f:	4	15	25	22	11	3	0	80

Example 8 *The number of monthly breakdowns of a computer is a RV having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month*

(a) without a breakdown,
(b) with only one breakdown and
(c) with atleast one breakdown.

Solution Let X denote the number of breakdowns of the computer in a month. X follows a Poisson distribution with mean (parameter) $\lambda = 1.8$.

$$\therefore \quad P\{X = r\} = \frac{e^{-\lambda}.\lambda^r}{r!} = \frac{e^{-1.8}.(1.8)^r}{r!}$$

(a) $P(X = 0) = e^{-1.8} = 0.1653$
(b) $P(X = 1) = e^{-1.8}\,(1.8) = 0.2975$
(c) $P(X \geq 1) = 1 - P(X = 0) = 0.8347$

Example 9 *It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the number of packets containing at least, exactly and at most 2 defective items in a consignment of 1000 packets using (i) binomial distribution and (ii) Poisson approximation to binomial distribution.*

Solution ***Use of binomial distribution***

p = probability that an item is defective = 0.05, $q = 0.95$ and n = No. of independent items (trials) considered = 20.

Let X denote the number of defectives in the n items considered.

$$P(X = r) = nC_r\, p^r\, q^{n-r}$$

(i) $\therefore \quad P(X = 2) = 20\,C_2\,(.05)^2\,(0.95)^{18}$
$= 0.1887$

If N is the number of sets (packets), each set (packet) containing 20 trials (items), then the number of sets containing exactly 2 successes (defectives) is given by

$$N(X = 2) = N \times P(X = 2)$$
$$= 1000 \times 0.1887 = 189, \text{ nearly}$$

(ii) $P(\text{at least 2 defectives}) = P(X \geq 2)$
$= 1 - \{P(X = 0) + P(X = 1)\}$
$= 1 - [20C_0\,(0.05)^0\,(0.95)^{20} + 20C_1\,(0.05)^1\,(0.95)^{19}]$
$= 1 - [0.3585 + 0.3774]$
$= 0.2641$

$\therefore \quad N(X \geq 2) = N \times P\,(X \geq 2)$
$= 1000 \times 0.2641 = 264$, nearly

(iii) $P(\text{at most 2 defectives}) = P(X \leq 2)$
$= P(X = 0) + P(X = 1) + P(X = 2)$

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Solution Let X denote the number of trials required by A to get his first success. Then X follows a geometric distribution given by

$$P(X = r) = p_1 q_1^{r-1} = \frac{3}{5} \cdot \left(\frac{2}{5}\right)^{r-1}; \quad r = 1, 2, ..., \infty$$

Let Y denote the number of trials required by B to get his first success. Then Y follows a geometric distribution given by

$$P(Y = r) = p_2 \cdot q_2^{r-1} = \frac{5}{7} \cdot \left(\frac{2}{7}\right)^{r-1}; \quad r = 1, 2, \ldots, \infty$$

$P\{B$ requires more trials to get his first success than A requires to get his first success$\}$

$$= \sum_{r=1}^{\infty} P\{X = r \text{ and } Y = r + 1 \text{ or } r + 2, \ldots, \infty\}$$

$$= \sum_{r=1}^{\infty} [P\{X = r\} \cdot P\{Y = r + 1 \text{ or } r + 2, \ldots, \infty\}]$$

(by independence)

$$= \sum_{r=1}^{\infty} \frac{3}{5} \cdot \left(\frac{2}{5}\right)^{r-1} \cdot \sum_{k=1}^{\infty} \frac{5}{7} \cdot \left(\frac{2}{7}\right)^{r+k-1}$$

$$= \sum_{r=1}^{\infty} \frac{3}{7} \cdot \left(\frac{4}{35}\right)^{r-1} \sum_{k=1}^{\infty} \left(\frac{2}{7}\right)^{k}$$

$$= \sum_{r=1}^{\infty} \frac{3}{7} \cdot \left(\frac{4}{35}\right)^{r-1} \cdot \left(\frac{\frac{2}{7}}{1 - \frac{2}{7}}\right)$$

$$= \frac{6}{35} \sum_{r=1}^{\infty} \left(\frac{4}{35}\right)^{r-1} = \frac{6}{35} \cdot \frac{1}{1 - \frac{4}{35}} = \frac{6}{31}.$$

Example 16 *A coin is tossed until the first head occurs. Assuming that the tosses are independent and the probability of a head occurring is p, find the value of p so that the probability that an odd number of tosses is required is equal to 0.6.*

Can you find a value of p so that the probability is 0.5 that an odd number of tosses is required?

Solution Let X denote the number of tosses required to get the first head (success). Then X follows a geometric distribution given by

$$P(X = r) = pq^{r-1}; \quad r = 1, 2, \ldots,$$

$\therefore P(X = \text{an odd number}) = P(X = 1 \text{ or } 3 \text{ or } 5, \ldots)$

$$= \sum_{r=1}^{\infty} P(X = 2r - 1) = \sum_{r=1}^{\infty} p\, q^{2r-2}$$

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7. Show that the largest value of the variance of a binomial distribution is $\frac{n}{4}$.
8. Find the mean and SD of the distribution whose moment generating functions is $(0.4\, e^t + 0.6)$.
9. When will the sum of 2 binomial variates having distributions $B(n_1, p_1)$ and $B(n_2, p_2)$ be also a binomial variate?
10. If X follows $B\left(3, \frac{1}{3}\right)$ and Y follows $B\left(5, \frac{1}{3}\right)$, find $P(X + Y \geq 1)$.
11. Write down the pmf of the Poisson distribution which is approximately equivalent to $B(100, 0.02)$.
12. If X is a Poisson variate such that $2P(X = 0) + P(X = 2) = 2P(X = 1)$, find $E(X)$.
13. If X is a poisson variate such that $E(X^2) = 6$, find $E(X)$.
14. If X is a Poisson variate such that $P(X = 0) = 0.5$, find $\text{Var}(X)$.
15. If X is a Poisson variate with parameter $\lambda > 0$, prove that
$$E(X^2) = \lambda E(X + 1).$$
16. If X is a Poisson variate with parameter λ, prove that
$$E(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda}).$$
17. If the MGF of a discrete RV X is $e^{4(et-1)}$, find $P(X = \mu + \sigma)$, where μ and σ are the mean and SD of X.
18. If X and Y are independent identical Poisson variates with mean 1, find $P(X + Y = 2)$.
19. Find the mean and variance of the discrete probability distribution given by
$P(X = r) = e^{-t}(1 - e^{-t})^{r-1} \quad r = 1, 2, 3, ..., \infty.$
20. If X is a geometric variate, taking values $1, 2, 3, \ldots, \infty$ find $P(X \text{ is odd})$.
21. Find the mean and variance of the distribution given by
$$P(X = r) = \frac{2}{3^r}, r = 1, 2, ..., \infty.$$
22. For the geometric distribution of X, which represents the number of Bernoulli's trials required to get the first success $\text{Var}(X) = 2E(X)$. Find the pmf of the distribution.
23. Find the MGF of the geometric distribution, given by $P(X = r) = q^{r-1}\, p$, $r = 1, 2, \ldots, \infty$.
24. If the MGF of a discrete RV X, taking values $1, 2, \ldots, \infty$, is $e^t(5 - 4e^t)^{-1}$, find the mean and variance of X.
25. Define hypergeometric distribution and give an example for the situation where it arises.
26. Write down the mean and variance of the hypergeometric distribution given by
$$P(X = r) = kC_r(N - k)\, C_{n-r}/NC_n,\ r = 0, 1, 2, \ldots.$$

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64. Among the 120 applicants for a job, only 80 are actually qualified. If 5 of the applicants are randomly selected for an in-depth interview, find the probability that only 2 of them will be qualified for the job using (i) hypergeometric probability and (ii) binomial approximation to hypergeometric probability.
65. If an auditor selects 5 returns from among 15 returns of which 9 contain illegitimate deductions, what is the probability that a majority of the selected returns contains illegitimate deductions?
66. In sampling a lot of 100 items, the sampling plan calls for inspection of 20 pieces. Find the probability of accepting a lot with 5 defectives, if we allow 1 defective in the sample.
67. If X and Y are independent binomial random variables having respective parameters (n, p) and (m, p), prove that the conditional probability mass function of X, given that $X + Y = k$, is that of a hypergeometric RV. [**Hint:** The joint probability mass function of X and Y is that of a binomial distribution with parameters $(m + n, p)$.]

Special Continuous Distributions

1. Uniform or rectangular distribution

Definition: A continuous RV X is said to follow *a uniform* or *rectangular distribution* in any finite interval, if its probability density function is a constant in that interval.

If X follows a uniform distribution in $a < x < b$, then $f(x) = \dfrac{1}{b-a}$ in $a < x < b$, as explained below:

When X follows a uniform distribution in (a, b),

$$f(x) = k.$$

By the basic property of a probability density function,

$$\int_{R_X} f(x)\,dx = 1$$

$$\therefore \qquad \int_a^b k\,dx = 1$$

$$\therefore \qquad k = \frac{1}{b-a}$$

Thus for a uniform distribution in (a, b),

$$f(x) = \frac{1}{b-a}.$$

When X follows a uniform distribution in (a, b), it is symbolically written as: X follows $U(a, b)$.

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$$= \frac{\lambda^k}{\Gamma(k)} \cdot \frac{1}{(\lambda - t)^k} \int_0^\infty y^{k-1} e^{-y}\, dy, \quad \text{(on putting } \lambda - t = y\text{)}.$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^k \quad [\because \text{ the integral} = \Gamma(k)]$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-k}$$

Now $M_{(X_1 + X_2 + \ldots + X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$ (since $X_1, X_2, \ldots, X_n$ are independent)

[Refer to property (4) of MGF given in section 4(d) of chapter (4)]

$$= \left(1 - \frac{t}{\lambda}\right)^{-k_1} \left(1 - \frac{t}{\lambda}\right)^{-k_2} \ldots \left(1 - \frac{t}{\lambda}\right)^{-k_n}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-(k_1 + k_2 + \ldots + k_n)}$$

which is the MGF of an Erlang variable with parameters $(\lambda, k_1 + k_2 + \ldots + k_n)$. Hence the reproductive property.

Relation Between the Distribution Functions (cdf) of the Erlang Distribution With $\lambda = 1$ (or Simple Gamma Distribution) and (Poisson Distribution)

If X is a Poisson random variable with mean λ,

then $$P(X \le K) = \sum_{r=0}^{k} \frac{e^{-\lambda} \lambda^r}{r!} \tag{1}$$

Differentiating both sides with respect to λ, we get

$$\frac{d}{d\lambda} P(X \le k) = \sum_{r=0}^{k} \frac{1}{r!} \{e^{-\lambda} \cdot r\lambda^{r-1} - e^{-\lambda} \times \lambda^r\}$$

$$= e^{-\lambda} \cdot \sum_{r=0}^{k} \left[\frac{\lambda^{r-1}}{(r-1)!} - \frac{\lambda^r}{r!}\right]$$

$$= e^{-\lambda} \left[-1 + \left(1 - \frac{\lambda}{1!}\right) + \left(\frac{\lambda}{1!} - \frac{\lambda^2}{2!}\right) + \cdots + \left\{\frac{\lambda^{k-1}}{(k-1)!} - \frac{\lambda^k}{k!}\right\}\right]$$

$$= -\frac{e^{-\lambda} \lambda^k}{k!} \tag{2}$$

Integrating both sides of (2) with respect to λ from λ to ∞, we get

$$\left[\sum_{r=0}^{K} \frac{e^{-\lambda} \lambda^r}{r!}\right]^\infty = -\int_\lambda^\infty \frac{1}{k!} e^{-\lambda} \lambda^k\, d\lambda$$

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2. Median and mode of the normal distribution N(μ, σ)

For the normal distribution $N(\mu, \sigma)$, the median M is given by

$$\int_M^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{2}$$

i.e., $$\int_M^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx + \int_{\mu}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{2}$$

i.e., $$\int_M^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx + \frac{1}{2} = \frac{1}{2},$$

$$\left\{ \text{since } \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx = 1 \text{ and the normal curve is symmetrical about } x = \mu \right\}$$

i.e., $$\int_M^{\mu} f(x)\, dx = 0$$

$\therefore \quad M = \mu$

For the normal distribution $N(\mu, \sigma)$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$$

$$\therefore \quad \log f(x) = k - \frac{1}{2\sigma^2}(x-\mu)^2$$

Differentiating with respect to x,

$$\frac{f'(x)}{f(x)} = -\frac{1}{\sigma^2}(x-\mu)$$

i.e., $$f'(x) = -\frac{1}{\sigma^2}(x-\mu) f(x)$$

$$= 0, \quad x = \mu$$

$$f''(x) = -\frac{1}{\sigma^2}\{(x-\mu) f'(x) + f(x)\}$$

$$\therefore [f''(x)]_{x=\mu} = -\frac{1}{\sigma^2} f(\mu) < 0$$

Therefore, $f(x)$ is maximum at $x = \mu$. That is, Mode of the distribution $N(\mu, \sigma) = \mu$.

Note *For the normal distribution, mean, median and mode are equal.*

3. Central moments of the normal distribution N(μ, σ)

Central moments μ_r of N(μ, σ) are given by $\mu_r = E(x-\mu)^r$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^r e^{-(x-\mu)^2/2\sigma^2}\, dx$$

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Deductions:

1. Putting $a_1 = a_2 = 1$ and $a_3 = a_4 = \ldots = a_n = 0$, we get the following result, in particular:
   If X_1 is $N(\mu_1, \sigma_1)$ and X_2 is $N(\mu_2, \sigma_2)$, then $X_1 + X_2$ is $N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}$. Similarly, $X_1 - X_2$ is $N(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$.
2. Putting $a_1 = a_2 = \ldots = a_n = \dfrac{1}{n}$ and assuming that each X_i is $N(\mu, \sigma)$, then

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ has a normal distribution}$$

$$N\left\{\frac{1}{n}\sum_{i=1}^{n}\mu, \sqrt{\sum_{i=1}^{n}\frac{1}{n^2}\sigma^2}\right\}$$

i.e., $N\{\mu, \sigma/\sqrt{n}\}$

Thus if X_i $(i = 1, 2, \ldots, n)$ are independent and identically distributed normal variables with mean μ and standard deviation σ, then their mean $\bar{X}$ is $N\{\mu, \sigma/\sqrt{n}\}$.

8. Normal distribution as limiting form of binomial distribution

When n is very large and neither p nor q is very small, the standard normal distribution can be regarded as the limiting form of the standardised binomial distribution.

When X follows the binomial distribution $B(n, p)$, the standardised binomial variable Z is given by $Z = \dfrac{X - np}{\sqrt{npq}}$. As X varies from 0 to n with step size 1, Z varies from $\dfrac{-np}{\sqrt{npq}}$ to $\dfrac{np}{\sqrt{npq}}$ with step size $\dfrac{1}{\sqrt{npq}}$. When neither p nor q is very small and n is very large, Z varies from $-\infty$ to ∞ with infinitesimally small step size. Hence, in the limit, the distribution of Z may be expected to be a continuous distribution extending from $-\infty$ to ∞ and having mean 0 and standard deviation 1. In fact the limiting form of the distribution of Z is standard normal distribution as seen below:

If X follows $B(n, p)$, then the MGF of X is given by $M_X(t) = (q + p\,e^t)^n$.

If $\qquad Z = \dfrac{X - np}{\sqrt{npq}}$, then

$$M_Z(t) = M_{\frac{1}{\sqrt{npq}}X - \frac{np}{\sqrt{npq}}}(t) = e^{\frac{-npt}{\sqrt{npq}}}\{q + p\,e^{t/\sqrt{npq}}\}^n$$

$$\therefore \quad \log M_Z(t) = -\frac{npt}{\sqrt{npq}} + n\log\{q + p\,e^{t/\sqrt{npq}}\}$$

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$$= \iint_{-1<x-y<1} f(x, y)\, dx\, dy$$

$$= \frac{1}{4} \iint_{-1<x-y<1} dx\, dy = \frac{1}{4}. \text{ Area of } ODEBGFO$$

$$= \frac{1}{2} \text{ Area of trapezium } ODEB, \text{ by symmetry}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\left(\sqrt{2} + 2\sqrt{2}\right)$$

$$= \frac{3}{4}$$

$$\therefore \qquad \{PQ > 1\} = \frac{1}{4}$$

Example 5 *If X is a RV with a continuous distribution function F(x), prove that Y = F(X) has a uniform distribution in (0, 1). Further,*

$$\text{if} \qquad f(x) = \begin{cases} \frac{1}{2}(x-1) & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

find the range of Y corresponding to the range $1.1 \le X \le 2.9$.

Solution The distribution function of Y is given by

$$G_Y(y) = P(Y \le y)$$

$$= P\{F(X) \le y\}$$

$$= P\{X \le F^{-1}(y)\} \text{ [The inverse exists, as } F(x) \text{ is non-decreasing and continuous]}$$

$$= F[F^{-1}(y)] \quad [\because P\{X \le x\} = F(x)]$$

$$= y$$

Therefore, the density function of Y is given by

$$g_Y(y) = \frac{d}{dy}[G_Y(y)] = 1$$

Also the range of Y is $0 \le y \le 1$, since the range of $F(x)$ is (0, 1).

Therefore, Y follows a uniform distribution in (0, 1).

Note *The converse of this problem has been worked in Example (14) of Worked Example (3).*

$$\text{When} \qquad f(x) = \begin{cases} \frac{1}{2}(x-1) & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

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$$= be^{ab}\left[\ \frac{a}{b}\ e^{-ab} + \frac{1}{b^2}e^{-ab}\ \right]$$

$$= a + \frac{1}{b} \tag{2}$$

$$E(X^2) = \int_a^\infty\ bx^2\ e^{-b\ (x\ -\ a)}\ \mathrm{d}x$$

$$= be^{ab}\left[x^2\left(\frac{e^{-bx}}{-b}\right) - 2x\left(\frac{e^{-bx}}{-b^2}\right) + 2\left(\frac{e^{-bx}}{-b^3}\right)\right]_a^\infty$$

$$= b\left[\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3}\right]$$

$$= \frac{1}{b^2}\ (a^2b^2 + 2ab + 2)$$

$$\mathrm{Var}(X\ ) = E(X^2) - \{E(X)\}^2$$

$$= \frac{1}{b^2}\ (a^2b^2 + 2ab + 2) - \left(\ a^2 + \frac{2a}{b} + \frac{1}{b^2}\ \right)$$

i.e., $\sigma^2 = \dfrac{1}{b^2}$ or $\sigma = \dfrac{1}{b}$ (3)

From (1) and (3), we get $b = c = \dfrac{1}{\sigma}$.

From (2) and (3), $\mu - \sigma = a$.

Example 12 *In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a RV having an Erlang distribution with parameters* $\lambda = \dfrac{1}{2}$ *and* $k = 3$. *If the power plant of this city has a daily capacity of 12 millions kilowatt-hours, what is the probability that this power supply will be inadequate on any given day.*

Solution Let X represent the daily consumption of electric power (in millions of kilowatt-hours). Then the density function of X is given as

$$f\,(x) = \frac{\left(\frac{1}{2}\right)^3}{\Gamma(3)}\ x^2\ e^{-x/2},\ x > 0$$

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Solution The density function of X is given by

$$f(x) = 2\alpha x\, e^{-\alpha x^2},\ x > 0 \quad [\because \beta = 2]$$

Now $P(X > 5) = \int_5^\infty 2\alpha x\, e^{-\alpha x^2}\, dx$

$$= \left(-e^{-\alpha x^2}\right)_5^\infty$$

$$= e^{-25a}$$

Given that $P(X > 5) = e^{-0.25}$

$\therefore \quad e^{-25\alpha} = e^{-0.25}$

$\therefore \quad \alpha = \dfrac{1}{100}$

For the Weibull distribution with parameters α and β, $E(X) = \alpha^{-1/\beta}\,\Gamma\left(\dfrac{1}{\beta} + 1\right)$

$\therefore$ Required mean $= \left(\dfrac{1}{100}\right)^{-\frac{1}{2}} \cdot \Gamma\left(\dfrac{3}{2}\right)$

$$= 10 \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= 5\sqrt{\pi}.$$

$$\text{Var}(X) = a^{-\frac{2}{\beta}}\left[\Gamma\left(\frac{2}{\beta} + 1\right) - \left\{\Gamma\left(\frac{1}{\beta} + 1\right)\right\}^2\right]$$

$$= \left(\frac{1}{100}\right)^{-1}\left[\Gamma(2) - \left\{\Gamma\left(\frac{3}{2}\right)\right\}^2\right]$$

$$= 100\left[1 - \left(\frac{1}{2}\sqrt{\pi}\right)^2\right]$$

$$= 100\left(1 - \frac{\pi}{4}\right)$$

Example 19 *If the time T to failure of a component follows a Weibull distribution with parameters α and β, find the hazard rate or conditional failure rate at time t of the component.*

Solution Refer to Example (19) in Worked Example 2(A).

If $f(t)$ is the density function of T and $h(t)$ is the hazard rate at time t, then

$$h(t) = \frac{f(t)}{1 - F(t)}$$

where $F(t)$ is the distribution function of T.

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$= 0.3413 + 0.2486,$ (using the table of areas)
$= 0.5899$

$$P(25 < X \le 30) = P\left\{ \frac{25-33}{3} < Z < \frac{30-33}{3} \right\}$$
$= P(-2.67 < Z < -1)$
$= P(1 < Z < 2\ 67)$, (by symmetry)
$= P(0 < Z < 2.67) - P(0 < Z < 1)$
$= 0.4962 - 0.3413$
$= 0.1549$

$$P(35 \le X < 40) = P\left\{ \frac{35-33}{3} < Z < \frac{40-33}{3} \right\}$$
$= P\{0.67 < Z < 2.33\}$
$= P\{0 < Z < 2.33\} - P\{0 < Z < 0.67\}$
$= 0.4901 - 0.2486$
$= 0.2415$

$\therefore$ $P\{(25 < X \le 30))$ or $(35 \le X < 40)\}$
$= P(25 < X \le 30) + P(35 \le X < 40)$,
(since the 2 ranges are mutually exclusive)
$= 0.1549 + 0.2415 = 0.3964$

$P\{X < 25$ or $X > 40\} = 1 - (0.5899 + 0.3964)$
$= 0.0137$

Profit/unit	*Probability*
Rs 100	0.5899
Rs 50	0.3964
Rs –60	0.0137

$\therefore$ E(Profit per unit) = Rs $(100 \times 0.5899 + 50 \times 0.3964 - 60 \times 0.0137)$
= Rs 79, approximately

Let the revised net profit per unit of the first category fuel be k.
Then E(Revised profit per unit)
$= \text{Rs } (k \times 0.5899 + 50 \times 0.3964 - 60 \times 0.0137)$
$= \text{Rs } (0.5899\,k + 18.998)$

E(Revised profit per unit) = Rs 79 + Rs 39.5, as per the manufacturer's wish

$\therefore$ $0.5899\,k + 18.998 = 118.5$

$$\therefore \quad k = \frac{118.5 - 18.998}{0.5899}$$
$= 168.68 \simeq$ Rs 169 nearly.

Example 26 *The marks obtained by the students in Mathematics, Physics and Chemistry in an examination are normally distributed with the means 52, 50 and 48 and with standard deviations 10, 8 and 6 respectively. Find the probability that a student selected at random has secured a total of (i) 180 or above and (ii) 135 or less.*

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14. Find the value of k, mean and variance of the normal distribution whose density function is $k \cdot 2^{-x^2}$ $-\infty < x < \infty$.
15. If X follows $N(30, 5)$ and Y follows $N(15, 10)$ show that $P(26 \le X \le 40) = P(7 \le Y \le 35)$.
16. If X follows $N(3, 2)$, find the value of k such that $P(|X - 3| > k) = 0.05$.
17. If $\log_{10}X$ follows $N(4, 2)$, find $P(1.202 < X < 83180000)$, given that $\log_{10}(1202) = 3.08$ and $\log_{10}(8318) = 3.92$.
18. For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. What are its mean and SD?
19. Show that, for a normal distribution, the quartile deviation, the mean deviation and the standard deviation are in the ratio 10 : 12 : 15.
20. If 2 normal universes A and B have the same total frequency, but the SD of A is k times the SD of B, prove that the maximum frequency of A is $\frac{1}{k}$ times that of B.
21. State the reproductive property of normal distribution.
22. If X and Y are independent RVs having $N(1, 2)$ and $N(2, 2)$ respectively find the density function of $(X + 2Y)$.
23. Why is normal distribution considered an important distribution?

Part B

24. X is uniformly distributed with mean 1 and variance $\frac{4}{3}$. If 3 independent observations of X are made, what is the probability that all of them are negative?
25. A point D is chosen on the line AB whose length is 1 and whose mid-point is C. If the distance X from D to A is a RV having a uniform distribution in (0, 1), what is the probability that AD, BD and AC will form a triangle?
26. A passenger arrives at a bus stop at 10 A.M., knowing that the bus will arrive at some time uniformly distributed between 10 A.M. and 10.30 A.M. What is the probability that he will have to wait longer than 10 min? If at 10.15 A.M. the bus has not yet arrived, what is the probability that he will have to wait at least 10 additional minutes?
27. A man and a woman agree to meet at a certain place between 10 A.M. and 11 A.M. They agree that the one arriving first will wait 15 min for the other to arrive. Assuming that the arrival times are independent and uniformly distributed, find the probability that they meet.
28. The RVs a and b are independently and uniformly distributed in the intervals (0, 6) and (0, 9) respectively. Find the probability that the roots of the exquation $x^2 - ax + b = 0$ are real.
29. If a, b, c are randomly chosen between 0 and 1, find the probability that the quadratic equation $ax^2 + bx + c = 0$ has real roots.

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