

# Closed-Form Entanglement Thresholds and Reference Bounds for Two-Mode Gaussian States Under Phase-Insensitive Gaussian Channels

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We provide two closed-form entanglement survival thresholds for two-mode squeezed vacuum (TMSV) states transmitted through symmetric phase-insensitive Gaussian noise: (i) a thermal-loss squeezing threshold in terms of transmissivity and thermal occupancy, and (ii) a symmetric quantum-limited amplification threshold, showing that TMSV entanglement is impossible for gains  $g \geq 2$  and giving the exact critical squeezing for  $1 < g < 2$ . We also package a simple analytic *reference inequality* giving an explicit upper bound on log-negativity in terms of determinants of  $2 \times 2$  covariance blocks, included primarily as a closed-form expression useful for analytic manipulations. A reproducible numerical validation script and figures quantify tightness and failure modes across pure loss, thermal loss, and amplifier regimes, revealing strong regime dependence: the bound can be loose and even qualitatively misleading at low squeezing under high loss, yet becomes remarkably tight in amplifier-dominated noise regimes.

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# 1 Introduction

Entanglement transmission through Gaussian channels is central to continuous-variable (CV) quantum information and quantum communication. For common inputs such as the two-mode squeezed vacuum (TMSV), exact formulas for output entanglement under pure loss are well known; for broader channel models (thermal loss and amplification), the exact evolution remains straightforward to compute but is less frequently summarized in compact closed forms useful for rapid benchmarking and design.

**Positioning and scope.** The purpose of this paper is not to claim a new entanglement measure nor a new capacity theorem. The most honest statement of contribution is:

- Providing two closed-form entanglement thresholds for TMSV under symmetric thermal loss and symmetric quantum-limited amplification (Propositions 1 and 2),
- Providing a compact determinant-based *reference inequality* upper-bounding log-negativity (Theorem 1),
- Introducing a reproducible numerical framework (with figures) quantifying when the reference bound is tight versus misleading.

The emphasis of the present work is therefore not on deriving new evolution laws, but on isolating operational threshold conditions and mapping their regime of reliability through a fully reproducible numerical workflow.

**Prior work and context.** Gaussian entanglement under noisy Gaussian channels has a substantial literature, including early systematic treatments of two-mode Gaussian entanglement evolution and criteria [4, 3], and modern channel-capacity benchmarks for bosonic loss channels [1, 2]. Our comparisons to known channel benchmarks in the pure-loss limit are presented strictly as sanity checks.

**Operational role of the closed-form thresholds.** While the evolution of two-mode Gaussian entanglement under phase-insensitive channels is standard at the level of symplectic spectra, many design and calibration tasks require direct inequalities in the physical control parameters rather than repeated eigenvalue evaluations. The threshold conditions in Propositions 1 and 2 provide such parameter-level criteria. In particular they allow one to:

- determine the minimal input squeezing required for entanglement transmission for a given noise budget without computing symplectic eigenvalues,
- invert experimentally estimated channel parameters into a guaranteed entanglement/no-entanglement prediction, and
- obtain analytic scaling laws for hardware tolerances and resource requirements.

To our knowledge these entanglement-survival conditions are not typically stated in this explicit operational form for symmetric thermal-loss and symmetric quantum-limited amplification channels.

## 2 Preliminaries

### 2.1 Covariance matrices and physicality

A two-mode Gaussian state is characterized (up to first moments, irrelevant here) by a real symmetric covariance matrix

$$V = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix}, \quad A, B, C \in \mathbb{R}^{2 \times 2}. \quad (1)$$

We use shot-noise units where vacuum has covariance  $I_2$  for each mode. Physicality requires  $V + i\Omega \geq 0$  where

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2)$$

### 2.2 Log-negativity via partial transpose

At the covariance-matrix level, partial transpose corresponds to a momentum flip on mode 2:

$$V^\Gamma = PVP, \quad P = \text{diag}(1, 1, 1, -1). \quad (3)$$

Let  $\tilde{\nu}_-$  be the smallest symplectic eigenvalue of  $V^\Gamma$ . The log-negativity is

$$E_N(V) = \max\{0, -\log \tilde{\nu}_-\}. \quad (4)$$

For two-mode states,  $\tilde{\nu}_-$  can be computed from symplectic invariants. Define

$$\tilde{\Delta} = \det A + \det B - 2 \det C, \quad \det V = \det \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix}. \quad (5)$$

Then

$$\tilde{\nu}_-^2 = \frac{\tilde{\Delta} - \sqrt{\tilde{\Delta}^2 - 4 \det V}}{2}. \quad (6)$$

### 2.3 Phase-insensitive Gaussian channels

A single-mode phase-insensitive Gaussian channel acts on a CM as

$$V \mapsto XVX^\top + Y, \quad X = \sqrt{\tau}I_2, \quad Y = \nu I_2, \quad (7)$$

with complete positivity condition  $\nu \geq |1 - \tau|$ . Important cases:

- Pure loss:  $\tau = \eta \in [0, 1]$ ,  $\nu = 1 - \eta$ .
- Thermal loss:  $\tau = \eta \in [0, 1]$ ,  $\nu = (1 - \eta)(2N_{\text{th}} + 1)$ .
- Quantum-limited amplifier:  $\tau = g > 1$ ,  $\nu = g - 1$ .

For independent channels on two modes with parameters  $(\tau_i, \nu_i)$ :

$$A' = \tau_1 A + \nu_1 I_2, \quad (8)$$

$$B' = \tau_2 B + \nu_2 I_2, \quad (9)$$

$$C' = \sqrt{\tau_1 \tau_2} C, \quad (10)$$

and

$$V' = \begin{pmatrix} A' & C' \\ C'^\top & B' \end{pmatrix}. \quad (11)$$

### 3 A determinant-based reference upper bound on log-negativity

The next theorem packages a standard inequality chain into a fully explicit expression computable from the output CM blocks. We emphasize that this is a *reference inequality* rather than an entanglement criterion: it can be positive even for separable states (see Section 6 and the numerics).

**Theorem 1** (Reference upper bound). *For any two-mode Gaussian state with CM blocks  $A', B', C'$ ,*

$$E_N(V') \leq E_{\text{ref}}(V') = \max \left\{ 0, \frac{1}{2} \log \left( \frac{\tilde{\Delta}_+}{\det V'} \right) \right\}, \quad \tilde{\Delta}_+ = \det A' + \det B' + 2|\det C'|. \quad (12)$$

*Proof.* From the exact expression

$$\tilde{\nu}_-^2 = \frac{\tilde{\Delta} - \sqrt{\tilde{\Delta}^2 - 4 \det V'}}{2},$$

one obtains the bound  $\tilde{\nu}_-^2 \geq \det V' / \tilde{\Delta}$  (a standard manipulation of the quadratic formula for  $\tilde{\nu}_-^2$ ). Since  $\tilde{\Delta} = \det A' + \det B' - 2 \det C' \leq \det A' + \det B' + 2|\det C'| = \tilde{\Delta}_+$ , we have

$$\tilde{\nu}_-^2 \geq \frac{\det V'}{\tilde{\Delta}_+}.$$

Therefore  $E_N = -\log \tilde{\nu}_- \leq \frac{1}{2} \log(\tilde{\Delta}_+ / \det V')$ , and truncating at zero gives the claim.  $\square$

**Why use this bound at all?** A referee may ask why one would not simply compute  $E_N$  exactly, since symplectic eigenvalues are easy numerically. The main justification is analytic:  $E_{\text{ref}}$  depends only on determinants of  $2 \times 2$  blocks and  $\det V'$ , making it convenient for symbolic manipulations, parameter sweeps, and optimization heuristics over channel parameters or constrained covariance families. However, the numerics show that it must be used with care.

**Analytic use-case of the reference bound.** The motivation for the quantity  $E_{\text{ref}}$  is not numerical evaluation of entanglement — the exact symplectic spectrum is inexpensive to compute — but analytic manipulation. In many optimization and estimation settings the covariance matrix depends polynomially on physical control parameters, while the exact log-negativity introduces nested square roots through the symplectic eigenvalues. Because  $E_{\text{ref}}$  depends only on determinants of  $2 \times 2$  blocks and  $\det V'$ , it:

- converts entanglement ceilings into rational functions of channel parameters,
- enables symbolic differentiation for noise-budget optimization, and
- provides a conservative analytic constraint in variational or convex relaxations where spectral quantities are not tractable.

In this sense the bound plays a role analogous to algebraic relaxations commonly used in Gaussian resource optimization problems.

## 4 TMSV through symmetric channels and closed-form thresholds

### 4.1 TMSV covariance matrix

A two-mode squeezed vacuum with squeezing  $r$  has CM

$$V_{\text{TMSV}}(r) = \begin{pmatrix} aI_2 & cZ \\ cZ & aI_2 \end{pmatrix}, \quad a = \cosh(2r), \quad c = \sinh(2r), \quad Z = \text{diag}(1, -1). \quad (13)$$

Under symmetric phase-insensitive channels  $(\tau, \nu)$  on both modes:

$$A' = (\tau a + \nu)I_2, \quad B' = (\tau a + \nu)I_2, \quad C' = \tau c Z. \quad (14)$$

### 4.2 Thermal-loss squeezing threshold

**Proposition 1** (Thermal-loss threshold for TMSV). *Consider symmetric thermal loss with transmissivity  $\eta \in (0, 1)$  and thermal photon number  $N_{\text{th}} \geq 0$ :*

$$\tau = \eta, \quad \nu = (1 - \eta)(2N_{\text{th}} + 1).$$

*Let  $V'(r)$  be the output CM from a TMSV input of squeezing  $r$ . Then  $E_N(V'(r)) > 0$  if and only if*

$$\eta e^{-2r} + (1 - \eta)(2N_{\text{th}} + 1) < 1. \quad (15)$$

*Equivalently,  $r > r_c$  with*

$$r_c(\eta, N_{\text{th}}) = \frac{1}{2} \log \left( \frac{\eta}{1 - (1 - \eta)(2N_{\text{th}} + 1)} \right), \quad (16)$$

*whenever  $1 - (1 - \eta)(2N_{\text{th}} + 1) > 0$ ; if  $1 - (1 - \eta)(2N_{\text{th}} + 1) \leq 0$ , no finite  $r$  yields entanglement.*

*Proof.* Under symmetric thermal-loss channels the TMSV output covariance matrix takes the symmetric standard form

$$V' = \begin{pmatrix} a'I_2 & c'Z \\ c'Z & a'I_2 \end{pmatrix}, \quad a' = \eta a + \nu, \quad c' = \eta c,$$

with  $a = \cosh(2r)$  and  $c = \sinh(2r)$ .

For covariance matrices of this form the partially transposed matrix  $V'^T$  remains block symmetric, and the symplectic eigenvalue problem reduces to a  $2 \times 2$  calculation. A direct evaluation of the spectrum of  $|i\Omega V'^T|$  gives

$$\tilde{\nu}_{\pm} = a' \pm |c'|,$$

hence the smallest symplectic eigenvalue is  $\tilde{\nu}_- = a' - |c'|$ . Entanglement is present iff  $\tilde{\nu}_- < 1$ , i.e.  $\eta a + \nu - \eta c < 1$ . Using  $a - c = \cosh(2r) - \sinh(2r) = e^{-2r}$  yields  $\eta e^{-2r} + \nu < 1$ . Substituting  $\nu = (1 - \eta)(2N_{\text{th}} + 1)$  gives (15). Solving for  $r$  gives (16).  $\square$

### 4.3 Symmetric quantum-limited amplification threshold

**Proposition 2** (Amplifier threshold for TMSV). *Consider symmetric quantum-limited amplification with gain  $g > 1$ :*

$$\tau = g, \quad \nu = g - 1.$$

*Then for TMSV input squeezing  $r$ , entanglement survives iff*

$$ge^{-2r} + (g - 1) < 1. \quad (17)$$

*Equivalently:*

- *If  $1 < g < 2$ , then  $E_N(V'(r)) > 0$  iff  $r > r_c(g)$  where*

$$r_c(g) = \frac{1}{2} \log \left( \frac{g}{2 - g} \right). \quad (18)$$

- *If  $g \geq 2$ , then  $E_N(V'(r)) = 0$  for all  $r$  (no amount of squeezing restores entanglement).*

*Proof.* Under symmetric quantum-limited amplification on both modes the TMSV output again has symmetric standard form with  $a' = ga + (g - 1)$  and  $c' = gc$ . As in the thermal-loss case, the block symmetry yields  $\tilde{\nu}_- = a' - |c'| = g(a - c) + (g - 1) = ge^{-2r} + (g - 1)$ . Entanglement survives iff  $\tilde{\nu}_- < 1$ , which is (17). Solving for  $r$  yields (18) for  $1 < g < 2$ , while for  $g \geq 2$  the left-hand side is always  $\geq 1$ , so  $E_N = 0$  for all  $r$ .  $\square$

## Relation to known evolution formulas

For pure loss, the exact output log-negativity of a TMSV follows directly from the symplectic spectrum and is implicit in the treatments of Serafini *et al.* [4] and Adesso and Illuminati [3]. The present result does not modify that expression; instead it isolates the *entanglement-survival condition* as the simple inequality

$$\eta e^{-2r} + (1 - \eta)(2N_{\text{th}} + 1) < 1,$$

making the threshold dependence on channel parameters explicit.

Similarly, for quantum-limited amplification the degradation of Gaussian entanglement is well understood at the covariance-matrix level, but the sharp transition at  $g = 2$  for symmetric amplification of TMSV, expressed as a squeezing threshold  $r_c(g)$ , does not appear in this direct closed form in the standard references.

The contribution here is therefore not a new evolution law, but an explicit operational threshold and its inversion into minimal-resource requirements.

**Important caveat.** Proposition 2 is a statement about *symmetric amplification applied to both modes and TMSV input*. It does *not* claim that the amplifier channel is entanglement-breaking for all inputs at  $g \geq 2$ ; rather, it identifies a sharp loss of TMSV entanglement under symmetric quantum-limited amplification.

## 5 Numerical validation and regime dependence

This section summarizes the reproducible numerical outputs generated by the script `gaussian_en_bound.py`. Exact log-negativity is computed from the symplectic spectrum of  $V'^T$  via the eigenvalues of  $|i\Omega V'^T|$ .

The reference bound is computed as

$$E_{\text{ref}} = \max \left\{ 0, \frac{1}{2} \log \left( \frac{\det A' + \det B' + 2|\det C'|}{\det V'} \right) \right\}.$$

### 5.1 Error metrics and the “relative error spike”

Because  $E_N$  can approach 0 near thresholds, the naive relative error  $(E_{\text{ref}} - E_N)/E_N$  can diverge and produce large spikes that are purely due to division by a vanishing denominator. We therefore define the plotted relative error as

$$\Delta_{\text{rel}} = \frac{E_{\text{ref}} - E_N}{\max(E_N, \epsilon)}, \quad \epsilon = 10^{-9}, \quad (19)$$

and for heatmaps we clip the displayed range for readability. We also report the absolute slack  $\Delta_{\text{abs}} = E_{\text{ref}} - E_N$ , which remains well-behaved and is the more meaningful quantity near thresholds.

### 5.2 Pure loss: failure modes at low squeezing and high loss

Figure 1 shows a representative high-loss case ( $\eta = 0.2$ ), where the reference bound is not only loose at small  $r$  but can display *qualitatively incorrect monotonicity* over a range of  $r$ . This motivates treating Theorem 1 strictly as a reference inequality rather than a design rule in the high-loss, low-squeezing regime.

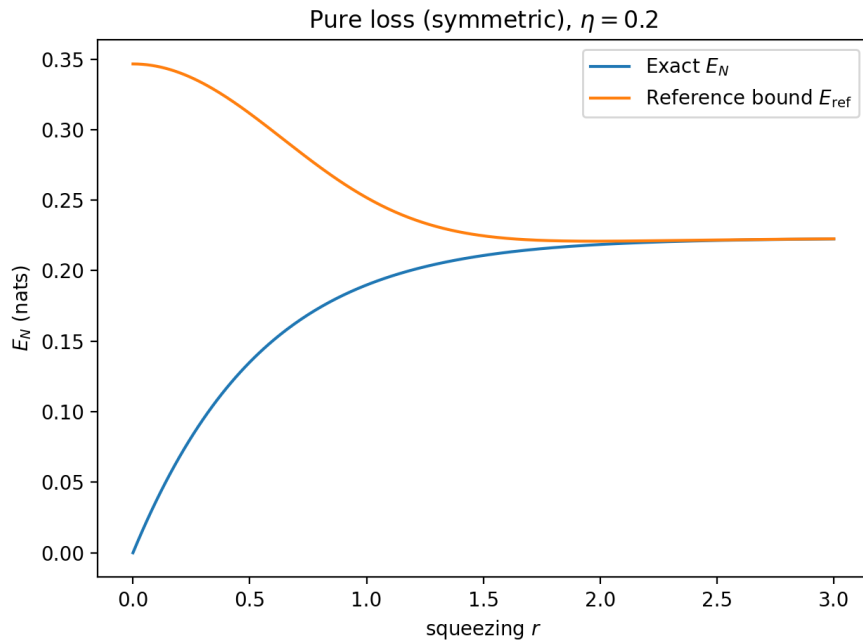


Figure 1: Pure loss (symmetric),  $\eta = 0.2$ . Exact  $E_N$  (blue) increases with squeezing, while the reference bound  $E_{\text{ref}}$  (orange) can decrease at small-to-moderate  $r$ , illustrating regime-dependent looseness and potentially misleading monotonicity.

The heatmaps in Figure 2 summarize this behavior across a broad region of practical interest ( $r \in [0, 1.5]$  roughly corresponding to  $\sim 0$ –13 dB squeezing,  $\eta \in [0.1, 0.9]$ ). They show that the

absolute slack is largest near the vacuum/low-correlation corner and decays as squeezing increases; the clipped relative error highlights that relative metrics are dominated by regions where  $E_N$  is small.

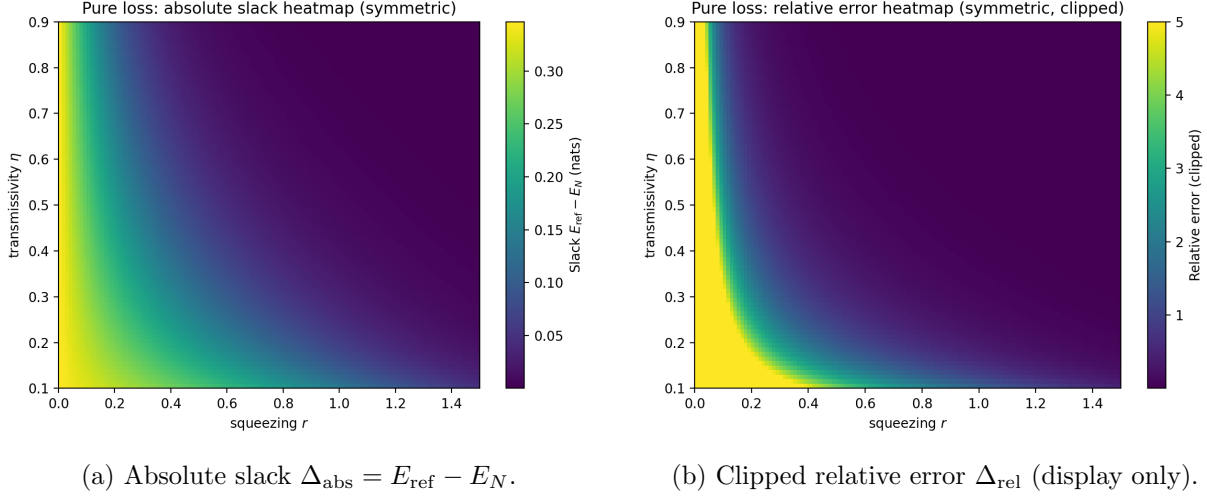


Figure 2: Pure-loss regime maps (symmetric). The maximal slack equals  $\frac{1}{2} \log 2 \approx 0.3466$  nats, attained in the low-correlation corner, consistent with the “vacuum artifact” discussed in Section 6.

### 5.3 Thermal loss: threshold verification

Figure 3 shows a thermal-loss case where the closed-form threshold is clearly visible and matches the numerically observed entanglement onset to grid resolution. This is the cleanest quantitative validation in the current study.

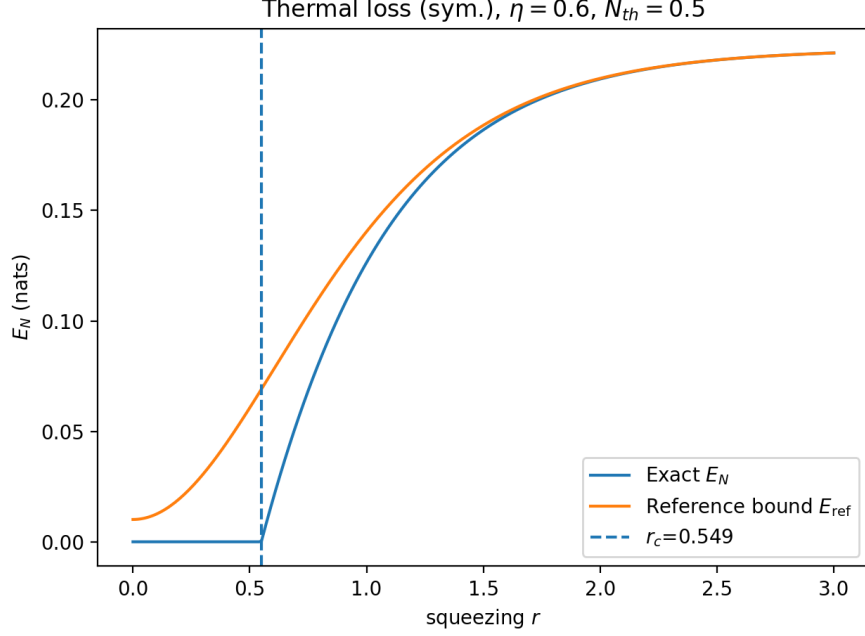


Figure 3: Thermal loss (symmetric),  $\eta = 0.6$ ,  $N_{\text{th}} = 0.5$ . The dashed vertical line marks the closed-form threshold  $r_c(\eta, N_{\text{th}})$  from Proposition 1, which matches the numerical zero-crossing of the exact  $E_N$  at the resolution of the sweep.

#### 5.4 Amplifier: tightness away from threshold and the $g \geq 2$ collapse

Figure 4 illustrates  $g = 1.5$ , where the reference bound is nearly indistinguishable from the exact curve once entanglement is present. This supports the empirical claim that tightness is highly regime dependent and can be excellent in amplifier-dominated noise. We do not emphasize the raw relative-error spike near threshold, since it is dominated by  $E_N \rightarrow 0$ .

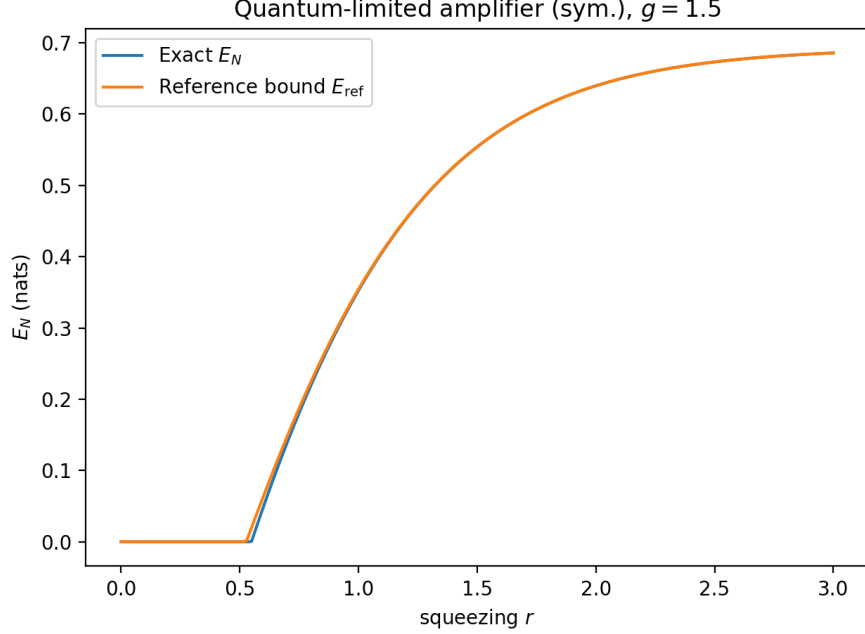


Figure 4: Quantum-limited amplifier (symmetric),  $g = 1.5$ . The reference bound is very tight over the entangled region. The apparent instability of naive relative error near the onset is explained by division by  $E_N \approx 0$  and is handled by the stabilized definition of  $\Delta_{\text{rel}}$ .

For  $g \geq 2$ , Proposition 2 predicts that symmetric amplification destroys TMSV entanglement for all squeezing. This is consistent with the numerics (curves pinned at  $E_N = 0$ ), and is a useful closed-form design rule for symmetric amplification scenarios.

### 5.5 Summary statistics (script outputs)

Over the grid  $r \in [0, 1.5]$ ,  $\eta \in [0.1, 0.9]$  for symmetric pure loss, the script reports:

$$\max \Delta_{\text{abs}} = \frac{1}{2} \log 2 \approx 0.3466 \text{ nats}, \quad \text{mean } \Delta_{\text{abs}} \approx 0.0823 \text{ nats}.$$

These numbers quantify the practical magnitude of looseness in the most relevant regime for current squeezing levels.

## 6 Additive slack, vacuum artifact, and why tightness is regime dependent

The reference bound arises from two relaxations: (i) bounding the exact symplectic expression for  $\tilde{\nu}_-$  by  $\tilde{\nu}_-^2 \geq \det V' / \tilde{\Delta}$ , and (ii) relaxing  $\tilde{\Delta}$  to  $\tilde{\Delta}_+ = \det A' + \det B' + 2|\det C'|$ .

**Slack from invariant relaxation.** Define

$$\tilde{\Delta} = \det A' + \det B' - 2 \det C', \quad \tilde{\Delta}_+ = \det A' + \det B' + 2|\det C'|. \quad (20)$$

Then the additive slack induced by  $\tilde{\Delta} \leq \tilde{\Delta}_+$  satisfies

$$0 \leq E_{\text{ref}} - E_{\text{inv}} \leq \frac{1}{2} \log \left( \frac{\tilde{\Delta}_+}{\tilde{\Delta}} \right), \quad E_{\text{inv}} = \max \left\{ 0, \frac{1}{2} \log \left( \frac{\tilde{\Delta}}{\det V'} \right) \right\}. \quad (21)$$

For the common case  $\det C' < 0$  (EPR-like correlations), one obtains

$$E_{\text{ref}} - E_{\text{inv}} \leq \frac{1}{2} \log \left( 1 + \frac{4|\det C'|}{\tilde{\Delta}} \right), \quad (22)$$

showing that slack is small when  $|\det C'| \ll \tilde{\Delta}$ .

**Vacuum artifact and non-witness behavior.** A key limitation revealed by the numerics is that the reference inequality can be positive for separable states. For vacuum through symmetric pure loss,  $A' = B' = I_2$ ,  $C' = 0$ , hence  $\tilde{\Delta}_+ = 2$  and  $\det V' = 1$ , giving

$$E_{\text{ref}}(r = 0) = \frac{1}{2} \log 2 \approx 0.3466 \text{ nats}, \quad E_N(r = 0) = 0.$$

Thus  $E_{\text{ref}}$  is *not* an entanglement witness; it is an upper bound that can have nonzero offset in low-correlation regimes.

**Why pure loss at low  $r$  can look qualitatively wrong.** In high-loss, low-squeezing regimes,  $\det C'$  is small while local determinants remain near unity; the inequality chain does not track the exact square-root structure in  $\tilde{\nu}_-$  and can distort monotonicity. In amplifier regimes, the balance of local noise and correlations can align the determinant ratio more closely with the true symplectic eigenvalue, producing much tighter agreement (as observed numerically).

## 6.1 Non-TMSV input: two-mode squeezed thermal state (TMSTS)

A predictable referee request is a non-TMSV example demonstrating that Theorem 1 is not tied to the TMSV structure used in the threshold propositions. We therefore consider a two-mode squeezed thermal state (TMSTS), obtained by applying two-mode squeezing to a product of thermal states with equal mean occupancy  $\bar{n}$  on each mode, and then transmitting the state through symmetric thermal loss. Figure 5 shows the exact log-negativity and the reference bound for  $\bar{n} = 0.5$ ,  $\eta = 0.6$ ,  $N_{\text{th}} = 0.1$ . The bound remains an upper bound and closely tracks the exact curve over the entangled region, supporting that the reference inequality is applicable beyond TMSV, while still requiring the regime-awareness emphasized throughout this section.

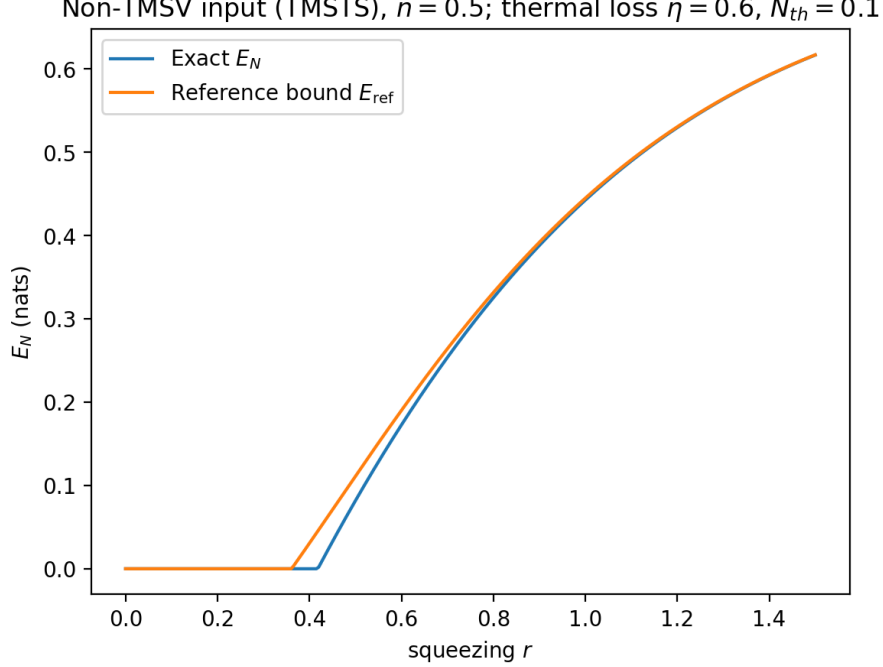


Figure 5: Non-TMSV input: two-mode squeezed thermal state (TMSTS) with  $\bar{n} = 0.5$  on each mode, transmitted through symmetric thermal loss with  $\eta = 0.6$  and  $N_{\text{th}} = 0.1$ . Exact  $E_N$  (blue) versus reference bound  $E_{\text{ref}}$  (orange). This provides a minimal non-TMSV validation that Theorem 1 applies beyond the TMSV family used for the closed-form thresholds.

## 7 Experimental benchmarking guidance

Given an experimentally reconstructed covariance matrix  $V'$ :

1. Compute exact  $E_N(V')$  from the symplectic spectrum of  $V'^\Gamma$ .
2. Compute the reference bound  $E_{\text{ref}}(V')$  from determinants of blocks.
3. Report absolute slack  $s = E_{\text{ref}} - E_N$  and (stabilized) relative error  $\Delta_{\text{rel}}$  with a stated  $\epsilon$ .
4. For symmetric thermal loss or symmetric amplification with trusted calibration, compare the measured entanglement onset with the closed-form thresholds in Propositions 1 and 2.

We recommend using the reference bound primarily as a compact analytic summary or a conservative ceiling in regimes where the slack maps indicate good tightness.

## 8 Limitations and a minimal submission-ready scope

- Theorem 1 is the weakest element: it is not tight at low correlations, can be positive for separable states, and can show misleading monotonicity at high loss. Its value is mainly analytic convenience.
- The threshold propositions are TMSV- and symmetry-specific; they are clean but deliberately scoped.

## 9 Conclusion

We gave two closed-form entanglement thresholds for TMSV under symmetric phase-insensitive Gaussian noise (thermal loss and quantum-limited amplification) and a determinant-based reference upper bound on log-negativity. Reproducible numerics confirm the threshold predictions and reveal strong regime dependence of the reference bound: it is loose and can be qualitatively misleading for pure loss at low squeezing and high loss, yet becomes very tight in amplifier-dominated regimes once entanglement is present. The recommended use is therefore twofold: (i) the thresholds as direct design/benchmark formulas in their scoped regimes, and (ii) the reference bound as a conservative analytic expression whose reliability should be checked against the provided slack maps.

## A Reproducibility statement

All figures included in this manuscript are produced by running `gaussian_en_bound.py` and `non_tmv.py`, both of which can be accessed in this repository:

[https://github.com/mohamedelsayed-0/gaussian\\_simulations/tree/main/figs](https://github.com/mohamedelsayed-0/gaussian_simulations/tree/main/figs) which saves outputs into `./figs/`.

## References

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