# Generative Adversarial Network (GAN)

Generative adversarial networks (GANs) are algorithmic architectures that use two neural networks, pitting one against the other (thus the "adversarial") in order to generate new, synthetic instances of data that can pass for real data. They are used widely in image generation, video generation, text generation and voice generation.

1. **Discriminative network D or (classification)**

   The rule of this network is to discriminate between two different types of data.

2. **Generative network G**

   Generator G is trained on data X sampled from some true distribution called D. which given some standard random distribution Z produces a distribution D' that is close to D according to some closeness metrics like L1 norm and L2 norm as shown in Fig 1.

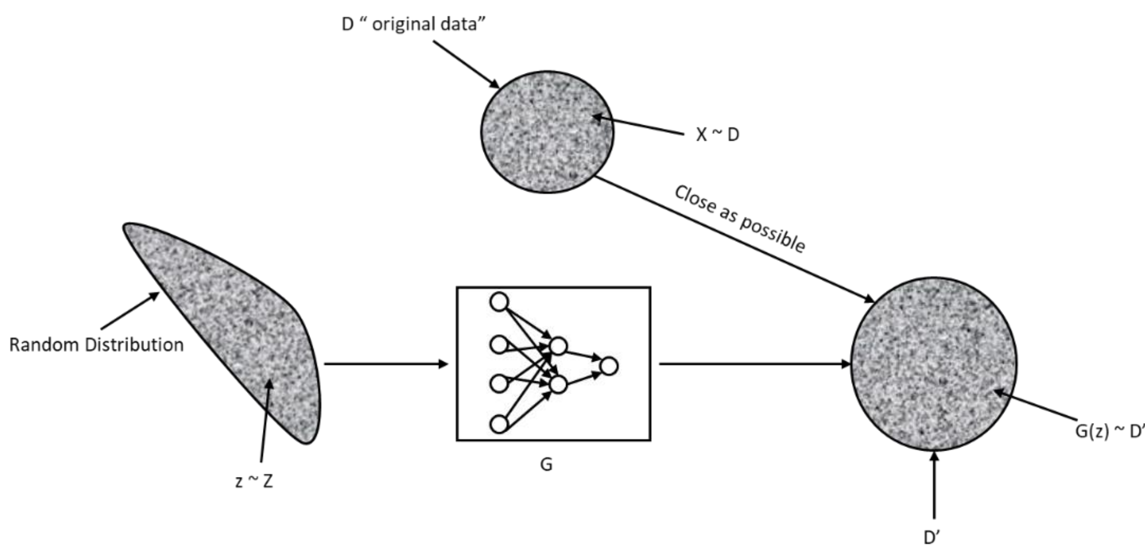


Fig 1.

## Basic Conventions to Understand Loss Function

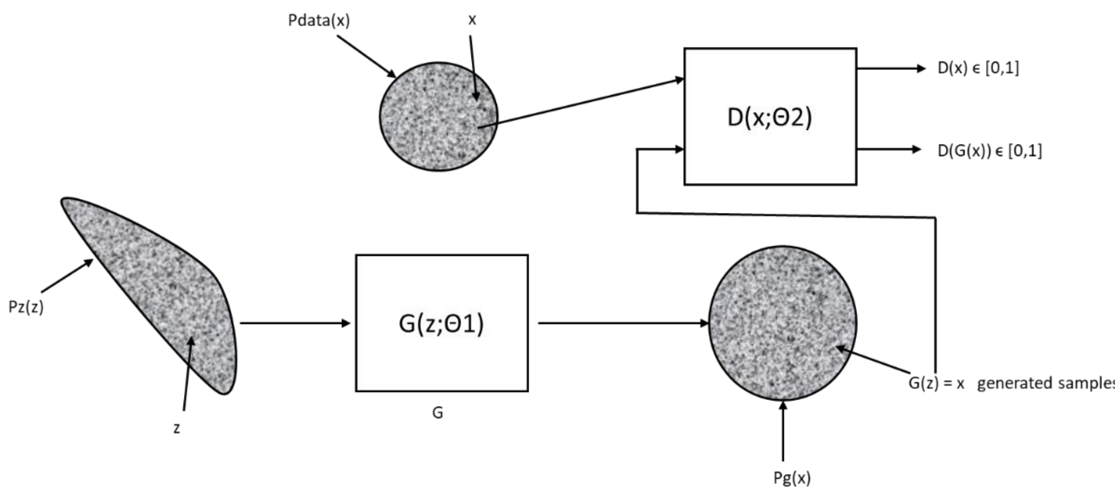


Fig 2

## Loss Function (Binary Cross-Entropy)

$$L(Y`, Y) = Y \log Y` + (1 - Y) \log(1 - Y`) \quad \textbf{... (1)}$$

Where Y` = reconstructed image and Y = original image.

The label coming from $p_{data}(x)$, y = 1 (Real Data) and Y` = D(x) by substituting in Eq 1.

$$L(D(x), 1) = \log D(x) + 0 \quad \textbf{. . . (2)}$$

The label coming from generated data Y = 0 (Fake Data) and Y` = D(G(x)) by substituting in Eq 1.

$$L(D(G(z), 0) = 0 + \log(1 - D(G(z))) \quad \textbf{. . . (3)}$$

Objective of the discriminative is to correctly classify fake vs real data for this 2 and 3 should be maximized.

2. Log (D(x))

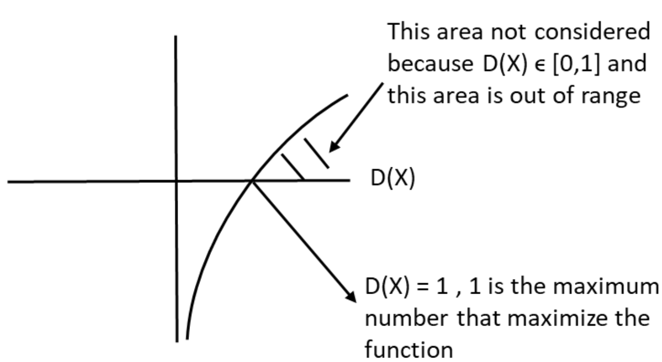


Fig 3

3. Log (1 – D(G(z)))

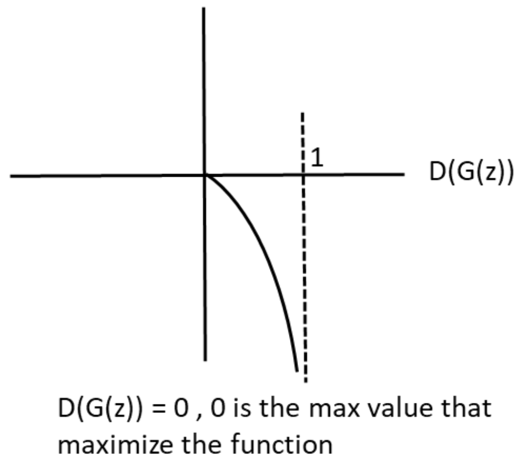


Fig 4

$$\max \log\big(D(x)\big) + \log\Big(1 - D\big(G(z)\big)\Big) = D$$

Objective of the generator G is to foul the discriminator D by generating fake samples looks similar to real data from equation 2 and 3.

If D(G(z)) = 1 means that generator G can foul the discriminator D as shown in Fig 4. In addition, to do that we have to minimize the function

$$\min\{\log\big(D(x)\big) + \log(1 - D\big(G(z)\big))\}$$

According to that the loss function equals to

$$\min G \max D\, V(D, G) = \min G \max D\, \{E_{x\sim P_{data}(x)}\,[\log D(X)] + E_{z\,\sim\,P_z(z)}[\log(1 - D(G(z))]\} \ldots \textbf{(4)}$$

## Finding the best D:

For fixed G, the optimal D formulated as follows:

$$D_G(x) = \frac{P_{data}(x)}{P_{data}(x) +\ P_g(x)}$$

The training criterion for the D, given any generator G is to maximize Eg 4. Hence, the optimal D for any G is denoted as $D_G = \arg\max V(D, G)$

**Note:**

$$E_{P(x)}[x] = \int_x x\, P_x(x) dx \ldots \textbf{(5)}$$

Applying equation 4 and 5.

$$V(D,G) = \int_x P_{data(x)} \log\big(D(x)\big) . dx + \int_z P_{z(z)} \log\big(1 - D(G(z))\big) . dz$$

In the original paper of GAN it is further written as.

$$V(D,G) = \int_x P_{data(x)} \log\big(D(x)\big) . dx + \int_x P_{g(x)} \log(1 - D(x)) . dx$$

To show how this substitution happened first we have to illustrate the concept of change of variable

## *Change of variable

The probability density function of random variable X is given as $P_x(x)$, it is possible to calculate the probability density function of some variable Y = G(x) this is called as change of variable and defined as

$$P_y(y) = P_x\left(G^{-1}(y)\right) \left| \frac{d}{dy} ( G^{-1}(y) ) \right|$$

For further information: [https://www.youtube.com/watch?v=OeD3RJpeb-w](https://www.youtube.com/watch?v=OeD3RJpeb-w)

Now we try to show how these 2 parts are equals

$$\int_z P_{z(z)} \log\big(1 - D(G(z))\big) . dz = \int_x P_{g(x)} \log(1 - D(x)) . dx$$

$$z \sim P_z(z) \rightarrow G \rightarrow (G(z) = x)$$

Assuming that G is invertible so $z = G^{-1}(x)$

$$\int_z P_{z(z)} \log\big(1 - D(G(z))\big) . dz = \int_x P_x(G^{-1}(x)) \log\big(1 - D(x)\big) . dG^{-1}(x)$$

Multiple and divide the equation by dx

$$\int_x \boldsymbol{P_x\left(G^{-1}(x)\right)\log\left(1- D(x)\right).\frac{d}{dx}\, G^{-1}(x)\, dx}$$

According to the rule of change of variable the highlight part can substituted by $P_g(x)$

$$\int_x \boldsymbol{P_g(x)\log\left(1- D(x)\right).dx}$$

So

$$\boldsymbol{V(D,G) = \int_x P_{data(x)}\log\left(D(x)\right).dx + \int_x P_{g(x)}\log(1- D(x)).dx}$$

The optimal D for a given G is obtained by maximize (log (1 – D(x) ))

The optimal G that minimize the loss function occurs when the D = the optimal D so we get the optimal G as

$$\boldsymbol{G = \arg\min(Optimal\ D\ , G)}$$

At this point, we must show that the optimization problem stated in A has unique solution for the optimal G and this solution satisfies when $P_g = P_{data}$

So our optimal D =

$$\boldsymbol{D_G(x) = \frac{P_{data}(x)}{P_{data}(x) + P_g(x)}}$$

By substituting the optimal value of D in

$$\boldsymbol{G = \arg\min(Optimal\ D, G)}$$

$$= \boldsymbol{\arg\min G\left[\int_x P_{data}(x)\log\left(Optimal\ D(x)\right) + P_g(x)\log(1 - Optimal\ D(x))\ .dx\right]}$$

$$= \boldsymbol{\arg\min G\left[\int_x P_{data}(x)\log\left(\frac{P_{data}(x)}{P_{data}(x)+ P_g(x)}\right) + P_g(x)\log(1 - \frac{P_{data}(x)}{P_{data}(x)+ P_g(x)})\ .dx\right]}$$

$$= \boldsymbol{\arg\min G\left[\int_x P_{data}(x)\log\left(\frac{P_{data}(x)}{P_{data}(x)+ P_g(x)}\right) + P_g(x)\log(\frac{P_g(x)}{P_{data}(x)+ P_g(x)})\ .dx\right]}$$

By adding and subtracting (log2) $P_{data}(x)$ and (log2) $p_g(x)$.

$$= \arg\min G \left[ \int_x \begin{matrix} (log2 - log2)P_{data}(x)\log\left(\frac{P_{data}(x)}{P_{data}(x)+P_g(x)}\right) + \\ (log2 - log2)P_g(x)\log(1 - \frac{P_{data}(x)}{P_{data}(x)+P_g(x)})\ .dx \end{matrix} \right]$$

By factoring and take –log2, $P_{data}(x)$, and $P_g(x)$ as common factor.

$$= \arg\min G \left[ \int_x \begin{matrix} -log2\left(P_{data}(x) + P_g(x)\right)\ .dx + P_{data}(x)\left[log2 + \log\left(\frac{P_{data}(x)}{P_{data}(x)+P_g(x)}\right)\right] + \\ P_g(x)\ [\ log2 + \log\left(\frac{P_g(x)}{P_{data}(x)+P_g(x)}\right)]\ \ .dx \end{matrix} \right]$$

By dividing this integral into 3 parts

$$G = \arg\min G \int_x -log2\left(P_{data}(x) + P_g(x)\right)\ .dx \rightarrow -log2(1+1) \rightarrow -log4$$

The integral of the probability density function $p_{data}(x)$ and $p_g(x)$ equal to 1

$$+ \arg\min G \int_x P_{data}(x)\left[log2 + \log(\frac{P_{data}(x)}{P_{data}(x) + P_g(x)})\right] .dx \rightarrow P_{data}(x) + \log(\frac{\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}}{2})$$

$$+ \arg\min G \int_x P_g(x)\left[log2 + \log(\frac{P_g(x)}{P_{data}(x) + P_g(x)})\right] .dx \rightarrow P_g(x) + \log(\frac{\frac{P_g(x)}{P_{data}(x) + P_g(x)}}{2})$$

To simplify this equation we recognize that

$$P_{data}(x) + \log\left(\frac{\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}}{2}\right) = KL\left[P_{data}(x)\ |\ \frac{P_{data}(x) + P_g(x)}{2}\right]$$

And

$$P_g(x) + \log\left(\frac{\frac{P_g(x)}{P_{data}(x) + P_g(x)}}{2}\right) = KL\left[P_g(x)\ |\ \frac{P_{data}(x) + P_g(x)}{2}\right]$$

Where KL is Kullback–Leibler divergence

$$\boldsymbol{G = \mathbf{arg\,min}\, G\ \left[-log4 + KL\left[P_{data}(x) \mid \frac{P_{data}(x) +\ P_g(x)}{2}\right] +\ KL\left[P_g(x) \mid \frac{P_{data}(x) +\ P_g(x)}{2}\right]\right]}$$

To simplify this equation again we have to know the JSD Jensen–Shannon divergence which formulated as

$$\boldsymbol{JSD = \frac{1}{2}\,\{KL(P|M) + KL(Q|M)\}\ where\ M = \ \frac{P+Q}{2}}$$

So

$$\boldsymbol{G = \mathbf{arg\,min}\, G\ [-log4 + 2\ JSD(P_{data}(x) \mid P_g(x)\ )]}$$

JSD becomes zero when $p_{data}(x) = P_g(x)$ which minimize the arguments and the value obtained = **-log4**

The goal of GAN process is to have $p_{data}(x) = P_g(x)$ so by substituting these values for optimal D = **1/2**